Topics in Harmonic Analysis

Themis Mitsis
Department of Mathematics, University of Crete, Greece

This set of notes was intended to supplement a graduate course in Harmonic Analysis that was planned to be given during my stay at the university of Jyväskylä as a Marie Curie fellow. For technical reasons, the course was never taught, so I am grateful to Pertti Mattila for the opportunity to publish these notes.

Most of the material is based on my personal notes from a series of lectures given by Tom Wolff at UW-Madison back in 1996. Anyone familiar with his mathematical preferences will recognize his style.

Wolff's expository article [**23**] and his own lecture notes [**25**] from a Caltech course (edited by Izabella Laba) are closely related to the subject matter of this work.

Contents

List of notation

$B(a,R)$: The disc $\{x \in \mathbb{R}^n : |x-a| < R\}$.
supp f : The support of the function (or distribution) f.
$\hat{f}$: The Fourier transform of f, $\hat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)dx$.
$\check{f}$: The inverse Fourier transform of f, $\check{f}(x) = \int e^{2\pi i x\cdot\xi} f(\xi)d\xi$.
$\hat{\mu}$: The Fourier transform of the measure μ, $\hat{\mu}(\xi) = \int e^{-2\pi i x\cdot\xi} d\mu(x)$.
$\check{\mu}$: By definition, $\check{\mu}(\xi) = \int e^{2\pi i x\cdot\xi} d\mu(x)$.
$D^\alpha f$: $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f$, where α is a multi-index, i.e., $\alpha = (\alpha_1, \ldots, \alpha_n)$, with α_j being natural numbers.
$|\alpha|$: The length of the multi-index α, $|\alpha| = \sum_{j=1}^n \alpha_j$.
ϕ_t : $\phi_t(x) = t^{-n}\phi(t^{-1}x)$, unless otherwise indicated.
$\mathcal{S}$: The Schwartz space.
$|E|$: Lebesgue measure (in the ambient Euclidean space), or cardinality of E, depending on the context.
$\dim E$: The lower Minkowski dimension of E.
∇f : The gradient of f, $\nabla f = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.
$\underset{D}{\operatorname{osc}} f$: The oscillation of f on D, $\underset{D}{\operatorname{osc}} f = \sup_{x,y\in D} |f(x) - f(y)|$.
S^{n-1} : The unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$.
$d\sigma$: Surface measure.
$d\mathcal{L}^k$: k-dimensional Lebesgue measure.
$C, C_1, \ldots$: Capital letters denote various constants whose values may change from line to line.
$\lesssim$: $x \lesssim y$ means "$x \le Cy$, where C is a constant".
$\simeq$: $x \simeq y$ means ($x \lesssim y$ & $y \lesssim x$).
$\Re(z)$, $\Im(z)$: The real and imaginary part of $z \in \mathbb{C}$.
χ_E : The characteristic function of E.
$\mathbb{E}$: Expectation.

CHAPTER 1

Some applications of Khinchin's inequality

In this chapter we will present, in the context of harmonic analysis, two typical applications of the following classical probabilistic inequality.

PROPOSITION 1.1 (Khinchin's inequality). *Let ε_j be independent random variables taking the values 1 and -1 with probability $1/2$ each. Then for any $p \in (0, \infty)$ and complex numbers $\{a_j\}_{j=1}^N$ we have*

$$\mathbb{E}\Big(\Big|\sum_{j=1}^N \varepsilon_j a_j\Big|^p\Big) \simeq \Big(\sum_{j=1}^N |a_j|^2\Big)^{p/2},$$

with bounds independent of N.

The proof of Khinchin's inequailty may be found in most books on elementary probabilty. What is important is that the bounds are independent of N, and that the right hand side depends only on $\{|a_j|\}$ and does not involve any cancelations.

Our first application concerns the most basic inequality for the L^p Fourier transform, namely the Hausdorff-Young theorem. Unlike the Plancherel theorem, Hausdorff-Young is not reversible. This fact may be proved in many ways. Let us prove it using Khinchin's inequality.

PROPOSITION 1.2. *($1 \leq p < 2$). For any $\varepsilon > 0$ there is a function $f \in \mathcal{S}$ with*

$$\|\hat{f}\|_{p'} < \varepsilon \|f\|_p.$$

PROOF. Let ϕ be a fixed Schwartz function with compact support. Let $\{x_j\}_{j=1}^N$ be a sequence of points in $\mathbb{R}^n$ such that the functions $\phi_j(x) := \phi(x - x_j)$ have disjoint supports. Then it is obvious that for any choice of $\varepsilon_j \in \{\pm 1\}$ ($j = 1, \dots N$) we have

$$\Big\|\sum_{j=1}^N \varepsilon_j \phi_j\Big\|_p^p = \sum_{j=1}^N \|\phi_j\|_p^p = N\|\phi\|_p^p,$$

and so,

$$\Big\|\sum_{j=1}^N \varepsilon_j \phi_j\Big\|_p = N^{1/p}\|\phi\|_p.$$

Moreover

$$\mathbb{E}\Big(\Big\|\sum_{j=1}^N \varepsilon_j \hat{\phi}_j\Big\|_{p'}^{p'}\Big) = \int \mathbb{E}\Big(\Big|\sum_{j=1}^N \varepsilon_j \hat{\phi}_j(\xi)\Big|^{p'}\Big) d\xi = \int \mathbb{E}\Big(|\hat{\phi}(\xi)|^{p'}\Big|\sum_{j=1}^N \varepsilon_j e^{2\pi i x_j \cdot \xi}\Big|^{p'}\Big) d\xi$$
$$\simeq N^{p'/2} \int |\hat{\phi}(\xi)|^{p'} d\xi,$$

where the last line follows by Khinchin's inequality with $a_j = e^{2\pi i x_j \cdot \xi}$. Therefore

$$\Big\| \sum_{j=1}^{N} \varepsilon_j \hat{\phi}_j \Big\|_{p'} \leq C_0 N^{1/2} \|\hat{\phi}\|_{p'},$$

for some choice of $\{\varepsilon_j\}$. Since $p < 2$ we can choose N_0 so that

$$C_0 N_0^{1/2} \|\hat{\phi}\|_{p'} < \varepsilon N_0^{1/p} \|\phi\|_p.$$

The result now follows on letting

$$f = \sum_{j=1}^{N_0} \varepsilon_j \phi_j.$$

□

Our second application is the basic result in Littlewood-Paley theory.

Let ϕ be a smooth function such that $\phi = 0$ on $B(0,1)$ and $\phi = 1$ outside $B(0,2)$, and let $\psi_j(x) = \phi(2^{-j}x) - \phi(2^{-j+1}x)$. Then ψ_j is supported in the annulus $\{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$ and

$$\sum_{j=-\infty}^{\infty} \psi_j(x) = 1, \text{ for all } x \neq 0.$$

For $f \in \mathcal{S}$, let $S_j f = (\psi_j \hat{f})\check{}$. Then, the *Littlewood-Paley square function* is defined by

$$S f = \Big(\sum_{j=-\infty}^{\infty} |S_j f|^2 \Big)^{1/2}.$$

Proposition 1.3 (Littlewood-Paley). *For any $f \in \mathcal{S}$ and $1 < p < \infty$, one has*

$$\|S f\|_p \simeq \|f\|_p.$$

For the proof, we need the following result (see [**20**] for a more sophisticated version).

Lemma 1.1 (Mikhlin multiplier theorem). *Let $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ satisfy $|D^\gamma m(\xi)| \lesssim |\xi|^{-\gamma}$, for all $\xi \neq 0$ and all multi-indices of length $|\gamma| \leq n+2$. Then*

$$\|(m\hat{f})\check{}\|_p \lesssim \|f\|_p$$

for any $f \in \mathcal{S}$ and $1 < p < \infty$.

Proof. Note that $(m\hat{f})\check{} = \check{m} * f$ and that the decay condition on m implies that, away from the origin, the distribution $\check{m}$ agrees with a function. So, if we let

$$m_j(\xi) = \psi_j(\xi) m(\xi),$$

and

$$K_N = \sum_{j=-N}^{N} \check{m}_j,$$

it is enough to show that the kernels K_N are Calderón-Zygmund uniformly in N. Namely, they satisfy

(i) $|K_N(x)| \lesssim |x|^{-n}$
(ii) $|\nabla K_N(x)| \lesssim |x|^{-n-1}$
(iii) $\|K_N * f\|_2 \lesssim \|f\|_2$

with bounds independent of N.

To prove (i), note that

$$\|D^\gamma m_j\|_1 \lesssim 2^{j(n-|\gamma|)}$$

and therefore

$$\|x^\gamma \check{m}_j\|_\infty \lesssim 2^{j(n-|\gamma|)}.$$

It follows that

$$|\check{m}_j(x)| \lesssim 2^{j(n-k)}|x|^{-k} \text{ for any } 0 \le k \le n+2.$$

Using this with $k = 0$ and $k = n + 2$, we conclude that

$$\begin{aligned} |K_N(x)| \le \sum_j |\check{m}_j(x)| &\le \sum_{2^j \le |x|^{-1}} |\check{m}_j(x)| + \sum_{2^j > |x|^{-1}} |\check{m}_j(x)| \\ &\lesssim \sum_{2^j \le |x|^{-1}} 2^{jn} + \sum_{2^j > |x|^{-1}} 2^{jn}(2^j|x|)^{-(n+2)} \lesssim |x|^{-n}. \end{aligned}$$

The proof of (ii) is similar and (iii) follows by Plancherel. □

To prove Proposition 1.3, let

$$m_N(\xi) = \sum_{j=-N}^{N} \varepsilon_j \psi_j(\xi)$$

and note that m_N satisfies the condition of Lemma 1.1 uniformly in N and uniformly in the realization of the random variables $\{\varepsilon_j\}$. Therefore, by the continuous version of Khinchin's inequality,

$$\int |(Sf)(x)|^p dx \lesssim \limsup_{N\to\infty} \mathbb{E}\Big[\int \Big| \sum_{j=-N}^{N} \varepsilon_j (S_j f)(x)\Big|^p dx \Big] \lesssim \|f\|_p^p.$$

To prove the lower bound, we use duality. Note that $\psi_j\psi_k = 0$ if $|j-k| > 1$, so by Parseval, Cauchy-Schwarz and Hölder,

$$\int f\overline{g} = \int \sum_{\{j,k:|j-k|\le 1\}} S_j f S_k \overline{g} \lesssim \|Sf\|_p \|S\overline{g}\|_{p'} \lesssim \|Sf\|_p \|g\|_{p'}$$

for all $g \in \mathcal{S}$. Therefore, $\|f\|_p \lesssim \|Sf\|_p$.

Note that Proposition 1.3 holds for arbitrary $f \in L^p$ (in which case we have to interpret $S_j f$ as $\check{\psi}_j * f$) by a standard limiting argument.

We will present other applications of Khinchin's inequality in Chapter 5 and Chapter 6.

CHAPTER 2

Stationary phase

Consider the oscillatory integral

$$I(\phi, a; \lambda) = \int_{-\infty}^{\infty} e^{i\lambda\phi(x)} a(x) dx,$$

where ϕ is a smooth function (the *phase*), and a is a smooth integrable function (the *amplitude*). Our objective is to study the behavior of I for large values of λ. In order to do that, we have to calculate its asymptotic expansion.

If $\phi(x) = -2\pi x$, then I is the Fourier transform of a, and integration by parts gives

$$|I(-2\pi x, a; \lambda)| = C\frac{|I(-2\pi x, a^{(k)}; \lambda)|}{|\lambda|^k} \leq C\frac{\|a^{(k)}\|_1}{|\lambda|^k},$$

for any k, provided that a has integrable derivatives up to order k. This shows that I is rapidly decreasing as $\lambda \to \infty$ and therefore the question of determining its asymptotic expansion is, in a sense, trivial.

