NORM ESTIMATES FOR A KAKEYA-TYPE MAXIMAL OPERATOR

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Abstract. We prove $L^p \to L^q$ estimates for the 2-dimensional analog of the Kakeya maximal function.

1. Introduction

Let $\mathcal{G}_{n,2}$ be the Grassmannian manifold of all 2-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the unique probability measure $\varphi_{n,2}$ which is invariant under the action of the orthogonal group (see, for example, [2]). For any locally integrable function f and $0 < \delta \ll 1$ we define

$$\mathcal{M}_\delta f : \mathcal{G}_{n,2} \to \mathbb{R}$$

by

$$\mathcal{M}_\delta f(\Pi) = \sup_{a\in\mathbb{R}^n} \frac{1}{|\Pi^\delta(a)|} \int_{\Pi^\delta(a)} |f(y)|dy,$$

where $\Pi^\delta(a)$ the $\delta/2$-neighborhood of the intersection of the plane $\Pi+a$ with the ball of radius $1/2$ centered at a, and $|\Pi^\delta(a)|$ is the volume of $\Pi^\delta(a)$. Thus, $\mathcal{M}_\delta f(\Pi)$ is, essentially, the maximal average of f over the neighborhoods of all 2-dimensional discs of unit diameter which are parallel to the plane Π. This operator is a natural variant of the Kakeya maximal function introduced by Bourgain [1]. In that context, one averages over neighborhoods of line segments of unit length. The mapping properties of the Kakeya maximal function have been the subject of a great deal of research. We refer the reader to the expository articles [4] and [5] for extensive accounts.

In this paper we are interested in proving $L^p \to L^q(\mathcal{G}_{n,2}, \varphi_{n,2})$ estimates for $\mathcal{M}_\delta$. To find the optimal range for p and q we argue as follows.

If f is the characteristic function of a ball of radius δ, then $\|f\|_p$ is comparable to $\delta^{n/p}$, and $\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\varphi_{n,2})}$ is comparable to δ^2. Therefore, it seems reasonable to expect a bound of the form

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\varphi_{n,2})} \leq C_{n,p,q}\delta^{2-n/p}\|f\|_p, \; p \leq n/2, \; q \geq 1.$$

On the other hand, if f is the characteristic function of a rectangle of dimensions $1 \times 1 \times \delta \times \cdots \times \delta$, then $\|f\|_p = \delta^{(n-2)/p}$ and $\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\varphi_{n,2})}$ is,

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up to a multiplicative constant, greater than $\delta^{2(n-2)/q}$. It follows that if the above estimate is true, we must have

$$\delta^{2(n-2)/q} \leq C_{n,p,q}\delta^{2-n/p}\delta^{(n-2)/p},$$

which implies $q \leq (n-2)p'$, where p' is the conjugate exponent of p.

These examples would reasonably lead to the following.

Conjecture. *For every $\epsilon > 0$ there exists a constant $C_{\epsilon,n,p,q} > 0$ such that*

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\varphi_{n,2})} \leq C_{\epsilon,n,p,q}\delta^{2-n/p-\epsilon}\|f\|_p,$$

where

$$1 \leq p \leq \frac{n}{2}, \quad q \leq (n-2)p'.$$

The purpose of this paper is to verify the conjecture when the range of p is a smaller interval. Namely, we use geometric-combinatorial ideas in the spirit of [1], to prove the following.

Theorem. *For every $\epsilon > 0$ there exists a constant $C_{\epsilon,n,p,q} > 0$ such that*

$$\|\mathcal{M}_\delta f\|_{L^q(\mathcal{G}_{n,2},\varphi_{n,2})} \leq C_{\epsilon,n,p,q}\delta^{2-n/p-\epsilon}\|f\|_p,$$

where

$$1 \leq p < \frac{n+1}{3}, \quad q \leq (n-2)p'.$$

2. Preliminaries

For the rest of the paper, the capital letter C will denote various positive constants whose values may change from line to line. Similarly C_ϵ will denote constants depending on ϵ. $x \lesssim y$ means $x \leq Cy$, and $x \simeq y$ means $(x \lesssim y \;\&\; y \lesssim x)$. Also, we will use the notation Π^δ for any set $\Pi^\delta(a)$, since the basepoint a is irrelevant in all our arguments. Further notational conventions follow below.

