ON NIKODYM-TYPE SETS IN HIGH DIMENSIONS

THEMIS MITSIS

Abstract. We prove that the complement of a higher dimensional Nikodym set must have full Hausdorff dimension.

1. Introduction

In [4] Nikodym constructed a subset F of the unit square in $\mathbb{R}^2$ such that F has planar measure 1, and for every point $x \in F$ there exists a line passing through x intersecting F in that single point. Such paradoxical sets are called Nikodym sets.

Falconer [3] extended Nikodym's result to higher dimensions. He proved that for every $n > 2$ there exists a set $F \subset \mathbb{R}^n$ such that the complement of F has Lebesgue measure zero, and for every $x \in F$ there is a hyperplane H so that $x \in H$ and $F \cap H = \{x\}$. We call such a set an n-Nikodym set.

The purpose of this paper is to show that the complement of an n-Nikodym set, even though is small in terms of Lebesgue measure, must be large in terms of Hausdorff dimension. Namely, we use ideas from [1] and [2] to prove the following.

Theorem. *The Hausdorff dimension of the complement of an n-Nikodym set is equal to n.*

A few remarks about our notation. $\mathcal{L}^k(\cdot)$ denotes k-dimensional Lebesgue measure and card$(\cdot)$ cardinality. $B(x, r)$ is the ball with center x and radius r. χ_A is the characteristic function of the set A. Finally, $x \lesssim y$ means $x \leq Cy$, where C is some positive constant not necessarily the same at each of its occurrences.

2. Proof of the theorem

Let E be the complement of an n-Nikodym set in $\mathbb{R}^n$. Without loss of generality we may assume that there is a subset A of the unit cube with $\mathcal{L}^n(A) > 0$ such that for every $x \in A$ there exists a set H_x with the following properties:

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(P1) H_x is a rotated translation of $\underbrace{[0,1]\times\cdots\times[0,1]}_{n-1}\times\{0\}$.

(P2) The center of H_x is the point x.

(P3) The normal vector to H_x makes an angle less than $\pi/100$ with the unit vector $e_n = (0,\dots,0,1)$.

(P4) $H_x \cap E = H_x \setminus \{x\}$, so in particular $\mathcal{L}^{n-1}(E\cap H_x) = 1$.

We will show that for every $\varepsilon > 0$ the $(n-\varepsilon)$-dimensional Hausdorff measure of E is not zero. Therefore, the Hausdorff dimension of E must equal n. To this end, fix a countable covering $\{B(x_j, r_j)\}$ of E, and for every integer k let

$$J_k = \left\{ j : 2^{-k} \le r_j \le 2^{-(k-1)} \right\},$$

$$E_k = E \cap \bigcup_{j\in J_k} B(x_j, r_j), \qquad \widetilde{E}_k = \bigcup_{j\in J_k} B(x_j, 2r_j).$$

We will bound $\sum_j r_j^{n-\varepsilon}$ from below by a constant depending only on ε.

Notice that for every $x \in A$ there exists an integer k_x such that

$$\mathcal{L}^{n-1}(E_{k_x} \cap H_x) \ge \frac{1}{4k_x^2}.$$

Indeed, if this were not the case for some $x \in A$, we would have

$$1 = \mathcal{L}^{n-1}(E\cap H_x) \le \sum_k \mathcal{L}^{n-1}(E_k \cap H_x) \le \sum_k \frac{1}{4k^2} < \frac{1}{2}.$$

Now let

$$A_k = \left\{ x \in A : \mathcal{L}^{n-1}(E_k \cap H_x) \ge \frac{1}{4k^2} \right\}. \tag{1}$$

Then

$$A = \bigcup_k A_k.$$

Therefore, there must be an integer N such that

$$\mathcal{L}^n(A_N) \ge \frac{\mathcal{L}^n(A)}{2N^2},$$

because otherwise we would have

$$\mathcal{L}^n(A) \le \sum_k \mathcal{L}^n(A_k) \le \sum_k \frac{\mathcal{L}^n(A)}{2k^2} < \mathcal{L}^n(A).$$

Next, we decompose the unit cube into a grid of small cubes, each of side 2^{-N}.

$$[0,1]^n = \bigcup_{i_1,\dots,i_n=1}^{2^N} \prod_{k=1}^{n} \left[(i_k-1)2^{-N}, i_k 2^{-N}\right] = \bigcup_{i_1,\dots,i_n=1}^{2^N} Q_{i_1\cdots i_n}.$$

