ON A PROBLEM RELATED TO SPHERE AND CIRCLE PACKING

THEMIS MITSIS

ABSTRACT. We prove that a set which contains spheres centered at all points of a set of Hausdorff dimension greater than 1 must have positive Lebesgue measure. We show by a counterexample that this is sharp. We also prove the corresponding result for circles provided that the set of centers has Hausdorff dimension greater than 3/2.

1. INTRODUCTION

The following geometric result is a consequence of the work of Stein on the spherical means maximal operator.

Theorem 1.1. Let $F \subset \mathbb{R}^d$, $d \geq 3$, be a set of positive Lebesgue measure. If $E \subset \mathbb{R}^d$ is a set which contains spheres centered at all points of F, then E has positive Lebesgue measure.

Bourgain [1], and, independently, Marstrand [7], proved the two - dimensional analogue, given as follows.

Theorem 1.2. Let $F \subset \mathbb{R}^2$ be a set of positive Lebesgue measure. If $E \subset \mathbb{R}^2$ is a set that contains circles centered at all points of F, then E has positive plane measure.

This should be contrasted with the following construction due to Talagrand [12].

Theorem 1.3. *There is a set of plane measure zero containing for each x on a given straight line, a circle centered at x.*

It is, therefore, natural to ask whether one can weaken the condition that the set F in the above theorems should be of postive measure. The main results in this paper are the following.

Theorem 1.4. Let $F \subset \mathbb{R}^d$, $d \ge 3$, be a Borel set of Hausdorff dimension s, s > 1. If $E \subset \mathbb{R}^d$ is a Borel set that contains spheres centered at each point of F, then E has positive Lebesgue measure.

Theorem 1.5. Let $F \subset \mathbb{R}^2$ be a Borel set of Hausdorff dimension s, s > 3/2. If $E \subset \mathbb{R}^2$ is a Borel set that contains circles centered at each point of F, then E has positive Lebesgue measure.

The paper is organized as follows. In Section 2 we state some geometric lemmas needed later on, in Section 3 we prove Theorem 1.4 and construct a counterexample related to it, in Section 4 we prove Theorem 1.5, and finally, in Section 5, we discuss the possibility of weakening the condition s > 3/2 in Theorem 1.5. Throughout this paper, $a \leq b$ means $a \leq Ab$ for some absolute constant A, and similarly with $a \geq b$ and $a \sim b$. We will denote Lebesgue measure by $|\cdot|$.

2. Background

We start with some notation.

B(x, r) is the open disk (or ball) with center x and radius r. C(x, r) is the circle (or sphere) with center x and radius r. $C^{\delta}(x, r)$ is the δ -neighborhood of the circle (or sphere) C(x, r), i.e., the set

$$\{y \in \mathbb{R}^d : r - \delta < |x - y| < r + \delta\}.$$

If C(x, r) and C(y, s) are circles, then we define

$$d((x, y), (r, s)) = |x - y| + |r - s|,$$

$$\Delta((x, y), (r, s)) = ||x - y| - |r - s||.$$

Note that $\Delta = 0$ if and only if the circles are internally tangent, that is, they are tangent and one is contained in the bounded component of the complement of the other. In what follows, we assume that the centers of all circles (or spheres) in question are contained in the disk B(0, 1/4) and that their radii are in the interval [1/2, 2].

The following lemma gives estimates on the size of the intersection of two annuli in terms of their relative position and their degree of tangency. The reader is referred to Wolff [14] for a proof.

Lemma 2.1. Suppose that C(x, r), C(y, s) are circles with $r \neq s$. Then for $0 < \delta < 1$ there exists an absolute constant A_0 such that

(1) $C^{\delta}(x,r) \cap C^{\delta}(y,s)$ is contained in a δ -neighborhood of arc length less than

$$A_0 \sqrt{\frac{\Delta + \delta}{|x - y| + \delta}},$$

centered at the point $x - r \cdot \text{sgn}(r - s) \frac{x - y}{|x - y|}$. (2) The area of intersection satisfies

$$|C^{\delta}(x,r) \cap C^{\delta}(y,s)| \le A_0 \frac{\delta^2}{\sqrt{(\delta + \Delta)(\delta + d)}}$$

The next result is essentially Marstrand's three circle lemma [7]. It is a quantitative version of the following fact known as the circles of Apollonius: given three circles which are not internally tangent at a single point, there are at most two other circles that are internally tangent to the given ones.

Lemma 2.2. There exists a constant A_1 such that if ε , $t, \lambda \in (0, 1)$ satisfy $\lambda \ge A_1 \sqrt{\frac{\varepsilon}{t}}$ then for three fixed circles $C(x_i, r_i), i = 1, 2, 3$ and for $\delta \le \varepsilon$ the set

$$\begin{aligned} \{(x,r) \in \mathbb{R}^2 \times \mathbb{R} : \Delta((x,r), (x_i, r_i)) < \varepsilon \ \forall i, \\ d((x,r), (x_i, r_i)) > t \ \forall i, \\ C^{\delta}(x,r) \cap C^{\delta}(x_i, r_i) \neq \emptyset \ \forall i, \\ \text{dist}(C^{\delta}(x,r) \cap C^{\delta}(x_i, r_i), C^{\delta}(x,r) \cap C^{\delta}(x_j, r_j)) \geq \lambda \\ \forall i, j : i \neq j \end{aligned}$$

is contained in the union of two ellipsoids in \mathbb{R}^3 each of diameter $\leq \frac{\varepsilon}{\lambda^2}$.

