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Master's thesis

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Hausdorff Dimension and Energy of Measures

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To my parents Eli and Sofoklis and my sister Elena

Contents

1 Prologue

J.Marstrand proved, in 1954 that:
If $E \subseteq \mathbb{R}^2$ is a Borel set with Hausdorff dimension $s \leq 1$, then the Hausdorff dimension of the projection of E on almost every line L through the origin has again Hausdorff dimension s, otherwise if $s > 1$ then E projects into a set of positive length in almost all directions.

Marstrand's original proof was based on the definition and basic properties of Hausdorff measures and depends heavily on delicate and, in places, complicated geometric and measure theoretic arguments.

In this thesis,we give R.Kauffman's proof(1968) of the theorem in $\mathbb{R}^d$. In his proof, he makes natural use of the potential theoretic characterization of Hausdorff dimension and Fourier transform methods.

Here we shall prove the theorem only for the projections of compact subsets of $\mathbb{R}^d$.

For the general case of projections of Borel sets, we need a generalization of **Frostman's Lemma** ,the proof of which,requires theory of Polish spaces, Souslin sets and capacities and its beyond the scope of this thesis. After the proof of Frostman's Lemma for **Borel** sets ,the general case of the main theorem is proved with very similar(almost the same)arguments used for the case of compact sets.

A proof of the general case of the Lemma can be found in **[5]**,**[7]**,**[8]**.

Marstrand's Theorem is also valid for projections of Borel sets on hyperplanes in $\mathbb{R}^d$ but we'll not prove that here. A proof of this, though, can be found in **[6]**.

2 Some notions from measure theory and weak convergence of measures

We denote $B(x,r)$ the open ball with center x and radius $r > 0$, and $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$ the distance between A and B where A, B are non-empty sets in a metric space (X,d).

Definition 2.1 (Metric outer measure). *Let μ^* be an outer measure on metric space (X,d). We say that μ^* is a* ***metric outer measure*** *if*

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

for every non-empty $A, B \subseteq X$ with $d(A,B) > 0$

If μ^* is an outer measure on X,we denote $\mathcal{A}_{\mu^*}$ the sigma algebra of μ^*- measurable subsets of X

Proposition 2.1 Let (X,d) be a metric space and μ^* an outer measure on X. Then, the measure μ which is induced by μ^* on $(X, \mathcal{A}_{\mu^*})$ is a Borel measure(i.e. all Borel sets in X are μ^*-measurable) if and only if μ^* is a metric outer measure.

Proof Suppose that all Borel sets in X are μ^*-measurable and take arbitrary non-empty $A, B \subseteq X$ with $d(A,B) > 0$. We consider $r = d(A,B)$ and the open set $U = \bigcup_{x \in A} B(x,r)$. It is clear that $A \subseteq U$ and $B \cap U = \emptyset$. Since U is μ^*-measurable we have

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap U) + \mu^*((A \cup B) \cap U^c) = \mu^*(A) + \mu^*(B)$$

Therefore, μ^* is a metric outer measure on X.

Now let μ^* be a metric outer measure and consider an open $U \subseteq X$. If A is a non-empty subset of U, we define

$$A_n = \{x \in A : d(x,y) \geq \frac{1}{n}, \forall y \notin U\}$$

It is obvious that $A_n \subseteq A_{n+1}$ for all n. If $x \in A \subseteq U$, there is $r > 0$ so that $B(x,r) \subseteq U$ and if we take $n \in \mathbb{N}$ so that $\frac{1}{n} \leq r$ then $x \in A_n$. Therefore $A = \bigcup_{n=1}^{\infty} A_n$. We define, now, $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$

and have that the sets $B_1, B_2, ...$ are pairwise disjoint and that $A = \bigcup_{n=1}^{\infty} B_n$. If $x \in A_n$ and $z \in B_{n+2}$, then $z \notin A_{n+1}$ and there is some $y \notin U$ so that $d(y,z) < \frac{1}{n+1}$. Then $d(x,z) \geq d(x,y) - d(y,z) > \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ Therefore, $d(A_n, B_{n+2}) \geq \frac{1}{n(n+1)}$ for every n. Since $A_{n+2} \supset A_n \cup B_{n+2}$, we find

$$\mu^*(A_{n+2}) \geq \mu^*(A_n \cup B_{n+2}) = \mu^*(A_n) + \mu^*(B_{n+2})$$

By induction we get

$$\mu^*(B_1) + \mu^*(B_3) + ... + \mu^*(B_{2n+1}) \leq \mu^*(A_{2n+1})$$

and

$$\mu^*(B_2) + \mu^*(B_4) + ... + \mu^*(B_{2n}) \leq \mu^*(A_{2n})$$

for all n. If at least one of the series $\mu^*(B_1)+\mu^*(B_3)+...$ and $\mu^*(B_2)+\mu^*(B_4)+$ ... diverges to $+\infty$, then either $\mu^*(A_{2n+1}) \to +\infty$ or $\mu^*(A_{2n}) \to +\infty$. Since the sequence $(\mu^*(A_n))$ is increasing, we get that in both cases it diverges to $+\infty$. Since, also $\mu^*(A_n) \leq \mu^*(A)$ for all n, we get that $\mu^*(A_n) \to \mu^*(A)$ If both series $\mu^*(B_1) + \mu^*(B_3) + ...$ and $\mu^*(B_2) + \mu^*(B_4) + ...$ converge, for every $\epsilon > 0$ there is n so that $\sum_{k=n+1}^{\infty} \mu^*(B_k) < \epsilon$. Now,

$$\mu^*(A) \leq \mu^*(A_n) + \sum_{k=n+1}^{\infty} \mu^*(B_k) < \mu^*(A_n) + \epsilon$$

This implies that $\mu^*(A_n) \to \mu^*(A)$. Therefore, in any case,$\mu^*(A_n) \to \mu^*(A)$

We consider an arbitrary $A \subseteq X$ and we take $C = A \cap U$. Since $A \cup U^c \subseteq U^c$,we have that $d(C_n, A \cap U^c) > 0$ for all n and hence,

$$\mu^*(A) \geq \mu^*(C_n \cup (A \cap U^c)) = \mu^*(C_n) + \mu^*(A \cap U^c)$$

for all n. Taking the limit as $n \to +\infty$ we find

$$\mu^*(A) \geq m^*(A \cap U) + \mu^*(A \cap U^c)$$

We conclude that every U open in X is μ^*- measurable and hence, every Borel set in X is μ^*-measurable. ■

Definition 2.2 (Support of a measure)**.** *Let μ be a Borel measure on a separable metric space (X,d). The support of μ is the set*

$$supp(\mu) := \{x \in X : \mu(B(x,r)) > 0, \forall r > 0\}$$

The support of μ is evidently closed: if $\{x_n\}_{n\in\mathbb{N}} \subseteq supp(\mu)$ is a sequence converging to some point $x \in X$, and $r > 0$, then $B(x,r)$ contains $B(x_m, r/2)$ for some $m \in \mathbb{N}$ large enough, and hence $\mu(B(x,r)) > 0$. This means that $x \in supp(\mu)$. Another common definition of $supp(\mu)$ is the following: $supp(\mu)$ is the smallest closest set F such that $\mu(X \setminus F) = 0$. These definitions agree on all separable metric spaces.

Definition 2.3 (Push-forward)**.** *Let μ be a measure on a space X, and let $f : X \to Y$ be a map, where Y is another arbitrary space. We define the push-forward of μ under f as the measure $f(\mu)$ defined by*

$$f(\mu)(A) := \mu(f^{-1}(A)),\ \ A \subseteq Y$$

Lemma 2.1. *Assume that $(X,d),(Y,s)$ are separable metric spaces, $f : X \to Y$ is continuous, and μ is a measure on X with compact support. Then*

$$supp(f(\mu)) = f(supp(\mu))$$

and in particular $supp(f(\mu))$ is compact. Moreover, if $g : Y \to [0,+\infty]$ is a non-negative Borel function, and μ is a Borel measure, then

$$\int_Y gd(f(\mu)) = \int_X (g \circ f)d\mu$$

Definition 2.4 (Weak convergence of measures)**.** *Let $\{\mu_j\}_{j\in\mathbb{N}}$ and μ be $locally finite$(gives finite measure to all compact sets) Borel measures on a metric space (X,d). We say that the measures μ_j converge weakly to μ if*

$$\int gd\mu_j \to \int gd\mu$$

for all $g \in C_c(X)$, i.e. the space of all continuous functions $g : X \to \mathbb{C}$ with compact support. In this case, we write $\mu_j \rightharpoonup \mu$.

