A CHARACTERIZATION OF VANISHING MEAN OSCILLATION

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Abstract. We show that a function has vanishing mean oscillation with respect to a nonatomic measure if and only if it satisfies an asymptotic reverse Jensen inequality.

1. introduction

Let μ be a positive, finite Borel measure on the unit circle $\mathbb{T}$. The space of functions of vanishing mean oscillation with respect to μ (VMO_μ) consists of all $f \in L^1_\mu(\mathbb{T})$ such that

$$\lim_{\delta\to 0} \sup_{\text{length}(I)<\delta} \frac{1}{\mu(I)} \int_I \left| f - \frac{1}{\mu(I)} \int_I f d\mu \right| d\mu = 0,$$

where the supremum is taken over all closed arcs $I \subset \mathbb{T}$ with length less than δ.

It is known that a sufficient condition in order for f to be in VMO_μ is

$$\lim_{\delta\to 0} \sup_{\text{length}(I)<\delta} \left(\frac{1}{\mu(I)} \int_I e^f d\mu\right)\left(\frac{1}{\mu(I)} \int_I e^{-f} d\mu\right) = 1. \tag{1}$$

Moreover, if μ is Lebesgue measure then (1) is also necessary. This is a consequence of the John-Nirenberg inequality. Therefore, (1) is in fact necessary for any measure satisfying such an inequality, in particular for nonatomic measures (see [4]). Note that (1) may be thought of as a limit Muckenhoupt A_2 condition (see, for example, [1] for the basic theory of weights).

It is also known that f has vanishing mean oscillation if and only if e^f satisfies an asymptotic reverse Cauchy-Schwarz inequality. This suggests an analogy between results relating usual Muckenhoupt weights to BMO (Bounded Mean Oscillation, see [1]), and results relating weights satisfying asymptotic conditions like (1) to VMO.

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The purpose of this paper is to push the analogy further by replacing (1) with an asymptotic reverse Jensen inequality. Namely, we prove the following.

Theorem. *Let μ be a positive, finite, nonatomic Borel measure on $\mathbb{T}$, and $f \in L^1_\mu(\mathbb{T})$. Then a necessary and sufficient condition in order for f to be a VMO_μ function is*

$$\lim_{\delta\to 0} \sup_{\text{length}(I)<\delta} \left(\frac{1}{\mu(I)} \int_I e^f d\mu\right) \exp\left(-\frac{1}{\mu(I)} \int_I f d\mu\right) = 1. \tag{2}$$

This parallels the familiar fact that

$$f \in BMO \Leftrightarrow e^{cf} \in A_\infty, \text{ for some } c > 0,$$

where A_∞ is the reverse Jensen class (see [1]). Note that if μ is atom-free then our result implies that (1) and (2) are equivalent.

Put it differently, A_2 and A_∞ coincide if one restricts to weights which tend to be constant on arbitrarily small arcs. Other structural properties of Muckenhoupt weights with respect to general measures may be found in [2], [3] and [5].

2. Proof of the theorem

To prove sufficiency, which is the main point, we fix $\varepsilon_0 > 0$ so that the number

$$c_0 := \frac{\sqrt{e} - 1 - \varepsilon_0}{(1 + \varepsilon_0)(e - 1)} \tag{3}$$

is positive and less than $1/2$. By (2), for any a with $1 < a < 1 + \varepsilon_0$ there exists $\delta_0 > 0$ so that for all arcs I with length less than δ_0 we have

$$\left(\frac{1}{\mu(I)} \int_I e^f d\mu\right) \exp\left(-\frac{1}{\mu(I)} \int_I f d\mu\right) < a. \tag{4}$$

Let I be such an arc. We will perform a "dyadic" decomposition of I in the spirit of Fefferman et al as follows: Since μ is nonatomic we can divide I into two adjacent subarcs of equal measure. We repeat the same division inside each of these subarcs to get four arcs of equal measure. We continue ad infinitum and obtain a family $\mathcal{J}$ of arcs. Letting $\mathcal{J}_k$ be the family of arcs obtained at k-th step in the construction of $\mathcal{J}$, we see that $\mathcal{J}$ has the following obvious properties:

- For every pair $J_1, J_2 \in \mathcal{J}$, either $J_1^\circ \cap J_2^\circ = \emptyset$, or one is contained in the other.
- Every $J \in \mathcal{J}$ splits into two subarcs J_+, J_- of equal μ-measure.

