THE WEIGHTED WEAK TYPE INEQUALITY FOR THE STRONG MAXIMAL FUNCTION

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Abstract. We prove the natural Fefferman-Stein weak type inequality for the strong maximal function in the plane, under the assumption that the weight satisfies a strong Muckenhoupt condition. This complements the corresponding strong type result due to Jawerth. It also extends the weighted weak type inequality for strong A_1 weights due to Bagby and Kurtz.

Let f be a locally integrable function in $\mathbb{R}^2$. The strong maximal function M is defined by

$$Mf(x) = \sup_R \frac{1}{|R|} \int_R |f|,$$

where $|E|$ denotes the two-dimensional Lebesgue measure of a set $E \subset \mathbb{R}^2$, and the supremum is taken over all rectangles $R \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, such that $x \in R$ (from now on by the term "rectangle" we will always mean a rectangle with sides parallel to the coordinate axes).

By a classical result of Jessen, Marcinkiewicz and Zygmund [7], M is bounded from $L(1 + \log^+ L)$ to weak L^1, that is

$$|\{Mf > \lambda\}| \leq C \int \frac{|f|}{\lambda} \left(1 + \log^+ \frac{|f|}{\lambda}\right), \ \lambda > 0, \tag{1}$$

which implies that M is bounded on every L^p, $p > 1$. The idea of their proof was to dominate M by iterates of the usual one-dimensional Hardy-Littlewood maximal function acting in different directions. A direct geometric proof was given much later by Córdoba and R. Fefferman [2]. The difficulty in a direct approach is that the Besicovitch covering lemma fails

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when applied to a family of rectangles having arbitrary eccentricities. The main contribution of [2] was exactly the discovery of a suitable substitute for the Besicovitch covering lemma.

As far as weighted inequalities are concerned, it is known that if w is a strong A_p weight, $p > 1$, that is, if there exists a constant $C > 0$ such that for all rectangles R we have

$$\left(\frac{1}{|R|}\int_R w\right)\left(\frac{1}{|R|}\int_R w^{-1/(p-1)}\right)^{p-1} \leq C,$$

then M is bounded on $L^p(w)$, namely

$$\int (Mf)^p w \leq C_p \int |f|^p w.$$

This, again, follows by an appeal to the one-dimensional theory. A different proof of a more general result may be found in Jawerth [5]. The endpoint case ($p = 1$) has been treated by Bagby and Kurtz [1]. They proved that if w is a strong A_1 weight, namely, if there exists a constant $C > 0$ such that for all rectangles R we have

$$\frac{1}{|R|}\int_R w \leq C \cdot \operatorname*{essinf}_{x\in R} w(x),$$

then

$$\text{(2)} \qquad \int_{\{Mf>\lambda\}} w \leq C' \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right) w, \ \ \lambda > 0.$$

The results above suggest an analogy between weighted inequalities for the strong maximal function and weighted inequalities for the usual Hardy-Littlewood maximal function. However, this analogy cannot be pushed too far unless we put some restrictions on the weight. For example, if we consider the "weighted" version of M, i.e.

$$M_w f(x) = \sup_{\substack{R \text{ rectangle} \\ x\in R}} \frac{1}{w(R)}\int_R |f| w, \text{ where } w(R) = \int_R w,$$

then R. Fefferman [4], using the idea of [2], has shown that if w belongs to a fixed strong A_r class, $r > 1$, then M_w is bounded on $L^p(w)$ for all $p > 1$ (see Jawerth and Torchinsky [6] for the endpoint). Note that if M is the

Hardy-Littlewood maximal function then M_w is bounded on every $L^p(w)$, $p > 1$, without any restriction on w. So, in this case, the analogy breaks down.

Under the same assumption on the weight as in [4], Jawerth [5] proved, by different methods, that M is bounded from $L^p(Mw)$ to $L^p(w)$, for all $p > 1$, i.e.

$$\int (Mf)^p w \le C_p \int |f|^p Mw. \tag{3}$$

As before, if M is the usual Hardy-Littlewood maximal function then (3) holds true for arbitrary w. This is due to C. Fefferman and Stein [3], and actually, (3) may be thought of as the "prototype" weighted maximal inequality.

