EMBEDDING B_∞ INTO MUCKENHOUPT CLASSES

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ABSTRACT. What is the smallest p for which a weight in the Reverse Hölder class B_∞ also belongs to the Muckenhoupt class A_p? We give an asymptotically sharp answer to this question.

1. INTRODUCTION

Let w be a weight, that is, a positive locally integrable function on $\mathbb{R}^n$. We say that w belongs to the B_∞ class if there exists a constant $C > 0$ such that for all cubes Q (here and in the rest of the paper, by the term cube we will always mean a cube with sides parallel to the coordinate axes) we have

$$w(x) \leq C \frac{1}{|Q|} \int_Q w, \text{ for almost all } x \in Q,$$

where $|\cdot|$ denotes Lebesgue measure. The smallest such C is called the B_∞ constant of w and is denoted by $B_\infty(w)$. Clearly, $B_\infty(w) \geq 1$.

This class was first defined by Franchi [3], and then, extensively studied by Cruz-Uribe and Neugebauer [2]. Earlier the B_∞ condition appeared in the papers by Andersen and Young [1] and Muckenhoupt [5].

Every B_∞ weight satisfies a reverse Hölder inequality (of arbitrary exponent) and consequently belongs to a Muckenhoupt class A_p, for a certain p, $1 < p < \infty$. Recall that A_p consists of those weights w for which there exists $C > 0$ such that for all cubes Q

$$\left(\frac{1}{|Q|} \int_Q w\right)\left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}\right)^{p-1} \leq C.$$

It is therefore of interest to try to determine the smallest possible p for which a B_∞ weight belongs to A_p. "Smallest" should be understood in the asymptotic sense, that is, as the B_∞ constant of w approaches 1.

The purpose of this paper is to prove the following asymptotically sharp result.

Theorem. *There exist positive constants $C_n, \widetilde{C}_n$, with $\widetilde{C}_n > 1$, such that for every $w \in B_\infty$ with $B_\infty(w) \leq \widetilde{C}_n$ we have*

$$w \in A_p, \text{ for all } p > 1 + C_n \log B_\infty(w).$$

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The conclusion of the Theorem conforms with our intuition. As $B_\infty(w)$ approaches 1, w tends to be a constant function, and therefore, one expects it to tend to belong to every A_p class.

To see why the Theorem is asymptotically sharp as well, consider the power weight

$$w_\rho(x_1,\dots,x_n) = |x_1|^\rho,\ \rho > 0.$$

A calculation shows that

$$\log B_\infty(w_\rho) \sim \rho,\ \text{for } \rho \leq 1.$$

Now, in order for w_ρ to belong to A_p we need $\rho < p-1$, which means

$$p > 1 + C\log B_\infty(w_\rho).$$

2. Notation, terminology and preliminaries

Throughout the paper the capital letter C and its variants $C_1, C_2, \dots$ denote various positive constants not necessarily the same at each of their occurrences. C_n denotes constants depending on the dimension n, again not necessarily the same each time they occur.

If dx is Lebesgue measure and w is a weight, then dw is the measure defined by $dw = wdx$. In that case, we write $w(A)$ instead of $dw(A)$ for any measurable set A.

We say that a weight w is doubling if there exists $C > 0$ such that for every pair of concentric cubes $Q \subset \widetilde{Q}$ with $|\widetilde{Q}| = 2^n|Q|$ we have

$$w(\widetilde{Q}) \leq Cw(Q).$$

The smallest such C is called the doubling constant of w, denoted by D_w. Every B_∞ weight is doubling.

Let w be a weight and f a dw-locally integrable function. f is said to be in the A_1^{dw} class if there exists $C > 0$ such that for all cubes Q

$$\frac{1}{w(Q)}\int_Q f\,dw \leq Cf(x),\ \text{for } dw\text{-almost all } x \in Q.$$

As before, the smallest C for which the above inequality holds is denoted by $A_1^{dw}(f)$.

Finally, we say that f is in $BMO(dw)$, if

$$\|f\|_{BMO(dw)} := \sup_Q \frac{1}{w(Q)}\int_Q |f - f_Q|\,dw < \infty,$$

where

$$f_Q = \frac{1}{w(Q)}\int_Q f\,dw$$

is the average of f on Q with respect to dw. The symbol dw in the notation introduced above will be suppressed if $w = 1$.

A quantitative form of the relation between $BMO(dw)$ and A_1^{dw} is given by the following result (see [4]).

Theorem 2.1. *Let w be a doubling weight, and $f \in A_1^{dw}$. Then* $\log f \in BMO(dw)$ *and*

$$\|\log f\|_{BMO(dw)} \le 2\log A_1^{dw}(f).$$

3. Proof of the Theorem

Note to begin with, that since $w \in B_\infty$, it is clear that $w^{-1} \in A_1^{dw}$ with $B_\infty(w) = A_1^{dw}(w^{-1})$. Consequently, by Theorem 2.1 we have

$$\|\log w\|_{BMO(dw)} \le C\log A_1^{dw}(w^{-1}) = C\log B_\infty(w). \tag{1}$$

To estimate the unweighted BMO norm of $\log w$, it is enough to choose, for every cube Q, a number c_Q and estimate the quantity

$$\sup_Q \frac{1}{|Q|}\int_Q |\log w - c_Q|.$$

So, let

$$c_Q = \frac{1}{w(Q)}\int_Q \log w dw,$$

and

$$I_Q = \frac{1}{|Q|}\int_Q |\log w - c_Q| = \frac{w(Q)}{|Q|}\left(\frac{1}{w(Q)}\int_Q |\log w - c_Q| w^{-1} dw\right). \tag{2}$$

