The Fourier Transform and applications

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- Products: $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{T} \times \mathbb{R}$, etc

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- Product of Haar measures on the components

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- Pontryagin duality: $\widehat{\widehat{G}} = G$.

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- Example: $G = \mathbb{Z}_m$ ("Discrete Fourier transform or DFT"):

$$\widehat{f}(k) = \frac{1}{m} \sum_{j=0}^{m-1} f(j) e^{-2\pi i k j / m}, \quad k \in \mathbb{Z}_m$$

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- $f, g \in L^1(G)$: their convolution is $f * g(x) = \int_G f(t) g(x-t)\, d\mu(t)$.
  Then $\|f * g\|_1 \le \|f\|_1 \|g\|_1$ and

$$\widehat{f * g}(\xi) = \widehat{f}(\xi) \cdot \widehat{g}(\xi), \quad \xi \in \widehat{G}$$

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- Fourier representation (inversion) in $\mathbb{Z}_m$: $G = \mathbb{Z}_m \Longrightarrow$ the m characters form a complete orthonormal set in $L^2(G)$:
$$f(x) = \sum_{k=0}^{m-1} \langle f(\cdot), e^{2\pi i k \cdot} \rangle e^{2\pi i k x} = \sum_{k=0}^{m-1} \widehat{f}(k) e^{2\pi i k x}$$

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- Fourier representation in $L^2(G)$: Compact G: The characters form a complete ONS. Since $C(G)$ is dense in $L^2(G)$:

$$f = \int_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi \, d\chi \quad \text{all } f \in L^2(G), \text{ convergence in } L^2(G)$$

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- Fourier representation in $L^2(G)$: Compact G: The characters form a complete ONS. Since $C(G)$ is dense in $L^2(G)$:

$$f = \int_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi \, d\chi \quad \text{all } f \in L^2(G), \text{ convergence in } L^2(G)$$

- $\widehat{G}$ necessarily discrete in this case

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$$\sum_{j=0}^{m-1} f(j)\overline{g(j)} = \sum_{k=0}^{m-1} \widehat{f}(k)\overline{\widehat{g}(k)}, \quad \text{all } f, g : \mathbb{Z}_m \to \mathbb{C}$$

Triple correlations in $\mathbb{Z}_p$: an application

- Problem of significance in (a) crystallography, (b) astrophysics: determine a subset $E \subseteq \mathbb{Z}_n$ from its triple correlation:

$$\begin{aligned} N_E(a,b) &= \#\{x \in \mathbb{Z}_n : x, x+a, x+b \in E\}, \quad a, b \in \mathbb{Z}_n \\ &= \sum_{x \in \mathbb{Z}_n} \mathbf{1}_E(x)\mathbf{1}_E(x+a)\mathbf{1}_E(x+b) \end{aligned}$$

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- Fourier transform of $N_E : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{R}$ is easily computed:

$$\widehat{N_E}(\xi,\eta) = \widehat{\mathbf{1}_E}(\xi)\widehat{\mathbf{1}_E}(\eta)\widehat{\mathbf{1}_E}(-(\xi+\eta)), \quad \xi,\eta \in \mathbb{Z}_n.$$

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$$\widehat{\mathbf{1}_E}(\xi) = \frac{1}{p}\sum_{s\in E}(\zeta^{\xi})^s, \quad \zeta = e^{-2\pi i/p} \text{ is a } p\text{-root of unity.} \tag{3}$$

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The basics of the FT on the torus (circle) $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

- $1 \le p \le q \Longleftrightarrow L^q(\mathbb{T}) \subseteq L^p(\mathbb{T})$: nested L^p spaces. True on compact groups.

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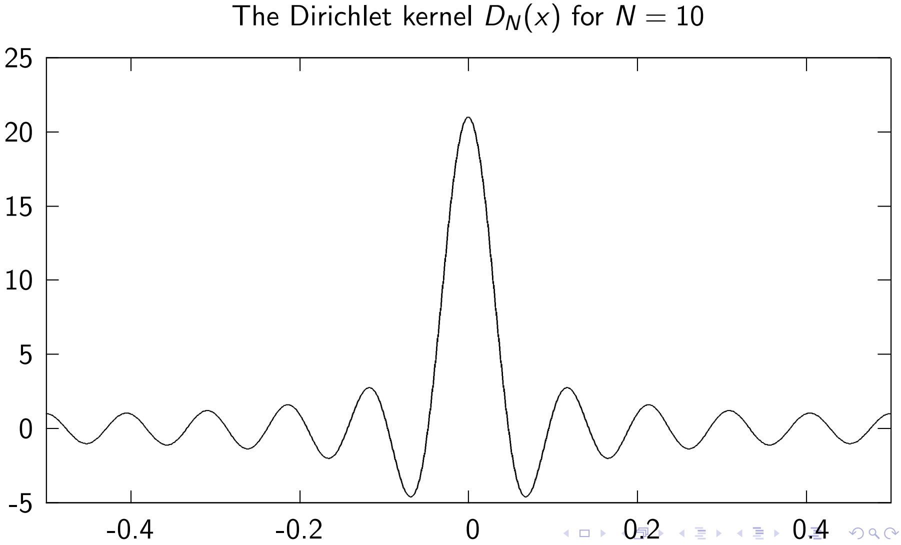
- From $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ we get easily $S_N(f; x) = f(x) * D_N(x)$, where

$$D_N(x) = \sum_{k=-N}^{N} e^{2\pi i k x} = \frac{\sin 2\pi(N + \frac{1}{2})x}{\sin \pi x}$$ (Dirichlet kernel of order N)