Let

$$I = \left\{(i_1,\dots,i_n) : Q_{i_1\cdots i_n} \cap A_N \neq \emptyset\right\}.$$

Notice that for each $(i_1,\dots,i_n) \in I$, property (P2) and (1) imply that there exists a rectangle $R_{i_1\cdots i_n}$ such that

- $R_{i_1\cdots i_n}$ has dimensions $\underbrace{1\times\cdots\times 1}_{n-1}\times 2^{-N}$.
- $R_{i_1\cdots i_n}$ is parallel to H_x for some $x \in Q_{i_1\cdots i_n}$.
- $R_{i_1\cdots i_n} \cap Q_{i_1\cdots i_n} \neq \emptyset$.
- $\mathcal{L}^n(\widetilde{E}_N \cap R_{i_1\cdots i_n}) \gtrsim N^{-2}2^{-N}$.

Now let

$$R'_{i_1\cdots i_n} = \begin{cases} R_{i_1\cdots i_n} & \text{if } (i_1,\dots,i_n)\in I \\ \emptyset & \text{otherwise} \end{cases}.$$

Then

$$\begin{aligned}
N^{-2}\mathcal{L}^n(A) \lesssim \mathcal{L}^n(A_N) &\leq \sum_{(i_1,\dots,i_n)\in I} 2^{-nN} = 2^{-(n-1)N}N^2 \sum_{(i_1,\dots,i_n)\in I} N^{-2}2^{-N} \\
&\lesssim 2^{-(n-1)N}N^2 \sum_{i_1,\dots,i_n=1}^{2^N} \mathcal{L}^n(\widetilde{E}_N \cap R'_{i_1\cdots i_n}) \\
&= 2^{-(n-1)N}N^2 \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\int_{\widetilde{E}_N} \sum_{i_n=1}^{2^N} \chi_{R'_{i_1\cdots i_n}} \Big) \\
&\leq 2^{-(n-1)N}N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\int \Big(\sum_{i_n=1}^{2^N} \chi_{R'_{i_1\cdots i_n}} \Big)^2 \Big)^{1/2} \\
&= 2^{-(n-1)N}N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\sum_{l,m=1}^{2^N} \int \chi_{R'_{i_1\cdots i_{n-1}l}} \chi_{R'_{i_1\cdots i_{n-1}m}} \Big)^{1/2} \\
&= 2^{-(n-1)N}N^2 \mathcal{L}^n(\widetilde{E}_N)^{1/2} \sum_{i_1,\dots,i_{n-1}=1}^{2^N} \Big(\sum_{l,m=1}^{2^N} \mathcal{L}^n(R'_{i_1\cdots i_{n-1}l} \cap R'_{i_1\cdots i_{n-1}m}) \Big)^{1/2}.
\end{aligned}$$

Now using property (P3), it is easy to show that for fixed $i_1,\dots,i_{n-1}$ we have

$$\mathcal{L}^n(R'_{i_1\cdots i_{n-1}l} \cap R'_{i_1\cdots i_{n-1}m}) \lesssim \frac{2^{-N}}{1+|m-l|}.$$

Consequently

$$\sum_{l,m=1}^{2^N} \mathcal{L}^n(R'_{i_1\cdots i_{n-1}l} \cap R'_{i_1\cdots i_{n-1}m}) \lesssim \log 2^N = N\log 2.$$

Therefore

$$N^{-2}\mathcal{L}^n(A) \lesssim 2^{-(n-1)N}N^2\mathcal{L}^n(\widetilde{E}_N)^{1/2}2^{(n-1)N}N^{1/2}$$
$$\Rightarrow \mathcal{L}^n(\widetilde{E}_N) \gtrsim N^{-9}\mathcal{L}^n(A)^2.$$

On the other hand, by the definition of $\widetilde{E}_N$ we have

$$\mathcal{L}^n(\widetilde{E}_N) \lesssim \mathrm{card}(J_N)2^{-nN}.$$

Hence

$$\mathrm{card}(J_N) \gtrsim 2^{nN}N^{-9}\mathcal{L}^n(A)^2.$$

We conclude that

$$\sum_j r_j^{n-\varepsilon} \gtrsim \mathrm{card}(J_N)(2^{-N})^{n-\varepsilon} \gtrsim 2^{N\varepsilon}N^{-9}\mathcal{L}^n(A)^2 \gtrsim C_\varepsilon.$$

The proof is complete.

References

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Department of Mathematics, University of Crete, Knossos Ave., 71409 Iraklio, Greece

E-mail address: `mitsis@fourier.math.uoc.gr`