A NOTE ON THE DISTANCE SET PROBLEM IN THE PLANE

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ABSTRACT. We use a simple geometric-combinatorial argument to establish a quantitative relation between the generalized Hausdorff measure of a set and its distance set, extending a result originally due to Falconer.

Let A be a subset of the plane. The *distance set $D(A)$* is defined as the set of all distances between points of A. Namely

$$D(A) = \{|x - y| : x, y \in A\}.$$

It follows from the work of Falconer [2] (see also Mattila [3]) that

$$\frac{1}{2} \le \dim(A) \le \frac{3}{2} \Rightarrow \dim(D(A)) \ge \dim(A) - \frac{1}{2}$$

and

$$\dim(A) > \frac{3}{2} \Rightarrow \mathcal{L}^1(D(A)) > 0,$$

where dim denotes Hausdorff dimension and $\mathcal{L}^1$ linear Lebesgue measure. This was improved by Wolff [4] who showed that

$$1 < \dim(A) \le \frac{4}{3} \Rightarrow \dim(D(A)) \ge \frac{3}{2}\dim(A) - 1$$

and

$$\dim(A) > \frac{4}{3} \Rightarrow \mathcal{L}^1(D(A)) > 0$$

Putting these results together, one has the following estimate for the Hausdorff dimension of the distance set

$$(1) \qquad \dim(D(A)) \ge \begin{cases} \dim(A) - \frac{1}{2} & \text{if } \frac{1}{2} \le \dim(A) \le 1; \\ \frac{3}{2}\dim(A) - 1 & \text{if } 1 < \dim(A) \le \frac{4}{3}. \end{cases}$$

It is a central open question in geometric measure theory whether $\dim(A) > 1$ implies that $D(A)$ must have positive Lebesgue measure.

The proof of (1) involves techniques from harmonic analysis, more specifically, Bessel function estimates of the Fourier transform of radial functions and decay estimates for the L^2 circular means of the Fourier transform of a

measure with finite energy. In this note we use a purely geometric argument to study the relation between a set and its distance set in terms of generalized Hausdorff measures. As a corollary, we obtain a sharpening of (1) in the case of sets of positive s-dimensional Hausdorff measure with $s < 1$.

Before proceeding with the statement of our result we make some definitions.

$B(x, r)$ is the open disc of radius r centered at x.

$Q(x, r)$ is the square $[a, b) \times [a, b)$ with center at x and diameter r.

If $A \subset \mathbb{R}^2$ is bounded, then its diameter is defined by

$$\text{diam}(A) = \sup\{|x - y| : x, y \in A\}.$$

For a finite set F, $\text{card}(F)$ denotes its cardinality.

A *measure function* is a non-decreasing function $h(r)$, $r \geq 0$, such that

$$\lim_{r \to 0} h(r) = 0.$$

The *generalized Hausdorff outer measure* Λ_h with respect to a measure function h is defined for $A \subset \mathbb{R}^2$ by

$$\Lambda_h(A) = \sup_{\delta > 0} \inf \left\{ \sum_j h(r_j) : A \subset \bigcup_j B(x_j, r_j), \ r_j < \delta \right\}.$$

When $h(r) = r^s$, Λ_h is the usual Hausdorff outer measure denoted by $\mathcal{H}^s$.

Our result is as follows.

Theorem. *Let C be a bounded Suslin subset of $\mathbb{R}^2$ and let h be a measure function. Suppose that $\Lambda_h(C) > 0$ and let*

$$\psi(r) = \left(h(r) \left(h(\text{diam}(C)) + \int_1^{1/r} h(1/s) ds \right) \right)^{1/2}.$$

Then $\Lambda_\psi(D(C)) > 0$ whenever ψ makes sense as a measure function.

Proof. By Frostman's Lemma (see Carleson [1]), there exists a measure μ supported in C such that

$$\mu(B(x, r)) \leq h(r) \quad \text{for all } x \in \text{spt}(\mu)$$

Let $a, b \in C$ be two points of density with respect to μ. Choose $R > 0$ such that $10R < |a - b|$ and put

$$A = B(a, R) \cap C, \ B = B(b, R) \cap C.$$

Now, define

$$D(A, B) = \{|x - y| : x \in A, y \in B\},$$

and let $\{(r_k, r_k + \varepsilon_k)\}_k$ be a countable covering of $D(A, B)$. Without loss of generality, we may assume that

$$\frac{8}{10}|a - b| \le r_k \le \frac{12}{10}|a - b|.$$

Then

(2) $$\mu(A)\mu(B) \le \sum_k (\mu \otimes \mu)\{(x, y) \in A \times B : r_k < |x - y| < r_k + \varepsilon_k\}.$$

Fix k and let $\{Q(x_i, \varepsilon_k)\}_{i \in I}$, $\{Q(y_j, \varepsilon_k)\}_{j \in J}$ be disjoint coverings of A and B respectively. Put

$$a_i = \mu(A \cap Q(x_i, \varepsilon_k)), \ \ b_j = \mu(B \cap Q(y_j, \varepsilon_k))$$

and consider the following sets of indices:

$$\mathcal{I} = \{(i, j) \in I \times J : r_k - 2\varepsilon_k < |x_i - y_j| < r_k + 3\varepsilon_k\},$$
$$\mathcal{A}(j) = \{i \in I : (i, j) \in \mathcal{I}\}, \ \ j \in J,$$
$$\mathcal{B}(i) = \{j \in J : (i, j) \in \mathcal{I}\}, \ \ i \in I.$$

