

# **$(n, 2)$ -SETS HAVE FULL HAUSDORFF DIMENSION**

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ABSTRACT. We prove that a set containing translates of every 2-plane must have full Hausdorff dimension.

## 1. INTRODUCTION

This is a continuation of [4] where a partial result on the problem under investigation was obtained. Since that paper is unpublished work, we will reproduce certain parts of it for the sake of completeness.

An  $(n, 2)$ -set in  $\mathbb{R}^n$  is a subset  $E \subset \mathbb{R}^n$  containing a translate of every 2-dimensional plane.

The natural question that arises is whether  $E$  must have positive Lebesgue measure. This turns out to be true in low dimensions. Marstrand [3] proved that  $(3, 2)$ -sets have positive measure. Bourgain [1] showed the same for  $(4, 2)$ -sets and made a connection with the Kakeya conjecture. In higher dimensions the question is open. However, it has been known for some time that if  $n > 4$  then  $\dim_H(E) \geq (2n + 2)/3$ , where  $\dim_H$  denotes Hausdorff dimension. This follows from the estimates for the 2-plane transform due to Christ [2]. In the present paper we modify the argument in [4], which in turn is based on geometric-combinatorial ideas very much in the spirit of Wolff [6], to obtain full dimension. Namely we prove the following.

**Theorem 1.1.** *Suppose  $n > 4$  and let  $E \subset \mathbb{R}^n$  be an  $(n, 2)$ -set. Then  $\dim_H(E) = n$ .*

## 2. TERMINOLOGY AND NOTATION

$S^{n-1} \subset \mathbb{R}^n$  is the  $(n - 1)$ -dimensional unit sphere.

$B(a, r)$  is the closed ball of radius  $r$  centered at the point  $a$ .

For  $X \subset \mathbb{R}^n$ ,  $X^\perp$  denotes its orthogonal complement.

If  $e \in S^{n-1}$ ,  $a \in \mathbb{R}^n$  then  $L_e(a) = \{a + te : t \in \mathbb{R}\}$  is the line in the  $e$ -direction passing through the point  $a$ .

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If  $e \in S^{n-1}$ ,  $a \in \mathbb{R}^n$ ,  $\beta > 0$  then  $T_e^\beta(a) = \{x \in \mathbb{R}^n : \text{dist}(x, L_e(a)) \leq \beta\}$  is the infinite tube with axis  $L_e(a)$  and cross-section radius  $\beta$ .

$\mathcal{L}^k$  denotes  $k$ -dimensional Lebesgue measure and  $\mathcal{L}^0$  counting measure. When the context is clear we will use the notation  $|\cdot|$  for all these measures.

Let  $\mathcal{G}_n$  be the Grassmannian manifold of all 2-dimensional linear subspaces of  $\mathbb{R}^n$  equipped with the unique probability measure  $\gamma_{n,2}$  which is invariant under the action of the orthogonal group. The elements of  $\mathcal{G}_n$  will be referred to as *direction planes*.

If  $P_1, P_2 \in \mathcal{G}_n$ , then their distance is defined by

$$d(P_1, P_2) = \|\text{proj}_{P_1} - \text{proj}_{P_2}\|$$

where  $\text{proj}_P : \mathbb{R}^n \rightarrow P$  is the orthogonal projection onto  $P$ .

A set of points or direction planes is called  $\rho$ -separated if the distance between any two of its elements is at least  $\rho$ .

If  $P \in \mathcal{G}_n$ ,  $1 \leq l \leq 4$ ,  $\delta > 0$  then  $P^{l,\delta}$  is a rectangle of dimensions  $l \times l \times \underbrace{\delta \times \cdots \times \delta}_{n-2}$ , that is, the image of  $[0, l] \times [0, l] \times [0, \delta] \times \cdots \times [0, \delta]$

under a rotation and a translation, such that its faces with dimensions  $l \times l$  are parallel to  $P$ . Such a set will be referred to as a  $\delta$ -plate or simply as a *plate*. When  $l = 1$  the superscript  $l$  will be suppressed.