The purpose of this paper is to prove the endpoint case ($p = 1$) of (3) which, as expected, turns out to be the weighted version of (1). Namely, we shall show the following.

Theorem. *Let w be a strong A_r weight for some fixed $r > 1$. Then*

$$w(\{Mf > \lambda\}) \le C \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right) Mw, \ \lambda > 0. \tag{4}$$

Proof. As usual $a \lesssim b$ means $a \le Cb$ for some constant $C > 0$ not necessarily the same each time it occurs.

Let M_d be the dyadic strong maximal function

$$M_d f(x) = \sup_R \frac{1}{|R|} \int_R |f|,$$

where the supremum is taken over all dyadic rectangles R (cartesian products of dyadic intervals) with $x \in R$. First, we shall prove the corresponding weak type estimate for M_d:

$$w(\{M_d f > \lambda\}) \lesssim \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right) M_d w, \ \lambda > 0. \tag{5}$$

So, pick a point x in $\{M_d f > \lambda\}$. Then there exists a dyadic rectangle R_x containing x such that

$$\int_{R_x} |f| > \lambda |R_x|.$$

Without loss of generality we may assume that $\{R_x\}_x$ is a finite family $\{R_i\}_{i=1}^L$. Now, fix a number $0 < \varepsilon_0 < 1$ to be determined later. By the Córdoba - R. Fefferman covering lemma [2], there exists a subfamily $\{R_i^*\}_{i=1}^M \subset \{R_i\}_{i=1}^L$ such that

$$|R_i^* \cap \bigcup_{j<i} R_j^*| \le \varepsilon_0 |R_i^*|, \ i = 1, \dots, M, \tag{6}$$

and

$$\bigcup_{i=1}^L R_i \subset \{M\chi_{\bigcup_{i=1}^M R_i^*} \ge \varepsilon_0\}. \tag{7}$$

Since w is a strong A_r weight, M is bounded from $L^r(w)$ to $L^r(w)$. So, (7) implies that

$$w(\bigcup_{i=1}^L R_i) \lesssim w(\bigcup_{i=1}^M R_i^*),$$

where the implicit constant depends on ε_0, r and the A_r-constant of w. Now, writing

$$\widetilde{R}_1 = R_M^*, \widetilde{R}_2 = R_{M-1}^*, \dots, \widetilde{R}_M = R_1^*$$

and applying the Córdoba - R. Fefferman covering lemma to $\{\widetilde{R}_i\}_{i=1}^M$ we get a subfamily $\{R_i^{**}\}_{i=1}^N \subset \{R_i^*\}_{i=1}^M$ such that

$$|R_i^{**} \cap \bigcup_{j\neq i} R_j^{**}| \le \varepsilon_0 |R_i^{**}|, \ i = 1, \dots, N, \tag{8}$$

and

$$\bigcup_{i=1}^M R_i^* \subset \{M\chi_{\bigcup_{i=1}^N R_i^{**}} \ge \varepsilon_0\}. \tag{9}$$

As before, (9) implies that

$$w(\bigcup_{i=1}^M R_i^*) \lesssim w(\bigcup_{i=1}^N R_i^{**}).$$

Therefore

$$w(\{M_d f > \lambda\}) \le w(\bigcup_{i=1}^L R_i) \lesssim w(\bigcup_{i=1}^N R_i^{**}). \tag{10}$$

Now, let μ and μ_w be the multiplicity and the "weighted" multiplicity functions, respectively, associated to the family $\{R_i^{**}\}_{i=1}^N$, i.e.

$$\mu(x) = \sum_{i=1}^N \chi_{R_i^{**}}(x), \quad \mu_w(x) = \sum_{i=1}^N \frac{w(R_i^{**})}{|R_i^{**}|}\chi_{R_i^{**}}(x),$$

and fix a number $0 < \delta_0 < 1$ to be chosen after ε_0. Then

$$\begin{aligned} w(\bigcup_{i=1}^N R_i^{**}) \le \sum_{i=1}^N w(R_i^{**}) &\le \delta_0 \sum_{i=1}^N \frac{w(R_i^{**})}{|R_i^{**}|}\int_{R_i^{**}} \frac{|f|}{\delta_0\lambda} \\ &= \delta_0 \int \mu_w (M_d w)^{-1} \frac{|f|}{\delta_0 \lambda} M_d w. \end{aligned}$$