S^{n-1} is the $(n-1)$-dimensional unit sphere.
$B(a,r)$ is the ball of radius r centered at a.
$A(a,r)$ is the annulus $B(a,2r) \setminus B(a,r)$.
$L_e(a)$ is the line in the direction $e \in S^{n-1}$ passing through the point a, i.e.

$$L_e(a) = \{a + te : t \in \mathbb{R}\}.$$

$T_e^{(r)(\beta)}(a)$ is the tube of length r, cross-section radius β, centered at a, and with axis in the direction $e \in S^{n-1}$, i.e.

$$T_e^{(r)(\beta)}(a) = \{x \in \mathbb{R}^n : \mathrm{dist}(x, L_e(a)) \leq \beta \text{ and } |\mathrm{proj}_{L_e(a)}(x) - a| \leq r/2\},$$

where $\mathrm{proj}_{L_e(a)}(x)$ is the orthogonal projection of x onto $L_e(a)$.
χ_E is the characteristic function of the set E.
$|\cdot|$ denotes Lebesgue measure or cardinality, depending on the context.

If $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$, then their distance θ is defined by

$$\theta(\Pi_1, \Pi_2) = \|\mathrm{proj}_{\Pi_1} - \mathrm{proj}_{\Pi_2}\|,$$

where $\|\cdot\|$ is the operator norm. $\varphi_{n,2}$ is a $2(n-2)$-dimensional Ahlfors-regular measure with respect to this distance (see [2]), in the sense that

$$\varphi_{n,2}(\{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, \Pi_0) \leq r\}) \simeq r^{2(n-2)}, \ \forall \Pi_0 \in \mathcal{G}_{n,2}, r < 1.$$

A finite subset of $\mathcal{G}_{n,2}$ is called δ-separated if the distance between any two of its elements is at least δ. So, if $\mathcal{B}$ is a maximal δ-separated subset of $\mathcal{A} \subset \mathcal{G}_{n,2}$, then

$$\varphi_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}|\delta^{2(n-2)}.$$

Moreover, if $\mathcal{A} \subset \mathcal{G}_{n,2}$ is δ-separated, and $\mathcal{B}$ is a maximal η-separated subset of $\mathcal{A}$ with $\eta \geq \delta$, then

$$|\mathcal{B}| \gtrsim |\mathcal{A}|(\delta/\eta)^{2(n-2)}.$$

For technical reasons, we introduce the following subset of $\mathcal{G}_{n,2}$.

$$\mathcal{A}_{n,2} := \{\Pi \in \mathcal{G}_{n,2} : \theta(\Pi, x_1x_2\text{-plane}) \leq 1/4\},$$

Notice that by invariance, it is enough to prove the Theorem for $\mathcal{M}_\delta$ restricted to $\mathcal{A}_{n,2}$.

We close this section with two lemmas that will allow us to control the intersection properties and the cardinality of a family of sets Π^δ containing a fixed line segment. They can be proved by fairly elementary arguments, so we omit the proofs.

Lemma 2.1. *Let $\Pi_1, \Pi_2 \in \mathcal{G}_{n,2}$ be such that $\theta(\Pi_1, \Pi_2) \leq 1/2$. Fix $a, b \in \mathbb{R}^n$, $\rho > 0$ with $\rho \leq |a-b| \leq 2\rho$. Then for any $\Pi_1^\delta, \Pi_2^\delta$ with $a, b \in \Pi_1^\delta \cap \Pi_2^\delta$ we have*

$$\Pi_1^\delta \cap \Pi_2^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)(\beta)}(a),$$

where $e = (a-b)/|a-b|$ and $\beta = C\delta/\theta(\Pi_1, \Pi_2)$.

Lemma 2.2. *Let $\{\Pi_j\}_{j=1}^M$ be a δ-separated set in $\mathcal{A}_{n,2}$. Fix $a, b \in \mathbb{R}^n$, $\rho \geq 4\delta$ with $|a-b| \geq \rho$. Suppose that for each j there exists Π_j^δ with $a, b \in \Pi_j^\delta$. Then, for any maximal ζ-separated subset $\mathcal{B} \subset \{\Pi_j\}_{j=1}^M$ with $\zeta \geq C\delta/\rho$, we have*

$$|\mathcal{B}| \gtrsim M(\rho\delta/\zeta)^{n-2}.$$

3. Proof of the Theorem

Let $E \subset \mathbb{R}^n$, $0 < \lambda \leq 1$, $\epsilon > 0$, and

$$A_\lambda = \{\Pi \in \mathcal{A}_{n,2} : \mathcal{M}_\delta \chi_E(\Pi) \geq \lambda\}.$$

By the standard interpolation theorems (see [3]), it is enough to prove the following restricted weak-type estimate at the endpoint.