If  $P_1^{l,\delta} \cap P_2^{l,\delta} \neq \emptyset$  and  $d(P_1, P_2) = r$  we will say that the plates *intersect at angle*  $\arcsin r$ .

The letter  $C$  will denote various positive constants whose values may change from line to line. Similarly,  $C_\epsilon$  will denote constants depending on  $\epsilon$ . If we need to keep track of the value of a constant through a calculation we will use subscripted letters  $C_1, C_2, \dots$  or the notation  $\bar{C}$ .  $x \lesssim y$  means  $x \leq Cx$  and  $x \simeq y$  means  $(x \lesssim y \text{ \& } y \lesssim x)$ .

Finally, note that

$$\gamma_{n,2}(\{P \in \mathcal{G}_n : d(P, P_0) \leq \delta\}) \simeq \delta^{2(n-2)} \text{ for all } P_0 \in \mathcal{G}_n, \delta \leq 1.$$

So if  $\mathcal{A} \subset \mathcal{G}_n$  and  $\mathcal{B}$  is a maximal  $\delta$ -separated subset of  $\mathcal{A}$  then

$$\gamma_{n,2}(\mathcal{A}) \lesssim |\mathcal{B}| \delta^{2(n-2)}.$$

Further, if  $\mathcal{A} \subset \mathcal{G}_n$  is  $\delta$ -separated and  $\mathcal{B}$  is a maximal  $\eta$ -separated subset of  $\mathcal{A}$  with  $\eta \geq \delta$  then

$$|\mathcal{B}| \gtrsim |\mathcal{A}| (\delta/\eta)^{2(n-2)}.$$

## 3. AUXILIARY LEMMAS

The following technical lemma allows us to control the intersection of two plates.

**Lemma 3.1.** *Let  $P_1^{l,\eta}, P_2^{l,\eta}$  be two plates such that  $d(P_1, P_2) \leq 1/2$ . Then there exists a tube  $T_e^\beta(a)$  with  $\beta = C\eta/d(P_1, P_2)$  such that*

$$P_1^{l,\eta} \cap P_2^{l,\eta} \subset T_e^\beta(a). \quad (1)$$

In particular

$$|P_1^{l,\eta} \cap P_2^{l,\eta}| \lesssim \frac{\eta^{n-1}}{d(P_1, P_2)}.$$

The proof of Theorem 1.1 will be, essentially, a reduction to the 3-dimensional case via the Radon transform. We give the definitions.

For a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfying the appropriate integrability conditions, the Radon transform

$$\mathcal{R}f : S^2 \times \mathbb{R} \rightarrow \mathbb{R}$$

is defined by

$$\mathcal{R}f(e, t) = \int_{\langle e, x \rangle = t} f(x) d\mathcal{L}^2(x).$$

It is proved in Oberlin and Stein [5] that for any measurable set  $E \subset \mathbb{R}^3$  one has the following estimate.

$$\|\mathcal{R}\chi_E\|_{3,\infty} \lesssim \|\chi_E\|_{3/2}$$

where

$$\|\mathcal{R}\chi_E\|_{3,\infty} = \left( \int_{S^2} \left( \sup_t \mathcal{R}\chi_E(e, t) \right)^3 d\sigma(e) \right)^{1/3}$$

and  $d\sigma$  is surface measure.

We can discretize this result as follows.

**Lemma 3.2.** *Suppose  $E$  is a set in  $\mathbb{R}^3$ ,  $\lambda \leq 1$  and let  $\{P_k\}_{k=1}^M$  be a  $\delta$ -separated set in  $\mathcal{G}_3$  such that for each  $k$  there is plate  $P_k^{l,C\delta}$  satisfying*

$$|P_k^{l,C\delta} \cap E| \gtrsim \lambda\delta.$$

Then

$$|E| \gtrsim \lambda^{3/2} M^{1/2} \delta.$$

*Proof.* For each  $e \in S^2$  let  $Q(e)$  be the plane with normal  $e$  passing through the origin. Then there is a  $\delta$ -separated set  $\{e_k\}_{k=1}^M$  on  $S^2$  such that  $P_k = Q(e_k)$ . Note that since  $1 \leq l \leq 4$ , for each  $e \in B(e_k, \delta/2) \cap S^2$  we have