Using the elementary inequality

$$st \le e^s + t(1+\log^+ t), \ \ s,t \ge 0$$

we get

$$\begin{aligned} w(\bigcup_{i=1}^N R_i^{**}) \le \delta_0 & \int_{\bigcup_{i=1}^N R_i^{**}} \exp(\mu_w (M_d w)^{-1}) M_d w \\ & + \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\delta_0\lambda}\right) M_d w \\ & \le \delta_0 \int_{\bigcup_{i=1}^N R_i^{**}} \exp(\mu_w (M_d w)^{-1}) M_d w \\ & + (1-\log\delta_0) \int \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right) M_d w. \end{aligned} \tag{11}$$

Now, let

$$Q = \int_{\bigcup_{i=1}^N R_i^{**}} \exp(\mu_w (M_d w)^{-1}) M_d w.$$

We claim that if we choose ε_0 small enough then

$$Q \lesssim w(\bigcup_{i=1}^N R_i^{**}). \tag{12}$$

To see this, we expand the exponential in a Taylor series. Then

$$Q = \sum_{k=0}^\infty \frac{1}{k!}\int \mu_w^k (M_d w)^{1-k} = \sum_{k=0}^\infty \frac{1}{k!}\int \mu_w \mu_w^{k-1} (M_d w)^{1-k}.$$

Since

$$\frac{w(R_i^{**})}{|R_i^{**}|}\chi_{R_i^{**}} \le M_d w,$$

we have

$$\begin{aligned} Q &\le \sum_{k=0}^{\infty} \frac{1}{k!} \int \mu_w \left(\sum_{i=1}^{N} \chi_{R_i^{**}} M_d w \right)^{k-1} (M_d w)^{1-k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int \mu_w \mu^{k-1} = \sum_{k=0}^{\infty} \frac{1}{k!} Q_k. \end{aligned}$$

To estimate Q_k we introduce the following notation: For $I \subset \{1, \dots, N\}$ we put

$$A_I = \bigcap_{i\in I} R_i^{**} \setminus \bigcup_{i\notin I} R_i^{**}.$$

Then the family $\{A_I : I \subset \{1, \dots, N\}\}$ is disjoint and moreover, for all i, n with $1 \le i, n \le N$ we have

$$R_i^{**} \cap \{\mu = n\} = \bigcup_{\substack{I\subset\{1,\dots,N\} \\ |I|=n-1 \\ i\notin I}} A_{\{i\}\cup I}. \tag{13}$$

So

$$Q_k = \sum_{n=1}^{N} \sum_{\substack{I\subset\{1,\dots,N\} \\ |I|=n}} \int_{A_I} \mu_w \mu^{k-1}.$$

Note that if $|I| = n$, then on A_I we have

$$\mu = n \quad \text{and} \quad \mu_w = \sum_{i\in I} \frac{w(R_i^{**})}{|R_i^{**}|}.$$

Therefore

$$Q_k = \sum_{n=1}^{N} n^{k-1} \sum_{\substack{I\subset\{1,\dots,N\} \\ |I|=n}} \sum_{i\in I} \frac{w(R_i^{**})}{|R_i^{**}|} |A_I|.$$

Rearranging the terms and then using (13) we get

$$\begin{aligned}
Q_k &= \sum_{n=1}^{N} n^{k-1} \sum_{i=1}^{N} \frac{w(R_i^{**})}{|R_i^{**}|} \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=n-1\\ i\notin I}} |A_{\{i\}\cup I}| \\
&= \sum_{n=1}^{N} n^{k-1} \sum_{i=1}^{N} \frac{w(R_i^{**})}{|R_i^{**}|} |R_i^{**} \cap \{\mu = n\}|.
\end{aligned}$$

Since the rectangles R_i^{**} satisfy (8), the argument in [2, p. 100] (this is the only point where we use the fact that the rectangles are two-dimensional and dyadic) shows that

$$|R_i^{**} \cap \{\mu = n\}| \lesssim \varepsilon_0^n |R_i^{**}|,$$

where the implicit constant depends on ε_0 (it is, actually, equal to $(\varepsilon_0(1-\varepsilon_0))^{-1}$). Consequently

$$Q_k \lesssim \sum_{n=1}^{N} n^{k-1} \varepsilon_0^n \sum_{i=1}^{N} w(R_i^{**}).$$

