University of Illinois at Urbana–Champaign
Department of Mathematics

Second Midterm Exam for **MATH-315** (Linear Algebra), Section Q1
Instructor: M.N. Kolountzakis

November 21, 1996

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 50
Duration: 1 hour and 20 minutes

*** Explain all your answers ****

GOOD LUCK!

1 (10 points) Compute the Gram-Schmidt QR factorization of the matrix

$$\begin{pmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & -2 \\ 4 & 0 & 0 \end{pmatrix}.$$

Show clearly that you understand how to do this, so that you get nearly full credit even in the presence of calculation errors.

2 (a) (3 points) Write the *normal equations* that one has to solve in order to find the *least squares* solution to the linear system

$$A\mathbf{x} = \mathbf{b},$$

where $A \in \mathbf{R}^{m\times n}$, $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$.

(b) (7 points) Find a least squares solutions to the system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 4 \\ -x_1 & + & x_2 & + & x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & 1 \\ x_1 & & & + & x_3 & = & 2 \end{array}$$

by solving the normal equations of (a).

3 (a) (7 points) Prove that if a $n \times n$ matrix has orthonormal columns then it also has orthonormal rows.
(HINT: Such a matrix must be non-singular.)

(b) (3 points) Give an example to show that this statement is not true for non-square matrices.

4 In $\mathbf{R}^n$ we define the following *norms*

$$\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|, \quad \|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad \|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|,$$

for the vector $\mathbf{x} = (x_1, \ldots, x_n)^\top$.

(a) (5 points) Show that for all $\mathbf{x} \in \mathbf{R}^n$: $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$.

(b) (5 points) Show that for all $\mathbf{x} \in \mathbf{R}^n$: $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$.

5 Let S be a subspace of an inner product space V, and let $n = \dim V$.

(a) (2 points) Define the space $S^\perp$.

(b) (2 points) Show that $S \cap S^\perp = \{\mathbf{0}\}$.

(c) (2 points) How are the numbers $n = \dim V$, $\dim S$ and $\dim S^\perp$ related. (State this; do not prove it.)

(d) (2 points) Use your answer in (c) to prove that

$$(S^\perp)^\perp = S.$$

(HINT: Clearly $S \subseteq S^{\perp\perp}$. What would happen to the dimension of $S^{\perp\perp}$ if $S^{\perp\perp}$ contained a vector not in S?)

(e) (2 points) Describe $S^\perp \subseteq \mathbf{R}^3$ when

$$S = \operatorname{span}\left\{(1\ 2\ 3)^\top\right\}.$$

What kind of geometric object is $S^\perp$?
If $(x\ y\ z)^\top \in S^\perp$ which relation do x, y, z satisfy?

SCRATCH PAGE