We also denote $C_0(\mathbb{R}^d)$ the space of continuous functions which vanish at infinity.

Theorem 2.2. *Let $\{\mu_j\}_{j\in\mathbb{N}}$ be a sequence of Borel measures on $\mathbb{R}^d$ satisfying*

$$\sup_{j\in\mathbb{N}} \mu_j(K) < +\infty$$

for all compact subsets $K \subseteq \mathbb{R}^d$.
Then, there exists a locally finite Borel measure μ, and a subsequence $\{\mu_{j_i}\}_{i\in\mathbb{N}}$, such that

$$\mu_{j_i} \rightharpoonup \mu \text{ as } i \to +\infty$$

Proof Let $||.||_\infty$ be the sup-norm in the space $C_c(\mathbb{R}^d)$. We will use the fact that $(C_c(\mathbb{R}^d), ||.||_\infty)$ is separable, that is, there exists a countable dense subset $\{g_k\}_{k\in\mathbb{N}} \subseteq C_c(\mathbb{R}^d)$
The sequence can also be chosen so that any function $g \in C_c(\mathbb{R}^d)$ supported in $B(0, M)$ can be approximated by functions g_k supported in $B(0, 2M)$, for $M \in \mathbb{N}$.

Then, for each $k \in \mathbb{N}$, choose a subsequence $\{j_i^k\}_{i\in\mathbb{N}}$ such that

$$a_k := \lim_{i\to\infty} \int_{\mathbb{R}^d} g_k d\mu_{j_i}^k$$

Such subsequences exist, because the set of real numbers

$$\left\{ \int_{\mathbb{R}^d} g_k d\mu_j : j \in \mathbb{N} \right\}$$

is bounded by hypothesis and recalling that g_k has compact support.

. Moreover, we may always take $\{j_i^{k+1}\}_{i\in\mathbb{N}}$ to be a subsequence of $\{j_i^k\}_{i\in\mathbb{N}}$ for any $k \in \mathbb{N}$, just by picking the sequences one at a time. Then, the sequence $\{j_m^m\}_{m\in\mathbb{N}}$ is an eventual subsequence of every sequence $\{j_i^k\}_{k\in\mathbb{N}}$: more precisely $j_m^m \in \{j_i^k\}_{i\in\mathbb{N}}$ for all $m \geq k$, because then $j_m^m \in \{j_i^m\}_{i\in\mathbb{N}} \subseteq \{j_i^k\}_{i\in\mathbb{N}}$.

It follows that

$$(1)\ a_k = \lim_{m\to\infty} \int_{\mathbb{R}^d} g_k d\mu_{j_m^m},\ k \in \mathbb{N}$$

Now, it follows from the density of the sequence $\{g_k\}_{k\in\mathbb{N}}$ that actually the limit

$$(2) \lim_{m\to\infty} \int_{\mathbb{R}^d} g d\mu_{j_m^m} =: T(g)$$

exists for every $g \in C_c(\mathbb{R}^d)$.
To see this, pick $g \in C_c(\mathbb{R}^d)$ with support in $B(0, M)$, and let $g_{k_1}, g_{k_2}, ...$ be

a subsequence of $\{g_k\}_{k\in\mathbb{N}}$ with $supp(g_{k_l}) \subseteq B(0, 2M)$ and $||g - g_{k_l}||_\infty \to 0$ as $l \to +\infty$. Then for any $l, L \in \mathbb{N}$,

$$|a_{k_l} - a_{k_L}| \leq \limsup_{m\to\infty} \left(\left| a_{k_l} - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| + \left| a_{k_L} - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| \right)$$

$$= \limsup_{m\to\infty} \left(\left| \int_{\mathbb{R}^d} g_{k_l} d\mu_{j_m^m} - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| + \left| \int_{\mathbb{R}^d} g_{k_L} d\mu_{j_m^m} - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| \right)$$

$$\leq \sup_{m\geqslant 0} \mu_{j_m^m}(B(0, 2M))(||g_{k_l} - g||_\infty + ||g_{k_L} - g||_\infty)$$

which shows that $\{a_{k_l}\}_{l\in\mathbb{N}}$ is a Cauchy sequence with a limit $b \in \mathbb{R}$. Finally, fix $\epsilon > 0$ and pick $k_l \in \mathbb{N}$ so large that $|b - a_{k_l}| < \epsilon$. Then, using the triangle inequality,(1) and hypothesis we obtain

$$\limsup_{m\to\infty} \left| b - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| \leq \limsup_{m\to\infty} \left| \int_{\mathbb{R}^d} g_{k_l} d\mu_{j_m^m} - \int_{\mathbb{R}^d} g d\mu_{j_m^m} \right| + \epsilon$$

$$\leq \limsup_{m\to\infty} \mu_{j_m^m}(B(0, 2M))||g_{k_l} - g||_\infty + \epsilon$$

$$\leq \sup_{m\geqslant 0} \mu_{j_m^m}(B(0, 2M))||g_{k_l} - g||_\infty + \epsilon$$

Letting $l \to +\infty$ and $\epsilon \to 0$ proves that the left hand side of (2) exists, and $T(g) = b$.

The operator $g \mapsto T(g)$ is clearly positive and linear: $T(g) \geq 0$ if $g \geq 0$, and $T(cg_1 + dg_2) = cT(g_1) + dT(g_2)$. The Riesz representation theorem (see [1]) now states that the functional is given by a positive Borel measure μ :

$$T(g) = \int_{\mathbb{R}^d} g d\mu, \; g \in C_c(\mathbb{R}^d)$$

The weak convergence $\mu_{j_m^m} \rightharpoonup \mu$ follows immediately from (2). ∎

Lemma 2.3. *Assume that $\{\mu_j\}_{j\in\mathbb{N}}$ is a sequence of locally finite Borel measures in a locally compact metric space (X, d) converging weakly to a locally finite Borel measure μ. If $K \subseteq X$ is compact, and $U \subseteq X$ is open and $\sigma-$compact, then the following inequalities hold:*

$$\mu(K) \geq \limsup_{j\to\infty} \mu_j(K)$$

$$\mu(U) \leq \liminf_{j\to\infty} \mu_j(U)$$

Proof We start with the first inequality. Let $\epsilon > 0$. Since K is compact and X is locally compact, there exists an open set $V \supset K$ with compact closure, and hence $\mu(V) < \infty$.
Then, it follows that

$$\mu(\{x : dist(x, K) < \frac{1}{j}\}) \to \mu(K) \text{ as } j \to \infty$$

(the $(\frac{1}{j})-$ neighbourhoods are contained in V for $j \in \mathbb{N}$ large enough), and consequently (using the continuity of the function $dist(x, K)$) we may find an open set $U \supset K$ with $\mu(K) \geq \mu(U) - \epsilon$.
Then, pick the function :

$$f(x) = \frac{dist(x, U^c)}{dist(x, K) + dist(x, U^c)}$$

which satisfies $1_K \leq f \leq 1_U$ and $f \in C_c(X)$.

By definition of weak convergence, we then have

$$\mu(K) \geq \mu(U) - \epsilon \geq \int f d\mu - \epsilon = \lim_{j\to\infty} \int f d\mu_j - \epsilon \geq \limsup_{j\to\infty} \mu_j(K) - \epsilon$$

Letting $\epsilon \to 0$,we obtain the first inequality.

