University Of Crete

Existence Of Fundamental Solution of a Differential Operator

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Abstract

A fundamental solution for a linear differential operator A is a distribution F, which satisfies the in-homogeneous equation $AF = \delta(x)$, where δ is the Dirac "delta function". The existence of a fundamental solution for any operator A with constant coefficients was shown by Bernard Malgrange and Leon Ehrenpreis.

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Chapter 1

Distributions

1.1 Introduction

1.1.1 Definition and Examples

We begin with the notion of a **multi-index**. This is any n-tuple of non-negative integers

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$$

The order of a multi-index is the quantity

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

If α is a multi-index, there is a partial differential operator D^α corresponding to it:

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

The space $C^\infty(\mathbb{R}^n)$ consists of all functions $\phi : \mathbb{R}^n \to \mathbb{C}$ such that $D^\alpha \phi \in C(\mathbb{R}^n)$ for each multi-index α. Thus, the mixed partial derivatives of ϕ of all orders exist and are continuous.

A vector space $\mathcal{D}(\mathbb{R}^n)$, called the **space of test functions**, is now introduced. Its elements are all the functions in $C^\infty(\mathbb{R}^n)$ having compact support. The support of a function ϕ is the closure of $\{x : \phi(x) \neq 0\}$ and it is denoted $supp(\phi)$. Another notation for $\mathcal{D}(\mathbb{R}^n)$ is $C_c^\infty(\mathbb{R}^n)$.

An element $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\phi \geq 0 \qquad \int_{\mathbb{R}^n} \phi(x)dx = 1 \qquad supp(\phi) \subseteq \{x : |x| \leq 1\}$$

is called a **mollifier**. If ϕ is a mollifier, then the scaled versions o ϕ, defined by

$$\phi_\delta(x) = \frac{1}{\delta^n}\phi\left(\frac{x}{\delta}\right) \qquad (\delta > 0)$$

play a role in certain arguments.

When we say that $\phi_j \to \phi$ for a sequence ϕ_j in $\mathcal{D}(\mathbb{R}^n)$ and a $\phi \in \mathcal{D}(\mathbb{R}^n)$ we mean that there is a compact set K so that $\mathrm{supp}(\phi_j) \subseteq K$ for every j and $D^a\phi_j \to D^a\phi$ uniformly in $\mathbb{R}^n$ for every multi-index a.

Theorem 1 For every multi-index α, D^α is a continuous linear transformation of $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$.

A **distribution** is a continuous linear functional on $\mathcal{D}$. Continuity of such a linear functional T is defined by this implication:

$$[\phi_j \in \mathcal{D}(\mathbb{R}^n) \;\;\&\;\; \phi_j \to 0] \Rightarrow T(\phi_j) \to 0$$

The space of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

Example 1 The **Dirac** distribution δ_ξ is defined by selecting $\xi \in \mathbb{R}^n$ and writing

$$\delta_\xi(\phi) = \phi(\xi)$$

It is a distribution, because firstly, it is linear:

$$\delta_\xi(\lambda_1\phi_1 + \lambda_2\phi_2) = (\lambda_1\phi_1 + \lambda_2\phi_2)(\xi) = \lambda_1\phi_1(\xi) + \lambda_2\phi_2(\xi) = \lambda_1\delta_\xi(\phi_1) + \lambda_2\delta_\xi(\phi_2)$$

Secondly, it is continuous because the condition $\phi_j \to 0$ implies that $\phi_j(\xi) \to 0$.

If we write δ without a subscript we refer to $\xi = 0$, i.e. $\delta = \delta_0$.

Example 2 The **Heaviside** distribution $\widetilde{H}$ is defined, when $n = 1$, by

$$\widetilde{H}(\phi) = \int_0^\infty \phi(x)dx \qquad (\phi \in \mathcal{D}(\mathbb{R}^n))$$

Example 3 Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable on every compact set in $\mathbb{R}^n$. We say that f is locally integrable or $f \in L^1_{loc}(\mathbb{R}^n)$. With f we associate a distribution $\widetilde{f}$ by means of the definition

$$\widetilde{f}(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx \qquad (\phi \in \mathcal{D}(\mathbb{R}^n))$$

The linearity of $\widetilde{f}$ is obvious. For the continuity, we observe that if $\phi_j \to 0$, then there is a compact K containing the supports of all ϕ_j. Then we have

$$\left|\widetilde{f}(\phi_j)\right| = \left|\int_K f(x)\phi_j(x)dx\right| \leq \sup_x |\phi_j(x)| \int_K |f(y)|\,dy \to 0$$

because $\phi_j \to 0$ entails $\sup_x |\phi_j(x)| \to 0$.

We say that $\widetilde{f}$ is the image of f in the space of distributions. Distributions of the form $\widetilde{f}$ are called regular distributions.

Example 4 If H is the Heaviside function, defined by the equation

$$H(x) = \begin{cases} 1, & if \quad x \geq 0 \\ 0, & if \quad x < 0 \end{cases}$$

then Example 2 above illustrates the principle in Example 3.

Example 5 Fix a multi-index α and define

$$T(\phi) = \int_{\mathbb{R}^n} D^\alpha \phi \qquad (\phi \in \mathcal{D}(\mathbb{R}^n))$$

This is a distribution.

1.1.2 Derivatives of Distributions

We have seen that the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions is very large; it contains (images of) all locally integrable functions in $\mathbb{R}^n$. Then, too, it contains functionals on $\mathcal{D}(\mathbb{R}^n)$ that are not readily identified with functions. Now we will define derivatives of distributions, taking care that the new notion of derivative will coincide with the classical one when both are meaningful.

Definition If T is a distribution and α is a multi-index, then $D^\alpha T$ is the distribution defined by

$$D^\alpha T = (-1)^{|\alpha|} T \circ D^\alpha$$

Example Let $\widetilde{H}$ be the Heaviside distribution, and then δ be the Dirac distribution at 0. Then with $n = 1$, $\alpha = 1$ and $D = \frac{d}{dx}$, we have $D\widetilde{H} = \delta$. Indeed, for any test function ϕ,

$$(D\widetilde{H})(\phi) = -\widetilde{H}(D\phi) = -\int_0^\infty \phi'(x)dx = \phi(0) - \phi(\infty) = \phi(0) = \delta(\phi).$$

Theorem Let $n = 1$, and let T be a distribution for which $DT = 0$. Then T is $\widetilde{c}$ for some constant c.

1.1.3 Convolutions

The **convolution** of two functions f and ϕ on $\mathbb{R}^n$ is a function $f * \phi$ whose defining equation is

$$(f * \phi)(x) = \int_{\mathbb{R}^n} f(y)\phi(x - y)dy \qquad (x \in \mathbb{R}^n) \tag{1.1}$$

The integral will certainly exist if $\phi \in \mathcal{D}(\mathbb{R}^n)$ and if $f \in L^1_{loc}(\mathbb{R}^n)$, because for each x, the integration takes place over a compact subset of $\mathbb{R}^n$. With a change of variable in the integral, $y = x - z$, one proves that

$$(f * \phi)(x) = \int_{\mathbb{R}^n} f(x-z)\phi(z)dz = (\phi * f)(x)$$

In taking the convolution of two functions, one can expect that some favorable properties of one factor will be inherited by the convolution function. This vague concept will be illustrated now in several ways. Suppose that f is merely integrable, while ϕ is a test function. In Equation (1.1), suppose that $n = 1$, and that we wish to differentiate $f * \phi$. On the right side of the equation, x appears only in the function ϕ, and consequently

$$(f * \phi)'(x) = \int_{-\infty}^{\infty} f(y)\phi'(x-y)dy$$

The diffentiability of the factor ϕ is inherited by the convolution product $f * \phi$. This phenomenon persists with higher derivatives and with many variables.

We shall see that convolutions are useful in approximating functions by smooth functions. Let ϕ be a mollifier; that is, $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\phi \geq 0$, $\int_{\mathbb{R}^n} \phi(x)dx = 1$, and $\phi(x) = 0$ when $|x| \geq 1$. With $\phi_\delta(x) = \frac{1}{\delta^n}\phi\left(\frac{x}{\delta}\right)$ it is easy to verify that $\int_{\mathbb{R}^n} \phi_\delta(x)dx = 1$. Then

$$\begin{aligned}
f(x) - (f * \phi_\delta)(x) &= f(x) - \int_{\mathbb{R}^n} f(x-z)\phi_\delta(z)dz \\
&= \int_{\mathbb{R}^n} f(x)\phi_\delta(z)dz - \int_{\mathbb{R}^n} f(x-z)\phi_\delta(z)dz \\
&= \int_{\mathbb{R}^n} [f(x) - f(x-z)]\phi_\delta(z)dz
\end{aligned}$$

Since $\phi(x)$ vanishes outside the unit ball in $\mathbb{R}^n$, $\phi_\delta(x)$ vanishes outside the ball of radius δ, as is easily verified. Hence in the equation above the only values of z that have any effect are those for which $|z| < \delta$. If f is uniformly continuous, the calculation shows that $f * \phi_\delta(x)$ is close to $f(x)$, and we have therefore approximated f by the smooth function $f * \phi_\delta$.