A proof of the preceding result can be found in Wolff [14].

We conclude this section with some facts from geometric measure theory. We refer the reader to Falconer [4] for definitions, proofs and details. In what follows, \mathcal{H}^s denotes *s*-dimensional Hausdorff measure.

Theorem 2.1. Let *E* be a Borel set in \mathbb{R}^d and let s > 0. Assume that $\mathcal{H}^s(E) > 0$. Then there exists a nontrivial finite measure μ supported on *E* such that $\mu(B(x, r)) \leq r^s$ for $x \in \mathbb{R}^n$ and r > 0.

If *E* is an *s*-set, i.e., $0 < \mathcal{H}^{s}(E) < \infty$, then a point $x \in E$ is called regular if the upper and the lower densities at *x* are equal to one; otherwise *x* is called irregular. An *s*-set *E* is said to be irregular if \mathcal{H}^{s} -almost all of its points are irregular. Irregular 1-sets are characterized by the following:

Theorem 2.2. A 1-set in \mathbb{R}^2 is irregular if and only if it has projections of linear Lebesgue measure zero in two distinct directions.

In fact, one can say much more.

Theorem 2.3. Let *E* be an irregular 1-set in \mathbb{R}^2 . Then $\operatorname{proj}_{\theta}(E)$ has linear Lebesgue measure zero for almost all $\theta \in [0, \pi)$, where $\operatorname{proj}_{\theta}$ denotes orthogonal projection from \mathbb{R}^2 onto the line through the origin making angle θ with some fixed axis.

3. The higher dimensional case

In this section we assume that $d \ge 3$. For $f : \mathbb{R}^d \to \mathbb{R}$, $\delta > 0$ small, we define $\mathcal{M}_{\delta} : B(0, 1/4) \to \mathbb{R}$ by

$$\mathcal{M}_{\delta}f(x) = \sup_{1/2 \le r \le 2} \frac{1}{|C^{\delta}(x,r)|} \int_{C^{\delta}(x,r)} |f(y)| dy.$$

Theorem 1.4 will be a consequence of the following $L^2 \rightarrow L^2$ maximal inequality.

Proposition 3.1. Let $F \subset B(0, 1/4)$ be a compact set in \mathbb{R}^d such that there exist s > 1 and a finite measure μ supported on F with $\mu(B(x, r)) \leq r^s$ for $x \in \mathbb{R}^d$ and r > 0. Then there exists a constant A that depends only on the measure of F and on s such that

$$\left(\int_F \left(\mathcal{M}_{\delta}f(x)\right)^2 d\mu(x)\right)^{1/2} \le A ||f||_2$$

for small $\delta > 0$ and all f.

The proof generally follows the lines of the proof of Theorem 1' in [6]. Even though the necessary modifications are not immediately obvious, we choose not to include the proof. A similar argument will appear in a forthcoming paper on the Kakeya maximal function.

Proof of Theorem 1.4. We may assume that $F \,\subset B(0, \frac{1}{4})$. Suppose that |E|=0 and choose *t* so that 1 < t < s. Then there exist a compact set $E_1 \subset E$, a compact set $F_1 \subset F$ with $\mathcal{H}^t(F_1) > 0$, and a positive number *r*, such that, for each $x \in F_1$, there is a sphere centered at *x* with radius $r(x) \in (r, 2r)$ which intersects E_1 in set of surface measure at least $r^{d-1}(x)$. Without loss of generality we may assume that r = 1. It follows that for all $x \in F_1$

$$\mathcal{M}_{\delta \chi_{E^{\delta}}}(x) \gtrsim 1,$$

where E_1^{δ} is the δ -neighborhood of the set E_1 .

By Theorem 2.1, there exists a nontrivial finite measure μ supported on F_1 such that $\mu(B(x, r)) \leq r^t$, for $x \in \mathbb{R}^d$, r > 0. Therefore, by Proposition 3.1, we have

$$\mu(F_1) \lesssim \int_{F_1} (\mathcal{M}_{\delta} \chi_{E_1^{\delta}}(x))^2 d\mu(x) \lesssim |E_1^{\delta}|.$$

The right-hand side of the above inequality tends to zero as $\delta \to 0$, so we get a contradiction.

We will show that we cannot drop the condition s > 1 in Theorem 1.4.

Proposition 3.2. There exists a set of d-dimensional Lebesgue measure zero containing for each $x \in [0, 1] \times \underbrace{\{0\} \times \cdots \times \{0\}}_{d=1}$, a sphere centered at x.

Proof. The idea, which goes back to Davies [3], is to parametrize the set of radii using a suitable irregular 1-set.

