THE BOUNDARY OF A SMOOTH SET HAS FULL HAUSDORFF DIMENSION

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Abstract. We prove that if the restriction of the Lebesgue measure to a set $A \subset [0, 1]$ with $0 < |A| < 1$ is a smooth measure, then the boundary of A must have full Hausdorff dimension.

1. Introduction

A continuous function $f : [0, 1] \to \mathbb{R}$ is called smooth in the sense of Zygmund (see [4]), if

$$\lim_{\delta \to 0^+} \sup_{\substack{x \in [0,1] \\ 0<h<\delta}} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h} = 0.$$

The set of all such functions is denoted by λ_*. Similarly, a positive, finite Borel measure μ on $[0, 1]$ is called smooth in the sense of Zygmund, if its distribution function $H(x) := \mu([0, x])$ is in λ_*.

It is a standard fact that one can construct smooth measures which are singular with respect to the Lebesgue measure (see, for example, [1], [2], [3]).

On the other hand, it is rather surprising that even an absolutely continuous smooth measure may be quite paradoxical. Indeed, a straightforward modification of the construction in [2] gives the following unexpected result.

Theorem 1 (Kahane)**.** *There exists a set* $A \subset [0, 1]$ *with* $0 < |A| < 1$*, such that the restriction of the Lebesgue measure to A, that is, the measure* μ *defined by* $\mu(E) = |A \cap E|$*, is a smooth measure.*

Any set with the properties stated in Theorem 1, is called a Z-set (or a smooth set).

It is reasonable to expect that Z-sets must have a rather complicated geometric structure. Indeed, the purpose of this paper is to show the following.

Theorem 2. *Let* $A \subset [0, 1]$ *be a Z-set. Then the Hausdorff dimension of the boundary of A equals* 1.

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2. Notation

A°, $\overline{A}$ and ∂A are, respectively, the interior, the closure and boundary of the set A.
$\dim A$ is the Hausdorff dimension of A.
By the term "interval" we will always mean "open subinterval of $[0,1]$".
If $I=(a,b)$, then we put

$$I_l=(a,(a+b)/2),\ I_r=((a+b)/2,b).$$

Note that $A\subset[0,1]$ is a Z-set if and only if

$$\lim_{\delta\to 0^+}\sup_{\substack{I\subset[0,1]\\|I|<\delta}}\frac{2}{|I|}\big||A\cap I_l|-|A\cap I_r|\big|=0.$$

Given small positive ε and δ, we say that $A\subset[0,1]$ is a $Z_{\varepsilon,\delta}$-set if

(1) A is open.
(2) $|\partial A|=0$.
(3) For each interval I with $|I|<\delta$, we have $\frac{2}{|I|}\big||A\cap I_l|-|A\cap I_r|\big|<\varepsilon$.

3. Proof of Theorem 2

We will need the following auxiliary result. Roughly speaking, it states that if a $Z_{\varepsilon,\delta}$-set cuts an interval into two equal pieces (in the measure-theoretic sense), then inside that interval, we can always find a disjoint family of smaller intervals which nearly cover the initial interval and so that the $Z_{\varepsilon,\delta}$-set still cuts them into two equal pieces.

Lemma 1. *Let $A\subset[0,1]$ be a $Z_{\varepsilon,\delta}$-set. Suppose that I is an interval with $|I|<\delta$ such that $|A\cap I|=\frac{1}{2}|I|$. Then there exists a disjoint family $\mathcal{D}_I$ of subintervals of I so that:*

(1) $\forall J\in\mathcal{D}_I,\ |A\cap J|=\frac{1}{2}|J|$.
(2) $\forall J\in\mathcal{D}_I,\ |J|\le\frac{1}{2}|I|$.
(3) $\displaystyle\sum_{J\in\mathcal{D}_I}|J|\ge(1-\varepsilon)|I|$.