We define special linear operators B and E_x by

$$\begin{aligned}
(E_x\phi)(y) &= \phi(y-x) \\
(B\phi)(y) &= \phi(-y)
\end{aligned}$$

We can write Equation (1.1) in the form

$$(f * \phi)(x) = \widetilde{f}(E_x B\phi) \tag{1.2}$$

For $f \in L^1_{loc}(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{aligned}\widetilde{E_x f}(\phi) &= \int_{\mathbb{R}^n} E_x f(y)\phi(y)dy = \int_{\mathbb{R}^n} f(y-x)\phi(y)dy = \int_{\mathbb{R}^n} f(z)\phi(z+x)dz \\ &= \widetilde{f}(E_{-x}\phi).\end{aligned}$$

i.e.

$$\widetilde{E_x f} = \widetilde{f} \circ E_{-x} \tag{1.3}$$

Based on (1.2) and (1.3) we have the following definition:

Definition If T is a distribution, we define the distribution $E_x T$ by

$$E_x T = T \circ E_{-x}$$

Also, if $\phi \in \mathcal{D}(\mathbb{R}^n)$ and T is a distribution, we define the function $T * \phi$ by

$$(T * \phi)(x) = T(E_x B\phi)$$

Lemma 1 For $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\mathbb{R}^n)$,

$$E_x(T * \phi) = (E_x T) * \phi = T * E_x\phi$$

Theorem If T is a distribution and if ϕ is a test function, then for each multi-index α,

$$D^\alpha(T * \phi) = (D^\alpha T) * \phi = T * D^\alpha\phi$$

1.2 Differential Operators

Definition A **linear differential operator** with constant coefficients is any finite sum of terms $c_a D^a$. Such an operator has the representation:

$$A = \sum_{|\alpha| \le m} c_\alpha D^\alpha$$

The constants c_α may be complex numbers. Clearly, A can be applied to any function in $C^m(\mathbb{R}^n)$.

Definition A distribution T is called a **fundamental solution** of the operator

$\sum_{|a|\leq m} c_a D^a$ if $\sum_{|a|\leq m} c_a D^a T$ is the Dirac distribution:

$$\sum_{|a|\leq m} c_a D^a T = \delta$$

Example 1 What are the fundamental solutions of the operator $D = \frac{d}{dx}$ in the case of $n = 1$? We seek all the distributions T that satisfy $DT = \delta$. We saw in the Example in Section 1.1.2 that $D\widetilde{H} = \delta$, where H is the Heaviside function. Thus $\widetilde{H}$ is one of the fundamental solutions. Since the distributions sought are exactly those for which $DT = D\widetilde{H}$, we see by the Theorem in Section 1.1.2 that $T = \widetilde{H} + \widetilde{c}$ for some constant c.

Theorem 1 Let A be a linear differential operator with constant coefficients, and T be a distribution that is a fundamental solution of A. Then for each test function ϕ,

$$A(T * \phi) = \phi$$

Proof Let $A = \sum_{|a|\leq m} c_a D^a$. Then $\sum_{|a|\leq m} c_a D^a T = \delta$. The theorem of Section 1.1.3 states that

$$D^a(T * \phi) = D^a T * \phi$$

From this we conclude that

$$A(T * \phi) = \sum c_a D^a (T * \phi) = (\sum c_a D^a T) * \phi = \delta * \phi = \phi$$

In the last step, we use the calculation

$$(\delta * \phi)(x) = \delta(E_x B_\phi) = (E_x B\phi)(0) = (B\phi)(0 - x) = \phi(x)$$

Example 2 We use the theory of distributions to find a solution of the differential equation $\frac{du}{dx} = \phi$, where ϕ is a test function. By Example 1, one fundamental solution of $\frac{d}{dx}$ is the distribution $\widetilde{H}$. By the preceding theorem, $\widetilde{H} * \phi$ will solve the differential equation. We have, with a simple change of variable,

$$u(x) = (\widetilde{H} * \phi)(x) = \int_{-\infty}^{\infty} H(y)\phi(x - y)dy = \int_{-\infty}^{x} \phi(z)dz$$

Example 3 Let us search for a solution of the differential equation

$$u' + au = \phi$$

using distribution theory. First, we try to discover a fundamental solution, i.e. a distribution T such that $DT + aT = \delta$. If T is such a distribution and if $v(x) = e^{ax}$, then

$$D(vT) = DvT + vDT = avT + v(\delta - aT) = v\delta = \delta$$

Consequently, by Example 1,

$$vT = \widetilde{H} + \widetilde{c}$$

and

$$T = \frac{1}{v}(\widetilde{H} + \widetilde{c})$$

Thus T is a regular distribution $\widetilde{f}$, and since c is arbitrary, we use $c = 0$, arriving at

$$f(x) = e^{-ax}H(x)$$

A solution to the differential equation is then given by

$$u(x) = (f * \phi)(x) = \int_{-\infty}^{\infty} e^{-ay}H(y)\phi(x-y)dy = \int_{0}^{\infty} e^{-ay}\phi(x-y)dy$$

This formula produces a solution if ϕ is bounded and of class C^1.

Let us introduce the **Laplace operator**, denoted by Δ and given by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

The following are easy to prove:

$$\frac{\partial}{\partial x_j}|x| = x_j\,|x|^{-1}, \qquad x \neq 0$$

$$\frac{\partial^2}{\partial x_j^2}|x| = |x|^{-1} - x_j^2\,|x|^{-3}, \qquad x \neq 0$$

Also, for $x \neq 0$ and $g \in C^2(0,\infty)$:

$$\Delta g(|x|) = g''(|x|) + (n-1)|x|^{-1}g'(|x|) \tag{1.4}$$

In order to find a fundamental solution to the Laplace operator, we require a function g (not a constant), such that $\Delta g(|x|) = 0$, throught $\mathbb{R}^n$, with the exception of the singular point $x = 0$. By (1.4), we see that g must satisfy the following differential equation, in which the notation $r = |x|$ has been introduced:

$$g''(r) + \frac{n-1}{r}g'(r) = 0$$

From this we get

$$g'(r) = cr^{1-n}$$

If $n \geq 3$, the last equation gives $g(r) = r^{2-n}$ as the desired solution. Thus, we have proved the following result:

Theorem 3 If $n \geq 3$, then $\Delta|x|^{2-n} = 0$ at all points of $\mathbb{R}^n$ except $x = 0$.

This theorem can be proved by a direct verification that $|x|^{2-n}$ satisfies the Laplace equation, except at 0. The fact that the Laplace equation is not satisfied at 0 is of special importance in what follows.

Let $f(x) = |x|^{2-n}$. As usual, $\widetilde{f}$ will denote the corresponding distribution. In accordance with the definition of derivative of a distribution, we have:

$$\Delta \widetilde{f} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \widetilde{f} = \sum_{i=1}^{n} (-1)^2 \widetilde{f} \circ \frac{\partial^2}{\partial x_i^2} = \widetilde{f} \circ \Delta$$

For any test function ϕ,

$$(\Delta \widetilde{f})(\phi) = \widetilde{f}(\Delta \phi) = \int_{\mathbb{R}^n} |x|^{2-n} (\Delta \phi)(x) dx \tag{1.5}$$

The integral on the right is improper because of the singular point at 0. It is therefore defined to be:

$$\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} |x|^{2-n} (\Delta \phi)(x) dx \tag{1.6}$$

For sufficiently small ε, the support of ϕ will be contained in $\{x : |x| < \varepsilon^{-1}\}$. The integral in (1.6) is over the set

$$A_\varepsilon = \left\{x : \varepsilon \leq |x| \leq \varepsilon^{-1}\right\}$$

An appeal will be made to Green's Second Identity, which states that for regions Ω satisfying certain mild hypotheses,

$$\int_\Omega (u(x)\Delta v(x) - v(x)\Delta u(x)) dx = \int_{\partial\Omega} (u(x)\nabla v(x) - v(x)\nabla u(x)) \cdot N(x) dS(x)$$

In the last formula, N denotes the unit normal vector to the surface $\partial\Omega$. Applying Green's formula to the integral in equation (1.6) , we notice that $\Delta|x|^{2-n} = 0$ in A_ε. Hence the integral is

$$\int_{A_\varepsilon} |x|^{2-n} \Delta \phi(x) dx = \int_{\partial A_\varepsilon} (|x|^{2-n} \nabla \phi(x) - \phi(x) \nabla |x|^{2-n}) \cdot N(x) dS(x) \tag{1.7}$$

The boundary of A_ε is the union of two spheres whose radii are ε and ε^{-1}. On the outer boundary, $\phi = \nabla\phi = 0$ because the support of ϕ is interior to A_ε. The following computation will also be needed for the points of the inner boundary:

$$\begin{aligned}\nabla|x|^{2-n}\cdot N(x) &= -\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}|x|^{2-n}\right)\frac{x_j}{|x|} = -\sum_{j=1}^{n}(2-n)|x|^{1-n}\left(\frac{x_j}{|x|}\right)^2 \\ &= (n-2)|x|^{1-n}\end{aligned}$$

The first term of the right side of equation (1.7) is estimated as follows on the inner boundary:

$$\begin{aligned}\int_{|x|=\varepsilon}\left||x|^{2-n}\nabla\phi(x)\cdot N(x)\right|dS(x) &\le \varepsilon^{2-n}\max_{|x|=\varepsilon}|\nabla\phi(x)|\int_{|x|=\varepsilon}dS(x) \\ &\le c\varepsilon^{2-n}\ \sigma_n\varepsilon^{n-1} = O(\varepsilon)\end{aligned}$$