Now let U be a $\sigma-$compact open set and $\epsilon > 0$.
Then we have that $U = \bigcup_{n=1}^{\infty} K_n$ where K_n are compact sets $\forall n \in \mathbb{N}$.
Setting $A_n = \bigcup_{m=1}^{n} K_m$,we have that $\mu(U) = \lim_{n\to\infty} \mu(A_n)$ thus $\exists n_0 \in \mathbb{N}$ such that $\mu(U) \leq \mu(A_{n_0}) + \epsilon$
The set $F = A_{n_0}$ is a compact subset of U, so as before we pick a function $f \in C_c(X)$ satisfying:

$$1_F \leq f \leq 1_U$$

We have:

$$\mu(U) \leq \mu(F) + \epsilon \leq \int f d\mu + \epsilon = \lim_{j\to\infty} \int f d\mu_j + \epsilon$$

$$= \liminf_{j\to\infty} \int f d\mu_j + \epsilon \leq \liminf_{j\to\infty} \mu_j(U) + \epsilon$$

Letting $\epsilon \to 0$ we obtain the second inequality. ∎

3 Concepts and theorems from harmonic analysis

The Euclidean norm and the Euclidean inner product are given by

$$|x| = \sqrt{x_1^2 + ... + x_d^2}$$

$$x \cdot y = x_1 y_1 + ... + x_d y_d$$

when $x = (x_1, ..., x_d), y = (y_1, ..., y_d) \in \mathbb{R}^d$

If $x = (x_1, ..., x_d) \in \mathbb{R}^d$ and $a = (a_1, ..., a_d), \beta = (\beta_1, ..., \beta_d) \in \mathbb{N}_0^d$ we write:

$$x^a = x_1^{a_1} \cdots x_d^{a_d}$$

$$|a| = a_1 + ... + a_d$$

$$a + \beta = (a_1 + \beta_1, ..., a_d + \beta_d)$$

$$a! = a_1! a_2! \cdots a_d!$$

$$D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \cdots \partial x_d^{a_d}}$$

and we write $a \leq \beta$ if $a_j \leq \beta_j, \forall j = 1, ..., d$.

The Lebesgue measure of a Lebesgue measurable $E \subseteq \mathbb{R}^d$ is denoted $m_d(E)$ and the space of all finite complex Borel measures on $\mathbb{R}^d$ is denoted $M(\mathbb{R}^d)$ with the norm $||\mu|| = |\mu|(\mathbb{R}^d)$, where $|\mu|$ is the absolute variation of μ and we have that $L^1(\mathbb{R}^d)$ is contained in $M(\mathbb{R}^d)$ via the identification $f \to \mu, d\mu = f dx$

We also denote $L^1_{loc}(\mathbb{R}^d)$ the space of locally integrable functions.

We define the convolution of f, g in $\mathbb{R}^d$ as follows:

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) dy$$

and if $\mu \in M(\mathbb{R}^d)$

$$(f * \mu)(x) = \int_{\mathbb{R}^d} f(x - y) d\mu(y)$$

whenever the above integrals make sense.

We recall the following properties of convolution:

1. If $f \in L^1(\mathbb{R}^d), g \in L^p(\mathbb{R}^d), 1 \leq p \leq \infty$,then $f * g \in L^p(\mathbb{R}^d)$ and

$$||f * g||_p \leq ||f||_1 ||g||_p$$

2. If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $f * g$ is continuous and

$$||f * g||_\infty \leq ||f||_p ||g||_q$$

3. If $\phi \in C_c^\infty(\mathbb{R}^d)$ and $f \in L^1_{loc}(\mathbb{R}^d)$ then $\phi * f \in C^\infty(\mathbb{R}^d)$ and

$$D^a(\phi * f) = (D^a \phi) * f$$

We now denote S the space of Schwartz functions $S(\mathbb{R}^d)$ which contains all functions $\phi \in C^\infty(\mathbb{R}^d)$ which satisfy:

$$\sup_{x \in \mathbb{R}^d} |x^\beta D^a \phi(x)| < +\infty, \forall a, \beta \in \mathbb{N}_0^d$$

If $\phi \in$ S and $k \in \mathbb{N}_0$,we define the quantity

$$p_k(\phi) = \sup_{x \in \mathbb{R}^d, |a| \leq k} (1 + |x|^2)^{\frac{k}{2}} |D^a \phi(x)|$$

We note that $\phi \in \mathrm{S} \Longleftrightarrow p_k(\phi) < +\infty, \forall k \in \mathbb{N}_0$

Indeed if $\phi \in$ S and $k \in \mathbb{N}_0$ then we denote $c_{a,\beta} = \sup_{x \in \mathbb{R}^d} |x^\beta D^a \phi(x)|$,for $a, \beta \in \mathbb{N}_0^d$ which are finite quantities.

We have that

$$(1 + |x|^2)^{\frac{k}{2}} = (1 + x_1^2 + ... + x_d^2)^{\frac{k}{2}}$$
$$\leq (1 + |x_1| + ... + |x_d|)^k = \sum_{|\beta| \leq k} \rho_\beta |x^\beta|$$

where ρ_β are nonnegative constants.

So

$$(1 + |x|^2)^{\frac{k}{2}} |D^a \phi(x)| \leq \sum_{|\beta| \leq k} \rho_\beta |x^\beta D^a \phi(x)| \leq \sum_{|\beta| \leq k} \rho_\beta c_{a,\beta}$$

Thus

$$p_k(\phi) \leq \sum_{|\beta| \leq k} \rho_\beta \max_{|a|, |\beta| \leq k} c_{a,\beta} < +\infty$$

Conversely if $a, \beta \in \mathbb{N}_0^d$,then

$$|x^\beta D^a \phi(x)| \leq |x|^k |D^a \phi(x)|$$

$$\leq (1+|x|^2)^{\frac{k}{2}} |D^a \phi(x)| \leq p_k(\phi)$$

for $k = \max\{|a|, |\beta|\}$. Note that $|x^\beta| \leq |x|^{|\beta|} \leq |x|^k$

Thus $\sup_{x \in \mathbb{R}^d} |x^\beta D^a \phi(x)| \leq p_k(\phi) < +\infty$

It can be easily proved that S is a subspace of $L^p(\mathbb{R}^d), \forall p \in [1, \infty]$ and if $\phi, \psi \in$ S then it can be easily proved that $\phi + \psi, \phi\psi, \phi * \psi, \phi_t \in$ S,where $\phi_t(x) = \frac{1}{t^d}\phi(\frac{x}{t}), t > 0$

Definition 3.1 (Radial function)**.** *A functions f defined on $\mathbb{R}^d$ is* ***radial*** *if $f(x) = g(|x|)$, where $g : [0, +\infty) \to \mathbb{R}$.*
Equivalently a function f is radial if and only if $f \circ T = f, \forall T \in SO(d)$,i.e. the space of all orthogonal matrices.

Now, let $f \in L^1(\mathbb{R}^d)$,then its Fourier transform is $\hat{f} : \mathbb{R}^d \longrightarrow \mathbb{C}$ defined by:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx$$

and we have the Fourier inversion formula which says that, if $\hat{f} \in L^1(\mathbb{R}^d)$,then

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{f}(\xi) d\xi$$

for almost every $x \in \mathbb{R}^d$

Given $f \in L^1(\mathbb{R}^d)$,we define $f^\vee(x) = \hat{f}(-x)$ for all $x \in \mathbb{R}^d$. The operation

$$f \mapsto f^\vee$$

is called the **inverse Fourier Transform**.