- Every decreasing sequence of arcs in $\mathcal{J}$ shrinks either to a point or to an arc of positive length and zero μ-measure. The family of these limit arcs is disjoint, hence countable, and therefore its union has μ-measure zero. Consequently, for μ-almost all points $x \in I$, there are (unique) $J_n^x \in \mathcal{J}_n$ such that $\bigcap\limits_n J_n^x = \{x\}$.

Now for $g \in L^1_\mu(I)$ consider the averaging operator

$$E_k(g) = \sum_{J \in \mathcal{J}_k} \left(\frac{1}{\mu(J)} \int_J g d\mu \right) \chi_J,$$

where χ_J is the characteristic function of J. By a standard argument, the corresponding "dyadic" maximal function

$$g \mapsto \sup_k E_k(|g|)$$

is of weak type $(1, 1)$ and therefore

$$\lim_k E_k(g) = g, \ \mu\text{-almost everywhere on } I.$$

(Alternatively, one may invoke the one-dimensional result of Sjögren [6] concerning the uncentered maximal function with respect to general measures.)

In particular

$$\begin{aligned} e^f &= \lim_k E_k\left(e^f\right) = \frac{1}{\mu(I)} \int_I e^f d\mu \prod_k \left(E_{k+1}\left(e^f\right)\right)\left(E_k\left(e^f\right)\right)^{-1} \\ &= \frac{1}{\mu(I)} \int_I e^f d\mu \prod_{J \in \mathcal{J}} \left(1 + \left(2 \frac{\int_{J_+} e^f d\mu}{\int_J e^f d\mu} - 1 \right) h_J \right) \\ &= \frac{1}{\mu(I)} \int_I e^f d\mu \prod_{J \in \mathcal{J}} (1 + \alpha_J h_J), \end{aligned} \tag{5}$$

where

$$\alpha_J = 2 \frac{\int_{J_+} e^f d\mu}{\int_J e^f d\mu} - 1, \ h_J(x) = \begin{cases} 1, & \text{if } x \in J_+ \\ -1, & \text{if } x \in J_- \\ 0, & \text{if } x \notin J \end{cases}.$$

To estimate α_J, note that

$$\begin{aligned}\sqrt{e} &= \exp\left(\frac{\mu(J_+)}{\mu(J)}\right)\\ &= \exp\left(\frac{1}{\mu(J)}\int_{J_+}(\chi_{J_+}+f)d\mu\right)\exp\left(-\frac{1}{\mu(J)}\int_J f d\mu\right)\\ &\le \frac{a}{\int_J e^f d\mu}\int_J \exp(\chi_{J_+}+f)d\mu,\end{aligned}$$

where we have used Jensen's inequality and (4). Consequently

$$\sqrt{e} \le \frac{a}{\int_J e^f d\mu}\left(e\int_{J_+} e^f d\mu + \int_{J\setminus J_+} e^f d\mu\right),$$

which, by (3), implies that

$$c_0 \le \frac{\sqrt{e}-a}{a(e-1)} \le \frac{\int_{J_+} e^f d\mu}{\int_J e^f d\mu}.$$

By symmetry, the same estimate holds with J_- in place of J_+. Therefore

$$|\alpha_J| \le 1-2c_0.$$

Now (5) yields

$$f = \log\left(\frac{1}{\mu(I)}\int_I e^f d\mu\right) + \sum_{J\in\mathcal{J}} \log(1+\alpha_J h_J). \tag{6}$$