Divide the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ into 16 disjoint squares of side 1/4. Let $S_{(i,j)} \ 1 \le i, j \le 4$ be those squares (indexed from bottom to top, left to right), and put

$$E_1 = S_{(1,2)} \cup S_{(1,4)} \cup S_{(4,1)} \cup S_{(4,3)}.$$

Apply the same procedure to each of $S_{(1,2)}$, $S_{(1,4)}$, $S_{(4,1)}$, $S_{(4,3)}$, and let E_2 be the union of the new squares. Continuing in the same manner we obtain a decreasing sequence of compact sets $\{E_n\}$. Let $E = \bigcap_{n=1}^{\infty} E_n$. Then *E* is a 1-set such that

$$\operatorname{proj}_{0}(E) = [0, 1], |\operatorname{proj}_{\pi/2}(E)| = 0, |\operatorname{proj}_{\pi/4}(E)| = 0$$

where $|\cdot|$ is linear Lebesgue measure, and proj₀, proj_{$\pi/2$}, proj_{$\pi/4$} denote, respectively, orthogonal projection onto the *x*-axis, the *y*-axis, and the line through the origin making angle $\pi/4$ with the *x*-axis. It follows from Theorem 2.2 that *E* is irregular. Let

$$A = \bigcup_{(a,b)\in E} \{(x_1,\ldots,x_d) : (x_1-a)^2 + x_2^2 + \cdots + x_d^2 = a^2 + b\}$$
$$= \bigcup_{(a,b)\in E} \{(x_1,\ldots,x_d) : x_d^2 = 2ax_1 + b - x_1^2 - \cdots - x_{d-1}^2\}.$$

Since $\text{proj}_0(E) = [0, 1]$, *A* contains a sphere centered at each point of $\{(a, 0, \dots, 0) : a \in [0, 1]\}$. Now fix s_1, \dots, s_{d-1} . Then

$$A \cap \{(x_1, \dots, x_d) : x_1 = s_1, x_2 = s_2, \dots, x_{d-1} = s_{d-1}\}$$

= {s_1} × {s_2} × \dots × {s_{d-1}} × B.

where

$$B = \{x_d : x_d^2 = 2as_1 + b - s_1^2 - s_2^2 - \dots - s_{d-1}^2, (a, b) \in E\}$$

B has measure zero if and only if $\{2as_1 + b : (a, b) \in E\}$ has measure zero. But $\mathcal{L}^1(\{2as_1 + b : (a, b) \in E\}) = 0$ for almost all $s_1 \in \mathbb{R}$ by Theorem 2.3. Therefore, by Fubini, *A* has *d*-dimensional measure zero.

4. The two-dimensional case

Before we proceed with the proof of Theorem 1.5, it might be instructive to discuss briefly the underlying ideas. It turns out that the two-dimensional problem can be reduced to estimating the measure of a family of thin annuli. By Lemma 2.1, the measure of the intersection of two annuli is large when the corresponding circles are internally tangent. It is, therefore, essential that we be able to control the total number of such tangencies. To this end, we employ Marstrand's three circle lemma together with a suitable counting argument. This approach was first used in Kolasa and Wolff [6], and, subsequently, in Schlag [8], [9]. We should, however, mention that, in contrast with the afforementioned authors, we do not make any cardinality estimates since these are not particularly useful in the case of general Hausdorff measures.

The motivation for the combinatorial part of the proof is the following observation (see [5] for more details).

Proposition 4.1. Let $\{C_j\}_{j=1}^N$ be a family of distinct circles such that no three are tangent at a single point. Then

$$card(\{(i, j) : C_i || C_j\}) \leq N^{5/3}$$

where $C_i \parallel C_j$ means that C_i and C_j are internally tangent.

Proof. Let $Q = \{(i, j_1, j_2, j_3) : C_i || C_{j_k}, k = 1, 2, 3\}$ and fix j_1, j_2, j_3 . Then, by the circles of Apollonius, there are at most two choices for *i*. Therefore,

$$card(Q) \le 2N(N-1)(N-2) < 2N^3.$$

On the other hand, if we let $n(i) = \operatorname{card}(\{j : C_i || C_j\})$, then

$$Q = \bigcup_{i=1}^{N} \left(\{i\} \times \{(j_1, j_2, j_3) : C_i \parallel C_{j_k}, \ k = 1, 2, 3\} \right).$$

Hence

$$\operatorname{card}(Q) \ge \sum_{i=1}^{N} n(i)(n(i) - 1)(n(i) - 2) \ge \sum_{i=1}^{N} (n(i) - 2)^3.$$

It follows that

$$\operatorname{card}(\{(i, j) : C_i \parallel C_j\}) = \sum_{i=1}^N n(i) = \sum_{i=1}^N (n(i) - 2) + 2N$$
$$\leq \left(\sum_{i=1}^N (n(i) - 2)^3\right)^{1/3} N^{2/3} + 2N$$
$$\leq (\operatorname{card}(Q))^{1/3} N^{2/3} + 2N \leq (2N^3)^{1/3} N^{2/3} + 2N \leq N^{5/3}.$$

The proof of Theorem 1.5 will be a quantitative version of Proposition 4.1, with Lemma 2.2 playing the role of the restriction imposed by the circles of Apollonius.