If ϕ is nonlinear with nonvanishing first derivative then by a change of variables, we get again a Fourier integral. So, the nontrivial case arises when the phase has critical points. We will restrict ourselves to phases with nondegenerate critical points. First, we estimate the so-called *Fresnel* integral:

$$I(x^2, a; \lambda) = \int e^{i\lambda x^2} a(x) dx.$$

Proposition 2.1. *Suppose a is a smooth function with compact support. Then for any $k \in \mathbb{N}$, we have*

$$I(x^2, a; \lambda) = \sqrt{\frac{\pi i}{\lambda}} \sum_{j=0}^{k-1} \frac{a^{(2j)}(0)}{j!} \left(\frac{i}{4\lambda}\right)^j + \frac{R_k(\lambda)}{(2\lambda)^k},$$

where the remainder satisfies the uniform estimate

$$|R_k(\lambda)| \leq \|a^{(2k)}\|_1.$$

Proof. Notice that

$$I(x^2, a; \lambda) = \lim_{\varepsilon \to 0^+} \int e^{(-\varepsilon + i\lambda)x^2} a(x) dx.$$

Now write

$$\int e^{(-\varepsilon+i\lambda)x^2} a(x) dx = \int e^{\mu x^2} a(x) dx = \int e^{\mu x^2} a(0) dx + \int e^{\mu x^2} x b(x) dx,$$

where

$$\mu = -\varepsilon + i\lambda,$$

and

$$b(x) = \frac{a(x) - a(0)}{x}.$$

The first term is

$$\int e^{\mu x^2} dx = \sqrt{\frac{\pi}{\varepsilon - i\lambda}}.$$

Integrating by parts the second term we get

$$\int e^{\mu x^2} x b(x) dx = -\frac{1}{2\mu} \int e^{\mu x^2} a_1(x) dx,$$

where $a_1 = b'$. Passing to the limit as $\varepsilon \to 0$ and repeatedly applying the above procedure, we get

$$I(x^2, a; \lambda) = \sqrt{\frac{\pi i}{\lambda}} \left(a(0) + \frac{a_1(0)}{-2i\lambda} + \cdots + \frac{a_{k-1}(0)}{(-2i\lambda)^{k-1}} \right) + \frac{I(x^2, a_k; \lambda)}{(-2i\lambda)^k},$$

where

$$a_j(0) = \frac{a^{(2j)}(0)}{2^j j!}.$$

The result follows since

$$|I(x^2, a_k; \lambda)| \leq \|a_k\|_1 \leq \|a^{(2k)}\|_1.$$

□

Now, suppose that the phase ϕ has finitely many critical points x_p in $\mathrm{supp}(a)$. Choose a smooth partition of unity

$$\sum h_\alpha = 1$$

in $\mathrm{supp}(a)$ such that for each α, $\mathrm{supp}(h_\alpha)$ contains exactly one critical point. Suppose that $x_p \in \mathrm{supp}(h_\alpha)$. Then, by the Morse Lemma, we have

$$I(\phi, h_\alpha; \lambda) = e^{i\lambda\phi(x_p)} I(\pm x^2, b_p; \lambda),$$

where b_p is a suitable smooth function with compact support, and

$$\pm = \mathrm{sgn}\phi''(x_p).$$

Proposition 2.1 then implies

$$I(\phi, a; \lambda) = \sqrt{\frac{\pi}{\lambda}} \sum_p e^{\pm\frac{\pi i}{4} + i\lambda\phi(x_p)} (a(x_p) + \cdots + \lambda^{-k} R_k(\lambda)).$$

Next, we turn to the higher dimensional case. Namely, we want to study the asymptotic behavior of the integral

$$I(\phi, a; \lambda) = \int_{\mathbb{R}^n} e^{2\pi i\lambda\phi(x)} a(x) dx.$$

Assume first that the phase is a quadratic form, i.e.

$$\phi(x) = \beta(x) = \frac{1}{2} \sum_{ij} \beta^{ij} x_i x_j.$$

Furthermore, suppose that β is nonsingular, i.e. $\det[\beta^{ij}] \neq 0$, and let

$$\beta^*(D) = \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

be the dual differential operator, where $[\beta_{ij}] = [\beta^{jk}]^{-1}$. Then we have the following.

Proposition 2.2. *If β is a nonsingular quadratic form, and a is a smooth function with compact support, then for any $k \in \mathbb{N}$, the following equation holds.*

$$\int_{\mathbb{R}^n} e^{2\pi i\lambda\beta(x)} a(x)dx = \frac{e^{\frac{\pi\sigma(\beta)i}{4}}}{\sqrt{\lambda^n|\mathrm{detHess}\beta|}} \sum_{j=0}^{k-1} \frac{\beta^*(D)^j a(0)}{j!}\left(\frac{i}{\lambda}\right)^j + \frac{R_k(\lambda)}{\lambda^k},$$

where $\sigma(\beta)$ is the signature of β, $\mathrm{Hess}\beta$ the Hessian matrix of β, and the remainder satisfies the uniform estimate

$$|R_k(\lambda)| \le \|(\beta^*(D))^k a\|_1.$$

Proof. Proposition 2.1 is the special case when $n = 1$. In the general case, take a linear transformation $x = A(y)$ such that

$$\beta(x) = \frac{1}{2}(y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2)$$

where $2p = \sigma(\beta) + n$. By Taylor's Theorem

$$a(y) = \sum_{|\alpha|=0}^{2k-1} \frac{D^\alpha a(0)}{\alpha!} y^\alpha + S_k(y).$$

Then

$$I(2\pi\beta(y) + \varepsilon i y^2; \lambda) = \sum_{|\alpha|=0}^{2k-1} \frac{D^\alpha a(0)}{\alpha!} \int e^{(2\pi i\beta(y) - \varepsilon y^2)\lambda} y^\alpha dy + \int e^{(2\pi i\beta(y) - \varepsilon y^2)\lambda} S_k(y)dy. \tag{2.1}$$

By Fubini's Theorem

$$\int e^{(2\pi i\beta(y) - \varepsilon y^2)\lambda} y^\alpha dy = \prod_{j=1}^{n} \int e^{(\pm 2iy_j^2 - \varepsilon y_j^2)\lambda} y_j^{\alpha_j} dy_j,$$

therefore, Proposition 2.1 applies. We can put the sum in (2.1) in the form $(P(D)a)(0)$, where

$$P(D) = \prod_{j=1}^{n} \frac{\exp(\frac{\pi\sigma(\beta_j)i}{4})}{\sqrt{\lambda|\mathrm{detHess}\beta_j|}} \exp\left(\frac{i\beta_j^*(\partial/\partial y_j)}{\lambda}\right)_k,$$

with

$$\beta_j = \pm\frac{1}{2}y_j^2.$$

Next, we calculate

$$\prod \exp\left(\frac{\pi\sigma(\beta_j)i}{4}\right) = \exp\left(\frac{\pi\sigma(\beta)i}{4}\right),$$

$$\prod \lambda|\mathrm{detHess}\beta_j| = \lambda^n|\mathrm{detHess}\beta|,$$

$$\prod \exp\left(\frac{i\beta_j^*(\partial/\partial y_j)}{\lambda}\right) = \exp\left(\frac{i\beta^*(D)}{\lambda}\right).$$

Finally,

$$|\mathrm{detHess}_y\beta| = |\det A|^2 |\mathrm{detHess}_x\beta| = |\det A|^2.$$

To conclude the proof, we estimate the remainder as in Proposition 2.1. □

If x_0 is a nondegenerate critical point of ϕ, we choose smooth local coordinates in a neighborhood of x_0 such that ϕ is a quadratic form in that coordinate system. Then we apply Proposition 2.2.

A good reference for the material in this chapter is [**13**].

CHAPTER 3

The uncertainty principle

In harmonic analysis, by the term *uncertainty principle*, we refer to the (rather vague) fact that a function and its Fourier transform cannot be both concentrated on small sets.

The simplest manifestation of this phenomenon is the fact that the Fourier transform of a compactly supported function cannot have compact support, unless the function is identically equal to zero.

Another way to understand the situation is the following.

PROPOSITION 3.1 (Bernstein's inequality). *Suppose* $f \in L^p$ $, 1 \le p \le \infty$ *and that* $\operatorname{supp}\hat{f} \subset B(0, R)$*. Then*

$$|D^\alpha f\|_p \lesssim R^{|\alpha|}\|f\|_p,$$

where the implicit constant depends only on the dimension n.

PROOF. If $p = 2$ this follows from Plancherel's Theorem.

$$\|D^\alpha f\|_2 = \|(D^\alpha f)\hat{}\|_2 = \|(2\pi i\xi)^\alpha \hat{f}\|_2 \lesssim R^{|\alpha|}\|\hat{f}\|_2 = R^{|\alpha|}\|f\|_2.$$

In the general case, fix a function $\phi \in \mathcal{S}$ with

$$\hat{\phi}(\xi) = 1 \ \forall \xi \in D(0, 1).$$

Then

$$(\phi_{R^{-1}})\hat{}\,\hat{f} = \hat{f},$$

and therefore

$$\phi_{R^{-1}} * f = f.$$

Now, suppose $|\alpha| = 1$. Then

$$D^\alpha f = D^\alpha \phi_{R^{-1}} * f = R(D^\alpha\phi)_{R^{-1}} * f.$$

So, by Minkowski's inequality

$$\|D^\alpha f\|_p \le R\|(D^\alpha\phi)_{R^{-1}}\|_1\|f\|_p = CR\|f\|_p.$$

The general case follows by induction on $|\alpha|$. □

COROLLARY 3.1. *Suppose that* $\operatorname{supp}\hat{f} \subset B(0, R)$*. Then*

$$\operatorname*{osc}_B f \lesssim \|f\|_\infty$$

for all discs B of radius R^{-1}*; here the implicit constant depends on n only.*

PROOF. By the Mean Value Theorem and Bernstein's inequality we have

$$\operatorname*{osc}_B f \lesssim R^{-1}\|\nabla f\|_\infty \lesssim R^{-1}R\|f\|_\infty = \|f\|_\infty.$$

□

So, if $\hat{f}$ is supported on a disc of radius R, then f is "essentially constant" on scale R^{-1}.

In higher dimensional harmonic analysis, it is often the case that the support of $\hat{f}$ is not contained in a disc, but in a set of high eccentricity. This case can be understood by starting from a disc and studying the behavior of the Fourier transform under linear maps.

If $\{e_j\}$ is a basis for $\mathbb{R}^n$ and $\{a_j\}$ are positive numbers, then the *ellipsoid* with axes $\{e_j\}$ and widths $\{a_j\}$ is the set

$$E = \Big\{\xi \in \mathbb{R}^n : \sum_j \Big(\frac{\langle \xi, e_j\rangle}{a_j}\Big)^2 \leq 1\Big\}.$$

Its *dual* ellipsoid is the set

$$E^* = \Big\{x \in \mathbb{R}^n : \sum_j \Big(a_j\langle x, a_j\rangle\Big)^2 \leq 1\Big\},$$

A basic fact from linear algebra is that there is always a linear transformation T such that

$$T(B(0,1)) = E^*,$$
$$(T^*)^{-1}(B(0,1)) = E,$$

where T^* is the transpose of T. Therefore, if f is function with $\operatorname{supp} \hat{f} \subset E$, then

$$\operatorname{supp}((f \circ T)\hat{\ }) = \operatorname{supp}(\hat{f} \circ (T^*)^{-1}) \subset B(0,1),$$

and since $f \circ T$ must be "essentially constant" on translates of $B(0,1)$ we conclude that f is "essentially constant" on translates of E^*. For example, by Corollary 3.1, we have

Corollary 3.2. *If* $\operatorname{supp} \hat{f} \subset E$, *then*

$$\operatorname*{osc}_{E^*+a} f \lesssim \|f\|_\infty,$$

for all $a \in \mathbb{R}^n$.

This can also be applied when E is the "rectangle"

$$\{\xi : |\langle \xi, e_j\rangle| \leq a_j\}$$

and E^* is the "dual rectangle"

$$\{x : |\langle x, e_j\rangle| \leq a_j^{-1}\},$$

since these rectangles are comparable to ellipsoids in the same way cubes are comparable to balls.

There are many other, and much deeper, forms of the uncertainty principle. We refer the reader to the monograph [**12**] for an extensive account.

CHAPTER 4

The restriction problem

This chapter is concerned with the following fundamental question which is still largely open.

When can one meaningfully restrict the Fourier transform of an L^p function to the surface of the unit sphere?

More quantitatively,

For what values of q is there an estimate

$$\|(f d\sigma)\check{}\|_q \lesssim \|f\|_\infty$$

for all $f \in L^\infty(S^{n-1})$?

To find the best possible q, one can take f to be a constant function. Then $(f d\sigma)\check{}$ may be evaluated using the technique of stationary phase (see [**20**], [**13**]). The result is as follows.

Proposition 4.1.

$$(d\sigma)\check{}(x) = \Re\left(A(x)e^{2\pi i(|x|-(n-1)/8)}\right), \quad \textit{for large } |x|,$$

where

$$C^{-1}|x|^{-(n-1)/2} \le |A(x)| \le C|x|^{-(n-1)/2},$$

$$|D^\alpha A(x)| \le C_\alpha |x|^{-(n-1)/2-|\alpha|}.$$

In particular, $(d\sigma)\check{} \in L^q$ precisely when $q > 2n/(n-1)$. The restriction conjecture of Stein is the statement that

$$(f d\sigma)\check{} \in L^q, \quad \text{for all } f \in L^\infty(S^{n-1}),\ q > \frac{2n}{n-1}.$$

To begin with, let's try to estimate the L^2 norm of $(f d\sigma)\check{}$ on large finite discs. To motivate the calculation, note that

$$\|(d\sigma)\check{}\|_{L^2(B(0,R))} \simeq R^{1/2}, \quad \text{for large } R,$$

by Proposition 4.1.

Proposition 4.2. *If $f \in L^\infty(S^{n-1})$, then*

$$\|(f d\sigma)\check{}\|_{L^2(B(a,R))} \lesssim R^{1/2}\|f\|_\infty.$$

Proof. We can assume $a = 0$; otherwise we replace f by

$$f_a(\xi) = e^{2\pi i a\cdot\xi} f(\xi),$$

which has the same L^∞ norm as f and satisfies

$$(f_a d\sigma)\check{}(x) = (f d\sigma)\check{}(x + a).$$

Let ϕ be a Schwartz function with the following properties.

$$\phi \geq 1 \text{ on } B(0,1) \quad \text{and} \quad \operatorname{supp}\hat{\phi} \subset B(0,1).$$

Let

$$\phi^R(x) = \phi(x/R).$$

Then

$$\|(f d\sigma)\check{}\|_{L^2(B(0,R))} \leq \|\phi^R (f d\sigma)\check{}\|_2 = \|(\phi^R)\hat{} * (f d\sigma)\|_2.$$

Now

$$(\phi^R)\hat{}(\xi) = R^n \hat{\phi}(R\xi),$$

and therefore

$$|(\phi^R)\hat{} * (f d\sigma)(\xi)| = R^n \Big| \int \hat{\phi}(\xi - R\eta) f(\eta) d\sigma(\eta) \Big| \lesssim R^n \|f\|_\infty \sigma(B(\xi, R^{-1})).$$

The latter quantity is $\lesssim R\|f\|_\infty$ for any ξ and is 0 if $\operatorname{dist}(\xi, S^{n-1}) > 1/R$. Accordingly,

$$\|(\phi^R)\hat{} * (f d\sigma)\|_2 \lesssim R\|f\|_\infty |\{\xi : \operatorname{dist}(\xi, S^{n-1}) < 1/R\}|^{1/2} \lesssim R^{1/2} \|f\|_\infty.$$

□

With a slightly more careful argument one can obtain the same estimate when f is just in L^2, i.e.,

$$\|(f d\sigma)\check{}\|_{L^2(B(a,R))} \lesssim R^{1/2} \|f\|_{L^2(S^{n-1})}.$$

The *Stein-Tomas theorem* is a bound of the form

$$\|(f d\sigma)\check{}\|_q \leq C\|f\|_{L^2(d\sigma)}$$

with an optimal value of q $(= 1(n+1)/(n-1))$ which is larger than $2n/(n-1)$, reflecting a difference between L^2 and L^∞ densities f. To understand this distinction, consider a function which has small support, so that its L^2 norm will be much smaller than its L^∞ norm. Indeed, consider the spherical cap

$$C_e^\delta = \{\xi \in S^{n-1} : |\xi - e| < \delta\}.$$

The smallest convex set containing C_e^δ is essentially a rectangle with width δ^2 in the e direction and δ in the perpendicular directions. Therefore, $|(f d\sigma)\check{}|$ should be essentially constant on translates of the dual rectangle

$$\tau_e^\delta(a) = \Big\{ x \in \mathbb{R}^n : |(x-a)\cdot e| < \frac{1}{2}\delta^{-2},\ |P_{e^\perp}(x-a)| < \delta^{-1} \Big\},$$

where

$$P_{e^\perp}(x) = x - \langle x, e\rangle e$$

is the projection of x on the line passing through the origin orthogonal to e.