$$\lambda \delta \lesssim |P_k^{l, C\delta} \cap E| \leq \int_{I_e} \mathcal{L}^2((Q(e) + x) \cap E) d\mathcal{L}^1(x)$$

where  $I_e$  is an interval on  $Q(e)^\perp$  with  $\mathcal{L}^1(I_e) \lesssim \delta$ . Therefore there exists  $x_e \in I_e$  such that

$$\lambda \lesssim \mathcal{L}^2((Q(e) + x_e) \cap E).$$

Hence

$$\lambda \lesssim \sup_t \mathcal{R}_{\chi_E}(e, t).$$

We conclude that

$$\begin{aligned} \lambda^3 \delta^2 M &\lesssim \sum_k \int_{B(e_k, \delta/2) \cap S^2} (\sup_t \mathcal{R}_{\chi_E}(e, t))^3 d\sigma(e) \\ &\leq \int_{S^2} (\sup_t \mathcal{R}_{\chi_E}(e, t))^3 d\sigma(e) \\ &= \|\mathcal{R}_{\chi_E}\|_{3, \infty}^3 \lesssim \|\chi_E\|_{3/2}^3 = |E|^2. \end{aligned}$$

□

This, in turn, gives rise to the following higher dimensional analogue.

**Lemma 3.3.** *Suppose  $E$  is a set in  $\mathbb{R}^n$ ,  $\lambda \leq 1$ ,  $\Pi \subset \mathbb{R}^n$  is a 3-plane and  $\{P_k\}_{k=1}^M$  is a  $\delta$ -separated set in  $\mathcal{G}_n$  such that for each  $k$  the plate  $P_k^\delta$  satisfies*

$$P_k^\delta \subset \Pi^{\tilde{C}\delta} \text{ and } |P_k^\delta \cap E| \geq \lambda |P_k^\delta|$$

where  $\Pi^{\tilde{C}\delta} = \{x \in \mathbb{R}^n : \text{dist}(x, \Pi) \leq \tilde{C}\delta\}$  is the  $\tilde{C}\delta$ -neighborhood of  $\Pi$ . Then

$$|E \cap \Pi^{\tilde{C}\delta}| \gtrsim \lambda^3 M^{1/2} \delta^{n-2}.$$

*Proof.* Whithout loss of generality we may assume that  $\Pi$  is the  $x_1 x_2 x_3$ -plane. Since  $P_k^\delta \subset \Pi^{\tilde{C}\delta}$  there is a direction plane  $Q_k \subset \Pi$  such that  $d(P_k, Q_k) \lesssim \delta$ . Therefore we can find a plate  $Q_k^{2, C_1\delta}$  with  $P_k^\delta \subset Q_k^{2, C_1\delta}$ . It follows that

$$|Q_k^{2, C_1\delta} \cap E \cap \Pi^{\tilde{C}\delta}| \gtrsim \lambda \delta^{n-2}.$$

Let  $\mathcal{B}$  be a maximal  $C_2\delta$ -separated subset of  $\{P_k\}_{k=1}^M$  and put  $\mathcal{B}' = \{Q_k : P_k \in \mathcal{B}\}$ . Then for  $Q_j, Q_k \in \mathcal{B}'$ ,  $j \neq k$ , we have

$$d(Q_j, Q_k) \geq d(P_j, P_k) - d(P_j, Q_j) - d(P_k, Q_k) \geq (C_2 - C)\delta \geq \delta$$

for  $C_2$  sufficiently large.

