NORM ESTIMATES FOR THE KAKEYA MAXIMAL FUNCTION WITH RESPECT TO GENERAL MEASURES

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Abstract. We generalize Bourgain's theorem on the Kakeya maximal function in the plane by proving norm estimates with respect to measures satisfying certain conditions. We use this to extend the classical result of Davies on the Hausdorff dimension of Kakeya sets in the plane.

1. Introduction

Let S^1 be the unit circle in the plane. If $0 < \delta \ll 1$, $e \in S^1$, $x \in \mathbb{R}^2$, then we define $T_e^\delta(x)$ to be the rectangle of dimensions $1 \times \delta$, centered at x such that its side with length 1 is in the e-direction. The Kakeya maximal function $\mathcal{K}_\delta : S^1 \to \mathbb{R}$ is defined for all locally integrable functions $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathcal{K}_\delta f(e) = \sup_{x\in\mathbb{R}^2} \frac{1}{|T_e^\delta(x)|} \int_{T_e^\delta(x)} |f(y)|dy.$$

$\mathcal{K}_\delta$ was introduced by Bourgain [1]. It is one of several similar maximal functions which have been studied by several authors going back at least to Córdoba [2]. Bourgain proved, using the Fourier transform, that $\mathcal{K}_\delta$ defines a bounded $L^2(\mathbb{R}^2) \to L^2(d\sigma)$ operator, where $d\sigma$ denotes arc length measure on S^1. Namely, there exists a constant $C > 0$, independent of δ, such that

$$||\mathcal{K}_\delta||_{L^2(d\sigma)} \leq C|\log\delta|^{1/2}||f||_{L^2(\mathbb{R}^2)}.$$

Interpolating with $||\mathcal{K}_\delta||_\infty \leq ||f||_\infty$ we get

$$||\mathcal{K}_\delta||_{L^p(d\sigma)} \leq C_p|\log\delta|^{1/p}||f||_{L^p(\mathbb{R}^2)}, \quad p \geq 2.$$

This estimate is sharp as can be seen, for example, by the Perron-tree construction due to Schoenberg [4].

In this paper we consider the problem of obtaining non trivial $L^p(\mathbb{R}^2) \to L^p(d\mu)$ estimates for $\mathcal{K}_\delta$, where μ is a measure supported on S^1. In particular, we study the influence of the geometric properties of μ on the operator

2000 *Mathematics Subject Classification.* Primary: 42B25. Secondary: 28A78.

This research has been supported by a Marie Curie Fellowship of the European Community programme "Improving Human potential and the Socio-economic Knowledge Base" under contract number HPMFCT-2000-00442.

norm of $\mathcal{K}_\delta$. It is clear that little can be said if μ is completely arbitrary. So, in order to get a meaningful problem, we have to impose certain restrictions on μ. It turns out that if μ satisfies the growth condition $\mu(B(e,r)) \leq \varphi(r)$, $e \in S^1$, $r > 0$, for some positive function φ, then $\mathcal{K}_\delta$ defines for $p \geq 2$ an $L^p(\mathbb{R}^2) \to L^p(d\mu)$ operator whose norm is bounded by a concrete function of δ. If we further assume that μ is Ahlfors regular, a notion to be defined in the next section, then our estimates are sharp. Fianlly, we use our results to give lower bounds on the generalized Hausdorff measure, also to be defined in the next section, of a wide class of Kakeya-type subsets of the plane.

2. Notation & Terminology

$B(x,r)$ is the open disk of radius r centered at x.

$|\cdot|$ denotes Lebesgue measure and $\dim(\cdot)$ Hausdorff dimension.

$x \lesssim y$ means $x \leq Ay$ for some absolute constant $A > 0$ and similarly with $x \simeq y$.

$\mathrm{spt}(\mu)$ is the support of the measure μ.

A Borel measure μ is said to be s-dimensional Ahlfors regular if

$$\mu(B(x,r)) \simeq r^s,$$

for all $x \in \mathrm{spt}(\mu)$, $r \leq 1$.

A measure function is a non-decreasing positive function $h(r)$, $r > 0$, such that

$$\lim_{r\to 0} h(r) = 0.$$

The generalized Hausdorff outer measure Λ_h with respect to a measure function h is defined for $A \subset \mathbb{R}^2$ by

$$\Lambda_h(A) = \sup_{\delta>0} \inf \left\{ \sum_j h(r_j) : A \subset \bigcup_j B(x_j, r_j),\ r_j < \delta \right\}.$$

When $h(r) = r^s$, Λ_h is the usual Hausdorff outer measure.

3. Measures satisfying a growth condition

The main result of this paper is the following.