Hence, when $\varepsilon \to 0$, this term approaches 0. The symbol σ_n represents the "area" of the unit sphere in $\mathbb{R}^n$. As for the other term,

$$\begin{aligned}\int_{|x|=\varepsilon}\left|[\phi(x)-\phi(0)]\nabla|x|^{2-n}\cdot N(x)\right|dS(x) &\le (n-2)\int_{|x|=\varepsilon}|x|^{1-n}|\phi(x)-\phi(0)|dS(x) \\ &\le (n-2)\varepsilon^{1-n}\max_{|x|=\varepsilon}|\phi(x)-\phi(0)|\int_{|x|=\varepsilon}dS(x) \\ &= (n-2)\varepsilon^{1-n}\omega(\varepsilon)\sigma_n\varepsilon^{1-n} \to 0\end{aligned}$$

In this calculation, $\omega(\varepsilon)$ is the maximum of $|\phi(x)-\phi(0)|$ on the sphere defined by $|x|=\varepsilon$. Obviously, $\omega(\varepsilon)\to 0$, because ϕ is continuous. Also,

$$\begin{aligned}-\phi(0)\int_{|x|=\varepsilon}\nabla|x|^{n-2}\cdot N(x)dS(x) &= (2-n)\phi(0)\int_{|x|=\varepsilon}|x|^{1-n}dS(x) = (2-n)\sigma_n\phi(0) \\ &= (2-n)\sigma_n\delta(\phi)\end{aligned}$$

Thus the integral in (1.7) is

$$(2-n)\sigma_n\phi(0) = (2-n)\sigma_n\delta(\phi)$$

Hence, this is the value of the integral in equation (1.5). We have established, therefore, that $\Delta\widetilde{f} = (2-n)\sigma_n\delta$. Summarizing, we have the following result:

Theorem 4 A fundamental solution of the Laplace operator in dimension $n \ge 3$ is the regular distribution corresponding to $\frac{|x|^{2-n}}{(2-n)\sigma_n}$, where σ_n denotes the area of the unit sphere in $\mathbb{R}^n$.

Example We will find a fundamental solution of the operator A defined (for $n=1$) by the equation

$$A\phi = \phi'' + 2a\phi' + b\phi$$

where a, b are constants. We seek a distribution T such that $AT = \delta$. Let us look for a regular distribution, $T = \widetilde{f}$. Using the definition of derivatives of distributions, we have

$$(A\widetilde{f})(\phi) = \widetilde{f}(\phi'' - 2a\phi' + b\phi) = \int_{-\infty}^{\infty} f(x)(\phi''(x) - 2a\phi'(x) + b\phi(x))dx$$

Guided by previous examples, we guess that f should have as its support the interval $[0, \infty)$. Then the integral above is restricted to the same interval.Using integration by parts, we obtain:

$$\begin{aligned}
&f\phi'|_0^\infty - \int_0^\infty f'(x)\phi'(x)dx - 2af\phi|_0^\infty + 2a\int_0^\infty f'(x)\phi(x)dx + b\int_0^\infty f(x)\phi(x)dx \\
&= -f(0)\phi'(0) - f'\phi|_0^\infty + \int_0^\infty f''(x)\phi(x)dx + 2af(0)\phi(0) \\
&+ \int_0^\infty (2af'(x) + bf(x))\phi(x)dx \\
&= -f(0)\phi'(0) + f'(0)\phi(0) + 2af(0)\phi(0) + \int_0^\infty (f''(x) + 2af'(x) + bf(x))\phi(x)dx
\end{aligned}$$

The easiest way to make this last expression simplify to $\phi(0)$ is to define f on $[0, \infty)$ in such way that:

1. $f'' + 2af' + bf = 0$
2. $f(0) = 0$
3. $f'(0) = 1$

This is an initial-value problem, which can be solved by writing down the general solution of the equation in (i) and adjusting the coefficients in it to achieve (ii) and (iii). The characteristic equation of the differential equation in (i) is:

$$\lambda^2 + 2a\lambda + b = 0$$

Its roots are $-a \pm \sqrt{a^2 - b}$. For example, if $a^2 > b$ and $d = \sqrt{a^2 - b}$, then the general solution of (i) is

$$c_1 e^{-ax} e^{dx} + c_2 e^{-ax} e^{-dx}$$

Upon imposing the conditions (ii) and (iii) we find that

$$f(x) = \begin{cases} d^{-1} e^{-ax} \sinh dx, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

A linear differential operator with non constant coefficients is typically of the form

$$A = \sum_{|\alpha| \leq m} c_\alpha D^\alpha \tag{1.8}$$

In order for this to interact properly with distributions, it is necessary to assume that $c_\alpha \in C^\infty(\mathbb{R}^n)$. Then AT is defined, when T is a distribution, by

$$AT = \sum_{|\alpha| \leq m} c_\alpha (D^\alpha T) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} c_\alpha (T \circ D^\alpha) \tag{1.9}$$

We notice that $T \circ D^\alpha$ is a distribution; multiplication of this distribution by the C^∞-function c_α is well-defined: multiplication of a distribution T by a function $f \in C^\infty(\mathbb{R}^n)$ is defined as the distribution fT given by

$$(fT)(\phi) = T(f\phi)$$

The result of applying (1.9) to a test function ϕ is therefore

$$(AT)(\phi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} (T \circ D^\alpha)(c_\alpha \phi)$$

The parentheses in (1.9) are necessary because $c_\alpha T \circ D$ is ambiguous; it could mean $(c_\alpha T) \circ D$.

It is useful to define the formal adjoint of the operator A in (1.8) . It is

$$A^*\phi = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (c_\alpha \phi) \qquad (\phi \in \mathcal{D}(\mathbb{R}^n))$$

This definition is in harmony with the definition of adjoint for operators on Hilbert space, for we have

$$(AT)(\phi) = T(A^*\phi) \qquad (T \in \mathcal{D}'(\mathbb{R}^n), \phi \in \mathcal{D}(\mathbb{R}^n))$$

Using the last Example, as a model, we can prove a theorem about fundamental solutions of ordinary differential operators in dimension $n = 1$.

Theorem 5 Consider the operator

$$A = \sum_{j=0}^{m} c_j(x) \frac{d^j}{dx^j}$$

in which $c_j \in C^\infty(\mathbb{R})$ and $c_m(x) \neq 0$ for all x. This operator has a fundamental solution which is a regular distribution.

Proof We find a function f defined on $[0, \infty)$ such that

1. $\sum_{j=0}^{m} c_j f^{(j)} = 0$
2. $c_{m-1}(0) f^{(m-1)}(0) = 1$
3. $c_j(0) f^{(j)}(0) = 0 \qquad (0 \leq j \leq m-2)$

Such a function exists by the theory of ordinary differential equations. In particular, an initial-value problem has a unique solution that is defined on any interval $[0, b]$, provided that the coefficient functions are continuous there and the leading coefficient does not have a zero in $[0, b]$. We also extend f to all of $\mathbb{R}$ by setting $f(x) = 0$ on the interval $(-\infty, 0)$. With the function f, we must verify that $A\widetilde{f} = \delta$. This is done as in the previous example.

Chapter 2

Fourier Transform

2.1 Introduction to Fourier Transform

2.1.1 Definitions and Basic Properties

In general, integral transforms are helpful in problems where there is a function f to be determined from an equation that it satisfies. A judiciously chosen transform is then applied to that equation, the result being a simpler equation in the transformed function F. After this simpler equation has been solved for F, the inverse transform is applied to obtain f. We illustrate with the Fourier transform.

We define a set of functions called **characters** e_y by the formula

$$e_y(x) = e^{2\pi i x\cdot y} \qquad x, y \in \mathbb{R}^n$$

Here we have written

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

where the x_i and y_i are the components of the vectors x and y.

The characters satisfy these equations:

1. $e_y(u + v) = e_y(u)e_y(v)$
2. $E_u e_y = e_y(-u)e_y, \qquad$ where $(E_u f)(x) = f(x - u)$
3. $e_y(x) = e_x(y)$
4. $e_y(\lambda x) = e_{\lambda y}(x) \qquad (\lambda \in \mathbb{R})$

The **Fourier transform** of a function f in $L^1(\mathbb{R}^n)$ is the function $\widehat{f}$ defined by the equation

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx \qquad (\xi \in \mathbb{R}^n)$$

The kernel $e^{-2\pi i x\cdot\xi}$ is obviously complex-valued, but x and ξ run over $\mathbb{R}^n$.

Theorem 1 We have

$$\widehat{E_y f} = e_{-y}\widehat{f}, \qquad \widehat{e_y f} = E_y \widehat{f}$$

Theorem 2 If f and g belong to $L^1(\mathbb{R}^n)$, then

$$\widehat{f * g} = \widehat{f}\widehat{g}$$

2.1.2 The Schwartz Space

The space $\mathcal{S}(\mathbb{R}^n)$, called Schwartz space, is the set of all ϕ in $C^\infty(\mathbb{R}^n)$ such that $P\,D^\alpha\phi$ is a bounded function, for each polynomial P and each multi-index α. If $P(x) = \sum_{|\alpha|\le m} c_\alpha x^\alpha$ is a polynomial, then $P(D)$ is defined to be the differential operator

$$P(D) = \sum_{|\alpha|\le m} c_\alpha D^\alpha$$

Lemma 1 The function $e_y(x)$ defined by $e_y(x) = e^{2\pi i x y}$ obeys the equation

$$P(D)e_y = P(2\pi i x)e_y$$

for any polynomial P, where $P^+(x) = P(2\pi i x)$.

