A GENERALIZATION OF A RESULT DUE TO HAVIN AND MAZYA

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Abstract. We generalize the classical result of Havin and Mazya which relates Bessel capacity and Hausdorff dimension.

Let

$$L^p_\alpha(\mathbb{R}^d) = \{f : f = G_\alpha * g,\ g \in L^p(\mathbb{R}^d)\},\ \alpha > 0,\ p > 1,$$

be the space of Bessel potentials, with norm

$$\|f\|_{\alpha,p} = \|g\|_p.$$

Here G_α is the Bessel kernel, i.e., the inverse Fourier transform of the function

$$\widehat{G_\alpha}(\xi) = (1 + |\xi|^2)^{-\alpha/2}.$$

The Bessel capacity of a set $E \subset \mathbb{R}^d$ is defined as

$$B_{\alpha,p}(E) = \inf\{\|f\|^p_{\alpha,p} :\ f \geq 1 \text{ on } E\}.$$

The relation between capacity and Hausdorff measure is given by the following result due to Havin and Mazya [2].

Theorem 1. *Let $E \subset \mathbb{R}^d$ be a Borel set. If $p > 1$, $\alpha p \leq d$, then*

$$B_{\alpha,p}(E) = 0 \Rightarrow \mathcal{H}^{d-\alpha p+\varepsilon}(E) = 0,\ \textit{for every } \varepsilon > 0.$$

This implies that the Hausdorff dimension of E is less than $d - \alpha p$. The original proof of Theorem 1 involved the Hardy-Littlewood maximal function and was rather indirect. A different proof based on Wolff's inequality may be found in [1]. The purpose of this short note is to give an easy direct proof of a more general result in the context of "mixed-norm" capacities, defined as follows. Let

$$L^{p_1,p_2}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}),\ p_1 > 1,\ p_2 > 1,$$

be the space of all functions with finite $\|\cdot\|_{p_1,p_2}$ norm, where

$$\|g\|_{p_1,p_2} = \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} |g(x_1,x_2)|^{p_1} dx_1\right)^{p_2/p_1} dx_2\right)^{1/p_2}.$$

For $\alpha > 0$, define the potential space

$$L^{p_1,p_2}_\alpha(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) = \{f : f = G_\alpha * g,\ g \in L^{p_1,p_2}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\},$$

with norm

$$\|f\|_{\alpha,p_1,p_2} = \|g\|_{p_1,p_2}.$$

Then the mixed-norm capacity of $E \subset \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is defined as

$$B_{\alpha,p_1,p_2}(E) = \inf\{\|f\|^{p_2}_{\alpha,p_1,p_2} : f \geq 1 \text{ on } E\}.$$

Now, we can state our result.

Theorem 2. *Let $E \subset \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be a Borel set.*

If $p_1 \leq p_2$ and $d_2 + d_1\frac{p_2}{p_1} - p_2\alpha \geq 0$ then

$$B_{\alpha,p_1,p_2}(E) = 0 \Rightarrow \mathcal{H}^{d_2+d_1\frac{p_2}{p_1}-p_2\alpha+\varepsilon}(E) = 0, \textit{ for every } \varepsilon > 0.$$

If $p_2 \leq p_1$ and $d_1 + d_2\frac{p_1}{p_2} - p_1\alpha \geq 0$ then

$$B_{\alpha,p_1,p_2}(E) = 0 \Rightarrow \mathcal{H}^{d_1+d_2\frac{p_1}{p_2}-p_1\alpha+\varepsilon}(E) = 0, \textit{ for every } \varepsilon > 0.$$

Proof. Without loss of generality we may assume that $E \subset [0,1]^d$. Let μ be a finite measure supported on E, and let u be a non-negative C_c^∞ function such that $u \geq 1$ on E. Then

$$\begin{aligned}\mu(E) &\leq \int u(x)d\mu(x) = \int G_\alpha * D^\alpha u(x)d\mu(x)\\ &= \int D^\alpha u(y) \int G_\alpha(x-y)d\mu(x)dy\\ &\leq \|u\|_{\alpha,p_1,p_2}\|G_\alpha * \mu\|_{q_1,q_2},\end{aligned}$$

where q_1, q_2 are the conjugate exponents of p_1, p_2 respectively, and $D^\alpha u$ is the fractional derivative operator acting on u, defined as the inverse Fourier transform of the function

$$(1+|\xi|^2)^{\alpha/2}\widehat{u}(\xi).$$

For each $n \geq 0$ we subdivide $\mathbb{R}^d$ into disjoint dyadic cubes of sidelength 2^{-n}, so that each cube of sidelength 2^{-k} is split into 2^d cubes of sidelength $2^{-(k+1)}$. If Q is such a dyadic cube then $l(Q)$ denotes its sidelength and $\widetilde{Q}$ the cube with the same center as Q and sidelength $3l(Q)$. Now, let

$$\widetilde{I}_\alpha(x) = \begin{cases} |x|^{\alpha-d}, & \text{if } 0 < |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}.$$

It follows from the properties of the Bessel kernel (see, e.g., [1], [3]) that there exist constants a and A such that

$$G_\alpha(x) \leq A\widetilde{I}_\alpha(x),\ 0 < |x| \leq 1,$$

and

$$G_\alpha(x) \leq Ae^{-a|x|},\ |x| > 1.$$

Therefore,

$$
\begin{aligned}
\|G_\alpha * \mu\|_{q_1,q_2} &\lesssim \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} (\widetilde{I_\alpha} * \mu(x_1,x_2))^{q_1} dx_1 \right)^{\frac{q_2}{q_1}} dx_2 \right)^{\frac{1}{q_2}} \\
&+ \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \left(\int_{|(x_1,x_2)-y|>1} e^{-a|(x_1,x_2)-y|} d\mu(y) \right)^{q_1} dx_1 \right)^{\frac{q_2}{q_1}} dx_2 \right)^{\frac{1}{q_2}} \\
&= B + B'.
\end{aligned}
$$