Then, letting

$$M_k = (\mu \otimes \mu)(\{(x, y) \in A \times B : r_k < |x - y| < r_k + \varepsilon_k\}),$$

we have

$$M_k \le \sum_{(i,j) \in \mathcal{I}} a_i b_j = \sum_{i \in I} a_i \sum_{j \in \mathcal{B}(i)} b_j$$

$$\le \left(\sum_{i \in I} a_i\right)^{1/2} \left(\sum_{i \in I} a_i \left(\sum_{j \in \mathcal{B}(i)} b_j\right)^2\right)^{1/2}$$

$$= \mu(A)^{1/2} \left(\sum_{i \in I} a_i \sum_{j_1, j_2 \in \mathcal{B}(i)} b_{j_1} b_{j_2}\right)^{1/2}$$

$$= \mu(A)^{1/2} \left(\sum_{j_1, j_2 \in J} b_{j_1} b_{j_2} \sum_{i \in \mathcal{A}(j_1) \cap \mathcal{A}(j_2)} a_i\right)^{1/2}.$$

Now for fixed $j_1, j_2 \in J$, the set

$$\{x_i : i \in \mathcal{A}(j_1) \cap \mathcal{A}(j_2)\}$$

is contained in the intersection of the two annuli

$$\{x : r_k - 2\varepsilon_k < |x - y_{j_1}| < r_k + 3\varepsilon_k\}$$

and

$$\{x : r_k - 2\varepsilon_k < |x - y_{j_2}| < r_k + 3\varepsilon_k\}.$$

Using geometry we can show that the area of this intersection is at most

$$\frac{100 r_k^2 \varepsilon_k^2}{|y_{j_1} - y_{j_2}| \sqrt{(2r_k)^2 - |y_{j_1} - y_{j_2}|^2}} \le \frac{100(\frac{12}{10}|a-b|)^2 \varepsilon_k^2}{|y_{j_1} - y_{j_2}| \sqrt{(16R)^2 - (4R)^2}}.$$

Note that $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$ are ε_k-separated sets of points in the plane. Therefore

$$\operatorname{card}(\{i \in \mathcal{A}(j_1) \cap \mathcal{A}(j_2)\}) \lesssim \frac{1}{\varepsilon_k + |y_{j_1} - y_{j_2}|}.$$

Furthermore, $a_i \le h(\varepsilon_k)$ because $Q(x_i, \varepsilon_k) \subset B(x_i, \varepsilon_k)$. It follows that

$$M_k \lesssim \mu(A)^{1/2} \left(\sum_{j_1, j_2 \in J} b_{j_1} b_{j_2} \frac{h(\varepsilon_k)}{\varepsilon_k + |y_{j_1} - y_{j_2}|} \right)^{1/2}$$

$$= \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \sum_{j_2 \in J} \frac{b_{j_2}}{\varepsilon_k + |y_{j_1} - y_{j_2}|} \right)^{1/2}$$

$$\lesssim \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \sum_{j_2 \in J} \int_{Q(y_{j_2}, \varepsilon_k)} \frac{d\mu(y)}{\varepsilon_k + |y_{j_1} - y|} \right)^{1/2}$$

$$= \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \int_B \frac{d\mu(y)}{\varepsilon_k + |y_{j_1} - y|} \right)^{1/2}$$

$$= \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \int_0^{1/\varepsilon_k} \mu(\{y \in B : \varepsilon_k + |y_{j_1} - y| < 1/s\}) ds \right)^{1/2}$$

$$\le \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \left(\int_0^1 \mu(B(y_{j_1}, 1/s)) ds + \int_1^{1/\varepsilon_k} h(1/s) ds \right) \right)^{1/2}$$

$$\le \mu(A)^{1/2} h(\varepsilon_k)^{1/2} \left(\sum_{j_1 \in J} b_{j_1} \left(h(\operatorname{diam}(C)) + \int_1^{1/\varepsilon_k} h(1/s) ds \right) \right)^{1/2}$$

$$= \mu(A)^{1/2} \mu(B)^{1/2} \psi(\varepsilon_k).$$

Hence, by (2)

$$\mu(A)^{1/2} \mu(B)^{1/2} \lesssim \sum_k \psi(\varepsilon_k).$$

This implies $\Lambda_\psi(D(C)) \ge \Lambda_\psi(D(A, B)) > 0$.   □

Taking $h(r) = r^s$, $1/2 < s < 1$, gives $\psi(r) \lesssim r^{s-1/2}$. Hence we obtain the following.

Corollary. *Let A be a subset of the plane such that $\mathcal{H}^s(A) > 0$, with $1/2 < s < 1$. Then $\mathcal{H}^{s-1/2}(D(A)) > 0$.*

Note that since an α-dimensional set may have zero $\mathcal{H}^\alpha$-measure, our corollary extends (1) in the indicated range of dimensions.

References

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