Proof. If $|A\cap I_l|=\frac{1}{2}|I_l|$ then we put $\mathcal{D}_I=\{I_l,I_r\}$ and we are done. So we may assume, without loss of generality, that $|A\cap I_l|<\frac{1}{2}|I_l|$. We will inductively construct a, possibly finite, sequence $\{\mathcal{B}_i\}_i$ of families of intervals in I_l with the following properties:

- $\bigcup_i\mathcal{B}_i$ is disjoint.
- $A\cap I_l\subset\bigcup_i\bigcup_{B\in\mathcal{B}_i}B$.
- For each $B\in\bigcup_i\mathcal{B}_i$, we have $|A\cap B|=\frac{1}{2}|B|$.

Since A is a $Z_{\varepsilon,\delta}$-set and $|A \cap I| = \frac{1}{2}|I|$, we have

$$\frac{1-\varepsilon}{2}|I_l| < |A \cap I_l| < \frac{1}{2}|I_l|. \tag{1}$$

In particular, $0 < |A \cap I_l| < |I_l|$, therefore there exists an interval $B_0 \subset I_l$ of maximum length, such that $|A \cap B_0| = \frac{1}{2}|B_0|$. Notice that if we put $(a, b) = B_0$, then $a, b \notin A \cap I_l$. Otherwise we would be able to find an interval B_0' such that $B_0 \subsetneq B_0' \subset I_l$ and $|A \cap B_0'| > \frac{1}{2}|B_0'|$. But then, since $|A \cap I_l| < \frac{1}{2}|I_l|$, there would be an interval B_0'' with $B_0' \subsetneq B_0'' \subset I_l$ such that $|A \cap B_0''| = \frac{1}{2}|B_0''|$, contradicting the maximality of the length of B_0.

Now let

$$\mathcal{B}_0 = \{B_0\},$$

and

$$\mathcal{C}_0 = \{K^\circ : K \text{ is a connected component of } I_l \setminus B_0 \text{ with } K^\circ \cap A \neq \emptyset\}.$$

If $\mathcal{C}_0 = \emptyset$, then $\mathcal{B}_0$ is a cover of $A \cap I_l$ with the required properties and the process terminates. If $\mathcal{C}_0 \neq \emptyset$, then for each $C \in \mathcal{C}_0$ we have $|A \cap C| < \frac{1}{2}|C|$. Indeed, if this were not the case, we would have $|A \cap (\overline{C \cup B_0})^\circ| \geq \frac{1}{2}|(\overline{C \cup B_0})^\circ|$, contradicting the maximality of the length of B_0 as before. Therefore, inside each $C \in \mathcal{C}_0$ we can find an interval B_C of maximum length such that $|A \cap B_C| = \frac{1}{2}|B_C|$. We let

$$\mathcal{B}_1 = \{B_C : C \in \mathcal{C}_0\},$$

and

$$\mathcal{C}_1 = \{K^\circ : K \text{ is a connected component of } I_l \setminus \bigcup_{j \leq 1} \bigcup_{B \in \mathcal{B}_j} B \text{ with } K^\circ \cap A \neq \emptyset\}.$$

As before, for each $(a, b) \in \mathcal{B}_1$, we have $a, b \notin A \cap I_l$. Now, if $\mathcal{C}_1 = \emptyset$ then we terminate the process, if not then we notice that $|A \cap C| < \frac{1}{2}|C|$ for each $C \in \mathcal{C}_1$ and continue as above.

Suppose now that $\mathcal{B}_i$ and $\mathcal{C}_i$ have been defined. By construction, $|A \cap C| < \frac{1}{2}|C|$ for each $C \in \mathcal{C}_i$, so there exists an interval $B_C \subset C$ of maximum length such that $|A \cap B_C| = \frac{1}{2}|B_C|$. Put

$$\mathcal{B}_{i+1} = \{B_C : C \in \mathcal{C}_i\},$$

and

$$\mathcal{C}_{i+1} = \{K^\circ : K \text{ is a connected component of } I_l \setminus \bigcup_{j \leq i+1} \bigcup_{B \in \mathcal{B}_j} B \text{ with } K^\circ \cap A \neq \emptyset\}.$$

This completes the inductive construction.

