University of Illinois at Urbana–Champaign
Department of Mathematics

Final Exam for **MATH-315** (Linear Algebra), Section Q1
Instructor: M.N. Kolountzakis

December 20, 1996

YOUR NAME:

YOUR CODE NAME:
Make up a not-too-long name your grade will be announced with. Keep private.

SCORE:

1	
2	
3	
4	
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10	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 100
Duration: 3 hours

*** Explain all your answers ****

GOOD LUCK!

1 (10 points) Let $A, B \in \mathbf{R}^{n\times n}$ and assume that

$$A\,B = B\,A.$$

If λ is an eigenvalue of A and the vector space E is defined as

$$E = \{\mathbf{x} \in \mathbf{R}^n :\ A\mathbf{x} = \lambda\mathbf{x}\}$$

show that B maps the vector space E into itself. In other words, show that

$$\mathbf{x} \in E \implies B\mathbf{x} \in E.$$

2 Let $A \in \mathbf{R}^{m\times n}$, $B \in \mathbf{R}^{n\times r}$ and $C = AB$. Show that

- (a) (3 points) $R(C)$, the column space of C, is a subspace of $R(A)$, the column space of A.
- (b) (3 points) $R(C^\top)$, the row space of C, is a subspace of $R(B^\top)$, the row space of B.
- (c) (4 points) $\operatorname{rank}(C) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

3 (10 points) Using row operations compute A^{-1} if

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

4 (10 points) Bring the following linear system to **row echelon** form and describe all its solutions, if any.

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 1 \\ -x_1 & - & x_2 & + & 4x_3 & - & x_4 & = & 6 \\ -2x_1 & - & 4x_2 & + & 7x_3 & - & x_4 & = & 1 \end{array}$$

5 In the matrices below the elements that do not appear are 0.

- (a) (3 points) If

$$M = \begin{pmatrix} a & & \\ & b & c \\ & d & e \end{pmatrix}$$

show that $\det M = a \begin{vmatrix} b & c \\ d & e \end{vmatrix}$.

- (b) (7 points) If

$$N = \begin{pmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ & & b_{11} & b_{12} \\ & & b_{21} & b_{22} \end{pmatrix}, \ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

use part (a) to show that

$$\det N = \det A \cdot \det B.$$

(HINT: Expand N along the first row.)

6 (10 points) Let

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and let L be the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ defined by

$$L(\mathbf{x}) = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + (x_1 + x_2)\mathbf{b}_3.$$

Find the 3×2 matrix A representing L with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbf{R}^2$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in $\mathbf{R}^3$.

7 (10 points) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

8

- (a) (2 points) Define the rank of a matrix A.

Assume that $A \in \mathbf{R}^{m \times n}$ has rank r. What are the dimensions of the following vector spaces and **why** (briefly).

- (b) (2 points) $R(A)$ (the column space of A).
- (c) (2 points) $R(A^\top)$ (the row space of A).
- (d) (2 points) $N(A)$ (the nullspace of A).
- (e) (2 points) $N(A^\top)$ (the nullspace of $A^\top$).

(HINT: Remember that $R(B)^\perp = N(B^\top)$. In words, the orthogonal complement of the column space of a matrix B is the nullspace of the transpose of B.)

9 (10 points) Assume that $A \in \mathbf{R}^{n \times n}$ and that $A^2 = A$ (this is called an idempotent matrix). Show that any eigenvalue of A must be either 0 or 1.

10 Let

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

- (a) (6 points) Find the transition matrix corresponding to the change of basis from $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.
- (b) (4 points) Find the coordinates of the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \mathbf{x}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

with respect to the basis $\mathcal{U}$.

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