Theorem 3.1. *Let μ be a positive Borel measure supported on S^1 such that*

$$\mu(B(x,r)) \leq \varphi(r),\ x \in S^1,\ 0 < r < 1,$$

for some positive function φ. Then for all $p \geq 2$ there exists a constant A_p such that

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \leq A_p C(\delta)^{1/p} \|f\|_{L^p(\mathbb{R}^2)},$$

where

$$C(\delta) = \mu(S^1) + \int_1^{1/\delta} \varphi(1/r)dr.$$

Proof. We will prove the $L^2(\mathbb{R}^2) \to L^2(d\mu)$ estimate. The theorem then follows by interpolation.

Without loss of generality we may assume that $\mathrm{spt}(\mu)$ is contained in the first quadrant. We cover $\mathrm{spt}(\mu)$ with a family $\{A_j\}$ of disjoint arcs each of length δ. Pick $e_j \in A \cap \mathrm{spt}(\mu)$ and let $a_j = \mu(A_j)$. Note that if $u, v \in A_j$ then for any $x \in \mathbb{R}^2$

$$T_u^\delta(x) \subset \widetilde{T}_v^\delta(x),$$

where $\widetilde{T}_v^\delta(x)$ is the rectangle with dimensions $2\times 4\delta$ and with the same center and orientation as $T_v^\delta(x)$. Therefore, for every $e \in A_j$

$$\mathcal{K}_\delta f(e) \lesssim \sup_{x\in\mathbb{R}^2} \frac{1}{\left|\widetilde{T}_{e_j}^\delta(x)\right|} \int_{\widetilde{T}_{e_j}(x)} |f(y)|dy \lesssim \frac{1}{\delta} \int_{\widetilde{T}_{e_j}(x_j)} |f(y)|dy,$$

for some $x_j \in \mathbb{R}^2$. We estimate $\|\mathcal{K}_\delta f\|_{L^2(d\mu)}$ by duality. Let $g \in L^2(d\mu)$ such that $\|g\|_{L^2(d\mu)} = 1$, and put

$$c_j = \left(\int_{A_j} |g(e)|^2 d\mu(e)\right)^{1/2}.$$

Then, letting

$$Q_g = \left|\int \mathcal{K}_\delta f(e) g(e) d\mu(e)\right|,$$

we have

$$\begin{aligned}
Q_g &= \left|\sum_j \int_{A_j} \mathcal{K}_\delta f(e) g(e) d\mu(e)\right| \lesssim \frac{1}{\delta} \sum_j \int_{\widetilde{T}_{e_j}(x_j)} |f(y)|dy \int_{A_j} |g(e)| d\mu(e) \\
&\le \frac{1}{\delta} \int |f(y)| \left(\sum_j a_j^{1/2} c_j \chi_{\widetilde{T}_{e_j}(x_j)}(y)\right) dy \\
&\le \frac{1}{\delta} \|f\|_{L^2(\mathbb{R}^2)} \left(\int \left(\sum_j a_j^{1/2} c_j \chi_{\widetilde{T}_{e_j}(x_j)}(y)\right)^2 dy\right)^{1/2} \\
&= \frac{1}{\delta} \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_{i,j} a_i^{1/2} a_j^{1/2} c_i c_j \left|\widetilde{T}_{e_i}(x_i) \cap \widetilde{T}_{e_j}(x_j)\right|\right)^{1/2} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_{i,j} \frac{a_i^{1/2} a_j^{1/2} c_i c_j}{\delta + |e_i - e_j|}\right)^{1/2},
\end{aligned}$$

where the last inequality follows by geometry. Now, let

$$F(i,j) = \frac{a_i^{1/2} a_j^{1/2} c_i c_j}{\delta + |e_i - e_j|},$$

and note that for all j

$$\sum_i a_i^{1/2} F(i,j) = a_j^{1/2} \sum_i \frac{a_i}{\delta + |e_i - e_j|} \lesssim a_j^{1/2} \sum_i \int_{A_i} \frac{d\mu(e)}{\delta + |e_i - e_j|}$$

$$= a_j^{1/2} \int \frac{d\mu(e)}{\delta + |e_i - e_j|} \leq a_j^{1/2} \int_0^{1/\delta} \mu(\{e : |e - e_j| < 1/r\}) dr$$

$$\leq a_j^{1/2} \left(\mu(S^1) + \int_1^{1/\delta} \mu(B(e_j, 1/r)) dr \right) \leq a_j^{1/2} C(\delta).$$