Also if $\mu \in M(\mathbb{R}^d)$ then we define its Fourier transform:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x)$$

Let $f, g \in L^1(\mathbb{R}^d)$. We recall some properties of Fourier transform:
1. $\hat{f}$ is uniformly continuous
2. $||\hat{f}||_\infty \leq ||f||_1$
3. $\widehat{f+g} = \hat{f} + \hat{g}$
4. $\hat{\tilde{f}} = \tilde{\hat{f}}$, where $\tilde{f}(x) = f(-x)$
5. $\hat{\bar{f}} = \bar{\tilde{\hat{f}}}$
6. $\widehat{s_y(f)}(\xi) = e^{-2\pi i y\cdot\xi}\hat{f}(\xi)$ where $s_y(f)(x) = f(x-y)$
7. $(\widehat{e^{2\pi i y\cdot x}f(x)})(\xi) = s_y(\hat{f})(\xi)$
8. $\hat{f}_t(\xi) = \hat{f}(t\xi)$,where $f_t(x) = \frac{1}{t^d}f(\frac{x}{t}), t > 0$
9. $\widehat{(D^a f)}(\xi) = (2\pi i\xi)^a \hat{f}(\xi)$
10. $(D^a \hat{f})(\xi) = (\widehat{(-2\pi ix)^a f(x)})(\xi)$
11. $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$
12. $\hat{f}, f^\vee \in$ S, if $f \in$ S
13. If $f, g \in L^1(\mathbb{R}^d)$ and $\mu, \nu \in M(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} \hat{\mu} d\nu = \int_{\mathbb{R}^d} \hat{\nu} d\mu$$

$$\int_{\mathbb{R}^d} \hat{f}(x)g(x)dx = \int_{\mathbb{R}^d} f(x)\hat{g}(x)dx \text{ (Duality Relation)}$$

14. $\lim_{|\xi|\to+\infty} \hat{f}(\xi) = 0$
15. The Fourier transform maps S onto S
16. The Fourier transform of a radial function is radial.
17. $\hat{\hat{f}} = \tilde{f}$

We state without proof the following theorem:

Theorem 3.1 (Plancherel). *(i) If $f, g \in$ S then $\int_{\mathbb{R}^d} \hat{f}\bar{\hat{g}} = \int_{\mathbb{R}^d} f\bar{g}$*
(ii) There is a unique bounded operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that $\mathcal{F}(f) = \hat{f}$ when $f \in$ S and $\mathcal{F}$ has the properties:
1.$\mathcal{F}$ is a unitary operator
2.$\mathcal{F}(f) = \hat{f}$ if $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

Statement 2 allows us to use the notation $\hat{f}$ for $\mathcal{F}(f)$ if $f \in L^2(\mathbb{R}^d)$ without any possible ambiguity.

Corollary 3.1 The follwing form of the duality relation is valid:

$$\int_{\mathbb{R}^d} \hat{\nu}\psi = \int_{\mathbb{R}^d} \hat{\psi} d\nu, \psi \in S$$

if $\nu = \mu + f dx, \mu \in M(\mathbb{R}^d), f \in L^2(\mathbb{R}^d)$

Proof If $f = 0$ then we are done,considering the measure $\lambda = \psi dx$

If $\mu = 0$,then it suffices to show that

$$\int_{\mathbb{R}^d} \mathcal{F}(f)\psi = \int_{\mathbb{R}^d} f\hat{\psi}$$

.
For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we have from duality relation that

$$\int_{\mathbb{R}^d} \hat{f}\psi = \int_{\mathbb{R}^d} f\hat{\psi}, \psi \in \mathrm{S}$$

If $f \in L^2(\mathbb{R}^d)$ then $\exists\ f_k \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $||f_k - f||_2 \to 0$.
We have

$$\left| \int_{\mathbb{R}^d} \hat{f}_k\psi - \int_{\mathbb{R}^d} \mathcal{F}(f)\psi \right| \leq \int_{\mathbb{R}^d} |\mathcal{F}(f_k - f)||\psi|$$
$$\leq ||\mathcal{F}(f_k - f)||_2^2 ||\psi||_2^2 = ||f_k - f||_2^2 ||\psi||_2^2 \to 0$$

Also from Cauchy-Schwartz inequality again, we have

$$\int_{\mathbb{R}^d} f_k\hat{\psi} \to \int_{\mathbb{R}^d} f\hat{\psi}$$

Thus $\int_{\mathbb{R}^d} \mathcal{F}(f)\psi = \int_{\mathbb{R}^d} f\hat{\psi}$. ■

Theorem 3.2. *If $\mu \in M(\mathbb{R}^d), f \in L^2(\mathbb{R}^d)$ and*

$$\hat{f} + \hat{\mu} = 0$$

then $\mu = -f dx$.In other words if $\mu \in M(\mathbb{R}^d)$ and $\hat{\mu} \in L^2(\mathbb{R}^d)$ then μ is absolutely continuous with respect to the Lebesgue measure with an L^2 density.

Proof By the Riesz representation theorem for measures on compact sets, the measure $\mu + f dx$ will be zero provided

$$(1) \quad \int_{\mathbb{R}^d} \phi d\mu + \phi f dx = 0, \forall \phi \in C_c(\mathbb{R}^d)$$

If $\phi \in C_c^\infty(\mathbb{R}^d)$ then (1) follows from **Corollary 3.1** for $\psi = \phi^\vee$
In general,if $\phi \in C_c(\mathbb{R}^d)$ we choose a sequence $\phi_k \in C_c^\infty(\mathbb{R}^d)$ such that:
1.$\phi_k \to \phi$ uniformly
2.$||\phi_k - \phi||_2 \to 0$
and by passing to the limit we have (1).

Now let $\hat{\mu} \in L^2(\mathbb{R}^d)$.
By Plancherel's theorem we can choose $g \in L^2(\mathbb{R}^d)$ such that $\hat{g} = \hat{\mu}$
Then $d\mu - gdx$ has Fourier transform zero, so by the first part of the proof $d\mu = gdx$.■

Theorem 3.3. *Ket μ be a finite Borel measure with compact support . If $\hat{\mu} \in L^1(\mathbb{R}^d)$, then μ is a continuous function.*

Proof Assume that $supp(\mu) \subseteq B(0, R)$ for some $R > 0$.
Let $\psi \in C^\infty(\mathbb{R}^d)$ with the properties:

$$\int_{\mathbb{R}^d} \psi = 1$$
$$\psi \geq 0$$
$$supp(\psi) \subseteq B(0, 1)$$

Denote $\psi_t = \frac{1}{t^d}\psi(\frac{x}{t})$ and $\mu_t(x) = (\psi_t * \mu)(x)$.We will prove that $\mu_t \in$ S
Let $1 > t > 0$.As an easy application of dominated convergence theorem ,since $\psi \in$ S and μ is finite,the identity $D^a(\psi_t * \mu) = (D^a\psi_t * \mu)$ holds for all $a \in \mathbb{N}_0^d$.

Note that $supp(D^a\psi_t) \subseteq B(0, t)$.
Let $k \in \mathbb{N}_0$ and $x \in \mathbb{R}^d$ and $a \in \mathbb{N}_0^d$ such that $|a| \leq k$.Then

$$(1 + |x|^2)^{\frac{k}{2}} D^a\mu_t(x) \leq \int_{\mathbb{R}^d} (1 + |x|^2)^{\frac{k}{2}} |D^a\phi(\frac{x-y}{t})| d|\mu|(y)$$

$$= \frac{1}{t^{d+|a|}} \int_{\mathbb{R}^d} \frac{(1+|x|^2)^{\frac{k}{2}}(1+|\frac{x-y}{t}|^2)^{\frac{k}{2}}|D^a\phi(\frac{x-y}{t})|}{(1+|\frac{x-y}{t}|^2)^{\frac{k}{2}}} d|\mu|(y) =: I(x)$$

If $y \notin B(x,t)$ then $\frac{|y-x|}{t} \geq 1$ thus $D^a\phi(\frac{|x-y|}{t}) = 0$, so

$$I(x) = \frac{1}{t^{d+|a|}} \int_{B(x,t)} \frac{(1+|x|^2)^{\frac{k}{2}}(1+|\frac{x-y}{t}|^2)^{\frac{k}{2}}|D^a\phi(\frac{x-y}{t})|}{(1+|\frac{x-y}{t}|^2)^{\frac{k}{2}}} d|\mu|(y)$$

$$\leq p_k(\phi)\frac{1}{t^{d+k}} \int_{B(x,t)} (1+|x|^2)^{\frac{k}{2}} d|\mu|(y)$$

if $|x| > R+t$ then $B(0,R) \cap B(x,t) = \emptyset$ thus

$$I(x) \leq \frac{1}{t^{d+|k|}}(1+(R+t)^2)^{\frac{k}{2}}||\mu|| < +\infty$$

which implies that $p_k(\mu_t) < +\infty$

Now

$$\hat{\psi}_t(\xi) = \hat{\psi}(t\xi) \to \hat{\psi}(0) = \int_{\mathbb{R}^d} \psi = 1 \text{ as } t \to 0$$

$$\mu_t(x) = \int_{\mathbb{R}^d} \hat{\mu}_t(\xi) e^{2\pi i\xi\cdot x} d\xi = \int_{\mathbb{R}^d} \hat{\psi}(t\xi)\hat{\mu}(\xi)e^{2\pi\imath\xi\cdot x}d\xi$$

$$\to^{t\to 0} \int_{\mathbb{R}^d} \hat{\mu}(\xi)e^{2\pi i\xi\cdot x}d\xi := g(x)$$