B' is easy to estimate. By Minkowski's inequality for integrals, we have

$$
\begin{aligned}
B' &\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} e^{-aq_1|(x_1,x_2)-y|} dx_1 \right)^{q_2/q_1} dx_2 \right)^{1/q_2} d\mu(y) \\
&\leq \mu(E) \left(\int_{\mathbb{R}^{d_2}} e^{-\frac{1}{\sqrt{2}} aq_2|x_2|} dx_2 \left(\int_{\mathbb{R}^{d_1}} e^{-\frac{1}{\sqrt{2}} aq_1|x_1|} dx_1 \right)^{q_2/q_1} \right)^{1/q_2} < \infty.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\widetilde{I_\alpha} * \mu(x) &= \int_{|x-y|\leq 1} \frac{d\mu(y)}{|x-y|^{d-\alpha}} = \sum_{n=0}^{\infty} \int_{2^{-(n+1)}<|x-y|\leq 2^{-n}} \frac{d\mu(y)}{|x-y|^{d-\alpha}} \\
&\leq \sum_{n=0}^{\infty} 2^{(n+1)(d-\alpha)} \mu(B(x,2^{-n})) \lesssim \sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})}{l(Q)^{d-\alpha}} \chi_Q(x) \\
&\lesssim \left(\sum_{n=0}^{\infty} 2^{-\delta p_1(n+1)} \right)^{1/p_1} \left(\sum_{l(Q)\leq 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} \chi_Q(x) \right)^{1/q_1},
\end{aligned}
$$

where δ is a positive number. Now let

$$\pi_2 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \longrightarrow \mathbb{R}^{d_2}$$

be the usual projection

$$\pi_2(x_1, x_2) = x_2.$$

Also, let

$$s = d_2 + d_1 \frac{p_2}{p_1} - p_2 \alpha,$$

and

$$t = d_1 + d_2 \frac{p_1}{p_2} - p_1 \alpha.$$

Suppose that $p_1 \leq p_2$ and that $\mathcal{H}^{s+\varepsilon}(E) > 0$ for some $\varepsilon > 0$. Then there exists a nontrivial finite measure μ supported on E such that

$$\mu(B(x,r)) \leq r^{s+\varepsilon}$$

for all $x \in \mathbb{R}^d$, $r > 0$. It follows that

$$\begin{aligned}
B^{q_2} &= \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} (\widetilde{I_\alpha} * \mu(x_1, x_2))^{q_1} dx_1 \right)^{q_2/q_1} dx_2 \\
&\lesssim \int_{\mathbb{R}^{d_2}} \left(\sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} l(Q)^{d_1} \chi_{\pi_2(Q)}(x_2) \right)^{q_2/q_1} dx_2 \\
&\lesssim \sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_2}}{l(Q)^{q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2}} \\
&= \sum_{n=0}^{\infty} 2^{n(q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2)} \sum_{l(Q)=2^{-n}} \mu(\widetilde{Q})\mu(\widetilde{Q})^{q_2-1} \\
&\lesssim \mu(E) \sum_{n=0}^{\infty} \frac{2^{n(q_2(d-\alpha+\delta)-d_1\frac{q_2}{q_1}-d_2)}}{2^{n(q_2-1)(s+\varepsilon)}} < \infty,
\end{aligned}$$

provided that δ has been chosen so that $p_2\delta < \varepsilon$.

Now suppose that $p_2 \le p_1$ and that $\mathcal{H}^{t+\varepsilon}(E) > 0$ for some $\varepsilon > 0$. Then, as before, there exists a nontrivial finite measure supported on E such that

$$\mu(B(x,r)) \le r^{t+\varepsilon},$$

for all $x \in \mathbb{R}^d$, $r > 0$. It follows that

$$\begin{aligned}
B^{q_1} &\lesssim \left(\int_{\mathbb{R}^{d_2}} \left(\sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)}} l(Q)^{d_1} \chi_{\pi_2(Q)}(x_2) \right)^{q_2/q_1} dx_2 \right)^{q_1/q_2} \\
&\lesssim \sum_{l(Q)\le 1} \frac{\mu(\widetilde{Q})^{q_1}}{l(Q)^{q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1}} \\
&= \sum_{n=0}^{\infty} 2^{n(q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1)} \sum_{l(Q)=2^{-n}} \mu(\widetilde{Q})\mu(\widetilde{Q})^{q_1-1} \\
&\lesssim \mu(E) \sum_{n=0}^{\infty} \frac{2^{n(q_1(d-\alpha+\delta)-d_2\frac{q_1}{q_2}-d_1)}}{2^{n(q_1-1)(t+\varepsilon)}} < \infty,
\end{aligned}$$

provided that $p_1\delta < \varepsilon$.

It follows that $\mu(E) \lesssim \|u\|_{\alpha,p_1,p_2}$. By assumption, $B_{\alpha,p_1,p_2}(E) = 0$. Therefore $\mu(E) = 0$ which is a contradiction. □

Of course, Theorem 2 implies Theorem 1, if we take $p_1 = p_2 = p$.

References

[1] D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, 1996.

[2] V. Havin, V. Mazya, Nonlinear potential theory, Russian Math. Surveys, 27 (1972), 71-148.

[3] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, 1970.

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