Now

$$\begin{aligned}
\sum_{i=1}^{N} w(R_i^{**}) &= \sum_{i=1}^{N} w(R_i^{**} \cap \bigcup_{j<i} R_j^{**}) + \sum_{i=1}^{N} w(R_i^{**} \setminus \bigcup_{j<i} R_j^{**}) \\
&= \sum_{i=1}^{N} w(R_i^{**} \cap \bigcup_{j<i} R_j^{**}) + w(\bigcup_{i=1}^{N} R_i^{**}).
\end{aligned}$$

Since w is a strong A_r weight, there exist constants $c_0 > 0$, $\eta_0 > 0$ such that for every rectangle R and every $E \subset R$ we have

$$\frac{w(E)}{w(R)} \le c_0 \left(\frac{|E|}{|R|} \right)^{\eta_0}.$$

In particular (6) implies

$$\frac{w(R_i^{**} \cap \bigcup_{j<i} R_j^{**})}{w(R_i^{**})} \le c_0 \left(\frac{|R_i^{**} \cap \bigcup_{j<i} R_j^{**}|}{|R_i^{**}|} \right)^{\eta_0} \le c_0 \varepsilon_0^{\eta_0}.$$

Therefore

$$\sum_{i=1}^{N} w(R_i^{**}) \le c_0 \varepsilon_0^{\eta_0} \sum_{i=1}^{N} w(R_i^{**}) + w(\bigcup_{i=1}^{N} R_i^{**}).$$

So, if ε_0 has been chosen small enough we have

$$\sum_{i=1}^{N} w(R_i^{**}) \lesssim w(\bigcup_{i=1}^{N} R_i^{**}).$$

This implies that

$$Q \lesssim w(\bigcup_{i=1}^{N} R_i^{**}) \sum_{n,k=0}^{\infty} \frac{\varepsilon_0^n n^k}{k!} \lesssim w(\bigcup_{i=1}^{N} R_i^{**}),$$

which proves the claim, for appropriately small ε_0.

Combining (11) and (12) we obtain

$$(1-\delta_0 C_{\varepsilon_0})w(\bigcup_{i=1}^{N} R_i^{**}) \leq (1-\log\delta_0) \int \frac{|f|}{\lambda}\left(1+\log^+\frac{|f|}{\lambda}\right) M_d w.$$

Choosing δ_0 small enough and then using (10) we get (5).

We now show that (5) holds with M_d replaced with M. Indeed if $x \in \{Mf > \lambda\}$ then there is a rectangle R containing x such that

$$\lambda < \frac{1}{|R|}\int_R |f|.$$

Notice that there exist four dyadic rectangles R_1', R_2', R_3', R_4' with measure comparable to the measure of R so that R is contained in their union. Then

$$\lambda < \sum_{k=1}^{4} \frac{|R_k'|}{|R|}\frac{1}{|R_k'|}\int_{R_k'} |f|,$$

which implies that for some k we have

$$\lambda \lesssim \frac{1}{|R_k'|}\int_{R_k'} |f|.$$

Therefore

$$R_k' \subset \{M_d f \gtrsim \lambda\}.$$

Hence

$$|R| \lesssim |R \cap \{M_d f \gtrsim \lambda\}|.$$

Consequently

$$M\chi_{\{M_d f \gtrsim \lambda\}}(x) \gtrsim 1.$$

We conclude that

$$\{Mf > \lambda\} \subset \{M\chi_{\{M_d f \gtrsim \lambda\}}(x) \gtrsim 1\}.$$

Since M is bounded on $L^r(w)$ we get that

$$w(\{Mf > \lambda\}) \lesssim w(\{M_d f \gtrsim \lambda\}),$$

which completes the proof. $\square$

Note that, by interpolation, (4) implies (3). Moreover, it implies (2) since a strong A_1 weight is a strong A_r weight, for every $r > 1$, and also satisfies $Mw \leq w$ almost everywhere. So, our result extends the corresponding results in [1] and [5].

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