Theorem 1 If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and if P is a polynomial, then

$$\widehat{P(D)\phi} = P^+\widehat{\phi}$$

Example 2 Let Δ denote the Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

Then $\Delta = P(D)$ if P is defined to be

$$P(x) = x_1^2 + \cdots + x_n^2 = |x|^2$$

Hence, for $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\widehat{\Delta\phi}(\xi) = \widehat{P(D)\phi}(\xi) = P^+(\xi)\widehat{\phi}(\xi) = -4\pi^2|\xi|^2\widehat{\phi}(\xi)$$

Theorem 2 If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and P is a polynomial, then

$$P(D)\widehat{\phi} = \widehat{P^-\phi}$$

where $P^{-}(x) = P(-2\pi ix)$.

2.1.3 The Inversion Theorems

Theorem 1 If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\phi(x) = \int_{\mathbb{R}^n} \widehat{\phi}(\xi) e^{2\pi i\xi\cdot x} d\xi$$

Theorem 2 If f and $\widehat{f}$ belong to $L^1(\mathbb{R}^n)$, then for almost all x,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i\xi\cdot x} d\xi$$

2.2 Applications of the Fourier Transform

We will give some representative examples to show how the Fourier Transform can be used to solve differential equations and integral equations.

Example 1 Let $n = 1$ and $D = \frac{d}{dx}$. If P is a polynomial, say $P(x) = \sum_{j=0}^{m} c_j x^j$, then $P(D)$ is a linear differential operator with constant coefficients:

$$P(D) = \sum_{j=0}^{m} c_j D^j$$

We consider the ordinary differential equation

$$P(D)u = g, \qquad -\infty < x < \infty \tag{2.1}$$

in which g is given and is assumed to be element of $L^1(\mathbb{R})$. Apply the Fourier Transform $\mathcal{F}$ to both sides of equation (2.1). Then use Theorem 1 in section (2.1.2) ,which asserts that if $u \in \mathcal{S}(\mathbb{R})$ then

$$\mathcal{F}[P(D)u] = P^{+}\mathcal{F}(u)$$

where $P^{+}(x) = P(2\pi ix)$. The transformed version of Equation (2.1) is therefore

$$P^{+}\mathcal{F}(u) = \mathcal{F}(g) \tag{2.2}$$

The solution of Equation (2.2) is

$$\mathcal{F}(u) = \mathcal{F}(g)/P^{+}$$

The function u is recovered by taking the inverse transformation, if it exists:

$$u = \mathcal{F}^{-1}[\mathcal{F}(g)/P^+] \tag{2.3}$$

Theorem 2 in Section 2.1.1 states that

$$\mathcal{F}(\phi * \psi) = \mathcal{F}(\phi)\mathcal{F}(\psi)$$

An equivalent formulation, in terms of $\mathcal{F}^{-1}$, is

$$\phi * \psi = \mathcal{F}^{-1}[\mathcal{F}(\phi)\mathcal{F}(\psi)] \tag{2.4}$$

If h is a function such that $\mathcal{F}(h) = 1/P^+$, then Equations (2.3) and (2.4) yield

$$u = \mathcal{F}^{-1}\left[\mathcal{F}(g)/P^+\right] = \mathcal{F}^{-1}[\mathcal{F}(g)\mathcal{F}(h)] = g * h$$

In detail,

$$u(x) = \int_{-\infty}^{\infty} g(y)h(x-y)dy$$

The function h must be obtained by the equation $h = \mathcal{F}^{-1}(1/P^+)$.

Example 2 This is a concrete case of Example 1, namely

$$u'(x) + bu(x) = e^{-|x|} \qquad (b > 0, \quad b \neq 1) \tag{2.5}$$

We will find the Fourier Transform of the function $k(x) = e^{-|x|}$: We have

$$k(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$$

Then, the Fourier Transform will have the form

$$\begin{aligned} \widehat{k}(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi i x\xi} k(x)dx = \int_{-\infty}^{0} e^{-2\pi i x\xi} e^{x} dx + \int_{0}^{\infty} e^{-2\pi i x\xi} e^{-x} dx \\ &= \left[\frac{e^{x(1-2\pi i\xi)}}{1-2\pi i\xi}\right]_{x\to-\infty}^{x=0} + \left[\frac{e^{x(-1-2\pi i\xi)}}{-1-2\pi i\xi}\right]_{x=0}^{x\to\infty} = \frac{1}{1-2\pi i\xi} + \frac{1}{1+2\pi i\xi} \\ &= \frac{2}{1+4\pi^2\xi^2} \end{aligned}$$

The Fourier Transform of Equation (2.5) is:

$$2\pi i\xi\widehat{u}(\xi) + b\widehat{u}(\xi) = 2/(1+4\pi^2\xi^2)$$

Solving for $\widehat{u}(\xi)$, we have

$$\widehat{u}(\xi) = \frac{2}{(1+4\pi^2\xi^2)(b+2\pi i\xi)}$$

By the Inversion Theorem,

$$u(x) = \int_{-\infty}^{\infty} \frac{2e^{2\pi ix\xi}}{(1+4\pi^2\xi^2)(b+2\pi i\xi)} d\xi$$

To simplify this, substitute ξ for $2\pi\xi$, to obtain

$$u(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+\xi^2)(b+i\xi)} d\xi$$

The integrand, call it $f(\xi)$, as a function of a complex variable ξ has poles at $\xi = +i, -i, ib$. In order to evaluate this integral, we use the residue calculus:

$$\pi u(x) = \int_{-\infty}^{\infty} f(\xi)d\xi = \begin{cases} 2\pi i(Res(f;i) + Res(f;ib)) & ,x > 0 \\ 2\pi i Res(f;-i) & ,x < 0 \end{cases} \tag{2.6}$$

Now we calculate:

$$Res(f;i) = \left[(\xi - i)\frac{e^{ix\xi}}{(\xi-i)(\xi+i)(b+i\xi)}\right]_{\xi=i} = \frac{e^{-x}}{2i(b-1)}$$

$$Res(f;-i) = \left[(\xi + i)\frac{e^{ix\xi}}{(\xi-i)(\xi+i)(b+i\xi)}\right]_{\xi=-i} = \frac{-e^{x}}{2i(b+1)}$$

$$Res(f;ib) = \left[(\xi - ib)\frac{e^{ix\xi}}{(\xi-i)(\xi+i)(b+i\xi)}\right]_{\xi=ib} = \frac{e^{-bx}}{i(1-b^2)}$$

Then Equation (2.6) gives

$$u(x) = \frac{e^{-x}}{b-1} + \frac{2e^{-bx}}{1-b^2}$$

for $x > 0$ and

$$u(x) = \frac{-e^{x}}{b+1}$$

for $x < 0$.

Lemma 2 If f is analytic in the horizontal zone $\{z \in \mathbb{C} \mid 0 \le Imz \le \eta\}$ and $|f(x+iy)| \le \frac{C}{x^2}$, where C does not depend on x and y, then

$$\int_{Imz=\eta} f(z)dz = \int_{Imz=0} f(z)dz$$

Proof The condition $|f(x+iy)| \leq \frac{C}{x^2}$ implies the existence of

$$\int_{Imz=\eta} f(z)dz = \int_{-\infty}^{+\infty} f(x+i\eta)dx$$

and of

$$\int_{Imz=0} f(z)dz = \int_{-\infty}^{+\infty} f(x)dx$$

Now we apply the theorem of Cauchy in the rectangle $[-R,R] \times [0,\eta]$:

$$\int_{-R}^{R} f(x)dx + \int_{0}^{\eta} f(R+iy)dy - \int_{-R}^{R} f(x+i\eta)dx - \int_{0}^{R} f(-R+iy)dy = 0$$

We have

$$\left|\int_{0}^{\eta} f(R+iy)dy\right| \leq \int_{0}^{\eta} |f(R+iy)|\, dy \leq \frac{C}{R^2}\eta \to 0 \quad as\ \ R \to \infty$$

and

$$\left|\int_{0}^{\eta} f(-R+iy)dy\right| \leq \int_{0}^{\eta} |f(-R+iy)|\, dy \leq \frac{C}{R^2}\eta \to 0 \quad as\ \ R \to \infty$$

Hence

$$\int_{-\infty}^{+\infty} f(x)dx - \int_{-\infty}^{+\infty} f(x+i\eta)dx = 0$$

Example 3 Consider the integral equation

$$\int_{-\infty}^{\infty} k(x-y)u(y)dy = g(x)$$

in which k and g are given, and u is an unknown function. We can write

$$u * k = g$$

After taking Fourier transforms and using Theorem 2 in Section 2.1.1 we have

$$\widehat{u}\widehat{k} = \widehat{g}$$

Hence $\widehat{u} = \widehat{g}/\widehat{k}$ and $u = \mathcal{F}^{-1}(\widehat{g}/\widehat{k})$. For a concrete case, contemplate this integral equation:

$$\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy = e^{-x^2/2}$$

Here, the functions k and g in the above discussion are

$$k(x) = e^{-|x|} \qquad\qquad g(x) = e^{-x^2/2}$$

The Fourier transform of k is given by the previous example as:

$$\widehat{k}(\xi) = \frac{2}{1 + 4\pi^2\xi^2}$$

We find $\widehat{g}(\xi)$:

$$\begin{aligned}\widehat{g}(\xi) &= \int_{-\infty}^{\infty} e^{-2\pi i x\xi} g(x)dx = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} e^{-x^2/2} dx \\ &= e^{-2\pi^2\xi^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+2\pi i\xi)^2} dx\end{aligned}$$