Proposition 4.3 (Knapp counterexample). *For any $e \in S^{n-1}$, $a \in \mathbb{R}^n$, and small positive δ, there is a function $f : S^{n-1} \to \mathbb{C}$ with*

$$\|f\|_\infty \leq 1,\ \operatorname{supp} f \subset C_e^\delta \text{ and } |(f d\sigma)\check{}| \gtrsim \delta^{n-1} \text{ on } \tau_e^\delta(a).$$

PROOF. Let $f(\xi) = e^{-2\pi i \xi\cdot a}$ if $\xi \in C_e^{C_0^{-1}\delta}$ and zero otherwise; here C_0 is a large positive constant depending on n only. If C_0 is large enough, then we have

$$|(\xi - e)\cdot(x-a)| < \frac{1}{100} \quad \forall \xi \in C_e^{C_0^{-1}\delta}, x \in \tau_e^\delta(a).$$

This is because

$$(\xi - e)\cdot(x-a) = (\xi - e)\cdot e(x-a)\cdot e + P_{e^\perp}(\xi - e)\cdot P_{e^\perp}(x-a),$$

with

$$|P_{e^\perp}(\xi - e)| < C_0^{-1}\delta,\ |P_{e^\perp}(x-a)| < \delta^{-1},\ |(\xi - e)\cdot e| \lesssim (C_0^{-1}\delta)^2,$$

and

$$|(x-a)\cdot e| \lesssim \delta^{-2}.$$

Accordingly, for $x \in \tau_e^\delta(a)$

$$\begin{aligned}
|(f d\sigma)\check{}(x)| &= \left| \int_{C_e^{C_0^{-1}\delta}} e^{2\pi i (x-a)\cdot\xi} d\sigma(\xi)\right| = \left| \int_{C_e^{C_0^{-1}\delta}} e^{2\pi i (x-a)\cdot(\xi - e)} d\sigma(\xi)\right| \\
&\geq \int_{C_e^{C_0^{-1}\delta}} \Re\left(e^{2\pi i (x-a)\cdot(\xi-e)}\right) d\sigma(\xi) \\
&\geq \cos\left(\frac{2\pi}{100}\right)\sigma\left(C_e^{C_0^{-1}\delta}\right) \simeq \delta^{n-1}.
\end{aligned}$$

□

With f as in Proposition 4.3, we have

$$\|f\|_{L^2(d\sigma)} \lesssim \delta^{\frac{n-1}{2}} \quad \text{and} \quad \|(f d\sigma)\check{}\|_q \gtrsim \delta^{n-1-\frac{n+1}{q}},$$

since the volume of $\tau_e^\delta(a)$ is approximately $\delta^{-(n+1)}$. Therefore, a bound of the form

$$\|(f d\sigma)\check{}\|_q \lesssim \|f\|_{L^2(d\sigma)}$$

can only hold if

$$\frac{n-1}{2} \leq n - 1 - \frac{n+1}{q},$$

which means

$$q \geq 2\frac{n+1}{n-1}.$$

PROPOSITION 4.4 (Stein-Tomas theorem). *If* $q \geq 2(n+1)/(n-1)$ *then*

$$\|(f d\sigma)\check{}\|_q \lesssim \|f\|_{L^2(d\sigma)}.$$

SKETCH OF THE PROOF. First show that the following three assertions are equivalent for any given $p < 2$, where $p' = p/(p-1)$.

(1) $\|(f d\sigma)\check{}\|_{p'} \leq C\|f\|_{L^2(d\sigma)}$ for all $f \in L^2(d\sigma)$
(2) $\|\hat{f}\|_{L^2(d\sigma)} \leq C\|f\|_p$ for all $f \in \mathcal{S}$
(3) $\|\check{\sigma} * f\|_{p'} \leq C^2\|f\|_p$ for all $f \in \mathcal{S}$

To prove (3) (hence (1)), let $\phi \in \mathcal{S}$ have compact support and satisfy

$$\phi(x) = 1\ \forall x \in B(0,1).$$

Put

$$\psi_j(x) = \phi(2^{-j}x) - \phi(2^{-(j-1)}x).$$

Show that if $f \in \mathcal{S}$, then

$$\sigma * f = (\phi\check{\sigma}) * f + \sum_{j\geq 1}(\psi_j\check{\sigma}) * f \quad \text{uniformly.}$$

Now prove the estimates

$$\|(\psi_j\check{\sigma}) * f\|_\infty \lesssim 2^{-j\frac{n-1}{2}}\|f\|_1,$$

and

$$\|(\psi_j\check{\sigma}) * f\|_2 \leq 2^j\|f\|_2.$$

Use these and Riesz-Thorin to obtain (3) in the case

$$p' > 2(n+1)/(n-1).$$

To prove the endpoint estimate, show that

$$\Big\|\sum_j 2^{(\frac{n-1}{2}+it)j}(\psi_j\check{\sigma}) * f\Big\|_\infty \lesssim \|f\|_1,$$

$$\Big\|\sum_j 2^{(-1+it)j}(\psi_j\check{\sigma}) * f\Big\|_2 \lesssim \|f\|_2,$$

and then use complex interpolation. □

A different proof, based on estimates for Fourier integral operators, may be found in [**20**].

CHAPTER 5

Kakeya sets

A basic fact proved by Besicovitch in the 20's is that for any $n \geq 2$ there is a compact set $E \subset \mathbb{R}^n$ with measure zero, which contains a unit line segment in every direction, i.e.,

$$\forall e \in S^{n-1}\ \exists a \in \mathbb{R}^n:\ a + te \in E\ \forall t \in [-1/2, 1/2].$$

Such sets are called *Besicovitch* or *Kakeya sets*.

There are many variants of this construction. We will present one of them (due to T. Wolff) and then discuss some further properties of Besicovitch sets.

LEMMA 5.1. *Let N be a large integer. Then there is a family of lines $\{l_a\}$, where a runs over the set of N^N numbers of the form*

$$a = \sum_{j=1}^{N} \frac{a_j}{N^j}, \quad a_j \in \{0, \ldots, N-1\},$$

such that the slope of l_a is a, and if we let l_a^t be the unique $y \in \mathbb{R}$ such that $(t, y) \in l_a$, then

(i) If $a < b$ then $l_a^1 < l_b^1$.

(ii) For each $t \in [0, 1]$, the set $\{y \in \mathbb{R} : |y - l_a^t| \leq N^{-N}$ for some $a\}$ has measure $\leq 4/N$.

PROOF. We define l_a by letting its y-intercept be

$$-\sum_{j=1}^{N} \frac{(j-1)a_j}{N^{j+1}},$$

and check that (i) and (ii) hold. We have

$$l_a^t = -\sum_{j=1}^{N} \frac{(j-1)a_j}{N^{j+1}} + t \sum_{j=1}^{N} \frac{a_j}{N^j} = \sum_{j=1}^{N} \frac{(Nt - j + 1)a_j}{N^{j+1}}.$$

Proof of (i). If $a > b$, then let k be the smallest index with $a_k \neq b_k$. Then $a_k - b_k \geq 1$, and $a_j - b_j \geq -(N-1)$ for $j > k$. So

$$\begin{aligned} l_a^1 - l_b^1 &= \frac{(n-k+1)(a_k - b_k)}{N^{k+1}} + \sum_{j>k} \frac{(N-j+1)(a_j - b_j)}{n^{j+1}} \\ &\geq \frac{N-k+1}{N^{k+1}} - (N-1) \sum_{j>k} \frac{N-j+1}{N^{j+1}} \\ &\geq \frac{N-k+1}{N^{k+1}} - (N-k+1)(N-1) \sum_{j>k} \frac{1}{N^{j+1}} > 0, \end{aligned}$$

by the formula for the sum of a geometric series.

Proof of (ii). Given $t \in [0,1]$, choose an integer k such that $(k-1)/N \leq t < k/N$. Suppose that $a_j = b_j$ for $j \leq k-1$. Then

$$|l_a^t - l_b^t| = \Big| \sum_{j \geq k} \frac{(Nt - j + 1)(a_j - b_j)}{N^{j+1}} \Big| \leq \sum_{j \geq k} \frac{(|k-j|+1)|a_j - b_j|}{N^{j+1}},$$

with the last inequality true since $Nt - j + 1 \in [k-j, k-j+1)$. So,

$$|l_a^t - l_b^t| \leq \sum_{j \geq k} \frac{j-k+1}{N^j} \leq 2N^{-k} \quad \text{(if } N \text{ is large).}$$

There are N^{k-1} choices for the sequence $\{a_j\}_{j=1}^{k-1}$, so the set $\{l_a^t\}$ is contained in the union of N^{k-1} intervals of length $2N^{-k}$. Hence $\{y : |y - l_a^t| \leq N^{-N}\}$ is contained in N^{k-1} intervals of length $2N^{-N} + 2N^{-k} \leq 4N^{-k}$, so has measure $\leq 4/N$. □

Now consider (for fixed a) the set

$$S_{l_a}^{\delta} = \{(t,y) : 0 \leq t \leq 1,\ \operatorname{dist}(y, l_a^t) \leq \delta\},$$

where $\delta = 1/2N^{-N}$. It evidently contains a line segment connecting $x = 0$ to $x = 1$ with slope m, for every m with $|m - a| \leq N^{-N}$. If $0 \leq m \leq 1$, then $|m - a| \leq N^{-N}$ for some $a = \sum_{j=1}^{N} \frac{a_j}{N^j}$. Now let

$$E = \bigcup_a S_{l_a}^{\delta}.$$

We will use the notation $E^t = \{y : (t,y) \in E\}$. The following is now obvious.

Corollary 5.1. *There is a set E_N with the following two properties.*

(i) E_N contains a line segment connecting $x = 0$ to $x = 1$ with slope m, for every $m \in [0,1]$.

(ii) $|E_N^t| \leq 4/N$ for every $t \in [0,1]$ (in particular, $|E_N| \leq 4/N$).

Remark. The preceding construction may be understood geometrically in terms of a variant on the Perron tree (cf [**10**]). Namely, start with a triangle. Cut it in N pieces by subdividing the vertical edge in N equal segments. Leave the top triangle alone and slide the others upward until the $x = 0$ intercepts coincide. Next, take each of the resulting triangles and subdivide it in N triangles as above. Thus, we have N families of N "small" triangles. Within each family, leave the top triangle alone and slide the others upward until the $x = 1/N$ intercepts coincide. Subdivide each of the N^2 resulting triangles obtaining N^2 families of N "smaller" triangles. Within each family, leave the top triangle alone and slide the others upward until the $x = 2/N$ intercepts coincide. Now repeat the process at abscissas $3/N, 4/N, \ldots$.

Proposition 5.1. *Besicovitch sets exist in $\mathbb{R}^n$, for any $n \geq 2$.*

Proof. To construct a Besicovitch set in $\mathbb{R}^2$, it suffices to construct a compact set with measure zero containing a line segment connecting $x = 0$ to $x = 1$ with slope m for every $m \in [0,1]$. This is done by passing to the limit as $N \to \infty$ in Corollary 5.1. However, one has to be careful about convergence.