The Dirichlet kernel



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- Consequence: In general $S_N(f;x)$ does not converge pointwise to $f(x)$, even for continuous f

Summability

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- $K_N(x)$ is an approximate identity:
(a) $\int_{\mathbb{T}} K_N(x)\,dx = \widehat{K_N}(0) = 1$,
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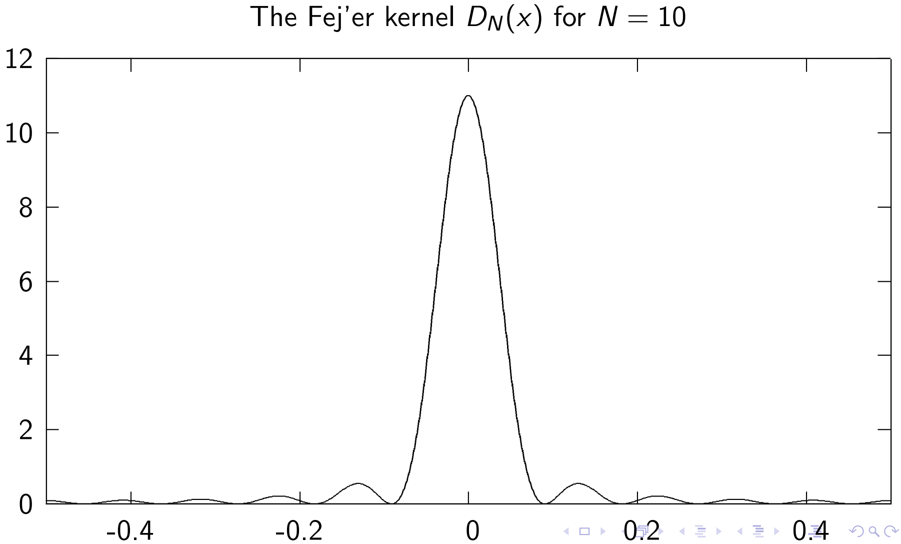
(c) for any $\epsilon > 0$ we have $\int_{|x|>\epsilon} |K_N(x)|\,dx \to 0$, as $N \to \infty$

The FEJÉR kernel



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- Another important summability kernel: the Poisson kernel

$$P(r,x) = \sum_{k\in\mathbb{Z}} r^k e^{2\pi ikx}, \quad 0 < r < 1: \text{ absolute convergence obvious}$$

Significant for the theory of analytic functions.

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- Another condition that imposes "decay":
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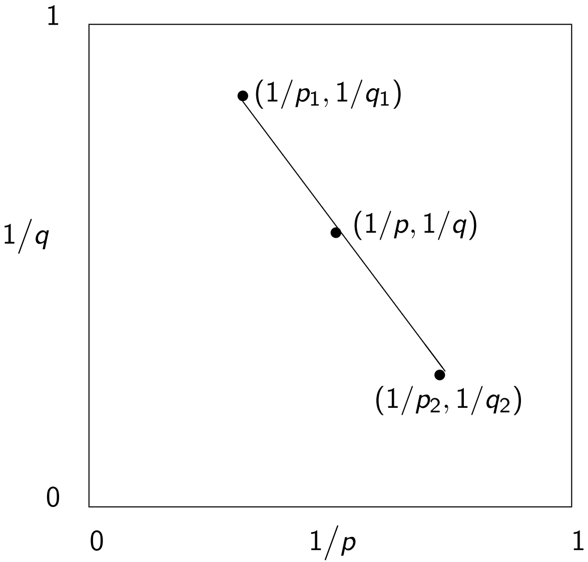
Interpolation of operators: the $1/p, 1/q$ plane



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- Use RIESZ-THORIN interpolation for $1 < p < 2$ for the operator $f \to \widehat{f}$ from $L^p(\mathbb{T}) \to L^q(\mathbb{Z})$.

An application: the isoperimetric inequality

- Suppose Γ is a simple closed curve in the plane with perimeter L enclosing area A.

$$A \leq \frac{1}{4\pi} L^2 \quad \text{(isoperimetric inequality)}$$

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- Equality in (4) precisely when $f(x) = \widehat{f}(-1) e^{-2\pi i x} + \widehat{f}(0) + \widehat{f}(1) e^{2\pi i x}$.

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- For equality must have $x(s) = a\cos 2\pi s + b \sin 2\pi s + c$,
  $y'(s) = 2\pi(x(s) - \widehat{x}(0))$. So $x(s)^2 + y(s)^2$ constant if $c = 0$.