We set $z = (x + 2\pi i\xi)/\sqrt{2}$ and, using Lemma 2, the last expression is

$$\sqrt{2}e^{-2\pi^2\xi^2} \int_{Imz=\sqrt{2}\pi\xi} e^{-z^2} dz = \sqrt{2}e^{-2\pi^2\xi^2} \int_{Imz=0} e^{-z^2} dz = \sqrt{2\pi}e^{-2\pi^2\xi^2}$$

Finally,

$$\widehat{g}(\xi) = \sqrt{2\pi}e^{-2\pi^2\xi^2}$$

We have

$$\widehat{u}(\xi) = \frac{\widehat{g}(\xi)}{\widehat{k}(\xi)} = \widehat{g}(\xi)\frac{1 + 4\pi^2\xi^2}{2}$$

We consider $P(x) = \frac{1-x^2}{2}$, so $P^+(x) = P(2\pi i x) = \frac{1+4\pi^2x^2}{2}$. Using Theorem 1 in section 2.1.2 , we get

$$\widehat{u} = P^+\widehat{g} = \widehat{P(D)g}$$

Finally,

$$u(x) = P(D)g(x) = \frac{1}{2}\left(g(x) - g''(x)\right) = \frac{1}{2}e^{-x^2/2}(2 - x^2)$$

2.3 Applications to Partial Differential Equations

Example 1 The simplest case of the heat equation is

$$u_{xx} = u_t \tag{2.7}$$

in which the subscripts denote partial derivatives. The distribution of heat in an infinite bar would obey this equation for $-\infty < x < \infty$ and $t \geq 0$. A fully defined practical problem would consist of the differential equation (2.7) and some auxiliary conditions. To illustrate, we consider (2.7) with initial condition

$$u(x, 0) = f(x) \qquad -\infty < x < \infty \tag{2.8}$$

The function f gives the initial temperature distribution in the bar. We define $\widehat{u}(\xi, t)$ to be the Fourier transform of u in the space variable. Thus,

$$\widehat{u}(\xi, t) = \int_{-\infty}^{\infty} u(x, t)e^{-2\pi i x\xi} dx$$

Taking the Fourier transform in Equations (2.7) and (2.8) with respect to the space variable, we obtain:

$$-4\pi^2\xi^2\widehat{u}(\xi, t) = \frac{d}{dt}\widehat{u}(\xi, t) \qquad \widehat{u}(\xi, 0) = \widehat{f}(\xi)$$

This is a first order ordinary differential equation in the time variable with initial condition and we obtain:

$$\widehat{u}(\xi, t) = \widehat{f}(\xi)e^{-4\pi^2\xi^2 t}$$

Also, we consider G given by

$$G(x, t) = \frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}$$

for $t > 0$ and we have $\widehat{G}(\xi, t) = e^{-4\pi^2\xi^2 t}$. Thus,

$$\widehat{u}(\xi, t) = \widehat{f}(\xi)\widehat{G}(\xi, t)$$

Consequently,

$$u(x, t) = (f * G(\cdot, t))(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(x - y)e^{-\frac{y^2}{4t}} dy \tag{2.9}$$

Example 2 We consider the problem

$$\begin{cases} u_{xx} = u_t & x \geq 0, t \geq 0 \\ u(x, 0) = f(x), \quad u(0, t) = 0 & x \geq 0, t \geq 0 \end{cases} \tag{2.10}$$

This is a minor modification of Example 1. The bar is "semi-infinite", and one end remains constantly at temperature zero. It is clear that f should have the property $f(0) = u(0, 0) = 0$. Suppose that we extend f somehow into the interval $(-\infty, 0)$ and then use the solution (2.9) of the previous example. Then at $x = 0$ we have

$$u(0, t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f(-y)e^{-\frac{y^2}{4t}} dy \tag{2.11}$$

The easiest way to ensure that this will be zero (and thus satisfy the boundary condition in our problem) is to extend f to be an odd function. Then the integrand in Equation (2.11) is odd, and $u(0, t) = 0$ automatically. We define $f(-x) = -f(x)$

for $x > 0$, and then equation (2.9) gives the solution for the problem (2.10).

Example 3 Again, we consider the heat equation with boundary conditions:

$$\begin{cases} u_{xx} = u_t & x \geq 0, t \geq 0 \\ u(x,0) = f(x), \quad u(0,t) = g(t) & x \geq 0, t \geq 0 \end{cases} \tag{2.12}$$

Because the differential equation is linear and homogeneous, the method of superposition can be applied. We solve two related problems:

$$v_{xx} = v_t \qquad v(x,0) = f(x) \qquad v(0,t) = 0 \tag{2.13}$$

$$w_{xx} = w_t \qquad w(x,0) = 0 \qquad w(0,t) = g(t) \tag{2.14}$$

The solution of (2.12) will be $u = v + w$. The problem in (2.13) is solved in Example 2. In (2.14), we take the sine transform of both sides in the space variable. The sine transform of a function $f(x)$ is defined by

$$f^S(\xi) = \int_0^\infty f(x)\sin(2\pi x\xi)dx$$

Then we have:

$$\begin{aligned} [w_{xx}(x,t)]^S(\xi) &= \int_0^\infty w_{xx}(x,t)\sin(2\pi x\xi)dx = -2\pi\xi \int_0^\infty w_x(x,t)\cos(2\pi x\xi)dx \\ &= 2\pi\xi w(0,t) - (2\pi\xi)^2 \int_0^\infty w(x,t)\sin(2\pi x\xi)dx \\ &= 2\pi\xi g(t) - 4\pi^2\xi^2 w^S(\xi,t) \end{aligned}$$

$$\begin{aligned} [w_t(x,t)]^S(\xi) &= \int_0^\infty w_t(x,t)\sin(2\pi x\xi)dx = \frac{d}{dt}\int_0^\infty w(x,t)\sin(2\pi x\xi)dx \\ &= \frac{d}{dt}w^S(\xi,t) \end{aligned}$$

$$w^S(\xi,0) = \int_0^\infty w(x,0)\sin(2\pi x\xi)dx = 0$$

Then, Equation (2.14) becomes:

$$2\pi\xi g(t) - 4\pi^2\xi^2 w^S(\xi,t) = \frac{d}{dt}w^S(\xi,t)$$

This is a first order ordinary differential equation with initial condition and its solution is easily found to be:

$$w^S(\xi,t) = 2\pi\xi e^{-4\pi^2\xi^2 t}\int_0^t e^{4\pi^2\xi^2\sigma}d\sigma$$

If w is made into an odd function in the space variable by setting $w(x,t) = -w(-x,t)$, when $x < 0$, then we know that the Fourier transform of w in the space variable must be as follows:

$$\begin{aligned}
\widehat{w}(\xi,t) &= \int_{-\infty}^{\infty} w(x,t)e^{-2\pi ix\xi}dx = \int_{-\infty}^{0} w(x,t)e^{-2\pi ix\xi}dx + \int_0^{\infty} w(x,t)e^{-2\pi ix\xi}dx \\
&= -\int_0^{\infty} w(-x,t)e^{2\pi ix\xi}dx + \int_0^{\infty} w(x,t)e^{-2\pi ix\xi}dx \\
&= \int_0^{\infty} w(x,t)(e^{-2\pi ix\xi} - e^{2\pi ix\xi})dx = -2i\int_0^{\infty} w(x,t)\sin(2\pi x\xi)dx \\
&= -2iw^S(\xi,t)
\end{aligned}$$

Therefore by the Inversion Theorem in Section (2.1.3)

$$w(x,t) = \int_{-\infty}^{\infty}\widehat{w}(\xi,t)e^{2\pi ix\xi}d\xi$$

and hence

$$w(x,t) = -4\pi i\int_{-\infty}^{\infty} e^{2\pi ix\xi}\xi e^{-4\pi^2\xi^2 t}\int_0^t e^{4\pi^2\xi^2\sigma}g(\sigma)d\sigma d\xi$$

To simplify this, we replace $2\pi\xi$ by ξ and get

$$w(x,t) = \frac{-i}{\pi}\int_{-\infty}^{\infty}\xi e^{ix\xi}\int_0^t e^{-\xi^2(t-\sigma)}g(\sigma)d\sigma d\xi$$