Notice that the A_1^{dw} condition for w^{-1} implies that the Hardy-Littlewood maximal function

$$M_{dw}^Q w^{-1}(x) = \sup_{x\in Q'\subset Q} \frac{|Q'|}{w(Q')}$$

is in $L^1(Q; dw)$ and in fact

$$\|M_{dw}^Q w^{-1}\|_{L^1(Q;dw)} \le A_1^{dw}(w^{-1})|Q| = B_\infty(w)|Q|.$$

So, by a well known result of Stein [6] (in the context of doubling weights), w^{-1} is in the $L\log L(Q; dw)$ class, and moreover

$$\|w^{-1}\|_{L\log L(Q;dw)} \le CD_w^2 \|M_{dw}^Q w^{-1}\|_{L^1(Q;dw)} \le CD_w^2 B_\infty(w)|Q|, \tag{3}$$

with C independent of Q.

On the other hand, since $\log w \in BMO(dw)$, the John-Nirenberg inequality (again, in the doubling context), implies that $\log w - c_Q$ is in the $\exp L\,(Q; dw/w(Q))$ class and

$$\|\log w - c_Q\|_{\exp L(Q;dw/w(Q))} \le CD_w^2 \|\log w\|_{BMO(dw)}, \tag{4}$$

where C is independent of Q.

Applying the generalized Hölder inequality to (2) and then using (3), (4) and (1) we obtain

$$\begin{aligned} I_Q &\le C\frac{w(Q)}{|Q|}\|\log w - c_Q\|_{\exp L(Q;dw/w(Q))}\left(\frac{1}{w(Q)}\|w^{-1}\|_{L\log L(Q;dw)}\right) \\ &\le CD_w^4 B_\infty(w)\|\log w\|_{BMO(dw)} \\ &\le CD_w^4 B_\infty(w)\log B_\infty(w). \end{aligned}$$

Therefore

$$\|\log w\|_{BMO} \le CD_w^4 B_\infty(w)\log B_\infty(w). \tag{5}$$

Now, we have to estimate the doubling constant of w in terms of its B_∞ constant. So, let $Q \subset \widetilde{Q}$ be a pair of concentric cubes with $|\widetilde{Q}| = 2^n|Q|$. Put

$$\lambda = \frac{2B_\infty(w)}{2B_\infty(w)-1}$$

and let $\{Q_j\}_{j=0}^{N+1}$ be a finite sequence of cubes such that

$$Q = Q_0 \subset Q_1 \subset \cdots \subset Q_N \subset \widetilde{Q} \subset Q_{N+1},$$

and

$$|Q_{j+1}| = \lambda|Q_j|, \ \ j = 0, 1, \dots, N.$$

Then

$$N \sim \frac{n\log 2}{\log\lambda}.$$

By the B_∞ condition we have

$$w(x) \le B_\infty(w)\frac{w(Q_{j+1})}{|Q_{j+1}|}, \text{ a.e. on } Q_{j+1}.$$

Integrating over $Q_{j+1}\setminus Q_j$ we obtain

$$w(Q_{j+1}\setminus Q_j) \le B_\infty(w)w(Q_{j+1})\frac{|Q_{j+1}\setminus Q_j|}{|Q_{j+1}|}.$$

Equivalently,

$$w(Q_j) \ge (1 - B_\infty(w)(1-1/\lambda))w(Q_{j+1}) = \frac{1}{2}w(Q_{j+1}).$$

Therefore

$$w(\widetilde{Q}) \le w(Q_{N+1}) \le 2^{N+1}w(Q) \le C_n^{B_\infty(w)}w(Q).$$

We conclude that

$$D_w \le C_n^{B_\infty(w)}. \tag{6}$$

Since $B_\infty(w) \le \widetilde{C}_n$, (5) and (6) imply

$$\|\log w\|_{BMO} \le C_n \log B_\infty(w). \tag{7}$$

To finish the proof of the theorem notice that by the John-Nirenberg inequality there exist $C > 0$, $C_n > 0$ such that if

$$0 < \lambda < \frac{C_n}{\|\log w\|_{BMO}}$$

then

$$\frac{1}{|Q|}\int_Q e^{\lambda|\log w-(\log w)_Q|} \leq C, \tag{8}$$

for all cubes Q. Note that (7) implies

$$\frac{C_n}{\|\log w\|_{BMO}} > 1,$$

provided that $\widetilde{C_n}$ has been chosen close enough to 1.

Now, for any $p > 1 + C_n \log B_\infty(w)$, choose λ with

$$1 < \lambda < \frac{C_n}{\|\log w\|_{BMO}}, \text{ and } 1 + \frac{1}{\lambda} < p.$$

Then (8) implies

$$e^{\pm\lambda(\log w)_Q}\frac{1}{|Q|}\int_Q e^{\mp\lambda\log w} \leq C.$$

Multiplying the $\pm$ estimates we get

$$\left(\frac{1}{|Q|}\int_Q w^\lambda\right)\left(\frac{1}{|Q|}\int_Q w^{-\lambda}\right) \leq C^2,$$

and by Hölder's inequality

$$\left(\frac{1}{|Q|}\int_Q w\right)^\lambda\left(\frac{1}{|Q|}\int_Q w^{-\lambda}\right) \leq C^2,$$

which means

$$\left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\lambda}\right)^{\frac{1}{\lambda}} \leq C^{\frac{2}{\lambda}}.$$

This shows that $w \in A_{1+\frac{1}{\lambda}}$, and hence $w \in A_p$.

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