Lemma 5.2. *Suppose F is a compact set with property (i) of Corollary 5.1 and that $\delta > 0$, $\varepsilon > 0$. Then there is another compact set $\tilde{F}$ with property (i), such that $\tilde{F} \subset \{z : dist(z, F) < \delta\}$ and $|\tilde{F}| < \varepsilon$.*

Proof. Note that the sets E_N in Corollary 5.1 are contained in $Q \stackrel{\text{def}}{=} [0,1] \times [-1,1]$. If l is a segment connecting $x = 0$ to $x = 1$ with slope m and y-intercept b then the affine map

$$A_l^\delta(x,y) = (x, \delta y + b + mx)$$

takes Q onto S_l^δ, and maps segments with slope μ to segments with slope $m + \delta\mu$. Accordingly, $A_l^\delta(E_N)$ is a subset of S_l^δ which contains segments with all slopes between m and $m + \delta$, and $|A_l^\delta(E_N)| \le 4\delta/N$. Now choose segments $l_j \subset F$ with

$$\text{slope}(l_j) = j\delta, \quad j = 0, \dots, [1/\delta].$$

Let

$$\tilde{F} = \bigcup_{j=0}^{[1/\delta]} A_{l_j}^\delta(E_N),$$

where N is sufficiently large. Then $\tilde{F} \subset \{z : \text{dist}(z, F) < \delta\}$ and $\tilde{F}$ contains segments with all slopes between 0 and 1. Moreover

$$|\tilde{F}| \le [1/\delta]\frac{4\delta}{N} \le \frac{4}{N} < \varepsilon,$$

provided N has been chosen $> 4/\varepsilon$. □

To finish the proof of the proposition when $n = 2$, we recursively choose a sequence of sets $\{F_j\}_{j=1}^\infty$ and numbers $\delta_j \to 0$ such that

(i) F_j has property (i) of Corollary 5.1, $\forall j \ge 1$.
(ii) $\{z : \text{dist}(z, F_j) \le \delta_j\} \subset \{z : \text{dist}(z, F_{j-1}) \le \delta_{j-1}\}$, $\forall j \ge 2$.
(iii) $|\{z : \text{dist}(z, F_j) \le \delta_j\}| < 2^{-j}$, $\forall j \ge 2$.

Namely, we can take F_1 to be E_N of Corollary 5.1 for any large enough N, and any sufficiently small number for δ_1. If F_j and δ_j have been chosen, then we choose F_{j+1} by Lemma 5.2 with $F = F_j$, $\delta = \delta_j$ and $\varepsilon = 2^{-(j+1)}$. If we then choose δ_{j+1} sufficiently small we will have (i)-(iii) for $j + 1$. Now let

$$F = \bigcap_j \{z : \text{dist}(z, F_j) \le \delta_j\}.$$

Then F is compact with measure zero, and a simple compactness argument shows that F has property (i) of Corollary 5.1. This completes the proof of the proposition when $n = 2$. For $n > 2$, it suffices to consider $E \times D$, where E is a Besicovitch set in $\mathbb{R}^2$, and D is a closed disc of radius 1 in $\mathbb{R}^{n-2}$. □

We now give another application of Lemma 5.1, which is needed for the disc multiplier counterexample argument.

Proposition 5.2. *Let* $\delta = 1/100N^{-N}$. *Then there is a collection of* $1/(100\delta)$ *rectangles* T_a *with dimensions* $1 \times \delta$, *so that*
(i) $|\bigcup_a T_a| \le 4/N$.
(ii) Let $\tilde{T}_a$ *be the rectangle obtained by translating* T_a *along its axis by* C_0 *units* $C_0 \ge 2$. *Then the rectangles* $\tilde{T}_a$ *are pairwise disjoint.*

Proof. Let l_a be one of the segments in Lemma 5.1. Form the rectangle T_a as follows: T_a has length 1, width δ, axis along l_a and its furthest left vertex is on the y-axis. One can check that T_a is contained in the set

$$\left\{(x,y) : 0 \le x \le 1 \text{ and } |y - l_a^x| < \frac{1}{100}N^{-N}\right\},$$

and furthermore $\tilde{T}_a$ is contained in

$$\Big\{(x,y) : x \geq \sqrt{2} \text{ and } |y - l_a^x| < \frac{1}{100}N^{-N}\Big\}.$$

Property (i) now follows from (ii) of Lemma 5.1. For (ii), suppose toward a contradiction that $a > b$ and $\tilde{T}_a \cap \tilde{T}_b \neq \emptyset$. Fix $(x,y) \in \tilde{T}_a \cap \tilde{T}_b$. Then $x > \sqrt{2}$ and

$$|l_a^x - l_b^x| \leq |l_a^x - y| + |y - l_b^x| < \frac{1}{50}N^{-N}.$$

On the other hand

$$l_a^x - l_b^x = (l_a^x - l_a^1) - (l_b^x - l_b^1) + (l_a^1 - l_b^1).$$

The last term is positive by (i) of Lemma 5.1, so

$$l_a^x - l_b^x \geq N^{-N}(\sqrt{2} - 1).$$

This is a contradiction since $\sqrt{2} - 1 > 1/50$. □

There is a basic open question about Besicovitch sets, which can be stated vaguely as “How small can they really be?”. In order to state a more precise question, we need a notion of “size”, or fractal dimension. One can work with the Hausdorff dimension, but to avoid technical complications, we use instead the “lower Minkowski dimension” (see [**17**] for several different notions of dimension) defined as follows: If $E \subset \mathbb{R}^n$ is compact then

$$\dim E = \sup\{\alpha : \exists C_a \text{ with } |E_\delta| \geq C_\alpha^{-1}\delta^{n-\alpha} \ \forall \delta \in (0,1]\},$$

where E_δ is by definition $\{z : \operatorname{dist}(z,E) < \delta\}$. Thus, $\dim E$ measures the rate at which $|E_\delta| \to 0$, as $\delta \to 0$. If E has positive measure then $\dim E = n$, if E is a point then $\dim E = 0$, if E is the Cantor set then $\dim E = \log 2/\log 3$.

Kakeya problem. If $E \subset \mathbb{R}^n$ is a Besicovitch set, then does it follow that the dimension of E is n?

If $n = 2$, then the answer is yes. This is due to R.O. Davies, see [**10**]. For general n, we refer the reader to [**21**], [**3**], [**14**] and [**15**].

In order to discuss this further, we need some notation. Since there is no distinction between segments pointing in the e and $-e$ direction, we let

$$\mathbb{P}^{n-1} = S^{n-1}/\{\pm 1\},$$

i.e., S^{n-1} with e and $-e$ identified, and define a distance on $\mathbb{P}^{n-1}$ by

$$\theta(e,f) = \cos^{-1}(|e - f|) \in [0, \pi/2],$$

thus, $\theta(e,f)$ is the unoriented angle subtended by e and f. For $e \in \mathbb{P}^{n-1}$, $a \in \mathbb{R}^n$, we let

$$T_e^\delta(a) = \Big\{x \in \mathbb{R}^n : |(x-a)\cdot e| < \frac{1}{2} \text{ and } |P_{e^\perp}(x-a)| < \delta\Big\}.$$

We will need the following purely geometrical fact, whose proof is left to the reader.

Lemma 5.3. *Assume that $e, f \in \mathbb{P}^{n-1}$ and $a, b \in \mathbb{R}^n$, then*
(i) $|T_e^\delta(a) \cap T_f^\delta(b)| \leq C\delta^n/(\theta(e,f) + \delta)$.
(ii) $\operatorname{diam}(T_e^\delta(a) \cap T_f^\delta(b)) \leq C\delta/(\theta(e,f) + \delta)$.

So, $T_e^\delta(a) \cap T_f^\delta(b)$ is contained in a rectangle of dimensions

$$C\delta \times \cdots \times C\delta \times \frac{C\delta}{\theta(e,f)}.$$

Note that the bounds are independent of a and b. We also define a δ-separated set in $\mathbb{P}^{n-1}$ to be a set $\{e_j\}$ such that $\theta(e_j, e_k) \geq \delta$ for all $j \neq k$. A maximal δ-separated set is a set which is δ-separated and is not contained in any larger δ-separated subset. If $\{e_j\}_{j=1}^M$ is a maximal δ-separated subset, then $M \simeq \delta^{-(n-1)}$. This may be seen by volume counting, since the discs $\{e \in \mathbb{P}^{n-1} : \theta(e, e_j) < \delta/2\}$ are disjoint (by δ-separateness), and the discs $\{e \in \mathbb{P}^{n-1} : \theta(e, e_j) < \delta\}$ cover $\mathbb{P}^{n-1}$ (by maximality).

We will now prove a partial result on the Kakeya problem.

Proposition 5.3. *If $E \subset \mathbb{R}^n$ is a Besicovitch set, then* $\dim E \geq (n+1)/2$.

Proof. The proof we give is due to Bourgain, see [**2**]. It is not the shortest possible, but it is the most illuminating. Note to begin with, that E_δ must contain the δ-neighborhood of a unit line segment in the e direction for every e. Thus

$$\forall e \in \mathbb{P}^{n-1}\ \exists a \in \mathbb{R}^n : T_e^\delta(a) \subset E_\delta. \tag{5.1}$$

Fix a maximal $C_0\delta$-separated subset $\{e_j\}_{j=1}^M$, $M \simeq \delta^{-(n-1)}$ and let T_j be the tube $T_{e_j}^\delta(a_j)$ given by (5.1). Here C_0 is a large constant. Let N be a large integer to be chosen later and consider two possibilities.

(i) There is no point $x \in \mathbb{R}^n$ such that x belongs to more than N T_j's.
(ii) There is at least one point $b \in \mathbb{R}^n$ which belongs to at least N T_j's.

In case (i) we have

$$|E_\delta| \geq \Big|\bigcup_j T_j\Big| \geq \frac{1}{N}\sum_j |T_j| \simeq \frac{1}{N}\delta^{-(n-1)}\delta^{n-1} = \frac{1}{N}.$$

In case (ii), fix a point b belonging to N T_j's. We can assume that these are $T_1, \ldots, T_N$. Consider the "outer halves" of the tubes, i.e., the sets

$$\tilde{T}_j = \Big\{x \in \mathbb{R}^n : |x-b| \geq \frac{1}{4}\Big\} \cap T_j, \quad j = 1, \ldots, N.$$

It is clear that $|\tilde{T}_j| \simeq |T_j| \simeq \delta^{n-1}$. On the other hand, the sets $\tilde{T}_j$ are pairwise disjoint, provided C_0 has been chosen large enough. This follows because, by Lemma 5.3, we have

$$\operatorname{diam}(T_j \cap T_k) \leq \frac{C}{C_0} < \frac{1}{4}, \quad \text{(if } C_0 \text{ is large)}$$

and $b \in T_j \cap T_k$, $|x-b| \geq 1/4\ \forall x \in \tilde{T}_j \cup \tilde{T}_k$. Therefore

$$|E_\delta| \geq \Big|\bigcup_j \tilde{T}_j\Big| = \sum_j |\tilde{T}_j| \simeq N\delta^{n-1}.$$

We conclude that

$$|E_\delta| \gtrsim \min\Big\{\frac{1}{N}, N\delta^{n-1}\Big\}$$

in all cases. Taking $N = [\delta^{-(n-1)/2}]$, this means that

$$|E_\delta| \gtrsim \delta^{(n-1)/2}, \quad \text{for all } \delta,$$

which is equivalent to $\dim E \geq (n+1)/2$. □

Note that Proposition 5.3 does not give the right bound when $n = 2$, since it is known that then $\dim E = 2$.

Proposition 5.4. *If $E \subset \mathbb{R}^2$ is a Besicovitch set, then* $\dim E = 2$.

Proof. The idea of this proof is based on an argument due to Córdoba [**7**], although he did not state the result this way. Fix $\delta > 0$, let $\{e_j\}_{j=1}^M$ be a maximal δ-separated set of directions in $\mathbb{P}^1$, and let $T_j = T_{e_j}^\delta(a_j)$ be a $1 \times \delta$ rectangle with axis in the e_j direction which is contained in E_δ. Note that if $\delta < \sigma < \pi/2$, then for each j, the set $\{k : \theta(e_j, e_k) \le \sigma\}$ has cardinality $\le C\sigma/\delta$. Therefore

$$\begin{aligned}1 \simeq \sum_{j=1}^M |T_j| = \Big\| \sum_{j=1}^M \chi_{T_j} \Big\|_1 &\le \Big\| \sum_{j=1}^M \chi_{T_j} \Big\|_2 |E_\delta|^{1/2} = \Big(\sum_{j,k} |T_j \cap T_k| \Big)^{1/2} |E_\delta|^{1/2} \\ &\lesssim \Big(M\delta + \sum_{j \neq k} |T_j \cap T_k| \Big)^{1/2} |E_\delta|^{1/2}. \end{aligned} \tag{5.2}$$

Fix j and consider $\sum_{k:k\neq j} |T_j \cap T_k|$. By Lemma 5.3,

$$|T_j \cap T_k| \le C\delta^2/\theta(e_j, e_k).$$

Hence

$$\begin{aligned}\sum_{k:k\neq j} |T_j \cap T_k| &\le \sum_{0\le m\le \log(1/\delta)} \operatorname{card}(\{k : \theta(e_j, e_k) \in [\delta 2^m, \delta 2^{m+1}]\}) \frac{\delta^2}{\delta 2^m} \\ &\le C \sum_{0\le m\le \log(1/\delta)} 2^m \frac{\delta^2}{\delta 2^m} \simeq \delta \log \frac{1}{\delta}.\end{aligned}$$

(5.2) now implies that

$$1 \lesssim \left(M\delta + M\delta \log \frac{1}{\delta} \right)^{1/2} |E_\delta|^{1/2} \lesssim \left(\log \frac{1}{\delta} \right)^{1/2} |E_\delta|^{1/2}.$$

Hence

$$|E_\delta| \ge (C \log(1/\delta))^{-1}.$$

Since $(\log(1/\delta))^{-1}$ goes to zero slower than any power of δ, this implies that $\dim E = 2$. □

Remark. The same proof works in any dimension, but gives the bound $\dim E \ge 2$, which is rather disappointing if $n > 2$.

It is a remarkable fact that the restriction conjecture implies the Kakeya conjecture. This is due to Bourgain, although a related construction was done earlier in [**4**]. Both constructions are variants on the argument in [**11**].