Example 4 The Helmholtz Equation is

$$\Delta u - gu = f$$

in which Δ is the Laplacian $\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$. The functions f and g are prescribed on $\mathbb{R}^n$, and u is the unknown function of n variables. We shall look at the special case when $g = 1$. To illustrate some variety in approaching such problems, let us simply try the hypothesis that the problem can be solved with an appropriate convolution: $u = f * h$. Substitution of this form for u in the differential equation leads to

$$\Delta(f * h) - f * h = f$$

Carrying out the differentiation under the integral that defines the convolution, we obtain

$$f * \Delta h - f * h = f$$

Is there a way to cancel the three occurrences of f in this equation? After all, $L^1(\mathbb{R}^n)$ is a Banach algebra, with multiplication defined by convolution. But there is no unit element and therefore there are no inverses. However, the Fourier transform converts the convolution into ordinary products, according to Theorem 2 in Section 2.1.1:

$$\widehat{f}\,\widehat{\Delta h} - \widehat{f}\,\widehat{h} = \widehat{f}$$

From this equation we cancel the factor $\widehat{f}$, and then express $\widehat{\Delta h}$ as in Example 1 in Section 2.1.2:

$$\begin{aligned} -4\pi^2|\xi|^2\widehat{h}(\xi) - \widehat{h}(\xi) &= 1 \\ \widehat{h}(\xi) &= \frac{-1}{1+4\pi^2|\xi|^2} \end{aligned}$$

The formula for h itself is obtained by use if the inverse Fourier transform, which leads to

$$h(x) = \pi^{n/2} \int_0^\infty t^{-n/2} e^{-t-\pi^2\frac{|x|^2}{t}}\,dt$$

Chapter 3

The Malgrange-Ehrenpreis Theorem

Malgrange and Ehrenpreis proved that every linear differential operator with constant coefficients has a fundamental solution. Let

$$L = \sum_{|\alpha| \le k} c_\alpha D^\alpha$$

When the dimension of the space is $n = 1$, we write

$$L = D^k + c_{k-1} D^{k-1} + \cdots + c_1 D + c_0$$

We will show that for every $f \in C_c^\infty(\mathbb{R})$ the equation

$$Lu = D^k u + c_{k-1} D^{k-1} u + \cdots + c_1 Du + c_0 u = f \tag{3.1}$$

has a solution $u \in C^\infty(\mathbb{R})$.

The natural tool for studying such operators is the Fourier transform:

$$\widehat{Lu}(\xi) = (2\pi i \xi)^k \widehat{u}(\xi) + c_{k-1}(2\pi i \xi)^{k-1} \widehat{u}(\xi) + \cdots + c_0 \widehat{u}(\xi) = P(\xi)\widehat{u}(\xi) \tag{3.2}$$

where,

$$P(\xi) = \sum_{|\alpha| \le k} c_\alpha (2\pi i \xi)^\alpha, \qquad c_k = 1$$

The polynomial P is called the symbol of L. In view of (3.2), if $f \in C_c^\infty(\mathbb{R})$, it would seem that we should be able to solve $Lu = f$ by taking

$$\widehat{u}(\xi) = \frac{\widehat{f}(\xi)}{P(\xi)}$$

That is,

$$u(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} \frac{\widehat{f}(\xi)}{P(\xi)} d\xi \tag{3.3}$$

The problem with this is that usually the polynomial P will have zeros, so that $\frac{\widehat{f}(\xi)}{P(\xi)}$ is not a locally integrable function and the integral (3.3) is not well-defined.

Let $\lambda_1, \dots, \lambda_k$ be the complex roots of $P(\xi)$, where

$$P(\xi) = (2\pi i)^k (\xi - \lambda_1) \cdots (\xi - \lambda_k)$$

We consider the $k+1$ intervals

$$[-k-1, -k+1), [-k+1, -k+3), \dots, [k-3, k-1), [k-1, k+1)$$

and the k real numbers $Im\lambda_1, \dots, Im\lambda_k$. Then at least one of the above intervals does not contain any of these numbers. If m is the center of such an interval, i.e. one of the numbers $-k, -k+2, \dots, k-2, k$, then,

$$|m - Im\lambda_j| \geq 1, \qquad j = 1, \dots, k$$

Lemma 1 Let $g(z)$ be a monic polynomial of degree k in the complex variable z such that $g(0) \neq 0$, and let $\lambda_1, \dots, \lambda_k$ be its zeros. Then

$$|g(0)| \geq (d/2)^k$$

where $d = \min |\lambda_j|$.

Proof We have $g(z) = (z - \lambda_1) \cdots (z - \lambda_k)$ and $g(0) = (-1)^k \lambda_1 \cdots \lambda_k$. So,

$$\left| \frac{g(z)}{g(0)} \right| = \left| 1 - \frac{z}{\lambda_1} \right| \cdots \left| 1 - \frac{z}{\lambda_k} \right|$$

When $|z| \leq d$ we get $\frac{|z|}{|\lambda_j|} \leq \frac{d}{d} = 1$ and consequently

$$\left| \frac{g(z)}{g(0)} \right| \leq 2^k$$

By the Cauchy integral formula,

$$k! = g^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=d} \frac{g(z)}{z^{k+1}} dz$$

Hence,

$$k! = \left| g^{(k)}(0) \right| =\leq \frac{k!}{2\pi} \max_{|z|=d} \frac{|g(z)|}{|z|^{k+1}} 2\pi d = \frac{k!}{d^k} \max_{|z|=d} |g(z)| \leq \frac{k!}{d^k} 2^k \, |g(0)|$$

Finally, $\left(\frac{d}{2}\right)^k \leq |g(0)|$.

We continue by taking the Fourier Transform of f:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)dx$$

where $\xi \in \mathbb{R}$.

Now we suppose that $\xi \in \mathbb{C}$ and $\xi = \eta + i\zeta$. Then, we have

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x(\eta+i\zeta)} f(x)dx = \int_{\mathbb{R}} e^{-2\pi i x\eta} e^{2\pi x\zeta} f(x)dx \tag{3.4}$$

We have to prove that the function in the last integral is integrable. Due to the fact that f has compact support, there is $M > 0$ such that the function f is equal to zero outside of the interval $[-M, M]$. Then:

$$\int_{\mathbb{R}} \left|e^{-2\pi i x(\eta+i\zeta)} f(x)\right| dx = \int_{-M}^{M} \left|e^{-2\pi i x(\eta+i\zeta)} f(x)\right| dx \leq e^{2\pi M|\zeta|} \int_{-M}^{M} |f(x)|\, dx < +\infty$$

Therefore, $e^{-2\pi i x(\eta+i\zeta)} f(x)$ is integrable in $\mathbb{R}$ and $\widehat{f}(\xi)$ is well-defined by (3.4).

The function $\widehat{f}(\xi)$ defined by (3.4) can be easily proven to be an analytic function in $\mathbb{C}$, because $e^{-2\pi i x\xi}$ is an analytic function of ξ. Consequently, the function

$$e^{2\pi i x\xi} \frac{\widehat{f}(\xi)}{P(\xi)} = e^{2\pi i x(\eta+i\zeta)} \frac{\widehat{f}(\eta+i\zeta)}{P(\eta+i\zeta)}$$

is an analytic function in $\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_k\}$.

Now, for every $j \in \mathbb{N}$, we have:

$$(-2\pi i\xi)^j \widehat{f}(\xi) = \int_{\mathbb{R}} D^j(e^{-2\pi i\xi x}) f(x)dx = (-1)^j \int_{\mathbb{R}} e^{-2\pi i\xi x} D^j f(x)dx$$

and hence

$$\begin{aligned}(2\pi|\xi|)^j |\widehat{f}(\xi)| &= \left|\int_{\mathbb{R}} e^{-2\pi i\xi x} D^j f(x)dx\right| \leq e^{2\pi M|\zeta|} \int_{-M}^{M} \left|D^j f(x)\right| dx \\ &\leq 2Me^{2\pi M|\zeta|} \left\|D^j f\right\|_\infty =: C(f, j, \zeta)\end{aligned}$$

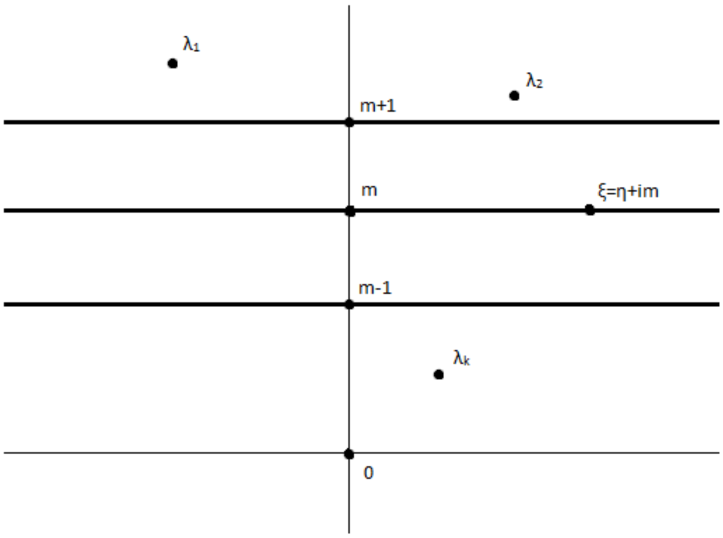
Finally,

$$\left|\widehat{f}(\xi)\right| \leq \frac{C^*(f, j, \zeta)}{|\xi|^j} \tag{3.5}$$

where the constant $C^*(f, j, \zeta) = C(f, j, \zeta)/(2\pi)^m$ depends on f, j and the imaginary part of ξ.