Now for each  $Q_k \in \mathcal{B}'$  let

$$L_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \leq \frac{\lambda\delta}{C_3} \right\},$$

$$H_k = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \geq \frac{\lambda\delta}{C_3} \right\}.$$

Note that

$$\mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) \lesssim \delta, \text{ for all } x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp.$$

Hence

$$\begin{aligned} \lambda\delta^{n-2} &\lesssim |Q_k^{2,C_1\delta} \cap E \cap \Pi^{\widetilde{C}\delta}| \\ &= \int_{B(0, \widetilde{C}\delta) \cap \Pi^\perp} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\ &= \int_{L_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\ &\quad + \int_{H_k} \mathcal{L}^3(Q_k^{2,C_1\delta} \cap E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\ &\leq \frac{\lambda\delta}{C_3} C\delta^{n-3} + C\delta \mathcal{L}^{n-3}(H_k). \end{aligned}$$

Therefore,  $\mathcal{L}^{n-3}(H_k) \gtrsim \lambda\delta^{n-3}$  for  $C_3$  sufficiently large.

Next, notice that  $|\mathcal{B}'| \simeq M$  and define

$$L = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| < \frac{\lambda M}{C_4} \right\},$$

$$H = \left\{ x \in B(0, \widetilde{C}\delta) \cap \Pi^\perp : |\{k : x \in H_k\}| \geq \frac{\lambda M}{C_4} \right\}.$$

Then

$$\begin{aligned} \lambda\delta^{n-3} M &\lesssim \sum_k \int \chi_{H_k} = \int_H \sum_k \chi_{H_k} + \int_L \sum_k \chi_{H_k} \\ &\leq M \mathcal{L}^{n-3}(H) + \frac{\lambda M}{C_4} \mathcal{L}^{n-3}(L) \\ &\leq M \mathcal{L}^{n-3}(H) + \frac{\lambda M}{C_4} C\delta^{n-3}. \end{aligned}$$

Therefore  $\mathcal{L}^{n-3}(H) \gtrsim \lambda\delta^{n-3}$  for  $C_4$  sufficiently large.

Note that for each  $x \in H$  there are at least  $\lambda M/C_4$  plates in  $\Pi + x$ , that is, plates in a copy of  $\mathbb{R}^3$ , with  $\delta$ -separated direction planes and such that

the 3-dimensional measure of their intersection with  $E \cap (\Pi + x)$  is at least  $C^{-1}\lambda\delta$ . Hence, by Lemma 3.2

$$\mathcal{L}^3(E \cap (\Pi + x)) \gtrsim \lambda^{3/2}(\lambda M)^{1/2}\delta.$$

We conclude that

$$\begin{aligned} |E \cap \Pi^{\tilde{C}\delta}| &\geq \int_H \mathcal{L}^3(E \cap (\Pi + x)) d\mathcal{L}^{n-3}(x) \\ &\gtrsim \lambda \delta^{n-3} \lambda^{3/2} (\lambda M)^{1/2} \delta \\ &= \lambda^3 M^{1/2} \delta^{n-2}. \end{aligned}$$

□

#### 4. THE MAIN ARGUMENT

By a modification of the argument in [1], Theorem 1.1 will be a consequence of the following.

**Proposition 4.1.** *Suppose  $E$  is a set in  $\mathbb{R}^n$ ,  $\lambda \leq 1$  and  $\{P_j\}_{j=1}^M$  is a  $\delta$ -separated set in  $\mathcal{G}_n$  with  $\text{diam}(\{P_j\}_{j=1}^M) \leq 1/2$ , such that for each  $j$  there is plate  $P_j^\delta$  satisfying*

$$|P_j^\delta \cap E| \geq \lambda |P_j^\delta|.$$

Then

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+2)/2} M^{1/2} \delta^{n-2}$$

*Proof.* We say that a point  $x \in E$  has multiplicity  $\mu$  if it belongs to exactly  $\mu$  plates  $P_j^\delta$ . We claim that there exists a set  $P_{j_0}^\delta \cap E$  such that the measure of the set of its points with multiplicity at least  $\frac{1}{2} M \lambda \delta^{n-2} / |E|$  is at least  $\frac{1}{2} |P_{j_0}^\delta \cap E|$ , because otherwise we would have

$$|E| \geq \left| \bigcup_{j=1}^M P_j^\delta \cap E \right| > \frac{2|E|}{M \lambda \delta^{n-2}} \sum_{j=1}^M \frac{1}{2} |P_j^\delta \cap E| \geq |E|.$$