Proof of Theorem 1.5. We may assume that $F \subset B(0, \frac{1}{4})$. Suppose that |E| = 0 and choose s_1 so that $3/2 < s_1 < s$. Then there exist a compact set $E_1 \subset E$, a compact set $F_1 \subset F$ with $\mathcal{H}^{s_1}(F_1) > 0$ and a positive number *r*, such that, for each $x \in F_1$, there is a circle centered at x with radius $r(x) \in (r, 2r)$ which intersects E_1 in a set of angle measure at least π . Without loss of generality we may assume that r = 1.

By Theorem 2.1, there exists a nontrivial finite measure μ supported on F_1 such that $\mu(B(x, r)) \leq r^{s_1}$, for $x \in \mathbb{R}^2$, r > 0.

Let $\{x_i\}_{i \in I}$ be a maximal δ -separated set of points in F_1 , and let $a_i = \mu(B(x_i, \delta))$. Choose $r_i > 0$ such that

$$|C^{\delta}(x_i, r_i) \cap E_1^{\delta}| \ge \frac{1}{2} |C^{\delta}(x_i, r_i)|, \tag{6}$$

where E_1^{δ} is the δ -neighborhood of E_1 . Let κ be the infimum of those $\lambda > 0$ such that there exists $J \subset I$ satisfying

$$\sum_{j\in J} a_j \ge \frac{1}{2}\mu(F_1),$$

and for all $j \in J$

$$\left|\left\{x \in C^{\delta}(x_j, r_j) \cap E_1^{\delta} : \sum_{i \in I} a_i \chi_{C^{\delta}(x_i, r_i)}(x) \le \lambda\right\}\right| \ge \frac{1}{4} |C^{\delta}(x_j, r_j)|.$$

Choose N large enough so that

$$s_1 - \frac{3}{2} > \frac{7 + 2s_1 - 1}{N}$$

and

$$\frac{N^2 - N}{N^2 - N - 2} < \frac{2s_1 + 1}{3}.$$

Let $C_2 > 1$ be a large constant to be determined later on and define

$$\beta: [\delta, 1] \times [\delta, 1] \to \mathbb{R}$$

by

$$\beta(t,\varepsilon) = \begin{cases} \frac{t^{\frac{1}{2}(s_1-1/2)}}{\varepsilon^{1/4}} C_2^{-\frac{3}{2}\frac{N}{N-2}}, & \text{if } t^{\frac{2s_1+1}{3}} < C_2^{2\frac{N}{N-2}}\varepsilon\\ \frac{\varepsilon^{1/N}}{t^{1/(N+1)}}, & \text{if } t^{\frac{2s_1+1}{3}} \ge C_2^{2\frac{N}{N-2}}\varepsilon \end{cases}.$$

Then, for small δ , β has the following properties:

$$\beta(t,\varepsilon) \ge C_2 \sqrt{\frac{\varepsilon}{t}} \Longrightarrow \beta(t,\varepsilon) = \frac{\varepsilon^{1/N}}{t^{1/(N+1)}},\tag{7}$$

$$\beta(t,\varepsilon) < C_2 \sqrt{\frac{\varepsilon}{t}} \Rightarrow \beta(t,\varepsilon) = \frac{t^{\frac{1}{2}(s_1-1/2)}}{\varepsilon^{1/4}} C_2^{-\frac{3}{2}\frac{N}{N-2}},$$
(8)

$$\sum_{\substack{\delta 2^k \le 1\\\delta 2^l \le 1}} \beta(\delta 2^k, \delta 2^l) < M,\tag{9}$$

where *M* is a constant that depends only on *N* and on s_1 .

Now for all $i, j \in I$ and $t, \varepsilon \in [\delta, 1]$, we define

$$\Delta_{ij} = \max\{\delta, ||x_i - x_j| - |r_i - r_j||\}$$

$$S_{t,\varepsilon}(j) = \{i \in I : C^{\delta}(x_i, r_i) \cap C^{\delta}(x_j, r_j) \neq \emptyset, \ t \le |x_i - x_j| \le 2t, \ \varepsilon \le \Delta_{ij} \le 2\varepsilon\},\$$

$$A_{t,\varepsilon}(j) = \Big\{ x \in C^{\delta}(x_j, r_j) : \sum_{i \in S_{t,\varepsilon}(j)} a_i \chi_{C^{\delta}(x_i, r_i)}(x) \ge \frac{1}{M} \beta(t, \varepsilon) \frac{\kappa}{2} \Big\}.$$

Claim 4.1. There exist $t, \varepsilon \in [\delta, 1]$ and a set of indices \overline{J} such that

$$|A_{t,\varepsilon}(j)| \ge \frac{1}{4M} \beta(t,\varepsilon) |C^{\delta}(x_j,r_j)|, \ \forall j \in \bar{J},$$

and

$$\sum_{j\in \bar{J}} a_j \ge \frac{1}{2M} \beta(t,\varepsilon) \mu(F_1).$$

Proof. Let

$$J_0 = \left\{ j \in I : \left| \left\{ x \in C^{\delta}(x_j, r_j) \cap E_1^{\delta} : \sum_{i \in I} a_i \chi_{C^{\delta}(x_i, r_i)}(x) \le \frac{\kappa}{2} \right\} \right| \ge \frac{1}{4} |C^{\delta}(x_j, r_j)| \right\}.$$