$$\varphi_{n,2}(A_\lambda) \leq C_\epsilon \left(\frac{1}{\delta}\right)^\epsilon \left(\frac{|E|}{\lambda^{(n+1)/3}\delta^{(n-2)/3}}\right)^3. \tag{3.1}$$

Now, let $\{\Pi_j\}_{j=1}^M$ be a maximal δ-separated subset of A_λ. Then proving (3.1) amounts to proving

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3} \delta^{n-2}. \tag{3.2}$$

Since $\Pi_j \in A_\lambda$, there exists Π_j^δ such that

$$|\Pi_j^\delta \cap E| \geq \frac{3}{4}\lambda |\Pi_j^\delta|. \tag{3.3}$$

Put $\gamma = \lambda^{1/2}(\log(1/\delta))^{-1/2}$ and note that (3.2) is trivial if $4\delta \geq \gamma$. Indeed, (3.3) implies

$$\begin{aligned}
|E| &\gtrsim \lambda\delta^{n-2} = \lambda^{(n+1)/3}\lambda^{-(n-2)/3}\delta^{n-2} \\
&\gtrsim \lambda^{(n+1)/3}(\delta^2 \log(1/\delta))^{-(n-2)/3}\delta^{n-2} \\
&= (\log(1/\delta))^{-(n-2)/3}\lambda^{(n+1)/3}((1/\delta)^{2(n-2)})^{1/3}\delta^{n-2} \\
&\geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3}\delta^{n-2}.
\end{aligned}$$

We may therefore assume that $4\delta \leq \gamma$.

Now, let δ_0 be a small constant to be determined later. Then for $\delta \geq \delta_0$, we have $M \lesssim 1$, and so (3.3) trivially implies (3.2) as before. Hence we can also assume that $\delta \leq \delta_0$.

After these preliminary reductions, we can proceed with the proof of (3.2). First, we find a large number of sets Π_j^δ so that the measure of their intersection with E is concentrated in annuli of fixed dimensions. More precisely, we claim that there exist a number $\rho \geq \gamma$ and a set $C \subset \{\Pi_j^\delta\}_{j=1}^M$ with

$$|C| \gtrsim (\log(C/\gamma))^{-2} M, \tag{3.4}$$

so that for each $\Pi_j^\delta \in C$ there is a set $P_j \subset \Pi_j^\delta$ of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2}\lambda|\Pi_j^\delta| \tag{3.5}$$

such that for each $z \in P_j$

$$|\Pi_j^\delta \cap E \cap B(z, 2\rho) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement| \gtrsim \lambda|\Pi_j^\delta| \quad \forall e \in S^{n-1}, r \in [\gamma, 1], \tag{3.6}$$

and

$$|\Pi_j^\delta \cap E \cap A(z,\rho)| \gtrsim (\log(C/\gamma))^{-1}\lambda|\Pi_j^\delta|. \tag{3.7}$$

To see this, note that for all $z \in \mathbb{R}^n$, $e \in S^{n-1}$ and $r \in [\gamma, 1]$ we have

$$
\begin{aligned}
|\Pi_j^\delta \cap E \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement| &= |\Pi_j^\delta \cap E| - |\Pi_j^\delta \cap E \cap T_e^{(4r)(\gamma^2/r)}(z)| \\
&\geq \frac{3}{4}\left(\lambda|\Pi_j^\delta| - Cr\frac{\gamma^2}{r}|\Pi_j^\delta|\right) \\
&= \frac{3}{4}\lambda(1 - C(\log(1/\delta))^{-1})|\Pi_j^\delta| \geq \frac{\lambda}{2}|\Pi_j^\delta|, \qquad (3.8)
\end{aligned}
$$

for δ_0 small enough.

Now, for each $1 \leq j \leq M$, $z \in \Pi_j^\delta \cap E$, $i \in \mathbb{N}$, consider the quantity

$$
Q(j, z, i) = \inf_{\substack{r \in [\gamma,1] \\ e \in S^{n-1}}} |\Pi_j^\delta \cap E \cap B(z, \gamma 2^i) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement|.
$$

Then

$$
Q(j, z, 0) \leq C\gamma^2\delta^{n-2} = C\lambda(\log(1/\delta))^{-1}\delta^{n-2} \leq \frac{\lambda}{10}|\Pi_j^\delta|,
$$

provided that δ_0 has been chosen small enough. On the other hand

$$
Q(j, z, \log(C/\gamma)) \geq \frac{\lambda}{2}|\Pi_j^\delta|.
$$

Therefore, there exists $i_{j,z}$ with $1 \leq i_{j,z} \leq \log(C/\gamma)$, such that

$$
Q(j, z, i_{j,z}) \geq \frac{\lambda}{4}|\Pi_j^\delta|, \quad \text{and} \quad Q(j, z, i_{j,z} - 1) < \frac{\lambda}{4}|\Pi_j^\delta|.
$$