So letting

$$\mu_0 = \frac{1}{2} \frac{M}{|E|} \lambda \delta^{n-2}, \tag{2}$$

we see that there is a plate  $P^\delta := P_{j_0}^\delta$  such that

$$|\{x \in P^\delta \cap E : |\{k : x \in P_k^\delta\}| \geq \mu_0\}| \geq \frac{\lambda}{2} \delta^{n-2}.$$

Note that for each  $x \in P^\delta \cap E$  with  $|\{k : x \in P_k^\delta\}| \geq \mu_0$  we have

$$\{k : x \in P_k^\delta\} = \bigcup_{i=1}^{\log(C/\delta)} \{k : x \in P_k^\delta \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}.$$

Therefore, by the pigeonhole principle, there is an integer  $i(x)$  with  $1 \leq i(x) \leq \log(C/\delta)$  such that

$$|\{k : x \in P_k^\delta \text{ and } \delta 2^{i(x)-1} \leq d(P_k, P) < \delta 2^{i(x)}\}| \geq (\log(C/\delta))^{-1} \mu_0.$$

And so,

$$\begin{aligned} & \{x \in P^\delta \cap E : |\{k : x \in P_k^\delta\}| \geq \mu_0\} \\ & \subset \bigcup_{i=1}^{\log(C/\delta)} \{x \in P^\delta \cap E : |\{k : x \in P_k^\delta \text{ and } \delta 2^{i-1} \leq d(P_k, P) < \delta 2^i\}| \\ & \qquad \qquad \qquad \geq (\log(C/\delta))^{-1} \mu_0\}. \end{aligned}$$

Applying the pigeonhole principle again, we see that there exists a number  $\rho := \delta 2^{i_0-1}$  and a set  $A \subset P^\delta \cap E$  of measure

$$|A| \gtrsim |\log \delta|^{-1} \lambda \delta^{n-2} \quad (3)$$

such that for every  $x \in A$

$$|\{k : x \in P_k^\delta \text{ and } \rho \leq d(P_k, P) < 2\rho\}| \gtrsim |\log \delta|^{-1} \mu_0. \quad (4)$$

Heuristically, (3) and (4) tell us that a large number of plates intersect  $P^\delta$  at approximately the same angle. We are going to estimate this number using the bound for the measure of their pairwise intersections. To do this, define

$$\mathcal{D} = \{P_k^\delta : P_k^\delta \cap P^\delta \neq \emptyset \text{ and } \rho \leq d(P_k, P) < 2\rho\}.$$

Then, by Lemma 3.1, we have

$$\begin{aligned} |\mathcal{D}| & \gtrsim \sum_{P_k^\delta \in \mathcal{D}} |P_k^\delta \cap P^\delta| \frac{\rho}{\delta^{n-1}} \\ & = \frac{\rho}{\delta^{n-1}} \int_{P^\delta} \sum_{P_k^\delta \in \mathcal{D}} \chi_{P_k^\delta} \\ & \geq \frac{\rho}{\delta^{n-1}} \int_A \sum_{P_k^\delta \in \mathcal{D}} \chi_{P_k^\delta} \\ & \gtrsim \frac{\rho}{\delta^{n-1}} |A| |\log \delta|^{-1} \mu_0 \\ & \gtrsim |\log \delta|^{-2} \lambda^2 \frac{\rho}{\delta} \frac{M}{|E|} \delta^{n-2}. \end{aligned} \quad (5)$$

Where the last inequality follows from (2) and (3) and the one before last from (4).

We are now in a position to carry out a geometric construction, in the spirit of [6], which will allow us to use Lemma 3.3. In order to help the reader understand our strategy, we first give an informal description.