By the minimality of κ , we have

$$\sum_{j\in J_0} < \frac{1}{2}\mu(F_1).$$

Therefore, if J' is the complement of J_0 , then

$$\sum_{j \in J'} a_j \ge \frac{1}{2}\mu(F_1) \tag{10}$$

and for all $j \in J'$

$$\left|\left\{C^{\delta}(x_j, r_j) \cap E_1^{\delta} : \sum_{i \in I} a_i \chi_{C^{\delta}(x_i, r_i)}(x) \le \frac{\kappa}{2}\right\}\right| < \frac{1}{4} |C^{\delta}(x_j, r_j)|$$

Hence, using (6) we obtain

$$\left|\left\{C^{\delta}(x_j, r_j) \cap E_1^{\delta} : \sum_{i \in I} a_i \chi_{C^{\delta}(x_i, r_i)}(x) > \frac{\kappa}{2}\right\}\right| \ge \frac{1}{4} |C^{\delta}(x_j, r_j)|.$$
(11)

For each $j \in J'$ let

$$B_{j} = \Big\{ x \in C^{\delta}(x_{j}, r_{j}) \cap E_{1}^{\delta} : \sum_{i \in I} a_{i} \chi_{C^{\delta}(x_{i}, r_{i})}(x) > \frac{\kappa}{2} \Big\}.$$
(12)

Then for all $j \in J'$

$$B_j \subset \bigcup_{k,l} A_{\delta 2^k, \delta 2^l}(j).$$

Indeed, suppose there existed $j \in J'$, $x \in B_j$ such that for all k, l with $\delta 2^k, \delta 2^l \leq 1$ we had $x \notin A_{\delta 2^k, \delta 2^l}(j)$. Then, by (9)

$$\sum_{i\in I} a_i \chi_{C^{\delta}(x_i,r_i)}(x) = \sum_{k,l} \sum_{i\in S} \sum_{\delta 2^k, \delta 2^l(j)} a_i \chi_{C^{\delta}(x_i,r_i)}(x) \le \frac{1}{M} \frac{\kappa}{2} \sum_{k,l} \beta(\delta 2^k, \delta 2^l) < \frac{\kappa}{2},$$

contradicting (12). It follows that for all $j \in J'$ there exist k, l such that

$$|A_{\delta 2^k, \delta 2^l}(j)| \ge \frac{1}{4M} \beta(\delta 2^k, \delta 2^l) |C^{\delta}(x_j, r_j)|.$$
(13)

In fact, if this were not the case, we would have that for some $j \in J'$

$$|B_j| \le \left| \bigcup_{k,l} A_{\delta 2^k, \delta 2^l}(j) \right| \le \sum_{k,l} |A_{\delta 2^k, \delta 2^l}(j)|$$

$$\leq \frac{1}{4M} |C^{\delta}(x_j, r_j)| \sum_{k,l} \beta(\delta 2^k, \delta 2^l) < \frac{1}{4} |C^{\delta}(x_j, r_j)|,$$

which contradicts (11). Finally, let

$$J(k,l) = \{ j \in J' : |A_{\delta 2^k, \delta 2^l}(j)| \ge \frac{1}{4M} \beta(\delta 2^k, \delta 2^l) |C^{\delta}(x_j, r_j)| \}.$$

Then, by (13)

$$J' = \bigcup_{k,l} J(k,l).$$

We claim that there exist $t = \delta 2^k$, $\varepsilon = \delta 2^l$ such that

$$\sum_{j \in J(k,l)} a_j \ge \frac{1}{2M} \beta(t,\varepsilon) \mu(F_1).$$

If not, then we would have

$$\sum_{j \in J'} a_j \le \sum_{k,l} \sum_{j \in J(k,l)} a_j < \frac{1}{2M} \mu(F_1) \sum_{k,l} \beta(\delta 2^k, \delta 2^l) < \frac{1}{2} \mu(F_1),$$

contradicting (10).

So, fix $t, \varepsilon \in [\delta, 1]$ as above. Then there are two cases.

Case 1:
$$\beta(t,\varepsilon) \ge C_2 \sqrt{\frac{\varepsilon}{t}}$$
.

It follows from the definition of κ that there exists a set of indices $J \subset I$ such that

$$\sum_{j\in J} a_j \ge \frac{1}{2}\mu(F_1),$$

and

$$\left|\left\{x \in C^{\delta}(x_j, r_j) \cap E_1^{\delta} : \sum_{i \in I} a_i \chi_{C^{\delta}(x_i, r_i)}(x) \le 2\kappa\right\}\right| \ge \frac{1}{4} |C^{\delta}(x_j, r_j)|,$$

for all $j \in J$. Now let

$$Q = \{ (j, j_1, j_2, j_3) : j \in \overline{J}, j_1, j_2, j_3 \in J, j_1, j_2, j_3 \in S_{t,\varepsilon}(j) \\ \operatorname{dist}(C^{\delta}(x_j, r_j) \cap C^{\delta}(x_{j_k}, r_{j_k}), C^{\delta}(x_j, r_j) \cap C^{\delta}(x_{j_l}, r_{j_l})) \ge \frac{\beta(t, \varepsilon)}{C_1 M} \\ \forall k, l \ k \neq l \}$$