Also $\mu_t(x)dx \rightharpoonup d\mu(x)$ as $t \to 0$. Indeed let $\epsilon > 0$ and $f \in C_c(\mathbb{R}^d)$. Then f is uniformly continuous so $\exists\delta > 0$ such that

$$|f(x+y) - f(y)| < \frac{\epsilon}{||\mu||},\ \forall x \in B(0,\delta)$$

For $t < \delta$ we have:

$$\left|\int_{\mathbb{R}^d} f(x)\mu_t(x)dx - \int_{\mathbb{R}^d} f(x)d\mu(x)\right| = \left|\int_{\mathbb{R}^d} f(x)(\psi_t * \mu)(x)dx - \int_{\mathbb{R}^d} f(y)d\mu(y)\right|$$

$$= \left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\psi_t(x-y)d\mu(y)dx - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\psi_t(x)dxd\mu(y)\right|$$

$$\leq \int_{\mathbb{R}^d} \int_{B(0,t)} |f(x+y) - f(y)| \psi_t(x) dx d|\mu|(y)$$

$$< \int_{\mathbb{R}^d} \int_{B(0,t)} \frac{\epsilon}{||\mu||} \psi_t(x) dx d|\mu|(y) \leq ||\mu|| ||\psi||_1 \frac{\epsilon}{||\mu||} = \epsilon$$

Finally $\mu_t(x)dx \rightharpoonup g(x)dx$. Indeed let $f \in C_c(\mathbb{R}^d)$ and $t_n \to 0$.

$$|\mu_{t_n}(x)| = \left| \int_{\mathbb{R}^d} \hat{\psi}(t_n \xi) \hat{\mu}(\xi) e^{2\pi \imath \xi \cdot x} d\xi \right| \leq ||\hat{\mu}||_1$$

so $|f(x)\mu_{t_n}(x)|, |g(x)f(x)| \leq ||\hat{\mu}||_1 |f(x)| \in L^1(\mathbb{R}^d)$
By dominated convergence theorem we have that

$$\int_{\mathbb{R}^d} f(x) \mu_{t_n}(x) dx \to \int_{\mathbb{R}^d} f(x) g(x) dx$$

. By uniqueness of weak limit $\mu = g$. ■

Definition 3.2. *(i)A function $f \in L^1_{loc}(\mathbb{R}^d)$ is a **tempered fuction** if*

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^N} dx < \infty$$

for some constant $N \in \mathbb{N}$.
*(ii) If f, g are tempered functions,we say that g is the **distributional Fourier transform** of f if*

$$\int_{\mathbb{R}^d} g\phi = \int_{\mathbb{R}^d} f\hat{\phi}, \ \forall \phi \in \mathrm{S}$$

Definition 3.3. *For a functions in $f \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ we have that $f = f_1 + f_2$ where $f_i \in L^i(\mathbb{R}^d)$. $i = 1, 2$ and we define its $(L^1 + L^2)-$ **Fourier Transform** by $\hat{f} = \hat{f}_1 + \hat{f}_2$*

Lemma 3.4. *Let $h_a = \frac{\Gamma(\frac{a}{2})}{\pi^{\frac{a}{2}}} |x|^{-a}$. Then $\hat{h_a} = h_{d-a}$ in the sence of $(L^1 + L^2)-$Fourier transforms if $\frac{d}{2} < Re(a) < d$, and in the sence of distributional Fourier transforms if $0 < Re(a) < d$.*

Here Γ is the gamma function,i.e.,

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

<u>Proof</u> Let $h(x) = |x|^{-a}$ where $a \in (\frac{d}{2}, d)$, then h is radial and $h \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ because $h(x) = h_1 + h_2$ where

$$h_1(x) = h(x)1_{\{|x|<1\}}(x) \in L^1(\mathbb{R}^d)$$

$$h_2(x) = h(x)1_{\{|x|\geq 1\}}(x) \in L^2(\mathbb{R}^d)$$

Now recall that if $f \in L^2(\mathbb{R}^d)$ and $\phi_n \in$ S such that $\phi_n \to^{L^2} f$, then $\hat{\phi_n} \to^{L^2} \mathcal{F}(f)$

Using this fact and simple changes of variables we can easily deduce that the L^2-Fourier transform $\mathcal{F}$ of a radial function $f \in L^2(\mathbb{R}^d)$ is radial. Thus the $(L^1 + L^2)-$Fourier transform of h is radial.

Also using the previous fact, again we have $\hat{h}(M\xi) = M^{-(d-a)}\hat{h}(\xi)$.
If $\xi \in \mathbb{R}^d$ then

$$\hat{h}(\xi) = \hat{h}(|\xi|\frac{\xi}{|\xi|}) = |\xi|^{-(n-a)}\hat{h}(\frac{\xi}{|\xi|})$$

So $\hat{h}(\xi) = c|\xi|^{-(d-a)}$ where $c = \hat{h}(x), \forall x \in \mathbb{S}^{d-1}$

Using the Duality relation(which is true for L^2 functions by approximation)we have

$$\int_{\mathbb{R}^d} |x|^{-a}e^{-\pi|x|^2}dx = c\int_{\mathbb{R}^d} |x|^{-(d-a)}e^{-\pi|x|^2}dx \ (1)$$

Here we used the identity: $\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2}$ where $e^{-\pi|x|^2} \in$ S
Polar Coordinates formula and appropriate changes of variables to on both sides of the equation (1) give us:

$$c = \frac{\Gamma(\frac{d-a}{2})\pi^{\frac{a}{2}}}{\Gamma(\frac{a}{2})\pi^{\frac{d-a}{2}}}$$

Hence $\hat{h_a} = h_{d-a}$.

Now for the general case, let $\phi \in \mathrm{S}, d > 2$ and

$$A(z) = \int_{\mathbb{R}^d} h_z \hat{\phi}$$

$$B(z) = \int_{\mathbb{R}^d} h_{d-z} \phi$$

We will show that $A(z), B(z)$ are holomorphic in the strip
$I = \{z : 0 < Re(z) < d-1\}$ and agree everywhere on I.
Note that $\frac{d}{2} < d - 1$

Recall that $\Gamma(z)$ is holomorphic in the region $\Omega = \{z : Re(z) > 0\}$ and has no zeroes, so the reciprocal gamma function $\frac{1}{\Gamma}(z)$
is holomorphic in Ω. So it suffices to show the holomorphy of the function $G : I \to \mathbb{C}$ where

$$G(z) = \int_{\mathbb{R}^d} |x|^{-z} \phi(x) dx$$

Let $z \in I$ and define $F(x,z) = |x|^{-z}\phi(x)$ and let $h_n \in \mathbb{C}$ such that $h_n \to 0$ and $|h_n| < \min\{1, \frac{d-Rez-1}{2}\}, \forall n \in \mathbb{N}$.
Then

$$\frac{F(x, z+h_n) - F(x,z)}{h_n} \to \frac{-\ln|x|}{x^z}\phi(x), \;\; a.e$$

and also

$$\left| \frac{F(x, z+h_n) - F(x,z)}{h_n} \right| = \left| \frac{1}{|h_n||x|^z} \left(\frac{1}{|x|^{h_n}} - 1 \right) \right| |\phi(x)| = \frac{|e^{-h_n \ln|x|} - 1|}{|h_n||x|^{Re(z)}} |\phi(x)|$$
$$= \frac{|e^{-h_n \ln|x|} - 1|}{|h_n||x|^{Re(z)}} |\phi(x)| 1_{\{|x| \le 1\}} + \frac{|e^{-h_n \ln|x|} - 1|}{|h_n||x|^{Re(z)}} |\phi(x)| 1_{\{|x| > 1\}}$$