Clearly, $\bigcup_i \mathcal{B}_i$ is disjoint, and $\forall B \in \bigcup_i \mathcal{B}_i$, $|A \cap B| = \frac{1}{2}|B|$. So, the only thing we have to check is that $\bigcup_i \mathcal{B}_i$ covers $A \cap I_l$. If $\mathcal{C}_{i_0} = \emptyset$ for some i_0, then, by the definition of $\mathcal{C}_{i_0}$, $\bigcup_{i \leq i_0} \mathcal{B}_i$ covers $A \cap I_l$. So, assume that $\mathcal{C}_i \neq \emptyset$

for each i. Furthermore, suppose toward a contradiction, that there is an $x \in A \cap I_l$ such that $x \notin \bigcup_i \bigcup_{B \in \mathcal{B}_i} B$. Let M be the connected component of $A \cap I_l$ containing x. Since for each $(a, b) \in \bigcup_i \mathcal{B}_i$ we have $a, b \notin A \cap I_l$, it follows that $M \cap \bigcup_i \bigcup_{B \in \mathcal{B}_i} B = \emptyset$. Therefore, there is a strictly decreasing sequence of intervals $C_i \in \mathcal{C}_i$ such that $M \subset C_i$. The corresponding intervals $B_{C_i} \in \mathcal{B}_{i+1}$ are disjoint, hence there exists an i_0 such that $|B_{C_{i_0}}| < |M|$. Notice that $|A \cap M| = |M|$ and $|A \cap C_{i_0}| < \frac{1}{2}|C_{i_0}|$. Consequently, we can find an interval M' with $M \subset M' \subset C_{i_0}$ such that $|A \cap M'| = \frac{1}{2}|M'|$. But this contradicts the maximality of the length of $B_{C_{i_0}}$.

We conclude that if we let $\mathcal{L}_I = \bigcup_i \mathcal{B}_i$, then

$$\sum_{J \in \mathcal{L}_I} |J| = 2 \sum_{J \in \mathcal{L}_I} |A \cap J| \geq 2|A \cap I_l| > (1 - \varepsilon)|I_l|,$$

where the last inequality follows from (1).

Now notice that $(A^{\complement})^\circ$ is a $Z_{\varepsilon,\delta}$-set with $|(A^{\complement})^\circ \cap I| = \frac{1}{2}|I|$ and $|(A^{\complement})^\circ \cap I_r| < \frac{1}{2}|I_r|$. So, the same procedure yields a disjoint family $\mathcal{R}_I$ of intervals in I_r such that $|(A^{\complement})^\circ \cap J| = \frac{1}{2}|J|$ (hence $|A \cap J| = \frac{1}{2}|J|$) for each $J \in \mathcal{R}_I$, and

$$\sum_{J \in \mathcal{R}_I} |J| > (1 - \varepsilon)|I_r|.$$

We let $\mathcal{D}_I = \mathcal{L}_I \cup \mathcal{R}_I$, and the proof of the Lemma is complete. □

Now, we can proceed with the proof of Theorem 2.

Let $A \subset [0, 1]$ be a Z-set. If $|\partial A| > 0$ then $\dim \partial A = 1$ and we are done. So we may assume that $|\partial A| = 0$. In that case, $|A^\circ| = |A| = |\overline{A}|$, so we may further assume that A is open.

Fix $0 < \varepsilon < \frac{1}{2}$. Then there exists $\delta > 0$ such that A is a $Z_{\varepsilon,\delta}$-set. Since $0 < |A| < 1$, we can find an interval I_0 such that $|I_0| < \delta$ and $|A \cap I_0| = \frac{1}{2}|I_0|$. Let

$$\mathcal{F}_0 = \{I_0\}.$$

By Lemma 1, we can inductively define

$$\mathcal{F}_{i+1} = \bigcup_{I \in \mathcal{F}_i} \mathcal{D}_I.$$

Note that $\forall I \in \mathcal{F}_i$, we have $|I| \leq 1/2^i$. Moreover, for each $I \in \mathcal{F}_{i+1}$ there is a unique $P_I \in \mathcal{F}_i$ (the "parent" of I) such that $I \subset P_I$.