By symmetry

$$\sum_j a_j^{1/2} F(i,j) \lesssim a_i^{1/2} C(\delta),$$

for all i. Therefore

$$Q_g \lesssim \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i c_i \sum_j c_j F(i,j) \right)^{1/2} \leq \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i \left(\sum_j c_j F(i,j) \right)^2 \right)^{1/4}$$

$$\leq \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i \left(\sum_j c_j^2 a_j^{-1/2} F(i,j) \right) \left(\sum_j a_j^{1/2} F(i,j) \right) \right)^{1/4}$$

$$\lesssim \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{1/4} \left(\sum_{i,j} c_j^2 a_j^{-1/2} F(i,j) a_i^{1/2} \right)^{1/4}$$

$$= \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{1/4} \left(\sum_j c_j^2 a_j^{-1/2} \sum_i a_i^{1/2} F(i,j) \right)^{1/4}$$

$$\lesssim \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{1/2} \left(\sum_j c_j^2 \right)^{1/4} = \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{1/2}.$$

We conclude that

$$\|\mathcal{K}_\delta f\|_{L^2(d\mu)} = \sup\{Q_g : \|g\|_{L^2(d\mu)} = 1\} \lesssim C(\delta)^{1/2} \|f\|_{L^2(\mathbb{R}^2)}.$$

□

Note that when $\varphi(r) = r$, that is, when μ is absolutely continuous with L^∞ density with respect to arc-length measure on S^1, we get $C(\delta) \lesssim |\log \delta|$ and thus recover Bourgain's result. Taking $\varphi(r) = r^s$, $0 < s < 1$, we obtain the following estimate for measures of fractal dimension.

Corollary 3.1. *Let μ be a positive Borel measure on S^1 such that*

$$\mu(B(x,r)) \leq r^s, \ x \in S^1, \ 0 < r < 1,$$

for some $s \in (0,1)$. *Then, for* $p \geq 2$

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \leq A_p \delta^{(s-1)/p} \|f\|_{L^p(\mathbb{R}^2)}.$$

Now, we show that the above estimate is sharp if μ satisfies an extra regularity condition.

Proposition 3.1. *Let* $s \in (0,1)$ *and* μ *be an* s*-dimensional Ahlfors regular measure on* S^1. *Then*

$$\|\mathcal{K}_\delta\| \simeq \delta^{(s-1)/p}.$$

Proof. Let $\{e_j\}_{j=1}^N$ be a maximal $\delta/2$-separated set of points in $\mathrm{spt}(\mu)$. For each j let A_j be an arc of length δ centered at e_j. Then

$$1 \leq \sum_{j=1}^N \chi_{a_j}(e) \leq 2, \quad \text{for all } e \in \mathrm{spt}(\mu).$$

Hence

$$\mu(S^1) \leq \sum_{j=1}^N \mu(A_j) \leq 2\mu(S^1). \tag{1}$$

Note that Ahlfors regularity implies that $\mu(A_j) \simeq \delta^s$. Therefore by (1) we get that $N \simeq \delta^{-s}$. Now let

$$E_\delta = \left\{ x \in \mathbb{R}^2 : 0 < |x| \leq 4,\ x|x|^{-1} \in \bigcup_j A_j \right\}.$$

Then

$$1 \lesssim \mathcal{K}_\delta \chi_{E_\delta}(e), \quad \text{for all } e \in \mathrm{spt}(\mu).$$

Note that

$$|E_\delta| \lesssim N\delta \simeq \delta^{1-s}.$$

Consequently

$$\|\mathcal{K}_\delta\| \geq \frac{\|\mathcal{K}_\delta \chi_{E_\delta}\|_{L^p(d\mu)}}{\|\chi_{E_\delta}\|_{L^p(\mathbb{R}^2)}} \gtrsim |E_\delta|^{-1/p} \gtrsim \delta^{(s-1)/p}.$$

□

4. An application to Kakeya-type sets

Let A be a subset of S^1. A set $E \subset \mathbb{R}^2$ is said to be an A-Kakeya set, if it contains a unit line segment in the e-direction for every $e \in A$. Using Theorem 3.1 and a suitable modification of the argument in [1], one can obtain the following generalization of the classical result of Davies [3] on the Hausdorff dimension of usual Kakeya sets in the plane.

Theorem 4.1. *Let A be a subset of S^1 such that $\Lambda_h(A) > 0$ for some measure function h. If E is an A-Kakeya set in the plane, then for every $\varepsilon > 0$ we have*

$$\Lambda_{\psi_\varepsilon}(E) > 0,$$

where ψ_ε is given by

$$\psi_\varepsilon(r) = r^{2-\varepsilon} \int_1^{2/r} h(1/u) du.$$

If we specialize to the case of the usual Hausdorff measure by taking $h(r) = r^s$, then we have the following.

Corollary 4.1. *Let E be an A-Kakeya set in the plane. Then*

$$\dim(E) \geq \dim(A) + 1.$$

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