University of Illinois at Urbana–Champaign
Department of Mathematics

First Midterm Exam for **MATH-315** (Linear Algebra), Section Q1
Instructor: M.N. Kolountzakis

October 3, 1996

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 50
Duration: 1 hour and 20 minutes

*** Explain all your answers ****

GOOD LUCK!

1 (a) (3 points) Describe the set of all solutions of the following linear system by reducing its augmented matrix to **reduced row echelon** form:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & + & x_3 & = & 2 \\ 4x_1 & + & 3x_2 & + & 3x_3 & = & 4 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 3 \end{array}$$

(b) (3 points) Find a 3×3 **non-zero** matrix A such that $A^2 = 0$. (HINT: Try a matrix with many zeros in it.)
(c) (4 points) A 2×2 matrix A has the property that it commutes with all 2×2 matrices. That is, for all 2×2 matrices B we have

$$AB = BA.$$

Prove that A is necessarily of the type

$$A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

for some real number x.
(HINT: Try multiplying A from both sides with matrices B that have only zeros as entries except for a single position where they have a one.)

2 (a) (3 points) Find all choices of c that make the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$$

singular.
(b) (2 points) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define the function

$$f(x) = \det(xI - A),$$

where I is the 2×2 identity matrix. Show that $f(x)$ is a polynomial of degree at most 2.
(c) (2 points) If the real number x_0 is such that $f(x_0) = 0$ then what can you say about the singularity of the matrix $x_0 I - A$?
(d) (3 points) Show that if $f(x_0) = 0$ then there is a **non-zero** vector $\mathbf{v} \in \mathbf{R}^2$ such that

$$A\mathbf{v} = x_0\mathbf{v}.$$

(HINT: Rewrite the above equality as: $x_0\, I\, \mathbf{v} - A\, \mathbf{v} = \mathbf{0}$.)

3 Consider the vector space P_4 of all polynomials of degree at most 3. Which of the following subsets are subspaces and why?
(2 points each)
(a) The polynomials $p(x)$ that vanish at $x = 2$.
(b) The polynomials $p(x)$ that **do not** vanish at $x = 2$.
(c) The polynomials $p(x)$ such that

$$\lim_{x \to +\infty} p(x) = +\infty.$$

(d) The polynomials $p(x)$ such that

$$\lim_{x \to +\infty} \frac{p(x)}{e^x} = 0.$$

(e) The polynomials $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ such that

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 \leq 1.$$

4 (a) (4 points) Prove that any non-empty subset of a linearly independent set of vectors is also linearly independent.
(b) (6 points) Assume that the sets

$$\left\{ \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right\}$$

are both bases for $\mathbf{R}^2$. Show that the set

$$\left\{ \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z \\ w \end{pmatrix} \right\}$$

is a basis for $\mathbf{R}^4$.

5 Let $\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{u_3} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$. Let $\mathcal{U}$ be the basis that consists of the vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ and $\mathcal{E}$ be the standard basis of $\mathbf{R}^3$.

(a) (4 points) Find the transition matrix A from $\mathcal{E} \to \mathcal{U}$ and the transition matrix B from $\mathcal{U} \to \mathcal{E}$.

(b) (2 points) What are the coordinates of the vector $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ with respect to $\mathcal{U}$?

(c) (2 points) Let

$$\mathcal{V} = \left\{ \mathbf{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v_3} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Find the transition matrix C from $\mathcal{V} \to \mathcal{U}$.

(d) If $\mathbf{x} = 3\mathbf{v_1} + 4\mathbf{v_2} + 5\mathbf{v_3}$ find the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{U}$.

SCRATCH PAGE