where $C_1 > 1$ is a constant to be determined before C_2 . Further, define the following sets of indices:

$$Q_1 = \{(j_1, j_2, j_3) : \exists j \text{ such that } (j, j_1, j_2, j_3) \in Q\},\$$

$$Q_2 = \{j_1 : \exists j_2, j_3 \text{ such that } (j_1, j_2, j_3) \in Q_1\}.$$

For $j_1 \in Q_2$ let

$$Q(j_1) = \{j_2 : \exists j_3 \text{ such that } (j_1, j_2, j_3) \in Q_1\}$$

= $\{j_3 : \exists j_2 \text{ such that } (j_1, j_2, j_3) \in Q_1\}.$

Now consider the quantity

$$R = \sum_{(j,j_1,j_2,j_3) \in Q} a_j a_{j_1} a_{j_2} a_{j_3}.$$

8

Note that if C_2 is large enough, then

$$\frac{\beta(t,\varepsilon)}{C_1M} \geq \frac{C_2}{C_1M} \sqrt{\frac{\varepsilon}{t}} \geq A_1 \sqrt{\frac{\varepsilon}{t}},$$

where A_1 is the constant in Lemma 2.2. It follows that if $(j_1, j_2, j_3) \in Q_1$ then the set $\{x_j : (j, j_1, j_2, j_3) \in Q\}$ is contained in the union of two ellipsoids of diameter $\lesssim \frac{\varepsilon}{\beta^2(t,\varepsilon)}$. Hence

$$R \lesssim \left(\frac{\varepsilon}{\beta^2(t,\varepsilon)}\right)^{s_1} \sum_{(j_1,j_2,j_3)\in Q_1} a_{j_1}a_{j_2}a_{j_3}.$$

Furthermore, if $j_1 \in Q_2$ and $j_2 \in Q(j_1)$ then there exists j such that $j_1, j_2 \in S_{t,\varepsilon}(j)$. Therefore,

$$|x_{j_1} - x_{j_2}| \le |x_{j_1} - x_j| + |x_j - x_{j_2}| \le 4t.$$

It follows that for fixed $j_1 \in Q_2$ the set $\{x_{j_2} : j_2 \in Q(j_1)\}$ is contained in a disk with center x_{j_1} and radius 4t. Hence

$$\sum_{j_2\in Q(j_1)}a_{j_2}\lesssim t^{s_1}.$$

Therefore,

$$R \lesssim \left(\frac{\varepsilon}{\beta^{2}(t,\varepsilon)}\right)^{s_{1}} \sum_{(j_{1},j_{2},j_{3})\in Q_{1}} a_{j_{1}}a_{j_{2}}a_{j_{3}} \le \left(\frac{\varepsilon}{\beta^{2}(t,\varepsilon)}\right)^{s_{1}} \sum_{j_{1}\in Q_{2}} a_{j_{1}} \left(\sum_{j_{2}\in Q(j_{1})} a_{j_{2}}\right)^{2}$$
$$\lesssim \mu(F_{1}) \left(\frac{\varepsilon}{\beta^{2}(t,\varepsilon)}\right)^{s_{1}} (t^{s_{1}})^{2}.$$
(14)

Now fix $j \in \overline{J}$.

Claim 4.2. There are three subsets D_1, D_2, D_3 of $A_{t,\varepsilon}(j)$ such that

$$dist(D_k, D_l) \ge \frac{2\beta(t, \varepsilon)}{C_1 M}, \ \forall k, l \ k \neq l,$$

and

$$|D_k| \gtrsim \delta \beta(t, \varepsilon), \ \forall k,$$

provided that C_1 is large enough.

Proof. We use complex notation. If $0 \le \theta_1 \le \theta_2 \le 2\pi$ let

$$G_{\theta_1,\theta_2} = A_{t,\varepsilon}(j) \cap \{x_j + re^{i\theta} \in C^{\delta}(x_j, r_j) : \theta_1 \le \theta \le \theta_2\}.$$

Then there exist $0 = \theta_1 < \cdots < \theta_7 = 2\pi$ such that

$$|G_{\theta_k,\theta_{k+1}}| = \frac{|A_{t,\varepsilon}(j)|}{6}, \ k = 1, ..., 6.$$

Let

$$D_k = G_{\theta_{2k-1}, \theta_{2k}}, \ k = 1, 2, 3.$$

Note that for all l

$$\frac{\beta(t,\varepsilon)}{24M}|C^{\delta}(x_j,r_j)| \le |G_{\theta_l,\theta_{l+1}}| \le \operatorname{diam}(G_{\theta_l,\theta_{l+1}})\delta.$$

Therefore,

diam(
$$G_{\theta_l,\theta_{l+1}}$$
) $\gtrsim \beta(t,\varepsilon)$.

It follows that if we choose C_1 large enough, then we have

dist
$$(D_k, D_l) \ge \frac{2\beta(t, \varepsilon)}{C_1 M}$$
, and $|D_k| \ge \beta(t, \varepsilon)\delta$.