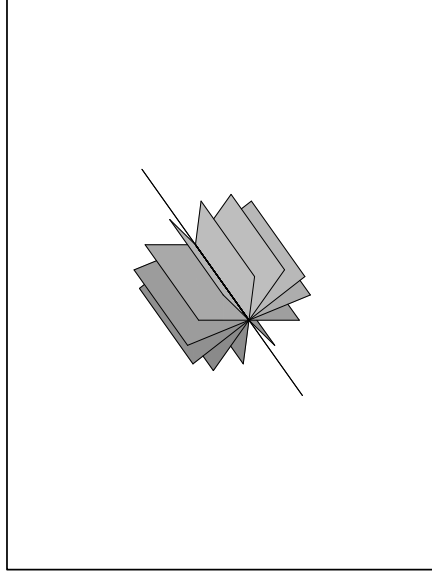


FIGURE 1. In this picture, the planes represent the 3-planes  $\Pi_i$  and the line represents the 2-plane  $c + P$ .

We know that the Radon transform estimate due to Oberlin and Stein is sharp in  $\mathbb{R}^3$ . So, we would like to slice our set with a number of thin neighborhoods of  $\mathbb{R}^3$  and then apply the higher dimensional discretized version of that estimate (as given by Lemma 3.3) to each of these neighborhoods. To this end, we pass  $(\rho/\delta)^{n-3}$  3-dimensional planes (these are the sets  $\Pi_i$  to be defined below) through the 2-dimensional plane which is parallel to the direction plane  $P$  and passes through the center  $c$  of the plate  $P^\delta$ . We do that in a “radial”, so to speak, fashion (see Figure 1). That is, each 3-plane  $\Pi_i$  is the translate of a 3-dimensional subspace spanned by  $P$  and a certain vector in the orthogonal complement of  $P$ . This ensures that every plate in  $\mathcal{D}$  belongs to some  $\Pi_i^{\tilde{C}\delta}$ , where  $\Pi_i^{\tilde{C}\delta}$  is the  $\tilde{C}\delta$ -neighborhood of  $\Pi_i$ . Our goal is to use Lemma 3.3 to estimate the measure of  $\Pi_i^{\tilde{C}\delta} \cap E$ , and then sum up these individual estimates to get a lower bound on the measure of our set. However, in order to do this efficiently, we have to take into account the overlap of the sets  $\Pi_i^{\tilde{C}\delta}$ . If there are “too many”  $\Pi_i^{\tilde{C}\delta}$ s, that is, if  $\rho/\delta \geq \lambda^{-1}|\log \delta|$  (this is case I below), we observe that their overlap increases as we approach the plane  $c + P$ . So we choose a suitable neighborhood  $\mathcal{X}$  of  $c + P$  in such a way that:

- The overlap of the sets  $\Pi_i^{\tilde{C}\delta} \cap \mathcal{X}^c$  is smaller.



- The measure of the intersection of every plate in  $\mathcal{D}$  with the reduced set  $E \cap \mathcal{X}^c$  is still large.

Then, we work with this reduced set  $E \cap \mathcal{X}^c$ . On the other hand, if there are not “too many”  $\Pi_i^{\tilde{C}\delta}$ ’s, that is, if  $\rho/\delta \leq \lambda^{-1}|\log \delta|$  (this is case II below), we just estimate their overlap with their number  $(\rho/\delta)^{n-3}$ .

We now proceed with the formal argument.

Let  $\{e_i\}_i$  be a maximal  $\delta/\rho$ -separated set of points on the  $(n-3)$ -dimensional unit sphere  $S^{n-1} \cap P^\perp$  and let

$$\Pi_i = c + \Pi'_i$$

where  $c$  is the center of  $P^\delta$  and  $\Pi'_i$  is the 3-dimensional space spanned by  $e_i$  and  $P$ . Then for each  $P_k^\delta \in \mathcal{D}$  there exists an  $i$  such that  $P_k^\delta \subset \Pi_i^{\tilde{C}\delta}$ , where  $\Pi_i^{\tilde{C}\delta}$  is the  $\tilde{C}\delta$ -neighborhood of  $\Pi_i$ . To see this, let  $y \in P_k^\delta$ , and pick  $w \in P_k^\delta \cap P^\delta$ . Then  $|y - w|$  is bounded by the diameter of  $P_k^\delta$  and belongs to a  $C\delta$ -neighborhood of the direction plane  $P_k$ . So, there exists a point  $z \in P_k$  (just take  $z$  to be the projection of  $y - w$  onto  $P_k$ ) with  $|z| \lesssim 1$  and  $|y - w - z| \lesssim \delta$ . Now write