Now, the solution of $Lu = f$ will be defined by changing formula (3.3). We consider the horizontal straight line parallel to the x-axis at height m, parametrized

Figure 3.1



The horizontal zone with vertical width equal to 2 *and median line* $\xi = \eta + im, \ \ \eta \in \mathbb{R}$.

by $\xi = \eta + im$, $\eta \in \mathbb{R}$. With this as the median line, we create the horizontal zone with vertical width equal to 2. The roots $\lambda_1, \ldots, \lambda_k$ lie outside of this zone (figure 3.1). So, for every $\xi = \eta + im$, $\eta \in \mathbb{R}$,

$$g(z) := \frac{1}{(2\pi i)^k} P(z + \xi)$$

is a monic polynomial of z and degree k. The roots of this polynomial are $\lambda_1^{'} = \lambda_1 - \xi, \ldots, \lambda_k^{'} = \lambda_k - \xi$. In addition, $|\lambda_1^{'}|, \ldots, |\lambda_k^{'}| \geq 1$, because the distance between $\lambda_1, \ldots, \lambda_k$ and ξ is not less than 1. Thus,

$$d = \min_{1 \leq j \leq k} |\lambda_j^{'}| \geq 1$$

Using Lemma 1 for the polynomial $g(z)$, we get:

$$|g(0)| \geq (d/2)^k \geq 1/2^k$$

Then,

$$|P(\xi)| \geq \pi^k \tag{3.6}$$

for every $\xi = \eta + im$ at the horizontal line.

Now we give the formula for the solution u of $Lu = f$. We define:

$$u(x) := \int_{\mathbb{R}} e^{2\pi i x(\eta + im)} \frac{\widehat{f}(\eta + im)}{P(\eta + im)} d\eta = \int_{Im\xi = m} e^{2\pi i x \xi} \frac{\widehat{f}(\xi)}{P(\xi)} d\xi \tag{3.7}$$

From (3.5) for $j = 2$, and from (3.6), we get:

$$\left| e^{2\pi ix(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)} \right| = e^{-2\pi xm} \left| \frac{\widehat{f}(\eta+im)}{P(\eta+im)} \right| \leq e^{-2\pi xm} \frac{C^*(f,2,m)}{\pi^k |\eta+im|^2} = \frac{C^{**}}{\eta^2+m^2}$$

where C^{**} is a constant independent of η. Thus,

$$e^{2\pi x(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)}$$

as a function of η is integrable on $\mathbb{R}$ and u is well-defined by (3.7).

Also, for every $j \in \mathbb{N}$:

$$\left| (2\pi i)^j (\eta+im)^j e^{2\pi ix(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)} \right| = (2\pi)^j e^{-2\pi xm} \frac{|\eta+im|^j |\widehat{f}(\eta+im)|}{|P(\eta+im)|}$$

From (3.5) for $j + 2$ instead of j, and from (3.6), we get:

$$(2\pi)^j e^{-2\pi xm} \frac{|\eta+im|^j |\widehat{f}(\eta+im)|}{|P(\eta+im)|} \leq (2\pi)^j e^{-2\pi xm} \frac{C^*(f,j+2,m)}{\pi^k |\eta+im|^2} = \frac{C^{**}}{\eta^2+m^2}$$

where C^{**} is a constant independent of η. Therefore,

$$(2\pi i)^j (\eta+im)^j e^{2\pi ix(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)}$$

is a function of η integrable on $\mathbb{R}$. Hence the j-th degree derivative of u exists:

$$\begin{aligned} D^j u(x) &= \int_{\mathbb{R}} D^j (e^{2\pi ix(\eta+im)}) \frac{\widehat{f}(\eta+im)}{P(\eta+im)} d\eta \\ &= \int_{\mathbb{R}} (2\pi i)^j (\eta+im)^j e^{2\pi ix(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)} d\eta \end{aligned}$$

We sum for $j = 0, 1, \ldots, k$:

$$\begin{aligned} Lu(x) &= \sum_{j=0}^{k} c_j D^j u(x) \\ &= \int_{\mathbb{R}} \left(\sum_{j=0}^{k} c_j (2\pi i)^j (\eta+im)^j \right) e^{2\pi ix(\eta+im)} \frac{\widehat{f}(\eta+im)}{P(\eta+im)} d\eta \end{aligned}$$

But $P(\xi) = \sum_{j=0}^{k} c_j (2\pi i)^j \xi^j$, so

$$Lu(x) = \int_{\mathbb{R}} e^{2\pi ix(\eta+im)} \widehat{f}(\eta+im) d\eta$$

Since $\widehat{f}$ satisfies (3.5) for $j = 2$ and is an analytic function in $\mathbb{C}$, we may use Lemma 2 in Section 2.2 and transfer the previous integral from the straight line at height m and parametrization $\xi = \eta + im$, to the line at height 0 and parametrization $\xi = \eta$, without changing the integral. Thus,

$$Lu(x) = \int_{\mathbb{R}} e^{2\pi i x\eta}\widehat{f}(\eta)d\eta = f(x)$$

We conclude that

$$u(x) = \int_{Im\xi=m} e^{2\pi i x\xi}\frac{\widehat{f}(\xi)}{P(\xi)}d\xi$$

is the solution of (3.1).

All the previous constitute the proof of the next theorem in the case $n = 1$:

Theorem If L is a differential operator with constant coefficients on $\mathbb{R}^n$ and $f \in C_c^\infty(\mathbb{R}^n)$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $Lu = f$.

Now we prove

The Malgrange-Ehrenpreis Theorem Every differential operator L with constant coefficients has a fundamental solution.

Proof Again in the case of dimension $n = 1$ and with notation as above, we define a linear functional K on $C_c^\infty(\mathbb{R}^n)$ by

$$K(f) := \int_{\mathbb{R}} \frac{\widehat{f}(-\eta - im)}{P(\eta + im)}d\eta = \int_{Im\xi=m} \frac{\widehat{f}(-\xi)}{P(\xi)}d\xi \tag{3.8}$$

Looking carefully at the constant $C^*(f, j, \zeta)$ of equation (3.5) for $j = 2$:

$$|\widehat{f}(-\xi)| \leq \frac{2Me^{2\pi M|m|}\,\|D^2 f\|_\infty}{4\pi^2|\xi|^2}$$

Combining (3.6) and (3.8) we have:

$$|K(f)| \leq \int_{\mathbb{R}} \frac{2Me^{2\pi M|m|}\,\|D^2 f\|_\infty}{4\pi^{k+2}|\eta + im|^2}d\eta = \int_{\mathbb{R}} \frac{1}{\eta^2 + m^2}d\eta\,\frac{Me^{2\pi M|m|}\|D^2 f\|_\infty}{2\pi^{k+2}} = C\|D^2 f\|_\infty$$

This implies that K is distribution.

Now from the definition of distributions :

$$\begin{aligned}
LK(f) &= \sum_{j=0}^{k} c_j D^j K(f) = \sum_{j=0}^{k} c_j(-1)^j K(D^j f) = \sum_{j=0}^{k} c_j(-1)^j \int_{Im\xi=m} \frac{\widehat{D^j f}(-\xi)}{P(\xi)}d\xi \\
&= \sum_{j=0}^{k} c_j(-1)^j \int_{Im\xi=m} \frac{(-2\pi i\xi)^j\widehat{f}(-\xi)}{P(\xi)}d\xi = \int_{Im\xi=m} \left[\sum_{j=0}^{k} c_j(2\pi i\xi)^j\right]\frac{\widehat{f}(-\xi)}{P(\xi)}d\xi \\
&= \int_{Im\xi=m} \widehat{f}(-\xi)d\xi = \int_{\mathbb{R}} \widehat{f}(-\eta - im)d\eta
\end{aligned}$$

We transfer again the integral from the horizontal line at height $-m$, to the straight line at height 0 without changing the integral:

$$LK(f) = \int_{\mathbb{R}} \widehat{f}(-\eta - im)d\eta = \int_{\mathbb{R}} \widehat{f}(-\eta)d\eta = f(0) = \delta(f)$$

Hence $LK(f) = \delta(f)$ for every $f \in C_c^\infty(\mathbb{R})$ and thus $LK = \delta$.

In case of more than one variables, on $\mathbb{R}^n$ with $n \geq 2$, we suppose that the differential operator is:

$$L = \sum_{|\alpha| \leq k} c_\alpha D^\alpha = \frac{\partial^k}{\partial x_n^k} + \sum_{|\alpha| \leq k, \alpha_n \leq k-1} c_\alpha D^\alpha$$

Then the equation $Lu = f$ with $u \in C^\infty(\mathbb{R}^n)$, $f \in C_c^\infty(\mathbb{R}^n)$ is equivalent to:

$$(2\pi i \xi_n)^k \,\widehat{u}(\xi) + \sum_{|\alpha| \leq k, \alpha_n \leq k-1} c_\alpha \,(2\pi i \xi)^\alpha \,\widehat{u}(\xi) = \widehat{f}(\xi)$$

More specifically,

$$P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$$

where $P(\xi)$ is the n-variables polynomial:

$$P(\xi) = P(\xi_1, \ldots, \xi_n) = (2\pi i)^k \,\xi_n^k + \sum_{|\alpha| \leq k, \alpha_n \leq k-1} c_\alpha \,(2\pi i)^{|\alpha|} \,\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

The polynomial $P(\xi_1, \ldots, \xi_n)$, when $\xi_1, \ldots, \xi_{n-1}$ are constants, is a k-th degree polynomial of ξ_n with maximum degree coefficient $(2\pi i)^k$. In addition,

$$P(\xi_1, \ldots, \xi_n) = (2\pi i)^k \xi_n^k + \sum_{j=0}^{k-1} P_j(\xi_1, \ldots, \xi_{n-1})\xi_n^j$$

where for all $j = 0, 1, \ldots, k-1$ the $P_j(\xi_1, \ldots, \xi_{n-1})$ is a polynomial of $(\xi_1, \ldots, \xi_{n-1})$, with constant coefficients.