Now, let $F_i = \bigcup_{I \in \mathcal{F}_i} I$ and $F = \bigcap_i F_i$, and notice that $F \subset \partial A$. We will show that

$$\dim F \geq 1 + \log_2(1 - \varepsilon).$$

This will be accomplished by recursively constructing a suitable sequence of measures μ_i, so that each of them is supported in F_i. This sequence will give rise to a "limit" measure μ supported in F with the property

$$\mu(I) < 6|I|^{1+\log_2(1-\varepsilon)} \tag{2}$$

for all intervals I. The Hausdorff dimension bound then follows by standard arguments.

The construction is as follows.

μ_0 is Lebesgue measure restricted to I_0.

Suppose that μ_i has been defined. Then for all $I \in \mathcal{F}_{i+1}$ define

$$\mu_{i+1}(I) = \frac{\mu_i(P_I)}{\sum\limits_{J \in \mathcal{D}_{P_I}} |J|} |I|,$$

and for any subset $E \subset [0, 1]$,

$$\mu_{i+1}(E) = \sum_{I \in \mathcal{F}_{i+1}} \mu_{i+1}(I) \frac{|E \cap I|}{|I|}.$$

It is clear that μ_i is supported in F_i. An easy induction shows that for each $I \in \mathcal{F}_i$ we have

$$\mu_i(I) < \frac{1}{(1-\varepsilon)^i} |I|,$$

and $\mu_j(I) = \mu_i(I)\ \forall j \geq i$. In particular $\mu_i(F_i) = |I_0|$.

Now, let $H_i(x) = \mu_i([0, x])$ be the distribution function of μ_i. If $x \notin F_i$ then $H_i(x) = H_{i+1}(x)$. On the other hand, if $x \in F_i$, let I_x^i be the unique interval in $\mathcal{F}_i$ such that $x \in I_x^i$. Then

$$|H_i(x) - H_{i+1}(x)| \leq \mu_i(I_x^i) < \frac{1}{(1-\varepsilon)^i} |I_x^i| \leq \frac{1}{(2(1-\varepsilon))^i}.$$

Since $\varepsilon < 1/2$, H_i converges uniformly to a continuous increasing function H. Let μ be the Borel measure whose distribution function is H. Then for every interval I, we have $\mu(I) = \lim \mu_i(I)$. Therefore, for each $I \in \mathcal{F}_i$, $\mu(I) = \mu_i(I)$. Consequently, μ is supported in F and $\mu(F) = |I_0|$. It remains to verify (2). So, let I be an interval, and i_0 an integer such that

$$\frac{1}{2^{i_0+1}} < |I| \leq \frac{1}{2^{i_0}}.$$

Then

$$\begin{aligned}
\mu(I) &\leq \mu\Big(\bigcup_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} J\Big) = \mu_{i_0+1}\Big(\bigcup_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} J\Big) = \sum_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} \mu_{i_0+1}(J) \\
&< \frac{1}{(1-\varepsilon)^{i_0+1}} \sum_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} |J| < \frac{2}{(1-\varepsilon)^{i_0}} \sum_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} |J|.
\end{aligned}$$

Note that each $J \in \mathcal{F}_{i_0+1}$ satisfies $|J| \leq 1/2^{i_0+1} < |I|$. Therefore

$$\sum_{\substack{J \in \mathcal{F}_{i_0+1} \\ J \cap I \neq \emptyset}} |J| \leq 3|I|.$$

Consequently

$$\mu(I) < \frac{6}{(1-\varepsilon)^{i_0}} |I|.$$

But $|I| \leq 1/2^{i_0}$ implies $1/(1-\varepsilon)^{i_0} \leq |I|^{\log_2(1-\varepsilon)}$ and we are done.

We conclude that $\dim \partial A \geq \dim F \geq 1 + \log_2(1-\varepsilon)$. Letting $\varepsilon \to 0$ we obtain $\dim \partial A = 1$.

References

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