Now, we use the pigeonhole principle: Since there are at most $\log(C/\gamma)$ possible $i_{j,z}$, there is an i_j and a set $P'_j \subset \Pi_j^\delta \cap E$ of measure

$$
|P'_j| \gtrsim (\log(C/\gamma))^{-1}\lambda|\Pi_j^\delta|
$$

such that for each $z \in P'_j$

$$
Q(j, z, i_j) \geq \frac{\lambda}{4}|\Pi_j^\delta|, \quad \text{and} \quad Q(j, z, i_j - 1) < \frac{\lambda}{4}|\Pi_j^\delta|.
$$

By the pigeonhole principle again, since there are M sets Π_j^δ and at most $\log(C/\gamma)$ possible i_j, there is an i_0 and a subset $C' \subset \{\Pi_j^\delta\}_{j=1}^M$ such that

$$
|C'| \gtrsim (\log(C/\gamma))^{-1}M,
$$

and for each $\Pi_j^\delta \in C'$ and each $z \in P'_j$

$$
|\Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0}) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement| \geq \frac{\lambda}{4}|\Pi_j^\delta| \quad \forall e \in S^{n-1}, r \in [\gamma, 1],
$$

and

$$
|\Pi_j^\delta \cap E \cap B(z, \gamma 2^{i_0 - 1}) \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement| < \frac{\lambda}{4}|\Pi_j^\delta|, \qquad (3.9)
$$

for some $e_{j,z} \in S^{n-1}$, $r_{j,z} \in [\gamma, 1]$.

Now, (3.8) and (3.9) imply that

$$\begin{aligned}\frac{\lambda}{4}|\Pi_j^\delta| &\leq |\Pi_j^\delta \cap E \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement| \\ &\quad - |\Pi_j^\delta \cap E \cap B(z,\gamma 2^{i_0-1}) \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement| \\ &= |\Pi_j^\delta \cap E \cap (B(z,\gamma 2^{i_0-1}))^\complement \cap (T_{e_{j,z}}^{(4r_{j,z})(\gamma^2/r_{j,z})}(z))^\complement| \\ &\leq |\Pi_j^\delta \cap E \cap (B(z,\gamma 2^{i_0-1}))^\complement| \\ &= \sum_{k=0}^{\log(C/\gamma)} |\Pi_j^\delta \cap E \cap A(z,\gamma 2^{i_0+k-1})|.\end{aligned}$$

Therefore, there is a $k_{j,z}$ such that

$$|\Pi_j^\delta \cap E \cap A(z,\gamma 2^{i_0+k_{j,z}-1})| \gtrsim (\log(C/\gamma))^{-1}\lambda|\Pi_j^\delta|.$$

Repeatedly using the pigeonhole principle as before, we conclude that there exist an integer k_0 and a set $\mathcal{C} \subset \mathcal{C}'$ with

$$|\mathcal{C}| \gtrsim (\log(C/\gamma))^{-2}M,$$

so that for each $\Pi_j^\delta \in \mathcal{C}$, there is a subset $P_j \subset P'_j$ of measure

$$|P_j| \gtrsim (\log(C/\gamma))^{-2}\lambda|\Pi_j^\delta|$$

such that for each $z \in P_j$

$$|\Pi_j^\delta \cap E \cap B(z,\gamma 2^{i_0}) \cap (T_e^{(4r)(\gamma^2/r)}(z))^\complement| \gtrsim \lambda|\Pi_j^\delta| \quad \forall e \in S^{n-1}, r \in [\gamma, 1],$$

and

$$|\Pi_j^\delta \cap E \cap A(z,\gamma 2^{i_0+k_0-1})| \gtrsim (\log(C/\gamma))^{-1}\lambda|\Pi_j^\delta|.$$

This proves the claim with $\rho := \gamma 2^{i_0+k_0-1}$.

Now we are in a position to carry out a "high-low multiplicity segment" argument (see [1]), as follows.

We fix a number N and consider two cases.

CASE I. For every $a \in \mathbb{R}^n$ we have $|\{j : a \in P_j\}| \leq N$.

CASE II. There exists $a \in \mathbb{R}^n$ such that $|\{j : a \in P_j\}| \geq N$.