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- Multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:
  (a) $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
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Fourier transform on $\mathbb{R}^n$

- Initially defined only for $f \in L^1(\mathbb{R}^n)$. $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x}\, dx$. Follows: $\left\|\widehat{f}\right\|_\infty \le \|f\|_1$. $\widehat{f}$ is continuous.
- Trig. polynomials are not dense anymore in the usual spaces.
- But Riemann-Lebesgue is true. First for indicator function of an interval

$$[a_1, b_1] \times \cdots \times [a_n, b_n].$$

  Then approximate an L^1 function by finite linear combinations of such.

- Multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:
  (a) $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
  (b) $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$
  (c) $\partial^\alpha = (\partial/\partial_1)^{\alpha_1} \cdots (\partial/\partial_n)^{\alpha_n}$
- Diff operators $D^j \phi := \frac{1}{2\pi i}(\partial/\partial x_j)$, $D^\alpha \phi = (1/2\pi i)^{|\alpha|} \partial^\alpha$.

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- Using Cauchy's integral formula for analytic functions we prove $\widehat{g}(\xi) = (2\pi)^{n/2} g(2\pi\xi)$. The Fourier inversion formula holds.
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FT on $L^2(\mathbb{R}^n)$

- Preservation of inner product: $\int \phi\overline{\psi} = \int \widehat{\phi}\overline{\widehat{\psi}}$, for $\phi, \psi \in \mathcal{S}$
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- By interpolation FT is defined on L^p, $1 \le p \le 2$, and satisfies the Hausdorff-Young inequality:

$$\left\|\widehat{f}\right\|_q \le C_p \|f\|_p, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

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- $u = x^\alpha$, $\widehat{u} = D^\alpha \delta_0$
- $u = \sum_{j=1}^J a_j \delta_{p_j}$, $\widehat{u}(\xi) = \sum_{j=1}^J a_j e^{2\pi i p_j \xi}$.

Examples of tempered distributions and their FT

- $u = \delta_0$, $\widehat{u} = 1$
- $u = D^\alpha \delta_0$. To find its FT

$$\widehat{D^\alpha \delta_0}(\phi) = (D^\alpha \delta_0)(\widehat{\phi}) = (-1)^{|\alpha|}\delta_0(D^\alpha \widehat{\phi}) = (-1)^{|\alpha|}\delta_0(\widehat{(-x)^\alpha \phi}(x))$$

$$= (-1)^{|\alpha|}\widehat{\delta_0}((-x)^\alpha \phi(x)) = \int x^\alpha \phi(x)$$

- So $\widehat{D^\alpha \delta_0} = x^\alpha$.
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- Poisson Summation Formula (PSF): $u = \sum_{k \in \mathbb{Z}^n} \delta_k$, $\widehat{u} = u$

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- $x = 0$ gives the PSF: $\sum_{k\in\mathbb{Z}^n} \phi(k) = \sum_{k\in\mathbb{Z}^n} \widehat{\phi}(k)$.

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- PALEY–WIENER: $f \in L^2(\mathbb{R})$. The following are equivalent:
(a) f is the restriction on $\mathbb{R}$ of a function F holomorphic in the strip $\{z : |\Im z| < a\}$ which satisfies
$$\int |F(x+iy)|^2\, dx \le C, \quad (|y| < a)$$
(b) $e^{a|\xi|}\widehat{f}(\xi) \in L^2(\mathbb{R})$.

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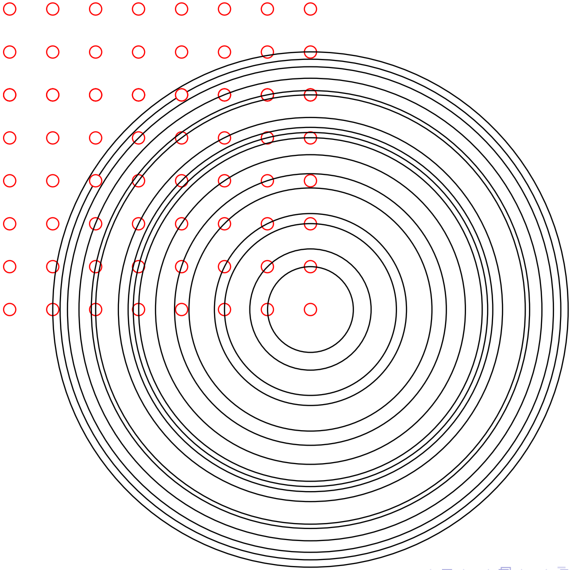
The circles on which $\widehat{\mathbf{1}_E}$ must vanish



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- But such a function can only have $O(R)$ zeros from 0 to R.

Zeros of entire functions of exponential type

- JENSEN's formula: F analytic in the disk $\{|z| \le R\}$, z_k are the zeros of F in that disk. Then

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- Such a function F can therefore have only $O(R)$ zeros in the disk $\{|z| \le R\}$.