Proposition 5.5. *If the restriction conjecture is true, then Besicovitch sets have dimension n.*

Proof. Let E be a Besicovitch set. Fix δ; then E_δ contains a tube $T_e^\delta(a_e)$ for every $e \in \mathbb{P}^{n-1}$. Let $\{e_j\}_{j=1}^M$ be a maximal C_0-separated subset of $\mathbb{P}^{n-1}$, and also regard $\{e_j\}_{j=1}^M$ as a set on the sphere S^{n-1} by choosing (arbitrarily) one of the two possible directions. Then, in the notation of Chapter 4, the spherical caps $C_{e_j}^\delta$ are disjoint, provided C_0 is large enough. Also, let τ_j be the tube obtained by dilating $T_{e_j}^\delta(a_{e_j})$ by a factor of δ^{-2}. Then, in the notation of Chapter 4, $\tau_j = \tau_{e_j}^\delta(\delta^{-2} a_{e_j})$. By the Knapp counterexample, there are functions $f_j : S^{n-1} \to \mathbb{C}$ such that

$$\operatorname{supp} f_j \subset C_{e_j}^\delta, \ \|f_j\|_\infty \le 1,$$

and

$$|(f_j d\sigma)\check{}(x)| \geq C^{-1}\delta^{n-1}, \ \forall x \in \tau_j. \tag{5.3}$$

Now suppose $\{\varepsilon_j\}_{j=1}^M$ are ± 1's. Then

$$\Big\| \sum_{j=1}^M \varepsilon_j f_j \Big\|_{L^\infty(S^{n-1})} \leq 1,$$

so, by the restriction conjecture

$$\Big\| \sum_{j=1}^M \varepsilon_j (f_j d\sigma)\check{} \Big\|_q \leq C_q, \ \forall q > \frac{2n}{n-1}. \tag{5.4}$$

On the other hand

$$\begin{aligned}
\mathbb{E}\Big(\Big\| \sum_{j=1}^M \varepsilon_j (f_j d\sigma)\check{} \Big\|_q^q\Big) &= \int \mathbb{E}\Big(\Big| \sum_{j=1}^M \varepsilon_j (f_j d\sigma)\check{}(x)\Big|^q\Big) dx \\
\text{(by Khinchin)} \qquad &\gtrsim \int \Big(\sum_{j=1}^M |(f_j d\sigma)\check{}(x)|^2\Big)^{q/2} dx \\
\text{(by (5.3))} \qquad &\gtrsim \delta^{(n-1)q} \int \Big|\sum \chi_{\tau_j}(x)\Big|^{q/2} dx \\
(x \mapsto \delta^2 x) \qquad &= \delta^{(n-1)q-2n} \int \Big|\sum \chi_{T_j}(x)\Big|^{q/2} dx,
\end{aligned}$$

where we have set $T_j = T_{e_j}^\delta(a_{e_j})$. Combining this with (5.4) we conclude that

$$\Big\| \sum \chi_{T_j} \Big\|_{q/2} \leq \delta^{4n/q-2(n-1)}, \ \forall q > \frac{2n}{n-1}.$$

On the other hand

$$1 \simeq \sum_{j=1}^M |T_j| = \Big\| \sum \chi_{T_j} \Big\|_1 \leq \Big\| \sum \chi_{T_j} \Big\|_{q/2} |E_\delta|^{1-2/q} \leq C_q \delta^{4n/q-2(n-1)} |E_\delta|^{1-2/q},$$

which means

$$|E_\delta| \gtrsim \delta^{2n-2q/(q-2)}$$

or

$$\dim E \geq 2q/(q-2) - n.$$

As $q \searrow 2n/(n-1)$, the number $2q/(q-2) - n \to n$, so this finishes the proof. $\square$

Problems

1. If you know what Hausdorff dimension is, then show that

$$\text{Hausdorff dimension of } E \le \dim E$$

for all compact sets E, and that strict inequality can hold.

2. (a) Suppose $0 < \lambda \le 1$, $E \subset \mathbb{R}^n$ and the following holds: for any $e \in \mathbb{P}^{n-1}$, there is a tube $T_e^\delta(a)$ such that $|E \cap T_e^\delta(a)| \ge \lambda |T_e^\delta(a)|$. By generalizing the argument in Proposition 5.3, show that then $|E| \ge C^{-1}\delta^{(n-1)/2}\lambda^{(n+1)/2}$.
(b) A further generalization is possible. Suppose $o < \lambda \le 1$, $E \subset \mathbb{R}^n$, $\Omega \subset \mathbb{P}^{n-1}$ and for each $e \in \Omega$ there is a tube $T_e^\delta(a)$ such that $|E \cap T_e^\delta(a)| \ge \lambda |T_e^\delta(a)|$. Then $|E| \ge C^{-1}(\delta^{n-1}|\Omega|)^{1/2}\lambda^{(n+1)/2}$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}$. The *Kakeya maximal function* $f_\delta^* : \mathbb{P}^{n-1} \to \mathbb{R}$, is defined by

$$f_\delta^*(e) \stackrel{\text{def}}{=} \sup_a \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f|.$$

There is another formulation of the Kakeya problem in terms of this maximal function, namely, that the estimate

$$\forall \varepsilon\ \exists C_\varepsilon : \|f_\delta^*\|_{L^p(\mathbb{P}^{n-1})} \le C_\varepsilon \delta^{-\varepsilon} \|f\|_p, \quad \text{where } p = n, \tag{5.5}$$

should hold.
(a) Show that this estimate, if true, would imply that Besicovitch sets have dimension n.
(b) Show that the estimate (5.5) cannot hold if $p < n$ (hint: let f be the characteristic function of a disc of radius δ).
(c) Using the preceding problem, one can prove the following estimate for f_δ^*.

$$\|f_\delta^*\|_q \le C\delta^{-(n/p-1)}\|f\|_p,$$

if $q = (n-1)p'$ and $p < (n+1)/2$ (hint: interpolate between a restricted weak type $L^{(n+1)/2} \to L^{n+1}$ estimate and an $L^1 \to L^\infty$ estimate).
(d) Using the proof of Proposition 5.4, it is possible to verify the conjecture (5.5) when $n = 2$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$. The *X-ray transform* of f is defined by

$$Xf(e, y) = \int_{l_e} f(x+y)\, d\mathcal{L}^1(x), \ e \in S^{n-1}, \ y \in l_e^\perp,$$

where l_e is the line through the origin in the e direction, and $l_e^\perp$ is the orthogonal complement of l_e. Christ [**5**] and Drury [**9**] proved the following estimate.

$$\|Xf\|_{L^{n+1}} \lesssim \|f\|_{\frac{n+1}{2}}, \tag{5.6}$$

where

$$\|Xf\|_{L^{n+1}}^{n+1} = \int_{S^{n-1}} \int_{l_e^\perp} |Xf(e,y)|^{n+1} d\mathcal{L}^{n-1}(y)\, d\sigma(e).$$

Show that (5.6) implies Proposition 5.3.

CHAPTER 6

Fefferman's counterexample

An L^p *multiplier* is a function m such that

$$\|(m\hat{f})\check{}\|_p \leq C\|f\|_p \quad \forall f \in \mathcal{S},$$

and its L^p *multiplier norm* $\|m\|_{M_p}$ is the smallest possible C in the above inequality. When $p = 2$, it is clear by the Plancherel theorem that any bounded function is an L^p multiplier, with $\|m\|_{M_p} \leq \|m\|_\infty$ (in fact, equality holds).

The characteristic function of the interval $[-1, 1]$ is an L^p multiplier. This follows from the boundedness of the Hilbert transform. Therefore, it seems natural to conjecture that, in any dimension, the characteristic function of the unit disc should be an L^p multiplier as well. It is a striking fact that the conjecture turns out to be false. The counterexample is due to C. Fefferman [**11**].

PROPOSITION 6.1. *If $p \neq 2$ and $n \geq 2$, then the characteristic function of the unit disc is not an L^p multiplier.*

This is based on the the existence of Besicovitch sets, more precisely on Proposition 5.2. One represents the operator as a convolution operator $(\chi\hat{f})\check{} = \check{\chi} * f$, where $\chi = \chi_{B(0,1)}$, and then uses the "sliding" argument plus asymptotics for $\check{\chi}$, i.e., the following.

LEMMA 6.1. *When $|x|$ is large,*

$$\check{\chi}(x) = 2\cos(2\pi(|x| + (n+1)/8))|x|^{-(n+1)/2} + B(x),$$

where $|B(x)| \leq C|x|^{-(n+3)/2}$.

PROOF. See [**13**]. □

The Knapp counterexample is replaced by the following.

LEMMA 6.2. *Suppose (in $\mathbb{R}^2$), $\tau = \tau_e^\delta(a)$ (notation as in Chapter 5) for some a, and let*

$$\tilde{\tau} = \tau_e^\delta(a + C_0\delta^{-2}e)$$

be the tube obtained by translating τ along its axis by distance $C_0\delta^{-2}$ where C_0 is a large constant. Then there is a Schwartz function g with

$$\operatorname{supp} g \subset \tilde{\tau} \text{ and } \|g\|_\infty \leq 1,$$

such that

$$|\check{\chi} * g(x)| \geq C^{-1} \text{ for all } x \in \tau.$$

PROOF. We assume for simplicity that $e = (1, 0)$, $a + C_0\delta^{-2}e = 0$, and we fix a Schwartz function ψ supported in $\tilde{\tau}$ with the following properties.

$$\|\psi\|_\infty \leq 1, \int \psi \geq C^{-1}\delta^{-3} \text{ and } \left\|\frac{d\psi}{dx_1}\right\|_\infty \leq C\delta^2.$$

Such a function may be obtained as follows: Start with any nonnegative Schwartz function ϕ supported in the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$ and consider the function $\psi(x) = \phi(\delta^2 x_1, \delta x_2)$. The estimates follow from the chain rule and change of variables formula. Now let

$$g(y) = e^{2\pi i(y_1 + (n+1)/8)}\psi(y).$$

We must check that $|\check{\chi} * g|$ is bounded below on τ. We have

$$\check{\chi} * g(x) = \int_{\tilde{\tau}} e^{-2\pi i(|x-y|+(n+1)/8)}|x-y|^{-3/2} g(y)dy + \int_{\tilde{\tau}} e^{2\pi i(|x-y|+(n+1)/8)}|x-y|^{-3/2} g(y)dy + \int_{\tilde{\tau}} B(x-y)g(y)dy \quad (6.1)$$

We claim that the last two terms are small and the first term is large. In fact, if $x \in \tau$ and $y \in \tilde{\tau}$ then $|x-y| \geq \delta^2$, so $|B(x-y)| \lesssim \delta^5$, hence the last term is $\lesssim \delta^5|\tilde{\tau}| \simeq \delta^2$. Next, consider the first term. It is

$$\int_{\tilde{\tau}} e^{2\pi i(y_1 - |x-y|)}|x-y|^{-3/2}\psi(y)dy \quad (6.2)$$

If $x \in \tau$, $y \in \tilde{\tau}$, then $y_1 > x_1 + (C_0 - 1)\delta^{-2}$, $|y_2| < \delta^{-1}$, $|x_2| < \delta^{-1}$. So

$$|x-y| = \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2} = (y_1 - x_1)\sqrt{1 + \frac{(y_2-x_2)^2}{(y_1-x_1)^2}}$$
$$= y_1 - x_1 + E,$$

where

$$|E| \leq \frac{1}{2}\frac{(y_2-x_2)^2}{y_1-x_1} < 2(C_0-1)^{-1}.$$

Accordingly,

$$\left|\int_{\tilde{\tau}} e^{2\pi i(x_1 - E)}|x-y|^{-3/2}\psi(y)dy\right| \gtrsim \int_{\tilde{\tau}} |x-y|^{-3/2}\psi(y)dy,$$

provided C_0 is large enough, so that $\cos(2(C_0-1)^{-1}) \geq 1/2$. Since

$$\int_{\tilde{\tau}} |x-y|^{-3/2}\psi(y)dy \gtrsim \delta^{-3}\int \psi(y)dy \simeq 1,$$

we conclude that the first term in (6.1) is $\geq C^{-1}$ in absolute value. The second term is

$$\int_{\tilde{\tau}} e^{2\pi i(y_1 + |x-y|)}|x-y|^{-3/2}\psi(y)dy. \quad (6.3)$$

Note that

$$\frac{\partial}{\partial y_1}(|x-y| + y_1) = \frac{y_1 - x_1}{|y-x|} + 1 \geq 1,$$

when $y \in \tilde{\tau}$, $x \in \tau$. Hence, if we integrate by parts with respect to y_1 we obtain

$$(6.3) = \frac{1}{2\pi i}\int_{\tilde{\tau}} \frac{\partial}{\partial y_1} e^{2\pi i(|x-y|+y_1)}\left(1 + \frac{y_1-x_1}{|y-x|}\right)^{-1}|x-y|^{-3/2}\psi(y)dy_1dy_2$$
$$= -\frac{1}{2\pi i}\int_{\tilde{\tau}} e^{2\pi i(|x-y|+y_1)}\frac{\partial}{\partial y_1}\left[\left(1 + \frac{y_1-x_1}{|y-x|}\right)^{-1}|x-y|^{-3/2}\psi(y)\right]dy_1dy_2.$$

One can check that

$$\left|\frac{d}{dy_1}\left[\left(1 + \frac{y_1-x_1}{|y-x|}\right)^{-1}|x-y|^{-3/2}\psi(y)\right]\right| \leq C\delta^5,$$

when $x \in \tau$, $y \in \tilde{\tau}$. We conclude that $|(6.3)| \leq C\delta^5|\tilde{\tau}| \leq C\delta^2$. Hence (6.1) is a sum of three terms, one of which is $\geq$ constant, and the other two are $\leq C\delta^2$. It follows that $|\check{\chi} * g|$ is bounded below by a constant on τ, provided δ is small. $\square$