In case I we have

$$\begin{aligned}|E| &\geq \Big|\bigcup_{j:\Pi_j^\delta \in \mathcal{C}} P_j\Big| \geq \frac{1}{N}\sum_{j:\Pi_j^\delta \in \mathcal{C}} |P_j| \\ &\gtrsim \frac{1}{N}|\mathcal{C}|\,(\log(C/\gamma))^{-2}\,\lambda\delta^{n-2} \gtrsim \frac{M}{N}\,(\log(C/\gamma))^{-4}\,\lambda\delta^{n-2}, \qquad (3.10)\end{aligned}$$

where the last two inequalities follow from (3.4) and (3.5).

In case II, we fix a number μ and consider two subcases.

$(\mathrm{II})_1$. For every $b \in A(a,\rho)$ we have $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \leq \mu$.
$(\mathrm{II})_2$. There exists $b \in A(a,\rho)$ such that $|\{j : a \in P_j, b \in \Pi_j^\delta\}| \geq \mu$.

In subcase $(\mathrm{II})_1$ we have

$$|E| \geq \Big| \bigcup_{j:a\in P_j} \Pi_j^\delta \cap E \cap A(a,\rho) \Big| \geq \frac{1}{\mu} \sum_{j:a\in P_j} |\Pi_j^\delta \cap E \cap A(a,\rho)|$$

$$\gtrsim \frac{N}{\mu} (\log(C/\gamma))^{-1} \lambda \delta^{n-2}, \tag{3.11}$$

where the last inequality follows from (3.7).

In subcase $(\mathrm{II})_2$ let $\mathcal{B}$ be a maximal $C\rho\delta/\gamma^2$-separated subset of $\{\Pi_j : a \in P_j, b \in \Pi_j^\delta\}$. Then for C large enough, Lemma 2.2 implies

$$|\mathcal{B}| \gtrsim \mu\gamma^{2(n-2)}. \tag{3.12}$$

Note that if $\Pi_j, \Pi_k \in \mathcal{B}$ then by Lemma 2.1

$$\Pi_j^\delta \cap \Pi_k^\delta \cap B(a, 2\rho) \subset T_e^{(4\rho)(\gamma^2/\rho)}(a),$$

where $e = (a-b)/|a-b|$, provided that C has been chosen large enough. Therefore the family

$$\left\{ \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement : \Pi_j \in \mathcal{B} \right\}$$

is disjoint. Consequently

$$|E| \geq \Big| \bigcup_{j:\Pi_j\in\mathcal{B}} \Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement \Big|$$

$$= \sum_{j:\Pi_j\in\mathcal{B}} |\Pi_j^\delta \cap E \cap B(a, 2\rho) \cap (T_e^{(4\rho)(\gamma^2/\rho)}(a))^\complement|$$

$$\gtrsim |\mathcal{B}|\lambda\delta^{n-2} \gtrsim \gamma^{2(n-2)}\mu\lambda\delta^{n-2}, \tag{3.13}$$

where the last two inequalities follow from (3.6) and (3.12). So, in case II we see that choosing

$$\mu = N^{1/2} (\log(C/\gamma))^{-1/2} \gamma^{-(n-2)},$$

(3.11) and (3.13) imply that

$$|E| \gtrsim (\log(C/\gamma))^{-1/2} \gamma^{n-2}\lambda N^{1/2}\delta^{n-2}. \tag{3.14}$$

In conclusion, we see that in case I, (3.10) holds, whereas in case II, (3.14) holds. So choosing

$$N = M^{2/3} (\log(C/\gamma))^{-7/3} \gamma^{-2(n-2)/3},$$

the right hand sides of (3.10) and (3.14) become equal. So, in both cases we have

$$\begin{aligned}|E| &\gtrsim \left(\log(C/\gamma)\right)^{-5/3}\gamma^{2(n-2)/3}\lambda M^{1/3}\delta^{n-2}\\ &= \left(\log(C/\gamma)\right)^{-5/3}\left(\lambda^{1/2}(\log(1/\delta))^{-1/2}\right)^{2(n-2)/3}\lambda M^{1/3}\delta^{n-2}\\ &\geq \left(\log(C/\delta)\right)^{-5/3}\left(\lambda^{1/2}(\log(1/\delta))^{-1/2}\right)^{2(n-2)/3}\lambda M^{1/3}\delta^{n-2}\\ &\geq C_\epsilon \delta^\epsilon \lambda^{(n+1)/3} M^{1/3}\delta^{n-2},\end{aligned}$$

where the inequality before the last one is true because $4\delta \leq \gamma$. The proof is complete.

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