(1) If $|x| \le 1$ then,

$$\frac{|e^{-h_n \ln|x|} - 1|}{|h_n||x|^{Re(z)}} 1_{\{|x| \le 1\}} |\phi(x)| \le \frac{|\ln|x|| |e^{|h_n||\ln|x||}}{|x|^{Re(z)}} 1_{\{|x| \le 1\}} |\phi(x)| = \frac{\ln \frac{1}{|x|} e^{|h_n| \ln \frac{1}{|x|}}}{|x|^{Re(z)}} 1_{\{|x| \le 1\}} |\phi(x)|$$

$$\leq \frac{1}{|x|^{Re(z)+1+|h_n|}} 1_{\{|x|\leq 1\}} |\phi(x)| \leq \frac{1}{|x|^{Re(z)+1+\frac{d-1-Re(z)}{2}}} 1_{\{|x|\leq 1\}} |\phi(x)| \in L^1(\mathbb{R}^d)$$

since ϕ is bounded everywhere.

(2) If $|x| > 1$ then

$$\frac{|e^{-h_n \ln|x|} - 1|}{|h_n||x|^{Re(z)}} |\phi(x)| 1_{\{|x|>1\}} \leq |x|^2 |\phi(x)| 1_{\{|x|>1\}} \in L^1(\mathbb{R}^d)$$

since $\phi \in$ S.

Using **(1),(2)** and the dominated convergence theorem we have that $A(z), B(z)$ are differentiable at z.
So A, B are differentiable at every $z \in I$ thus holomorphic in I
By Corollary 3.1 ,

$$A(z) = B(z), \forall z \in (\frac{d}{2}, d) \supset (\frac{d}{2}, d-1)$$

Thus by identity theorem $A(z) = B(z), \forall z \in I$
If $Re(z) > \frac{d}{2}$ then $h_a \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$, so its $L^1 + L^2$ and distributional Fourier transforms coincide. $\blacksquare$.

4 Hausdorff dimension and Frostman's Lemma for compact sets

In this chapter we introduce notion of the Hausdorff measure and dimension in $\mathbb{R}^d$ and we prove the Frostman's Lemma which is one of the most important tools for the proof of the main theorem, because it leads us to the potential-theoretic characterization of the Hausdorff dimension in the next chapter.

Let $a > 0$, and $E \subseteq \mathbb{R}^d$. For $\delta \in (0, +\infty)$, one defines

$$H_a^\delta(E) = \inf\left(\sum_j r_j^a\right)$$

where the infimum is taken over all countable coverings of E by balls $B(x_j, r_j)$ with $r_j < \delta$.
It is clear that $H_a^\delta(E)$ increases as δ decreases, and we define

$$H_a(E) = \lim_{\delta \to 0} H_a^\delta(E)$$

It is also clear that $H_a^\delta(E) \leq H_\beta^\delta(E)$ if $a > \beta$ and $\delta \leq 1$, thus $H_a(E)$ is a nonincreasing function of a.

We also denote $H_a^\infty(\cdot)$ the set function $H_a^\delta(\cdot)$ when $\delta = +\infty$.

Remark H_a is a metric outer measure on $\mathbb{R}^d$, and hence,by Proposition 2.1, all Borel sets in $\mathbb{R}^d$ are H_a-measurable.

Indeed it is not difficult to see that H_a is an outer measure.Now let $A, B \subseteq \mathbb{R}^d$ with $d(A, B) > 0$ If $H_a(A \cup B) = +\infty$ then the equality $H_a(A \cup B) = H_a(A) + H_a(B)$ is clearly true. We suppose that $H_a(A \cup B) < +\infty$ and hence, $H_a^\delta(A \cup B) < +\infty$ for every $\delta > 0$. We take arbitrary $\delta < \frac{d(A,B)}{2}$ and an arbitrary covering $A \cup B \subseteq \bigcup_{j=1}^\infty B(x_j, r_j)$ with $r_j < \delta$ for every j. It is obvious that each $B(x_j, r_j)$ intersects at most one of the A and B. We set $C_j = B(x_j, r_j)$ when $B(x_j, r_j)$intersects A and $C_j = \emptyset$ otherwise and, similarly, $D_j = B(x_j, r_j)$ when $B(x_j, r_j)$ intersects B and $D_j = \emptyset$ otherwise. Then, $A \subseteq \bigcup_{j=1}^\infty C_j$ and $B \subseteq \bigcup_{j=1}^\infty D_j$.We denote $r(C_j), r(D_j)$ the radii of the balls C_j, D_j respectively. So $H_a^\delta(A) \leq \sum_{j=1}^\infty (r(C_j))^a$ and $H_a^\delta(B) \leq$

$\sum_{j=1}^{\infty}(r(D_j))^a$.Adding,we find $H_a^\delta(A) + H_a^\delta(B) \leq \sum_{j=1}^{\infty} r_j^a$ and, taking the infimum of the right side, $H_a^\delta(A) + H_a^\delta(B) \leq H_a^\delta(A \cup B)$ Taking the limit as $\delta \to 0^+$ we find $H_a(A) + H_a(B) \leq H_a(A \cup B)$ and since the opposite inequality is obvious, we conclude that $H_a(A) + H_a(B) = H_a(A \cup B)$

Theorem 4.1. *There is a unique number a_0, called the Hausdorff dimension of E or **dim**(E), such that $H_a(E) = \infty$ if $a < a_0$ and $H_a(E) = 0$ if $a > a_0$*

Proof Let $E \subseteq \mathbb{R}^d$ Borel. We claim that if $0 < a_1 < a_2 < \infty$ and if $H_{a_1}(E) < \infty$ then $H_{a_2}(E) = 0$
Indeed,since $H_{a_1}(E) < \infty$ we have that $H_{a_1}^\delta(E) < \infty, \forall \delta > 0$. We fix such a δ and consider a covering $E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j)$ with $r_j \leq \delta$, for all j so that $\sum_j r_j^{a_1} < H_{a_1}^\delta(E) + 1 \leq H_{a_1}(E) + 1$
Therefore

$$H_{a_2}^\delta(E) \leq \sum_j r_j^{a_2} \leq \delta^{a_2 - a_1} \sum_j r_j^{a_1} \leq (H_{a_1}^\delta(E) + 1)\delta^{a_2 - a_1}$$

and taking the limit as $\delta \to 0^+$, we find $H_{a_2}(E) = 0$

Now we consider various cases

1.$H_a(E) = 0$ for every $a > 0$. In this case we set $a_0 = 0$.
2.$H_a(E) = +\infty$ for every $a > 0$. We, now, set $a_0 = +\infty$
3. There are a_1 and a_2 in $(0, +\infty)$ so that $H_{a_1}(E) > 0$ and $H_{a_2}(E) < +\infty$.
The above claim implies that $a_1 \leq a_2$ and that $H_a(E) = +\infty$ for every $a \in (0, a_1)$ and $H_a(E) = 0$ for every $a \in (a_2, +\infty)$.
We consider the set $\{a \in (0, +\infty) : H_a(E) = +\infty\}$ and its supremum $a_0 \in [a_1, a_2]$. Again,our claim implies that $H_a(E) = +\infty$ for every $a \in (0, a_0)$ and $H_a(E) = 0$ for every $a \in (a_0, +\infty)$. ■

From the previous theorem we can easily prove that

$$\mathbf{dim}(E) = \sup\{a \geq 0 : H_a(E) = +\infty\} = \inf\{a \geq 0 : H_a(E) = 0\}$$

We now state without proof some simple properties of the Hausdorff dimension in $\mathbb{R}^d$:

1. $E \subseteq F \Longrightarrow \mathbf{dim}(E) \leq \mathbf{dim}(F)$
2. For an open subset $U \subseteq \mathbb{R}^d, \mathbf{dim}(U) = d$

3. $\mathbf{dim}(F) = 0$ if F is finite or countable.
4. $\mathbf{dim}(\bigcup_{m=1}^{\infty} F_m) = \sup_m\{\mathbf{dim}(F_m)\}$
5. If $f : E \to \mathbb{R}^m$ is Lipschitz, then $\mathbf{dim}(f(E)) \leq \mathbf{dim}(E)$
6. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a **similarity with ratio C** (i.e. for some $C > 0$, $|f(x)-f(y)| = C|x-y|\ \forall x, y$) or an affine transformation, then $\mathbf{dim}(f(E)) = \mathbf{dim}(E)$

Lemma 4.2. *Let $E \subseteq \mathbb{R}^d$ Borel. Then $H_a^{\infty}(E) = 0 \Longleftrightarrow H_a(E) = 0$*

Proof ($\Longleftarrow$) $\forall \delta > 0$ we have $H_a^{\infty}(E) \leq H_a^{\delta}(E) \Longrightarrow H_a^{\infty}(E) = 0$.