Proof of Proposition 6.1 ($p > 2$, $n = 2$). Let $\delta = 1/100N^{-N}$, where N is large. By Proposition 5.2 and a dilation by a factor of δ^{-2}, we can find $\simeq 1/\delta$ rectangles τ_j with dimensions $\delta^{-2} \times \delta^{-1}$, so that $\{\tilde{\tau}\}$ are disjoint and $|\bigcup_j \tau_j| \lesssim \delta^{-4}/N$. Choose a funtion g_j corresponding to τ_j by Lemma 6.2, and consider $\sum \varepsilon_j g_j$ where each ε_j is ± 1. Then, for any choice of ε_j's, we have

$$\Big\| \sum_j \varepsilon_j g_j \Big\|_p^p = \sum_j \|g_j\|_p^p \leq \frac{1}{\delta}\delta^{-3} = \delta^{-4}. \tag{6.4}$$

On the other hand, $(\chi \hat{f})\check{} = \check{\chi} * f$, and

$$\begin{aligned} \mathbb{E}\Big(\Big\| \sum_j \varepsilon_j \check{\chi} * g_j \Big\|_p^p \Big) &= \int \mathbb{E}\Big(\Big| \sum_j \varepsilon_j \check{\chi} * g_j(x) \Big|^p \Big) dx \\ \text{(by Khinchin)} \qquad &\simeq \int \Big(\sum_j |\check{\chi} * g_j(x)|^2 \Big)^{p/2} dx \\ &\gtrsim \int \Big| \sum_j \chi_{\tau_j}(x) \Big|^{p/2} dx. \end{aligned}$$

By Hölder's inequality, we have

$$\delta^{-4} \simeq \sum_j |\tau_j| = \Big\| \sum_j \chi_{\tau_j} \Big\|_1 \leq \Big| \bigcup_j \tau_j \Big|^{1-2/p} \Big\| \sum_j \chi_{\tau_j} \Big\|_{p/2},$$

and therefore

$$\Big\| \sum_j \chi_{\tau_j} \Big\|_{p/2}^{p/2} \geq \delta^{-4} N^{p/2-1}.$$

So, for some choice of $\{\varepsilon_j\}$,

$$\Big\| \check{\chi} * \sum_j \varepsilon_j g_j \Big\|_p^p \gtrsim \delta^{-4} N^{p/2-1}.$$

Together with (6.4), this implies $\|\chi\|_{M_p} \gtrsim N^{1-2/p}$, and since $p > 2$, this can be made arbitrarily large by choosing N appropriately. The proof is complete. $\square$

Remark. The characteristic function of a regular polygon in the plane is an L^p multiplier. Its norm, however, tends to infinity (logarithmically) as the number of the sides of the polygon goes to infinity. This was shown by Córdoba [**8**] using a Kakeya-type argument.

CHAPTER 7

Some topics from combinatorial geometry

In this chapter we consider a discrete problem which should clearly be related to the Kakeya problem.

DEFINITION. *A line l is said to be incident to a point p if p lies on l.*

Given points $\{p_j\}_{j=1}^k$ and lines $\{l_i\}_{i=1}^n$, how many pairs (i, j) can there be such that l_i is incident to p_j?

THEOREM 7.1 (Szemerédi-Trotter). *In $\mathbb{R}^2$, the number of incidences between k points and n lines is $\leq C((kn)^{2/3} + k + n)$.*

We will give a proof from [**6**]. First we discuss a certain partial result.

LEMMA 7.1. *Assume (a_{ij}) is an $n \times m$ $(0, 1)$-matrix and that (a_{ij}) has no 2×2 submatrix consisting of 1's. Then (a_{ij}) contains at most $C(mn^{1/2} + n)$ 1's altogether (The assumption means that there do not exist i_1, i_2, j_1, j_2 such that $a_{i_1 j_1} = a_{i_2 j_1} = a_{i_1 j_2} = a_{i_2 j_2} = 1$).*

PROOF. Let

$$I = |\{(i, j) : a_{ij} = 1\}|, \text{ (total number of 1's)},$$

and

$$m_i = |\{j : a_{ij} = 1\}|, \text{ (number of 1's in the } i\text{-th row)}.$$

Let

$$\mathcal{J} = \{(i, j, k) : j \neq k \text{ and } a_{ij} = a_{ik} = 1\}.$$

We will count $\mathcal{J}$ in two different ways:

$$|\mathcal{J}| = \sum_i m_i(m_i - 1) \geq \sum_{i:m_i \geq 2} \frac{1}{2} m_i^2,$$

since for fixed i, there are $m_i(m_i - 1)$ choices for $j \neq k$ with $a_{ij} = a_{ik} = 1$. On the other hand, if j and k have been chosen, there can be at most one choice for i, otherwise we would violate the assumption. Therefore

$$|\mathcal{J}| \leq m(m - 1).$$

Hence

$$\sum_{i:m_i \geq 2} m_i^2 \leq 2m^2,$$

and then

$$I = \sum_{i:m_i \leq 1} m_i + \sum_{i:m_i \geq 2} m_i \leq n + n^{1/2} \left(\sum_{i:m_i \geq 2} m_i^2 \right)^{1/2} \leq n + (2n)^{1/2} m.$$

□

Remark. If instead of "no 2×2 submatrix", we assume "no $s \times s$ submatrix", then the bound is $I \le C_s(mn^{2-1/s} + n)$ and is proved the same way, except that one now defines

$$\mathcal{J}_s = \{(i, j_1, \dots, j_s) : j_1, \dots, j_s \text{ are distinct and } a_{ij_1} = \dots = a_{ij_s} = 1\},$$

and uses Hölder's inequality instead of Cauchy-Schwarz at the last step. We will need the case $s = 3$ later on.

Corollary 7.1. *The number of incidences between k points and n lines in the plane is $\le C(kn^{1/2} + n)$. In particular, If $k = n$ this gives $I \le Cn^{3/2}$, whereas Szemerédi-Trotter gives $I \le Cn^{4/3}$.*

Proof. Two lines intersect in at most one point, so if we form a (0,1)-matrix via

$$a_{ij} = \begin{cases} 1, & \text{if } p_j \in l_i \\ 0, & \text{otherwise} \end{cases}$$

then (a_{ij}) has no 2×2-submatrix of 1's. Therefore, by Lemma 7.1, it has $\lesssim kn^{1/2} + n$ 1's altogether. ◻

In [**6**], this type of bound is called a *Canham threshold*. Lemma 7.1 is sharp (when $n \ge m$; if $n \le m$ one does better by reversing the roles of n and m). Here is an example when $n = m$ (the same example works in general). To describe it, we need the following number theoretic result.

Lemma 7.2. *Let $n = p^2$, where p is an odd prime. Then there is a subset $\Lambda \subset \{0, \dots, n-1\}$ such that*
(i) $|\Lambda| = n^{1/2}$.
(ii) The numbers $\lambda + \mu$ with $\lambda, \mu \in \Lambda$ and $\lambda \le \mu$ are all distinct.

Proof. Let $[m]_p$ be the remainder on dividing m by p. Define $\lambda_k = kp + [k^2]_p$, $0 \le k \le p-1$, and $\Lambda = \{\lambda_k\}_{k=0}^{p-1}$. Then property (i) is obvious. For (ii) suppose that $\lambda_i + \lambda_j = \lambda_k + \lambda_l$. Taking $\mod p$, we see that

$$i^2 + j^2 = k^2 + l^2 \mod p,$$

hence $i + j = k + l$. Then

$$i^2 - k^2 = l^2 - j^2 \mod p,$$

$$i - k = l - j \mod p.$$

So, by dividing $\mod p$, we get

$$i + k = l + j \mod p \text{ (unless } i = k).$$

Then

$$i - k = l - j \mod p,$$

$$i + k = l + j \mod p,$$

so

$$i = l \mod p,$$

and therefore $i = l$. ◻

Now, let $n = p^2$ and $\Lambda = \{\lambda_k\}_{k=0}^{p-1}$ be the set given by Lemma 7.2. Define $\{a_{ij}\}_{i,j=1}^n$ via

$$a_{ij} = \begin{cases} 1, & \text{if } i - j \in \Lambda \\ 0, & \text{if } i - j \notin \Lambda \end{cases}$$

Then, (a_{ij}) has no 2×2 submatrix of 1's: suppose that $i_1 - j_1 = \lambda_1$, $i_1 - j_2 = \lambda_2$, $i_2 - j_2 = \lambda_3$, $i_2 - j_2 = \lambda_4$, with $i_1 \neq i_2$ and $j_1 \neq j_2$. Then $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3$ contradicting the distinct sums property of Λ. On the other hand, suppose we fix i with $i \leq p^2$. Then $a_{ii+\lambda}$ for any $\lambda \in \Lambda$. So the i-th row of the matrix (a_{ij}) has p 1's and there are at least p^3 1's altogether, with $p^3 = n^{3/2}$.

One therefore needs a different type of argument to prove Szemerédi-Trotter. This will be the *cell decomposition* technique from [**6**]. First, some terminology.

- A *line arrangement* is a family of non vertical lines $l_1, \dots, l_r$ in $\mathbb{R}^2$.
- A *vertex* of the arrangement is a point where two or more lines intersect.
- A *cell* is a polygon in $\mathbb{R}^2$ (possibly unbounded) with no more than four sides. We take a cell to be an open set.
- A *triangulation of a line arrangement* is a decomposition of the components of $\mathbb{R}^2 \setminus \{l_1, \dots, l_r\}$ into cells together with their boundaries.

It is clear that, in general, it will require at least r^2 cells to triangulate an arrangement of r lines, since the r lines may split $\mathbb{R}^2$ into r^2 regions. For example, consider $r/2$ lines parallel to each of two given lines. Evidently, there are $(r/2 + 1)^2$ complementary components.

Lemma 7.3. *It is possible to triangulate a line arrangement using $\lesssim r^2$ cells. This can be done by an algorithm*

$$\{l_1, \dots, l_r\} \longrightarrow \triangle(\{l_1, \dots, l_r\})$$

where $\triangle(\{l_1, \dots, l_r\})$ is the set of cells forming a triangulation. Furthermore, this algorithm has the following property: each cell of $\triangle(\{l_1, \dots, l_r\})$ is also a cell of $\triangle(\{l_{j_1}, \dots, l_{j_4}\})$ for some 4-element subset $\{l_{j_1}, \dots, l_{j_4}\} \subset \{l_1, \dots, l_r\}$.

Proof. We let $\{p_k\}$ be the set of vertices of the arrangement, and for each k, we form the maximal segments m_k^+ and m_k^- which extend vertically up and down from p_k and do not intersect any line l_j. These segments together with the l_j's subdivide $\mathbb{R}^2$ into polygons. Each polygon has ≤ 4 sides. We leave the proof to the reader, the idea is that each polygon has a "top" and a "bottom". Also, each polygon has at most two vertical sides, i.e., four sides in all.

The last property in the statement of the Lemma is then clear, since in order to find $l_{j_1}, \dots, l_{j_4}$ to produce a cell Ω in the triangulation, we need only choose the lines from the top and bottom of Ω and two other lines whose intersection with the top and bottom yields vertices whose corresponding segments $m_k^\pm$ form the vertical sides. We have to show that $\triangle(\{l_1, \dots, l_r\})$ contains $\lesssim r^2$ cells. However, there are at most r^2 vertices, hence at most $2r^2$ vertical segments $m_k^\pm$. Each can be part of the boundary of at most two cells, so there are $\lesssim 4r^2$ cells which have a vertical boundary line. How many cells can there be which do not have a vertical boundary line? One can see that if Ω has no vertical boundary line, then the boundary of Ω is contained in at most two lines l_j. Furthermore, for each pair of lines l_{j_1}, l_{j_2} there are at most two cells whose boundary is contained in $l_{j_1} \cup l_{j_2}$, and therefore the number of cells with no vertical boundary line is at most twice the number of pairs of lines l_j, i.e., $2r^2$. This finishes the proof. □

If $\Omega \subset \mathbb{R}^2$ is an open set and l is a line, then we say that l *enters* Ω if $l \cap \Omega \neq \emptyset$.

LEMMA 7.4. *Let $C = \{l_1, \dots, l_n\}$ be a set of n lines, and fix $r < n$. Then it is possible to subdivide $\mathbb{R}^2$ into $\lesssim r^2$ cells in such a way that no more than $A(n \log n)/r$ lines $l_j \in C$ enter each given cell. In fact, if we choose at random r of the lines l_j and apply Lemma 7.3, then with probability $\geq 3/4$ (say) we obtain a cell decomposition with these properties.*

PROOF. First of all, there are clearly at most Cn^4 open sets $\Omega \subset \mathbb{R}^2$ which can be a cell in the decomposition obtained by choosing four of the lines l_j and applying the algorithm of Lemma 7.3 (there are n^4 choices of the four lines, and for each such choice, there are a bounded number of cells).

By the last statement of Lemma 7.3, there are at most Cn^4 open sets which can be a cell in the decomposition obtained by choosing r of the lines l_j and applying the algorithm. For each such set Ω let $P(\Omega)$ be the probability that Ω is actually a cell when the lines are chosen at random, and let $n(\Omega)$ be the cardinality of the set of lines in C which enter Ω. Then we claim that

$$P(\Omega) \leq \left(1 - \frac{n(\Omega)}{n}\right)^r,$$

and in fact this is clear, since in order for Ω to be a cell, it is necessary that none of the $n(\Omega)$ lines which enter Ω belongs to the random sample. Therefore, if ν is a fixed number, then

$$\begin{aligned}\operatorname{Prob}(n(\Omega) \geq \nu, \text{ for some } \Omega \in \triangle(\{l_1, \dots, l_r\})) &\leq \sum_{\Omega : n(\Omega) \geq \nu} P(\Omega) \\ &\leq \sum_{\Omega : n(\Omega) \geq \nu} \left(1 - \frac{n(\Omega)}{n}\right)^r \leq Cn^4 \left(1 - \frac{\nu}{n}\right)^r,\end{aligned}$$

which is small if $\nu = A(n \log n)/r$ with A large. □

PROOF OF THE SZEMERÉDI-TROTTER THEOREM. We will assume $k = n$ and will prove only a slightly weaker statement: Let I be the number of incidences between n lines $\{l_i\}_{i=1}^n$ and n points $\{p_j\}_{j=1}^n$. Then

$$I \leq Cn^{4/3} (\log n)^{1/3}, \text{ (instead of } I \leq Cn^{4/3}).$$

The assumption $k = n$ is easily removed using the same argument. However, to avoid losing a logarithmic factor, one needs a refinement of Lemma 7.4.