$$z = z_1 + z_2 \in P \oplus P^\perp,$$

$$c - w = w_1 + w_2 \in P \oplus P^\perp.$$

Since  $d(P, P_k) \simeq \rho$ , we have  $|z_2| \lesssim \rho$ , and since  $c - w$  belongs to a  $C\delta$ -neighborhood of the direction plane  $P$  we get  $|w_2| \lesssim \delta$ . Now  $z_2/|z_2|$  belongs to the unit sphere of  $P^\perp$ , so we can find an  $e_i$  such that  $|z_2/|z_2| - e_i| \leq \delta/\rho$ . Therefore,

$$|z_2 - |z_2|e_i| \leq \frac{\delta}{\rho}|z_2| \lesssim \delta.$$

Finally, notice that

$$y = [(y - w - z) + (z_2 - |z_2|e_i) - w_2] + [z_1 - w_1 + |z_2|e_i] + c,$$

where the vector in the first square bracket has length at most  $C\delta$  and the vector in the second square bracket belongs to  $\Pi'_i$ . We conclude that  $y \in \Pi_i^{\tilde{C}\delta}$ .

Therefore, if we let

$$\mathcal{D}_i = \{P_k^\delta \in \mathcal{D} : P_k^\delta \subset \Pi_i^{\tilde{C}\delta}\}$$

then

$$\mathcal{D} = \bigcup_i \mathcal{D}_i.$$

Now let  $\gamma = \lambda|\log \delta|^{-1}$  and consider two cases.

CASE I.  $\delta \leq \gamma\rho$ .

CASE II.  $\delta \geq \gamma\rho$ .

In case I let

$$\mathcal{X} = \{x \in \mathbb{R}^n : \text{dist}(x, c + P) \leq \gamma\rho\}.$$

First, we show that each  $P_k^\delta \in \mathcal{D}$  has large intersection with  $E \cap \mathcal{X}^c$ . Indeed, notice that

$$P_k^\delta \cap \mathcal{X} \subset P_k^{2\gamma\rho} \cap \mathcal{X}.$$

Hence, by (1) in Lemma 3.1,  $P_k^{2\gamma\rho} \cap \mathcal{X}$  is contained in a tube of cross-section radius  $C\gamma$ . Now, the intersection of a tube of cross-section radius  $C\gamma$  with the plate  $P_k^\delta$  is contained in the intersection of two rectangles of dimensions  $\infty \times C\gamma \times C\gamma \cdots \times C\gamma$  and  $1 \times 1 \times \delta \times \cdots \times \delta$ , and therefore has volume at most  $C\gamma\delta^{n-2}$  (recall that  $\delta \leq \gamma\rho \leq \gamma$ ). We conclude that the volume of  $P_k^\delta \cap \mathcal{X}$  is at most  $C\lambda|\log \delta|^{-1}\delta^{n-2}$ . Consequently

$$\begin{aligned} |P_k^\delta \cap (E \cap \mathcal{X}^c)| &= |P_k^\delta \cap E| - |P_k^\delta \cap E \cap \mathcal{X}| \\ &\geq |P_k^\delta \cap E| - |P_k^\delta \cap \mathcal{X}| \\ &\geq \lambda\delta^{n-2} - C\lambda|\log \delta|^{-1}\delta^{n-2} \\ &\geq \frac{\lambda}{2}\delta^{n-2} \end{aligned}$$

for  $\delta$  sufficiently small.