By Taylor's theorem

$$\log(1+\alpha_J h_J) = \alpha_J h_J - \frac{1}{2}\alpha_J^2 \chi_J (1+\varphi_J)^{-2},$$

where φ_J is a function supported in J with $|\varphi_J| \le |\alpha_J|$. Hence

$$f = \log\left(\frac{1}{\mu(I)}\int_I e^f d\mu\right) + \sum_{J\in\mathcal{J}} \alpha_J h_J - \sum_{J\in\mathcal{J}} \alpha_J^2 \psi_J,$$

where ψ_J is a function supported in J with $|\psi_J| \le 1/(8c_0^2)$. Using the above representation of f we get that

$$\begin{aligned}\mathrm{Osc}_f(I) &:= \frac{1}{\mu(I)}\int_I \left| f - \frac{1}{\mu(I)}\int_I f d\mu\right| d\mu\\ &\le \frac{2}{\mu(I)}\int_I \left|\sum_{J\in\mathcal{J}} \alpha_J h_J\right| d\mu + \frac{2}{\mu(I)}\int_I \left(\sum_{J\in\mathcal{J}} \alpha_J^2 |\psi_J|\right) d\mu\\ &\lesssim \left(\frac{1}{\mu(I)}\int_I \left(\sum_{J\in\mathcal{J}} \alpha_J h_J\right)^2 d\mu\right)^{1/2} + \frac{1}{\mu(I)}\sum_{J\in\mathcal{J}} \alpha_J^2 \mu(J),\end{aligned}$$

where the implicit constant depends only on c_0.

Since the functions h_J are orthogonal we have

$$\int_I \left(\sum_{J \in \mathcal{J}} \alpha_J h_J \right)^2 d\mu = \sum_{J \in \mathcal{J}} \alpha_J^2 \mu(J).$$

Consequently

$$\mathrm{Osc}_f(I) \lesssim \left(\frac{1}{\mu(I)} \sum_{J \in \mathcal{J}} \alpha_J^2 \mu(J) \right)^{1/2} + \frac{1}{\mu(I)} \sum_{J \in \mathcal{J}} \alpha_J^2 \mu(J).$$

To estimate the expression above, notice that (4) and (6) imply

$$\begin{aligned}
-\log a &\leq -\log \left(\frac{1}{\mu(I)} \int_I e^f d\mu \right) + \frac{1}{\mu(I)} \int_I f d\mu \\
&= \frac{1}{\mu(I)} \sum_{J \in \mathcal{J}} \int_J \log(1 + \alpha_J h_J) d\mu \\
&= \frac{1}{2\mu(I)} \sum_{J \in \mathcal{J}} (\log(1 + \alpha_J) + \log(1 - \alpha_J)) \mu(J) \\
&\lesssim -\frac{1}{\mu(I)} \sum_{J \in \mathcal{J}} \alpha_J^2 \mu(J).
\end{aligned}$$

Therefore

$$\mathrm{Osc}_f(I) \lesssim \sqrt{\log a} + \log a,$$

which proves that $f \in VMO_\mu$.

To prove necessity, as we pointed out in the introduction, one uses a standard argument involving the John-Nirenberg inequality for nonatomic measures (see [4]), so we omit the proof.

3. Remarks

The result of this paper is new even in the case of the one-dimensional Lebesgue measure. Moreover, it is also valid in the case of the higher dimensional Lebesgue measure (in this situation we replace "arcs" and "length" with "cubes" and "diameter", respectively, in the statement of the theorem). Indeed, we may repeat the argument almost verbatim by performing a canonical dyadic decomposition of a cube into rectangles of bounded eccentricities. Then everything goes through since all the rectangles which appear in the decomposition are comparable to cubes. On the other hand, in the context of arbitrary nonatomic measures, our methods do not seem to generalize to higher dimensions. The reason is that if we "dyadically" decompose a cube with respect to μ, we will get a family of rectangles with arbitrary eccentricities, in which case (2) is of no use, unless we make further assumptions on μ. However, the "only if" part of the theorem is valid in any dimension by the the general John-Nirenberg inequality of [4].

Finally, let us observe that the assumption that μ should be nonatomic is needed at least for the "only if" part of the theorem. This may be seen by considering the example from [4]: Fix a sequence $\theta_n \searrow 0$ and let $x_n = e^{i\theta_n}$. Now if we put

$$\mu = \sum_n \frac{1}{2^{n^2}} \delta_{x_n}, \ f(x_n) = 2^n$$

then it easy to show that $f \in VMO_\mu$. But e^f is not μ-integrable so it cannot satisfy (2).

On the other hand, we do not know whether the assumption on μ is needed for the "if" part of the theorem.

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