Let $r = n^{1/3} (\log n)^{1/3}$ and apply Lemma 7.4. We may assume that none of the points p_j lies on the vertical cell boundaries-otherwise we change the definition of "vertical" slightly. Let $\{\Omega_k\}_{k=1}^R$ be the resulting cell decomposition, $R \leq Cr^2$, and put

$$I(\Omega_k) = \text{cardinality}(\{(l_i, p_j) : p_j \in l_i, \text{ and } p_j \in \Omega_k\}).$$

Also, let $l_{i_1}, \dots, l_{i_r}$ be the lines in the random sample, and

$$I(l_{i_k}) = \text{cardinality}(\{(l_i, p_j) : p_j \in l_i, \text{ and } p_j \in l_{i_k}\}).$$

Then it is clear that

$$I \leq \sum_{k=1}^R I(\Omega_k) + \sum_{k=1}^r I(l_{i_k}).$$

Now by the "Canham threshold", i.e., Corollary 7.1, we have

$$I(\Omega_k) \lesssim n_k^{1/2} m_k + n_k,$$

where

$$m_k = \# \text{ of points } p_j \text{ which belong to } \Omega_k,$$

$$n_k = \text{\# of lines } l_i \text{ which enter } \Omega_k.$$

Since $n_k \leq A(n \log n)/r$, we conclude that

$$\begin{aligned}\sum_{k=1}^{R} I(\Omega_k) &\lesssim \sum_{k=1}^{R} \left(\frac{n \log n}{r}\right)^{1/2} m_k + \frac{n \log n}{r} R \\ &\lesssim \left(\frac{n \log n}{r}\right)^{1/2} n + nr \log n.\end{aligned}$$

Also, for each k we have

$$I(l_{i_k}) \leq n + (n-1)$$

since, of course, l_{i_k} is incident to $\leq n$ points p_j, and any other line l_i is incident to at most one point on l_{i_k}. Therefore

$$\sum_k I(l_{i_k}) \leq (2n-1)r.$$

We conclude that

$$I \lesssim \left(\frac{n \log n}{r}\right)^{1/2} n + nr \log n + nr \lesssim n^{4/3} (\log n)^{1/3},$$

by choice of r. □

The following example of Erdös shows that the theorem is sharp. Fix n and consider the n^2 lines l connecting a point $(0, k_0)$, $1 \leq k_0 \leq n$ to a point $(1, k_1)$, $1 \leq k_1 \leq n$, here k_0 and k_1 are integers. The equations of the lines are $y = xk_0 + (1-x)k_1$, hence if x is a rational with denominator q, then so is y. It follows that there are $\leq nq$ possibilities for y. Now fix a number B. There are q integers between B and $2B$ and for each of these, there are $\simeq B$ rationals with denominator q. Accordingly, there are B^2 rationals with denominator between B and $2B$. Consider the set of $\simeq B^3 n$ points of the form $(p/q, y)$ and incident to at least one line l. Each line is incident to $\simeq B^2$ such points, since the line must contain a point with any given x coordinate. Hence there are $\simeq B^2 n^2$ incidences between the lines l and the point p. Since $B^2 n^2 = (B^3 n n^2))^{2/3}$, we get the result.

Remarks. (1) In the situation of the Kakeya problem, the idea of the preceding construction can be used to show the following: For any $0 < \alpha < 1$, there is a compact set $E \subset \mathbb{R}^2$ with $\dim E \leq 1/2(1+3\alpha)$ such that for every $e \in S^{n-1}$, there is line l in the e direction with $\dim(E \cap l) \geq \alpha$. It is an interesting question whether the number $1/2(1+3\alpha)$ is sharp, and if not, what is the sharp number to replace it. Various partial results can be proved without much difficulty, for example, $\dim E \geq 1/2$ and $\dim E \geq 2\alpha$.

(2) There is a famous question called the "unit distance problem" which can be stated in the following (equivalent) ways.

(*) How many incidences can there be between n points in the plane and n circles of radius 1?

(**) Given n points p_i in the plane, how many pairs (p_i, p_j) can there be such that $|p_i - p_j| = 1$?

The proofs of the Szemerédi-Trotter theorem also apply to this problem and give the bound $Cn^{4/3}$. However, this bound is not known to be sharp. In fact, Erdös conjectured $C_\varepsilon n^{1+\varepsilon}$.

CHAPTER 8

Besicovitch-Rado-Kinney sets

A BRK (Besicovitch-Rado-Kinney) set is a compact set in the plane with measure zero, containing a circle of every radius between 1 and 2. Such sets can be constructed , as was done by Besicovitch-Rado and by Kinney, by modifying the construction of Besicovitch sets, and it is also possible to prove they exist by using the existence of Besicovitch sets, see [**10**] and [**17**]. The latter possibility may be understood in terms of the fact that lines are just circles passing through a fixed point if one works on the sphere.

One can ask the same dimension question in this context. We will discuss the following result from [**22**].

Proposition 8.1. *Any BRK set has dimension 2.*

This also has a maximal function formulation. Here we want to average over δ-neighborhoods of circles, and the role played by the direction of a line in the case of the Kakeya problem is now played by the radius of the circle. Therefore, if $f : \mathbb{R}^2 \to \mathbb{R}$ and $\delta > 0$, then we define $M_\delta f : [1, 2] \to \mathbb{R}$ by

$$M_\delta f(r) = \sup_x \frac{1}{|C_\delta(x, r)|} \int_{C_\delta(x,r)} |f|,$$

where $C_\delta(x, r) = \{y : r - \delta/2 < |x - y| < r + \delta/2\}$.

The existence of BRK sets shows that there can be no estimate of the form

$$||M_\delta f||_{L^p([1,2])} \leq C||f||_p$$

unless $p = \infty$. Therefore, we look for an estimate

$$\forall\varepsilon\ \exists C_\varepsilon :\ ||M_\delta f||_{L^p([1,2])} \leq C_\varepsilon \delta^{-\varepsilon} ||f||_p, \tag{$\star$}_p$$

and $(\star)_p$ for any $p < \infty$ will suffice to prove Proposition 8.1. To find the right value for p, consider $f = \chi_{R_\delta}$, where R_δ is a rectangle with dimensions $\delta^{1/2} \times \delta$. It is easy to see that for any $r \in [1, 2]$ there is a point x such that $C_\delta(x, r)$ contains a fixed portion of R_δ, i.e.,

$$|C_\delta(x, r) \cap R_\delta| \geq C^{-1}|R_\delta|.$$

Therefore

$$M_\delta f(r) \gtrsim \frac{|R_\delta|}{|C_\delta(x, r)|} \simeq \delta^{1/2},$$

and if $(\star)_p$ holds, then

$$\delta^{1/2} \simeq ||M_\delta f||_p \leq C_\varepsilon \delta^{-\varepsilon} |R_\delta|^{1/p} = C_\varepsilon \delta^{-\varepsilon+3/(2p)},$$

i.e., $p \geq 3$.

Proposition 8.2. $(\star)_3$ *holds.*

The idea of the proof is as follows. There is a related discrete problem which can be understood using the techniques of [**6**] described in Chapter 7. Then, one passes to the continuous problem by replacing circles with annuli and keeping track of various error terms. We will mainly discuss the discrete problem, since the actual proof of Proposition 8.2 is quite technical.

Let us say that two circles $C(x,\rho) = \{y : |x-y| = \rho\}$ and $C(\tilde{x},\tilde{\rho})$ are *internally tangent* (written $C(x,\rho) \parallel C(\tilde{x},\tilde{\rho})$) if they are tangent and one is contained in the bounded component of the complement of the other. Analytically, this means that $|x-\tilde{x}| = |\rho-\tilde{\rho}|$.

One can ask the following question: Given a set of n circles $\mathcal{C} = \{C(x_i,\rho_i)\}_{i=1}^n$, how many pairs $C(x_i,\rho_i)$ and $C(x_j,\rho_j)$ can there be so that $C(x_i,\rho_i) \parallel C(x_j,\rho_j)$?

This question has the obvious answer n^2, since one can consider the "shell" configuration, where any two circles are tangent. In order to get a meaningful question, one has to add an assumption which rules out this type of configuration.

Tangency counting problem. With $\mathcal{C} = \{C(x_i,\rho_i)\}_{i=1}^n$, assume that no three circles $C(x_i,\rho_i)$ are tangent at a point. Then how many pairs $C(x_i,\rho_i)$ and $C(x_j,\rho_j)$ can there be with $C(x_i,\rho_i) \parallel C(x_j,\rho_j)$?

We do not know the answer but will prove the following which is what is needed for Proposition 8.2.

Proposition 8.3. *For any $\varepsilon > 0$, there is a bound of the form $C_\varepsilon n^{3/2+\varepsilon}$ in the tangency counting problem.*

The proof is closely related to [**6**]. Observe to begin with, that one can think of a circle $C(x_i,\rho_i)$ in any of three ways:

- As a circle!
- As a point $(x_i,\rho_i) \in \mathbb{R}^3$.
- As a light cone $\Gamma(x_i,\rho_i) = \{(x,\rho) : |x-x_i| = |\rho-\rho_i|\} \subset \mathbb{R}^3$.

Note that

$$C(x_i,\rho_i) \parallel C(x_j,\rho_j) \Leftrightarrow (x_i,\rho_i) \text{ is incident to } \Gamma(x_j,\rho_j).$$

Therefore, our problem is an incidence problem between points and surfaces in $\mathbb{R}^3$, and we need the 3-dimensional version of the technique in [**6**], which is in the same paper.

First, the Canham type bound, which is $n^{5/3}$ in this case.

Lemma 8.1. *Suppose that $\{C(x_i,\rho_i)\}_{i=1}^n$ and $\{C(y_j,s_j)\}_{j=1}^k$ are collections of circles and that no three $C(x_i,\rho_i)$'s are tangent at a point. Then there are $\lesssim kn^{2/3} + n$ pairs (i,j) such that $C(x_i,\rho_i) \parallel C(y_j,s_j)$.*

Proof. The "Circles of Appolonius" says that if $C(x_1,\rho_1)$, $C(x_2,\rho_2)$ and $C(x_3,\rho_3)$ are not tangent at a point, then there are at most two circles which are internally tangent to all three. In other words, the (0,1)-matrix

$$a_{ij} = \begin{cases} 1, & \text{if } C(x_i,\rho_i) \parallel C(y_j,s_j) \\ 0, & \text{otherwise} \end{cases}$$

has no 3×3 submatrix of 1's. Now use Lemma 7.1. □

Next, the cell decomposition. A *cone arrangement* is a family of r light cones $\Gamma(x_i,\rho_i) \subset \mathbb{R}^3$. A *cell* is an open set $\Omega \subset \mathbb{R}^3$ whose bounadry is contained in the union of ≤ 6 surfaces which are algebraic of degree ≤ 2. We want to triangulate the cone arrangement, i.e., subdivide the components of $\mathbb{R}^3 \setminus \bigcup_i \Gamma(x_i, \rho_i)$ into cells, using as few cells as possible. At least r^3 cells are needed, since $\mathbb{R}^3 \setminus \bigcup_i \Gamma(x_i, \rho_i)$ may have $\simeq r^3$ components.

LEMMA 8.2. *It is possible to triangulate a cone arrangement using $\lesssim r^3 \log r$ cells. In fact, there is an algorithm for doing this, and if $\{\Gamma(x_i, \rho_i)\}_{i=1}^n$ is a family of light cones, $r < n$, and this algorithm is applied to a random sample of r of the $\Gamma(x_i, \rho_i)$'s, then with probability at least $3/4$, at most $A(n \log n)/r$ $\Gamma(x_i, \rho_i)$'s enter any given cell.*

PROOF. This is similar to the proof of Lemma 7.4, see [**6**]. □

PROOF OF PROPOSITION 8.3. Let $r = n^{1/4}$ and $\{\Gamma(x_{j_k}, \rho_{j_k})\}_{k=1}^r$ be a suitable random sample. As with the Szemerédi-Trotter theorem, we may assume that each point (x_i, ρ_i) lies either on one of the cones in the random sample, or else in one of the cells from Lemma 8.2. We let

$$C^* = \{(x_i, \rho_i) : (x_i, \rho_i) \in \Gamma(x_{j_k}, \rho_{j_k}), \text{ for some } k\},$$
$$C_k = \{(x_i, \rho_i) : (x_i, \rho_i) \in \Omega_k\}.$$

Claim. With probability at least $3/4$, $|C^*| \leq C_0 r n^{2/3}$.