Next, we show that the sets  $\Pi_i^{\tilde{C}\delta} \cap \mathcal{X}^c$  have small overlap. Namely, we claim that if  $\text{dist}(x, c + P) \geq \gamma\rho$ , then  $x$  belongs to at most  $C\gamma^{-(n-3)}$  sets  $\Pi_i^{\tilde{C}\delta}$ . To see this, we can clearly assume that  $c = \mathbf{0}$ . Now suppose that  $x \in \Pi_i^{\tilde{C}\delta}$  and write  $x = u + w \in P \oplus P^\perp$ . Then  $|w - \langle w, e_i \rangle e_i| \lesssim \delta$ . Therefore, by simple algebra, either  $|w - |w|e_i| \lesssim \delta$ , or  $|w + |w|e_i| \lesssim \delta$ . On the other hand,  $\text{dist}(x, P) \geq \gamma\rho$  implies that  $|w| \geq \gamma\rho$ . Consequently we have either  $|e_i - w/|w|| \lesssim \delta/(\gamma\rho)$ , or  $|e_i + w/|w|| \lesssim \delta/(\gamma\rho)$ . It follows that

$$\{e_i : x \in \Pi_i^{\tilde{C}\delta}\} \subset B(w/|w|, C\delta/(\gamma\rho)) \cup B(-w/|w|, C\delta/(\gamma\rho)).$$

Since the  $e_i$ 's are  $\delta/\rho$ -separated points on an  $(n-3)$ -dimensional unit sphere, we conclude that

$$\text{card}(\{e_i : x \in \Pi_i^{\tilde{C}\delta}\}) \lesssim \left( \frac{\delta/(\gamma\rho)}{\delta/\rho} \right)^{n-3} = \gamma^{-(n-3)}.$$

Hence

$$\begin{aligned} |E| &\geq \left| \bigcup_i (E \cap \mathcal{X}^c) \cap \Pi_i^{\tilde{C}\delta} \right| \\ &\gtrsim \gamma^{n-3} \sum_i |(E \cap \mathcal{X}^c) \cap \Pi_i^{\tilde{C}\delta}| \end{aligned}$$

$$\gtrsim \gamma^{n-3} \lambda^3 \delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}$$

where the last inequality follows from Lemma 3.3 applied, for each  $i$ , to the set  $E \cap \mathcal{X}^c$ , the plates in  $\mathcal{D}_i$  and the 3-plane  $\Pi_i$ .

In case II, since  $|\{\Pi_i\}_i| \lesssim (\rho/\delta)^{n-3}$ , we have

$$\begin{aligned} |E| &\geq \left| \bigcup_i E \cap \Pi_i^{\tilde{C}\delta} \right| \\ &\gtrsim (\delta/\rho)^{n-3} \sum_i |E \cap \Pi_i^{\tilde{C}\delta}| \\ &\geq \gamma^{n-3} \sum_i |E \cap \Pi_i^{\tilde{C}\delta}| \\ &\gtrsim \gamma^{n-3} \lambda^3 \delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2} \end{aligned}$$

with the last inequality true by Lemma 3.3 applied, for each  $i$ , to the set  $E$ , the plates in  $\mathcal{D}_i$  and the 3-plane  $\Pi_i$ .

We conclude that in either case

$$|E| \gtrsim \gamma^{n-3} \lambda^3 \delta^{n-2} \sum_i |\mathcal{D}_i|^{1/2}. \quad (6)$$

To estimate the sum above, note that  $\Pi_i^{\tilde{C}\delta}$ , being the  $\tilde{C}\delta$ -neighborhood of a copy of  $\mathbb{R}^3$ , can contain at most  $C(\rho/\delta)^2$  plates whose direction planes are  $\delta$ -separated and at distance approximately  $\rho$  from  $P$ . Therefore

$$|\mathcal{D}| \leq \sum_i |\mathcal{D}_i| \lesssim \frac{\rho}{\delta} \sum_i |\mathcal{D}_i|^{1/2}. \quad (7)$$

Combining (5), (6) and (7) we obtain

$$|E| \geq C_\epsilon \delta^{2\epsilon} \lambda^{n+2} \frac{M}{|E|} \delta^{2(n-2)}$$

where the logarithmic factors have been absorbed into  $C_\epsilon \delta^{2\epsilon}$ . Consequently

$$|E| \geq C_\epsilon \delta^\epsilon \lambda^{(n+2)/2} M^{1/2} \delta^{n-2}$$

proving the proposition.  $\square$

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