For each k let

$$S_k = \{i \in S_{t,\varepsilon}(j) : D_k \cap C^{\delta}(x_i, r_i) \neq \emptyset\}.$$

Then

$$\begin{split} \kappa \beta^2(t,\varepsilon)\delta &\lesssim \int_{D_k} \frac{\beta(t,\varepsilon)}{M} \frac{\kappa}{2} dx \leq \sum_{i \in S_k} a_i |D_k \cap C^{\delta}(x_i,r_i)| \\ &\leq \sum_{i \in S_k} a_i |C^{\delta}(x_j,r_j) \cap C^{\delta}(x_i,r_i)| \lesssim \sum_{i \in S_k} a_i \frac{\delta^2}{\sqrt{t\varepsilon}}, \end{split}$$

where the last inequality follows from Lemma 2.1. Therefore,

$$\sum_{i\in D_k} a_i \gtrsim \frac{1}{\delta} \kappa \beta^2(t,\varepsilon) \sqrt{t\varepsilon}.$$

By Lemma 2.1, if $i \in S_{t,\varepsilon}(j)$ then

diam
$$(C^{\delta}(x_j, r_j) \cap C^{\delta}(x_i, r_i)) \leq A \sqrt{\frac{\varepsilon}{t}} \leq \frac{A\beta(t, \varepsilon)}{C_2}.$$

Therefore, $i_1 \in S_k$, $i_2 \in S_l$, $k \neq l$ implies that

$$dist(C^{\delta}(x_{i_1}, r_{i_1}) \cap C^{\delta}(x_j, r_j), C^{\delta}(x_{i_2}, r_{i_2}) \cap C^{\delta}(x_j, r_j))$$

$$\geq \frac{2\beta(t, \varepsilon)}{C_1 M} - \frac{2A\beta(t, \varepsilon)}{C_2} \geq \frac{\beta(t, \varepsilon)}{C_1 M},$$

provided that C_2 is sufficiently large. It follows that if $j_k \in S_k$, k = 1, 2, 3, then $(j, j_1, j_2, j_3) \in Q$. Hence

$$R \geq \sum_{j \in \bar{J}} a_j \sum_{\substack{j_1 \in S_1 \\ j_2 \in S_2 \\ j_3 \in S_3}} a_{j_1} a_{j_2} a_{j_3} \gtrsim \beta(t,\varepsilon) \left(\frac{1}{\delta} \kappa \beta^2(t,\varepsilon) \sqrt{t\varepsilon}\right)^3.$$

If we compare the above equation with (14), and then use (7), we obtain

$$\kappa^{3} \lesssim \delta^{3} \frac{\varepsilon^{s_{1}-3/2} t^{2s_{1}-3/2}}{\beta^{2s_{1}+7}(t,\varepsilon)} \lesssim \delta^{3}.$$

Case 2: $\beta(t,\varepsilon) \leq C_{2} \sqrt{\frac{\varepsilon}{t}}.$

Fix $j \in \overline{J}$. Then we have

$$\begin{split} \kappa \delta \beta^2(t,\varepsilon) &\lesssim \kappa \frac{\beta(t,\varepsilon)}{M} |A_{t,\varepsilon}(j)| = \int_{A_{t,\varepsilon}(j)} \kappa \frac{\beta(t,\varepsilon)}{M} dx \\ &\lesssim \sum_{i \in S_{t,\varepsilon}(j)} a_i |C^{\delta}(x_i,r_i) \cap C^{\delta}(x_j,r_j)| \lesssim \frac{\delta^2}{\sqrt{t\varepsilon}} \sum_{i \in S_{t,\varepsilon}(j)} a_i, \end{split}$$

where we have used Lemma 2.1 and the definition of $A_{t,\varepsilon}(j)$.

Note that the set $\{x_i : i \in S_{t,\varepsilon}(j)\}$ is contained in a disk of radius 2*t*. Therefore,

$$\sum_{i\in S_{t,\varepsilon}(j)}a_i \lesssim t^{s_1}.$$

It follows that

$$\kappa \lesssim \delta \frac{t^{s_1 - 1/2}}{\varepsilon^{1/2} \beta^2(t, \varepsilon)}.$$

Using (8), we obtain $\kappa \leq \delta$. We conclude that, in either case

$$\kappa \lesssim \delta.$$
 (15)

To complete the proof, notice that

$$\frac{1}{2}\mu(F_{1}) \leq \sum_{j \in J} a_{j} = \frac{1}{\delta} \sum_{j \in J} a_{j}\delta$$

$$\leq \frac{1}{\delta} \sum_{j \in J} a_{j} \Big| \Big\{ x \in C^{\delta}(x_{j}, r_{j}) \cap E_{1}^{\delta} : \sum_{i \in I} a_{i}\chi_{C^{\delta}(x_{i}, r_{i})}(x) \leq 2\kappa \Big\} \Big|$$

$$\leq \frac{1}{\delta} \int_{\{x \in E_{1}^{\delta} : \sum_{j \in J} a_{j}\chi_{C^{\delta}(x_{j}, r_{j})}(x) \leq 2\kappa \}} \Big(\sum_{j \in J} a_{j}\chi_{C^{\delta}(x_{j}, r_{j})}(x) \Big) dx$$

$$\leq \frac{1}{\delta} \kappa |E_{1}^{\delta}| \leq |E_{1}^{\delta}|, \qquad (16)$$

where the last inequality follows from (15).