PROOF OF THE CLAIM.

$$\begin{aligned}
\mathbb{E}(|C^*|) &\leq \sum_j \operatorname{Prob}(j \in \{j_1, \ldots, j_r\}) |\{C(x_i, \rho_i) : C(x_i, \rho_i) \parallel C(x_j, \rho_j)\}| \\
&= \frac{r}{n} \sum_j |\{C(x_i, \rho_i) : C(x_i, \rho_i) \parallel C(x_j, \rho_j)\}| \\
&\lesssim \frac{r}{n} n^{5/3},
\end{aligned}$$

by Lemma 8.1 □

Now, let us denote

$$I(C, C) = |\{(i, j) : C(x_i, \rho_i) \parallel C(x_j, \rho_j\}|,$$

and

$$I(C^*, C^*) = |\{(i, j) : (x_i, \rho_i), (x_j, \rho_j) \in C^* \text{ and } C(x_i, \rho_i) \parallel C(x_j, \rho_j)\}|.$$

We claim that

$$I(C, C) \leq C_1 n^{3/2} \log^2 n + I(C^*, C^*). \tag{8.1}$$

Namely, let

$$n_k = \#\text{ of } \Gamma(x_i, \rho_i) \text{ which enter } \Omega_k,$$
$$m_k = |C_k|,$$
$$I_k = |\{(i, j) : (x_i, \rho_i) \in C_k \text{ and } C(x_i, \rho_i) \parallel C(x_j, \rho_j)\}|.$$

Then

$$\begin{aligned}
I(C, C) &\leq \sum_{k=1}^R I_k + I(C^*, C^*) \\
&\lesssim \sum_{k=1}^R m_k n_k^{2/3} + n_k + I(C^*, C^*) \\
&\lesssim \left(\frac{n \log n}{r}\right)^{2/3} \sum_{k=1}^R m_k + \frac{n \log n}{r} R + I(C^*, C^*)
\end{aligned}$$

$$\lesssim n^{3/2}\log^2 n + I(C^*, C^*),$$

by choice or r. This proves (8.1). Because of the claim above, we can use induction to finish the proof. We will show that $I(\mathcal{C}, \mathcal{C}) \leq An^{3/2}\log^2 n$ for a suitable constant A. If A is large, then this is obvious for small values of n. Suppose it has been proved for $n \leq n_0$. We will prove it for $n \leq (n_0/C_0)^{12/11}$, where C_0 is the constant in the claim (note that this number is $> n_0 + 1$ if n_0 is large, so this completes the induction). Since $|C^*| \leq C_0 n^{11/12}$, the inductive hypothesis implies

$$I(C^*, C^*) \leq A|C^*|^{3/2}\log^2 |C^*| \leq A(C_0 n^{11/12})^{3/2}\log^2(C_0 n^{11/12}) \leq n^{3/2},$$

if n_0 is large. So, inequality (8.1) implies

$$I(\mathcal{C}, \mathcal{C}) \leq C_1 n^{3/2}\log^2 n + n^{3/2},$$

and now we are done, provided A has been chosen $\geq 2C_1$. $\square$

Now, a brief, heuristic sketch of the proof of Proposition 8.1. Roughly speaking, two annuli $C_\delta(x, \rho)$ and $C_\delta(\tilde{x}, \tilde{\rho})$ can intersect either tangentially, in which case

$$|C_\delta(x, \rho) \cap C_\delta(\tilde{x}, \tilde{\rho})| \simeq \delta^{3/2},$$

or transversely, where we have

$$|C_\delta(x, \rho) \cap C_\delta(\tilde{x}, \tilde{\rho})| \simeq \delta^2,$$

the former case being "worse" since $\delta^{3/2} > \delta^2$.

Now suppose that E is a BRK set. Let $\{p_j\}_{j=1}^M$, $M \simeq 1/\delta$, be a maximal δ-separated subset of $[1, 2]$, and for each j, choose an annulus $C_\delta(x_j, \rho_j) \subset E_\delta$. Let

$$m(x) = \sum_j \chi_{C_\delta(x_j, \rho_j)}(x),$$

and define μ ("multiplicity") to be the smallest integer such that for $1/2M$ choices of j, we have

$$|C_\delta(x_j, \rho_j) \cap \{x : m(x) \geq \mu\}| \leq \frac{1}{2}|C_\delta(x_j, \rho_j)|.$$

Lemma 8.3. *In order to prove Proposition 8.1, it suffices to prove that*

$$\forall\varepsilon\ \exists C_\varepsilon : \mu \leq C_\varepsilon \delta^{-\varepsilon}.$$

Proof. Let

$$\tilde{E}_\delta = \{x \in E_\delta : m(x) \leq \mu\}.$$

Then

$$|E_\delta| \geq |\tilde{E}_\delta| \geq \mu^{-1}\sum_j |C_\delta(x_j, \rho_j) \cap \tilde{E}_\delta| \gtrsim \mu^{-1}M\delta \simeq \mu^{-1},$$

and the lemma follows. $\square$

Now, we have

$$\mu \simeq \mu\delta M \lesssim \sum_j \int_{C_\delta(x_j, \rho_j)} m(x)dx = \sum_{i,j} |C_\delta(x_i, \rho_i) \cap C_\delta(x_j, \rho_j)|.$$

Pretend that two circles must be either tangent, or sufficiently transverse, and that the $\delta^{3/2}$ and δ^2 numbers for the measure of the intersection can be justified. Then

$$\sum_{i,j} |C_\delta(x_i, \rho_i) \cap C_\delta(x_j, \rho_j)| \lesssim I(\mathcal{C}, \mathcal{C})\delta^{3/2} + M^2\delta^2,$$

where $\mathcal{C} = \{C(x_j, \rho_j)\}_{j=1}^{M}$. If we further pretend that $\mathcal{C}$ satisfies the "no three circles tangent at a point" condition, then we can apply Proposition 8.3 to obtain

$$\sum_{i,j} |C_\delta(x_i, \rho_i) \cap C_\delta(x_j, \rho_j)| \lessapprox \delta^{-(3/2+\varepsilon)}\delta^{3/2} + \delta^{-2}\delta^2 \lessapprox \delta^{-\varepsilon},$$

and the "proof" is complete.

CHAPTER 9

Averages over circles

Let σ be linear Lebesgue measure on the unit circle in the plane. There is a standard sharp estimate for the corresponding averaging operator. Namely

$$\|\sigma * f\|_3 \lesssim \|f\|_{3/2}.$$

Equivalently, if σ_δ is normalized planar measure on the annulus $C_\delta(0,1)$, then

$$\|\sigma_\delta * f\|_3 \lesssim \|f\|_{3/2},$$

where the implicit constant is independent of δ.

This estimate may be proved using the Fourier transform and complex interpolation (see [**20**]). We will show that counting arguments in the spirit of Chapter 7 can be used to obtain the corresponding restricted weak-type inequality.

Proposition 9.1. *Let E be a subset of $[0,1]\times[0,1]$. For $\lambda > 0$ define*

$$F = \{x : (\chi_E * \sigma_\delta)(x) > \lambda\}.$$

Then

$$|F| \lesssim \lambda^{-3}|E|^2.$$

Proof. The argument we present is from [**19**]. Divide $[0,1]\times[0,1]$ into a family of squares Q_j of sidelength δ and for each integer k let

$$J_k = \{j : 2^{-k}\delta^2 < |Q_j \cap E| \le 2^{-k+1}\delta^2\},$$

$$E_k = \bigcup_{j\in J_k} Q_j \cap E, \quad \tilde{E}_k = \bigcup_{j\in J_k} Q_j, \quad F_k = \{x : (\chi_{\tilde{E}_k} * \sigma_{3\delta})(x) \ge C^{-1}2^k k^{-2}\lambda\}.$$

Then, for suitable C, we have

$$F \subset \bigcup_k F_k.$$

Now, for fixed k, let $\{x_i\}_{i=1}^M$ and $\{y_p\}_{p=1}^N$ be maximal δ-separated sets in F_k and $\tilde{E}_k$ respectively, and put

$$\lambda_k = 2^k k^{-2}\lambda.$$

We can clearly assume that $\lambda_k \gtrsim \delta$. Also notice that

$$|\tilde{E}_k| \simeq N\delta^2.$$

Moreover, each x_i satisfies

$$|C_{4\delta}(x_i,1)\cap E| \gtrsim \lambda_k\delta, \tag{9.1}$$

since

$$F_k \subset \{x : (\chi_{\tilde{E}_k} * \sigma_{4\delta})(x) \gtrsim \lambda_k\}.$$

Now consider the following set of indices.

$$Q = \{(i, p_1, p_2) : ||x_i - y_{p1}| - 1| < \delta,\ ||x_i - y_{p2}| - 1| < \delta,\ |y_{p1} - y_{p2}| > C^{-1}\lambda_k - \delta\}.$$

We will count Q in two different ways. For given p_1, p_2, there are at most λ_k^{-1} annuli $C_\delta(x_i, 1)$ passing through the points y_{p1} and y_{p2}. Therefore

$$|Q| \lesssim N^2 \lambda_k^{-1}.$$

On the other hand, for every p_1 and every p_2 there is at least one choice of i. So, (9.1) implies

$$|Q| \gtrsim M(\lambda_k \delta^{-1})^2.$$

Consequently

$$M \lesssim \lambda_k^{-3} \delta^2 N^2,$$

or, equivalently

$$|F_k| \lesssim \lambda_k^{-3} |\tilde{E}_k|^2.$$

Summing over k we obtain

$$|F| \leq \sum_k |F_k| \lesssim \sum_k \lambda_k^{-3} |\tilde{E}_k|^2 \lesssim \sum_k \lambda^{-3} 2^{-3k} k^6 2^{2k} |E|^2 \lesssim \lambda^{-3} |E|^2.$$

This completes the proof. □

One can consider the corresponding maximal operator as well. Namely, define

$$\mathcal{M}_\delta f : \mathbb{R}^2 \to \mathbb{R}$$

by

$$\mathcal{M}_\delta f(x) = \sup_{1 \leq r \leq 2} \int_{C_\delta(x,r)} f(y) dy.$$

Then for all $p > 2$ we have

$$\|\mathcal{M}_\delta f\|_p \lesssim \|f\|_p. \tag{9.2}$$

This estimate was originally proved by Bourgain [**1**]. Schlag [**18**] used techniques in the spirit of Chapter 8 to obtain a purely combinatorial proof.

Notice that (9.2) has the following geometric consequence, which was proved, independently, by Marstrand [**16**]: Suppose that $B \subset \mathbb{R}^2$ is a union of circles of arbitrary radii, and let A be the set of their centers. Then

$$|A| > 0 \Rightarrow |B| > 0.$$

This can be shown by reducing to the case when B is compact and the radii of the circles are in the interval $[1, 2]$, and then letting $f = \chi_{B_\delta}$ in (9.2). The best possible result was proved in [**24**]:

$$\dim(A) > 1 \Rightarrow |B| > 0.$$

This is sharp by a construction due to Talagrand, see [**17**].

Bibliography

[1] J. Bourgain. Averages over convex curves and maximal operators. *J. Anal. Math.* **47** (1986), 69-85.

[2] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis. *Geom. Funct. Analysis* **1** (1991), no. 2, 147-187.

[3] J. Bourgain. On the dimension of Kakeya sets and related maximal inequalities. *Geom. Funct. Anal.* **9** (1999), no. 2, 256-287.

[4] W. Beckner, A. Carbery, S. Semmes, F. Soria. A note on restriction of Fourier transforms to spheres. *Bull. London Math. Soc.* **21** (1989), 394-398.

[5] M. Christ. Estimates for the k-plane transform. *Indiana Univ. Math. J.* **33** (1984), 891-910.

[6] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, E. Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.* **5** (1990), 99-160.

[7] A. Córdoba. The Kakeya maximal function and spherical summation multipliers. *Amer. J. Math.* **99** (1977), 1-22.

[8] A. Córdoba. The multiplier problem for the polygon. *Ann. of Math. (2)* **105** (1977), no. 3, 581-588.

[9] S. Drury. L^p estimates for the X-ray transform. *Illinois J. Math.* **27** (1983), 125-129.

[10] K. J. Falconer. *The geometry of fractal sets.* Cambridge University Press, 1985.

[11] C. Fefferman. The multiplier problem for the ball. *Ann. of Math.* **94** (1971), 330-336.

[12] V. Havin, B. Jöricke. *The uncertainty principle in harmonic analysis.* Springer-Verlag, 1994.

[13] L. Hörmander. *The analysis of linear partial differential operators.* Springer-Verlag, 1983.

[14] N. Katz, I. Laba, T. Tao. An improved bound on the Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$. *Ann. of Math.* **152** (2002), no.2, 383-446.

[15] I. Laba, T. Tao. An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension. *Geom. Funct. Anal.* **11** (2001), no. 4, 773-806.

[16] J. M. Marstrand. Packing circles in the plane. *Proc. London Math. Soc. (3)* **55** (1987), no. 1, 37-58.

[17] P. Mattila. *Geometry of sets and measures in Euclidean spaces.* Cambridge University Press, 1995.

[18] W. Schlag. A geometric proof of the circular maximal function. *Duke Math. J.* **93** (1998), 505-533.

[19] W. Schlag. On continuum incidence problems related to harmonic analysis. Preprint.

[20] E. M. Stein. *Harmonic Analysis.* Princeton University Press, 1993.

[21] T. Wolff. An improved bound for Kakeya type maximal functions. *Rev. Mat. Iberoamericana.* **11** (1995), no.3, 651-674.

[22] T. Wolff. A Kakeya-type problem for circles. *Amer. J. Math.* **119** (1997), no. 5, 985-1026.

[23] T. Wolff. Recent work connected with the Kakeya problem. *Prospects in Mathematics (Princeton, NJ, 1996)*, 129-162, *Amer. Math. Soc., Providence, RI*, 1999.

[24] T. Wolff. Local smoothing estimates on L^p for large p. *Geom. Funct. Anal.* **10** (2000), no. 5, 1237-1288.

[25] T. Wolff. Lectures in Harmonic Analysis. To appear in the AMS *University Lectures* series.