For each fixed $\xi' = (\xi_1, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, we consider $P(\xi_1, \ldots, \xi_n) = P(\xi', \xi_n)$ as a polynomial in the single complex variable ξ_n. Let $\lambda_1(\xi'), \ldots, \lambda_k(\xi')$ be the roots of this polynomial. Then there is $m \in \{-k, -k+2, \ldots, k-2, k\}$ such that

$$|m - Im\lambda_j(\xi')| \geq 1, \qquad j = 1, \ldots, k$$

where m depends on ξ', $m = m(\xi')$.

Lemma 2 If the coefficients of the polynomial $P_u(z) = \sum_{j=0}^k c_j(u) z^j$ are continuous functions of u, then its roots sre also continuous functions of u.

Proof Let z_0 be a root of $P_{u_0}(z)$ for a specific value u_0 of u. We take a small radius $\varepsilon > 0$ so that the only root of $P_{u_0}(z)$ in the disk $|z - z_0| \le \varepsilon$ is z_0. By the continuity of $c_m(u)$ at u_0, there is a $\delta > 0$ so that

$$|P_u(z) - P_{u_0}(z)| \le a$$

for $|z - z_0| = \varepsilon$ and $|u - u_0| \le \delta$, where $a = \min_{|z-z_0|=\varepsilon} |P_{u_0}(z)|$.
Therefore,

$$|P_u(z) - P_{u_0}(z)| \le |P_{u_0}(z)|$$

for $|z - z_0| = \varepsilon$ and $|u - u_0| \le \delta$.
We apply the theorem of Rouche and we get that $P_u(z)$ has the same number of roots in the disk $|z - z_0| \le \varepsilon$ as $P_{u_0}(z)$. If we apply this to every root of $P_{u_0}(z)$, we see that the roots of $P_u(z)$ are all in a small disk of radius ϵ around the roots of $P_{u_0}(z)$, if $|u - u_0| \le \delta$. Hence, the roots of $P_u(z)$ are continuous functions of u.

Lemma 3 There is a Borel measurable function $m : \mathbb{R}^{n-1} \to \{-k, \ldots, k\}$ such that for all $\xi' \in \mathbb{R}^{n-1}$:

$$|m(\xi') - Im\lambda_j(\xi')| \ge 1, \qquad j = 1, \ldots, k$$

Proof Let

$$V_m = \left\{\xi' \in \mathbb{R}^{n-1} \ : \ |m - Im\lambda_j(\xi')| \ge 1, \ \ j = 1, \ldots, k\right\}$$

Since the functions $Im\lambda_j(\xi')$ are continuous functions of ξ', each V_m is a closed set.
Also,

$$\mathbb{R}^n = V_{-k} \cup V_{-k+2} \cup \cdots \cup V_{k-2} \cup V_k$$

and we define

$$m(\xi') = \begin{cases} -k, & \xi' \in V_{-k} \\ -k+1, & \xi' \in V_{-k+1} \backslash V_{-k} \\ -k+2, & \xi' \in V_{-k+2} \backslash (V_{-k+1} \cup V_{-k}) \\ \vdots & \vdots \end{cases}$$

Now the function $m(\xi')$ has the properties we need.

Now, imitating the formula for solution of $Lu = f$ and for the fundamental solution K we found in the case $n = 1$, we give the corresponding formulas in the general case:

$$u(x) = \int_{\mathbb{R}^{n-1}} \left(\int_{Im\xi_n = m(\xi')} e^{2\pi i(x_1\xi_1 + \cdots + x_n\xi_n)} \frac{\widehat{f}(\xi_1, \ldots, \xi_n)}{P(\xi_1, \ldots, \xi_n)} d\xi_n \right) d\xi_1 \ldots d\xi_{n-1}$$

and

$$K(f) = \int_{\mathbb{R}^{n-1}} \left(\int_{Im\xi_n = m(\xi')} \frac{\widehat{f}(-\xi_1, \ldots, -\xi_n)}{P(\xi_1, \ldots, \xi_n)} d\xi_n \right) d\xi_1 \ldots d\xi_{n-1}$$

Based on the last two formulas, the proof in the case $n = 1$ for $Lu = f$ and $LK = \delta$ can be repeated for the case of general n.

Example 1 We consider the wave equation in dimension 2:

$$u_{tt} - u_{xx} = f(t, x)$$

The corresponding differential operator is:

$$L = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$$

and the corresponding polynomial is:

$$P(\eta, \xi) = -4\pi^2(\eta^2 - \xi^2)$$

When ξ is constant, the polynomial in the η variable has the roots:

$$\eta = \pm\xi$$

To avoid these roots we will consider η-integrals on the line with equation $\eta + i$, $\eta \in \mathbb{R}$. Then the fundamental solution is:

$$\begin{aligned} K(f) &= -\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\widehat{f}(-\eta - i, -\xi)}{(\eta + i)^2 - \xi^2} d\eta d\xi \\ &= -\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{2\pi i((\eta+i)t+\xi x)}}{(\eta + i)^2 - \xi^2} f(t, x) dt dx \right) d\eta d\xi \\ &= -\frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{e^{2\pi i\eta t}}{(\eta + i)^2 - \xi^2} d\eta \right) e^{2\pi i\xi x} d\xi e^{-2\pi t} f(t, x) dt dx \end{aligned}$$

By the method of residues we find:

$$\int_{\mathbb{R}} \frac{e^{2\pi i\eta t}}{(\eta + i)^2 - \xi^2} d\eta = \begin{cases} 0, & t > 0 \\ \frac{2\pi}{\xi} e^{2\pi t} \sin(2\pi t\xi), & t < 0 \end{cases}$$

Therefore,

$$K(\phi) = -\frac{1}{2\pi} \int_{\mathbb{R}} \int_{-\infty}^{0} \left(\int_{\mathbb{R}} \frac{\sin(2\pi t\xi)}{\xi} e^{2\pi i\xi x} d\xi \right) f(t, x) dt dx$$

Again, by the method of residues we find when $t < 0$:

$$\int_{\mathbb{R}} \frac{\sin(2\pi t\xi)}{\xi} e^{2\pi i\xi x} d\xi = \begin{cases} 0, & |x| > -t \\ -\pi, & |x| < -t \end{cases}$$

Hence, the fundamental solution is given by:

$$K(f) = \frac{1}{2} \int_{-\infty}^{0} \int_{-t}^{t} f(t,x) dx dt, \qquad f \in C_c^\infty(\mathbb{R}^2)$$

Example 2 Now we take the heat equation in dimension 2:

$$u_t - u_{xx} = f(t,x)$$

The corresponding differential operator is:

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

and the corresponding polynomial is:

$$P(\eta, \xi) = -2\pi i \left(\eta + 2\pi i \xi^2\right)$$

Now if ξ is constant, the polynomial in the η variable has one root:

$$\eta = -2\pi i \xi^2$$

with non-positive imaginary part. To avoid the root we will consider η integrals on the line with equation $\eta + 2\pi i$, $\eta \in \mathbb{R}$. The fundamental solution is:

$$\begin{aligned} K(f) &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\widehat{f}(-\eta - 2\pi i, -\xi)}{\eta + 2\pi(1+\xi^2)i} d\eta d\xi \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{2\pi i((\eta+2\pi i)t + \xi x)}}{\eta + 2\pi(1+\xi^2)i} f(t,x) dt dx \right) d\eta d\xi \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{e^{2\pi i\eta t}}{\eta + 2\pi(1+\xi^2)i} d\eta \right) e^{2\pi i\xi x} d\xi e^{-4\pi^2 t} f(t,x) dt dx \end{aligned}$$

By the method of residues we find:

$$\int_{\mathbb{R}} \frac{e^{2\pi i\eta t}}{\eta + 2\pi(1+\xi^2)i} d\eta = \begin{cases} 0, & t > 0 \\ -2\pi i e^{4\pi^2 t(1+\xi^2)}, & t < 0 \end{cases}$$

Thus,

$$K(f) = \int_{-\infty}^{0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{4\pi^2 t\xi^2} e^{2\pi i \xi x} d\xi \right) f(t,x) dx dt$$

Now we have for $t < 0$:

$$\int_{\mathbb{R}} e^{4\pi^2 t\xi^2} e^{2\pi i \xi x} d\xi = \frac{1}{2\sqrt{\pi |t|}} e^{\frac{x^2}{4t}}$$

Therefore, the fundamental solution is:

$$K(f) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{0} \int_{\mathbb{R}} \frac{1}{\sqrt{|t|}} e^{\frac{x^2}{4t}} f(t,x) dx dt, \qquad f \in C_c^{\infty}(\mathbb{R}^2)$$