If we let $\delta \to 0$ then the right-hand side of (16) tends to zero, which is a contradiction.

5. Possible improvements

As we discussed at the beginning of Section 4, the proof of Theorem 1.5 was motivated by a result of combinatorial nature, namely Proposition 4.1, which asserts that if one is given a family of N circles such that no three of them are internally tangent at a point, then there is a bound of the form $CN^{5/3}$ on the total number of tangencies.

This, however, is far from being sharp. Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [2] developed a technique which leads to a bound of the form

$$C_{\varepsilon} N^{3/2+\varepsilon} \ \forall \varepsilon > 0,$$

suggesting that it might be possible to weaken the condition s > 3/2 in Theorem 1.5. Indeed, Wolff [13] proved the following $L^3 \rightarrow L^3$ maximal inequality.

Theorem 5.1. *For* $x_1 \in \mathbb{R}$ *, let*

$$M_{\delta}f(x_1) = \sup_{\substack{r \in [1/2,2]\\x_2 \in \mathbb{R}}} \frac{1}{|C^{\delta}(x,r)|} \int_{C^{\delta}(x,r)} |f|,$$

where $x = (x_1, x_2)$. Then

$$\forall \varepsilon > 0 \; \exists A_{\varepsilon} : \|M_{\delta}f\|_{L^{3}(\mathbb{R})} \leq A_{\varepsilon}\delta^{-\varepsilon}\|f\|_{3}.$$

Using this, he proved, in the same paper, the following.

Theorem 5.2. If $\alpha \leq 1$ and if *E* is a set in the plane which contains circles centered at all points of a set with Hausdorff dimension at least α , then *E* has Hausdorff dimension at least $1 + \alpha$.

The preceding result suggests that a set *E* as in the statement of Theorem 1.5 has to be fairly large. In view of this and the analogy between Proposition 3.1 and the spherical means maximal theorem, it seems reasonable to make the following conjecture which would imply that Theorem 1.5 is true for all s > 1.

Conjecture 5.1. For $\delta > 0$ small, $f : \mathbb{R}^2 \to \mathbb{R}$, define $\mathcal{M}_{\delta} : B(0, 1/4) \to \mathbb{R}$, by

$$\mathcal{M}_{\delta}f(x) = \sup_{1/2 \le r \le 2} \frac{1}{|C^{\delta}(x,r)|} \int_{C^{\delta}(x,r)} |f(y)| dy$$

Let $F \subset B(0, 1/4)$ be a compact set in \mathbb{R}^2 such that there exist s > 1 and a finite measure μ supported on F with $\mu(B(x, r)) \leq r^s$, for $x \in \mathbb{R}^2$ and r > 0. Then there exists a constant A that depends only on the measure of F and on s, such that

$$\left(\int_{F} (\mathcal{M}_{\delta}f(x))^{p(s)} d\mu(x)\right)^{1/p(s)} \le A ||f||_{p(s)}.$$
(15)

Note that in order for the above inequality to hold, it is necessary that $p(s) \ge 4 - s$. To see that, let I = [-1/8, 1/8], and let $E \subset I$ be a Cantor set of Hausdorff dimension s - 1. Then $\mathcal{H}^{s-1}(E \cap B(0, \delta)) \sim \delta^{s-1}$. Define

$$F_{\delta} = I \times (E \cap B(0, \delta^{1/2})),$$

and

$$R_{\delta} = [1 - \delta, 1 + \delta] \times [-2\delta^{1/2}, 2\delta^{1/2}].$$

Notice that

$$x \in F_{\delta} \Rightarrow \mathcal{M}_{\delta}\chi_{R_{\delta}}(x) \gtrsim \delta^{1/2}$$

Therefore, using (15)

$$\delta^{1/2}(\mathcal{H}^{s}(F_{\delta}))^{1/p(s)} \lesssim \left(\int_{F_{\delta}} (\mathcal{M}_{\delta} \chi_{R_{\delta}})^{p(s)}(x) d\mathcal{H}^{s}(x) \right)^{1/p(s)}$$
$$\lesssim \|\chi_{R_{\delta}}\|_{p(s)} = \delta^{3p(s)/2}.$$

On the other hand

$$\mathcal{H}^{s}(F_{\delta}) \sim \mathcal{H}^{s-1}(E \cap B(0, \delta^{1/2})) \sim \delta^{(s-1)/2}.$$

Hence

$$\delta^{\frac{1}{2}}\delta^{\frac{(s-1)}{2p(s)}} \lesssim \delta^{\frac{3}{2p(s)}},$$

which is possible only if $p(s) \ge 4 - s$.

We conclude by mentioning an observation made by Schlag: if the local smoothing conjecture due to Sogge [10] is correct, then Theorem 1.5 is true for all s > 1.

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12

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DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA *E-mail address*: themis@cco.caltech.edu