($\Longrightarrow$)Let $H_a^{\infty}(E) = 0$ and $\delta > 0$.
Then there exists a covering $\{B(x_j, r_j) : j \in \mathbb{N}\}$ of E such that $\sum_j r_j^a < \delta^a$

$$r_j^a \leq \sum_j r_j^a < \delta^a, \forall j \in \mathbb{N} \Longrightarrow r_j < \delta, \forall j \in \mathbb{N}$$

thus

$$H_a^{\delta}(E) \leq \sum_j r_j^a < \delta^a$$

Letting $\delta \to 0$, we have $H_a(E) = 0$. ■

For $n \in \mathbb{Z}$, We denote

$$\mathcal{D}_n = \{\prod_{k=1}^{d} [\frac{j_k}{2^n}, \frac{j_k+1}{2^n}) : j_1, j_2, ..., j_d \in \mathbb{Z}\}$$

the dyadic cubes in $\mathbb{R}^d$ with side length $\frac{1}{2^n}$. and $\mathcal{D} = \bigcup_{n\in\mathbb{Z}} \mathcal{D}_n$

(A) If $Q, Q' \in \mathcal{D}$ intersect,then either $Q \subseteq Q'$ or $Q' \subseteq Q$
(B) If $\mathcal{D}^{max} \subseteq \mathcal{B}$ is the family of maximal sets (with respect to set inclusion) in $\mathcal{B}$, where $\mathcal{B}$ is a familly of dyadic cubes$\overline{\ }_{,then}$D^{max} consists of disjoint sets.

Definition 4.1 (Frostman measure)**.** *A Borel measure μ on $\mathbb{R}^d$ is called an $s-$Frostman measure, if there is a constant $C > 0$, such that $\mu(B(x, r)) \leq Cr^s, \forall x \in \mathbb{R}^d, \forall r > 0$.*

Lemma 4.3 (Frostman's Lemma)**.** *Assume that $E \subseteq \mathbb{R}^d$ is a compact. Then $H_s(E) > 0$ if and only if there exists a non-zero $s-$Frostman measure μ with $supp(\mu) \subseteq E$.*

<u>Proof</u> ($\Longleftarrow$) Let $\delta > 0$ and $\{B(x_j, r_j) : j \in \mathbb{N}\}$ a covering of E by balls with $r_j < \delta, \forall j \in \mathbb{N}$.
Then $0 < \mu(E) \leq \sum_j \mu(B(x_j, r_j)) \leq C \sum_j r_j^s$.

This is true for every such covering of E thus $H_s^\delta(E) \geq \frac{1}{C}\mu(E)$.
Since $\delta > 0$ was arbitrary, $H_s^\delta(E) \geq \frac{1}{C}\mu(E) > 0, \forall \delta > 0$.
As $\delta \to 0$, we have that $H_s(E) > 0$.

($\Longrightarrow$) By lemma 4.1 since $H_s(E) > 0$, then $H_s^\infty(E) > 0$.
We'll find an $s-$Frostman measure μ with $supp(\mu) \subseteq E$ and $\mu(\mathbb{R}^d) = \mu(E) \geq N_d H_s^\infty(E) > 0$,where N_d is a positive constant that depends only on the dimension d.

Assume that the lemma has already been proven for all compact $E \subseteq [0,1)^d$. Then let $E \subseteq \mathbb{R}^d$ a compact set with $H_s^\infty(E) > 0$ and a cube $Q \supset E$ with side length $M > 0$.
So $Q = \prod_{k=1}^d [a_k, a_k + M), a_1, ..., a_d \in \mathbb{R}$. Define $T : Q \to [0,1)^d$ as

$$T(x_1, x_2, ..., x_d) = \frac{1}{M}(x_1 - a_1, x_2 - a_2, ..., x_d - a_d)$$

Then T has the properties:

$$|T(x) - T(y)| = \frac{|x - y|}{M}, \forall x, y \in Q$$

$$T(Q) = [0,1)^d$$

So it is easy to check that $H_s^\infty(T(E)) = \frac{H_s^\infty(E)}{M^s}$, and we may find (by assumption) an $s-$Frostman measure μ_0 with $supp(\mu_0) \subseteq T(E)$ and $\mu_0(T(E)) \geq N_d H_s^\infty(T(E)) = N_d \frac{H_s^\infty(E)}{M^s}$. Finally consider the measure $\mu := M^s \mu_0(T(\cdot))$ where $supp(\mu) \subseteq E$. Then

$$\mu(E) = M^s \mu_0(T(E)) \geq N_d H_s^\infty(E)$$

$$\mu(B(x,r)) = M^s \mu_0(T(B(x,r)) \leq C_d M^s \frac{r^s}{M^s} = C_d r^s$$

So, μ is the desired measure.

Now, we prove the lemma under the assumption $E \subseteq [0,1)^d$.
Let $\delta = \frac{1}{2^n}$ for some $n \in \mathbb{N}$, and let $\mathcal{D}_\delta$ be the collection of dyadic cubes of side-length $l(Q) = \delta$, which are contained in $[0,1)^d$.
Also, let $\mathcal{D}_\delta(E) := \{Q \in \mathcal{D}_\delta : Q \cap E \neq \emptyset\}$, and write

$$E_\delta = \bigcup_{Q \in \mathcal{D}_\delta(E)} Q \subseteq [0,1)^d$$

We will first construct a measure μ_δ with $supp(\mu_\delta) \subseteq E_\delta$, and satisfying

$$\mu_\delta(B(x,r)) \leq C_d r^s, \forall x \in \mathbb{R}^d, \forall r \in [\delta, +\infty)$$

for some $C_d > 0$.
For $Q \in \mathcal{D}_\delta$, we start by defining the set functions μ_δ^0 such that

$$\mu_\delta^0(Q) := \begin{cases} l(Q)^s & \text{If } Q \in \mathcal{D}_\delta(E) \\ 0 & \text{If } Q \cap E = \emptyset \end{cases}$$

and also for $Q \in \mathcal{D}_\delta(E)$ define $\mu_\delta^0 \restriction_Q$ to be a suitably weighted copy of Lebesgue measure on Q so that its mass on Q is $\mu_\delta^0(Q)$. Clearly μ_δ^0 is a measure supported on $\overline{E_\delta}$ and satisfies:

$$\mu_\delta^0(B(x,r)) \leq 5^d r^s, \forall x \in \mathbb{R}^d, \forall r \in [\delta, 2\delta)$$

Indeed, if $x \in \mathbb{R}^d$ and $r \in [\delta, 2\delta)$ then we can cover $B(x,r)$ with 5^d dyadic cubes with side-length δ and by definition of μ_δ^0 we have that $\mu_\delta^0(Q) \leq l(Q)^s, \forall Q \in \mathcal{D}_\delta$. So

$$\mu_\delta^0(B(x,r)) \leq \sum_{j=1}^{5^d} \mu_\delta^0(Q_j) \leq 5^d \delta^s \leq 5^d r^s$$

