University of Illinois at Urbana–Champaign
Department of Mathematics

Second Midterm Exam for **MATH/CS-313** (Combinatorial Mathematics), Section F1
Instructor: M.N. Kolountzakis

April 17, 1997

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 50
Duration: 1 hour

*** Explain all your answers ****

GOOD LUCK!

1 (10 points) $A \subseteq \mathbf{N}$ is an infinite set of positive integers. Prove that there is an infinite $B \subseteq A$ which has one of the two following properties.

1. If $x, y \in B$ and $x < y$ then x divides y.
2. If $x, y \in B$ and $x < y$ then x does not divide y.

2

(a) (5 points) Show by induction that a tree (a connected graph without circuits) on n vertices has $n-1$ edges.

(b) (5 points) Show that any connected graph on n vertices with $n-1$ edges is a tree (i.e., it has no circuits).

3 (10 points) Prove Lagrange's Theorem: If G is a finite group and $H \subseteq G$ is a subgroup then $|H|$ divides $|G|$.

HINT: Show that the relation $x \sim y$ if and only if $x^{-1}y \in H$ is an equivalence relation on G. How large are its equivalence classes?

4 (10 points) The finite group G acts on the finite set X. You are reminded that the size of the orbit of $x \in X$ (call this orbit $o(x)$) is

$$|o(x)| \;=\; |G : G_x| \;=\; \frac{|G|}{|G_x|},$$

where G_x is the subgroup consisting of those elements g of G for which $xg = x$ (G_x is called the stabilizer subgroup of x).

Show that the number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{fix}(g)|,$$

where $\mathrm{fix}(g) = \{x \in X : \; xg = x\}$.

HINT: The number of orbits is equal to $\displaystyle\sum_{x \in X} \frac{1}{|o(x)|}$.

5 (10 points) Let P be a partially ordered set (a *poset*) whose maximum chain has size r. Show that P can be partitioned into r antichains.

SCRATCH PAGE