Now in order to have control for $\mu_\delta^0(B(x,r))$ when $r \geq 2\delta$ we need to modify μ_δ^0 on scales larger than δ. We'll do this by induction.
Assume that μ_δ^k has already been defined for some $k \geq 0$ and
1. μ_δ^k is a measure.
2. $\mu_\delta^k(B(x,r)) \leq 5^d r^s, \forall x \in \mathbb{R}^d, \forall r \in [\delta, 2^{k+1}\delta)$
3. $\mu_\delta^k(Q) \leq l(Q)^s, \forall Q \in \mathcal{D}_{2^k\delta}$

Let $Q \in \mathcal{D}_{2^{k+1}\delta}$

(A) If $\mu_\delta^k(Q) \leq l(Q)^s = (2^{k+1}\delta)^s$ then set

$$\mu_\delta^{k+1} \restriction_Q := \mu_\delta^k \restriction_Q$$

(B) If $\mu_\delta^k(Q) > l(Q)^s$ then set

$$\mu_\delta^{k+1} \restriction_Q := \frac{l(Q)^s}{\mu_\delta^k(Q)} \mu_\delta^k \restriction_Q$$

so that now $\mu_\delta^{k+1}(Q) = l(Q)^s$.
Clearly μ_δ^{k+1} is a measure and $\mu_\delta^{k+1}(Q) \leq l(Q)^s, \forall Q \in \mathcal{D}_{2^{k+1}\delta}$ and

$$\mu_\delta^{k+1}(Q) \leq 5^d r^s, \forall x \in \mathbb{R}^d, \forall r \in [\delta, 2^{k+2}\delta)$$

This completes the definition of μ_δ^{k+1}.Note that

$$\mu_\delta^{k+1}(A) \leq \mu_\delta^k(A), \forall A \in \mathbb{R}^d, \forall k \geq 0$$

since $\frac{l(Q)^s}{\mu_\delta^k(Q)} < 1$ in case **(B)**.

For $k = n$ we have that $2^n\delta = 1$. Set $\mu_\delta := \mu_\delta^n$.
Then $\mu_\delta([0,1)^d) \leq 1$ by construction,and since $\mu_\delta(\mathbb{R}^d \setminus [0,1)^d) = 0$, we also have $\mu_\delta(Q) \leq l(Q)^s$ for all dyadic Q with $l(Q) \geq 1$.

Fix a cube $Q \in \mathcal{D}_{2^k\delta}$ for some $k \geq 0$.If $k \geq n$ then $\mu_\delta(Q) \leq l(Q)^s$,since $l(Q) \geq 1$.
If $k < n$ then $\mu_\delta(Q) \leq \mu_\delta^k(Q) \leq l(Q)^s$ by construction.So we conclude that $\mu_\delta(Q) \leq l(Q)^s$ for all dyadic cubes with side-length larger than δ.
If $x \in \mathbb{R}^d$ and $r \geq \delta$ then $\exists m \geq 0$ such that $2^m\delta \leq r < 2^{m+1}\delta$ thus $\mu_\delta(B(x,r)) \leq 5^d r^s$.

Now we'll show that $\mu_\delta(E) \geq N_d H_s^\infty(E)$ for some $N_d > 0$. We have that $\forall x \in E_\delta$ exists $Q_x \ni x$ such that $\mu_\delta(Q_x) = l(Q_x)^s$
Indeed this is the biggest cube $Q \ni x$ for which case **(B)** occurred, because then $\mu_\delta(Q) = l(Q)^s$ and if case **(B)** never occurred for cubes containing x, then the x lies in some $Q \in \mathcal{D}_\delta$ where $\mu_\delta^0(Q) = l(Q)^s$ thus $\mu_\delta(Q) = l(Q)^s$ since μ_δ, by construction, extends μ_δ^0 to cubes with side-length $\geq \delta$.

Denote M the set of maximal elements of $\{Q_x : x \in E_\delta\}$. Then M consists of disjoint cubes which they also cover E_δ, so

$$\mu_\delta(E_\delta) = \sum_{Q\in M} \mu_\delta(Q) = \sum_{Q\in M} l(Q)^s \geq \frac{1}{(\sqrt{d})^s} \sum_{Q\in M} diam(Q)^s \geq \frac{1}{(\sqrt{d})^s} H_s^\infty(E)$$

since every $Q \in M$ is a subset of the ball $B(s, diam(Q))$ where s is the center of Q and those balls cover E.

Now the sequence $\{\mu_{\frac{1}{2^n}} : n \in \mathbb{N}_0\}$ satisfies the hypothesis of theorem 2.2 so there exists a subsequence $\{\mu_{k_n}\}_{n\in\mathbb{N}}$ and a locally finite Borel measure μ such that $\mu_{k_n} \rightharpoonup \mu$ as $n \to +\infty$.
Each measure μ_{k_n} is supported on $E_{k_n} := \overline{E_{2^{-k_n}}}$ and $\mathbb{R}^d \setminus E = \bigcup_{n=1}^\infty (\mathbb{R}^d \setminus E_{k_n})$. Thus by lemma 2.3

$$\mu(\mathbb{R}^d \setminus E) = \lim_{n\to+\infty} \mu(\mathbb{R}^d \setminus E_{k_n}) \leq \lim_{n\to+\infty} \liminf_{j\to+\infty} \mu_{k_j}(\mathbb{R}^d \setminus E_{k_n}) = 0$$

since $\mu_{k_j}(\mathbb{R}^d \setminus E_{k_n}) = 0, \forall j \geq n$. So we proved that $supp(\mu) \subseteq E$.

Also since E is compact, $\exists M > 0$ such that $E \subseteq \overline{B(0, M)} =: B$ and so

$$\mu(\mathbb{R}^d) = \mu(B) \geq \limsup_{j\to+\infty} \mu_{k_j}(B) \geq \limsup_{j\to+\infty} \mu_{k_j}(E_{k_j}) \geq \frac{1}{(\sqrt{d})^s} H_s^\infty(E) > 0$$

By construction of the measures $\mu_{k_j}, j \in \mathbb{N}$ and by lemma 2.3 it is clear that $\mu(B(x, r)) \leq 5^d r^s, \forall r > 0, \forall x \in \mathbb{R}^d$. $\blacksquare$

5 Riesz energy and its connection to the Hausdorff dimension and Fourier transform

We now define the $a-$**dimensional Riesz energy** of a (positive) measure μ with compact support(the compact support assumption is not needed; it is included to simplify the presentation) by the formula

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{-a} d\mu(x)d\mu(y)$$

We always assume that $0 < a < d$ and we also define the **Riesz potential**

$$\mathbb{V}_\mu^a(x) = \int_{\mathbb{R}^d} |x-y|^{-a} d\mu(y)$$

(which is the convolution of $|x|^{-a}$ with μ). Thus

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d} \mathbb{V}_\mu^a d\mu$$

Lemma 5.1. (i) *If μ is a probability $a-$Frostman measure with compact support then $\mathbb{I}_\beta(\mu) < \infty,\ \forall \beta < a$.*
(ii) *Conversely,if μ is a probability measure with compact support and with $\mathbb{I}_a(\mu) < \infty$ then there exists a probability $a-$Frostman measure ν such that $\nu(X) \leq 2\mu(X)$ for all sets $X \subseteq \mathbb{R}^d$*

<u>Proof</u> (i)We have that $supp(\mu) \subseteq B(0,M),\ M > 0$. Let $x \in supp(\mu)$ and $\beta < a$.Then from the monotone convergence theorem:

$$\mathbb{V}_\mu^\beta(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^\beta} d\mu(y) = \sum_{j=0}^{\infty} \int_{\left\{\frac{2M}{2^{j+1}} \leq |x-y| < \frac{2M}{2^j}\right\}} \frac{1}{|x-y|^\beta} d\mu(y)$$

$$\leq \frac{2^\beta}{M^\beta} \sum_{j=0}^{\infty} 2^{j\beta} \int_{\left\{\frac{2M}{2^{j+1}} \leq |x-y| < \frac{2M}{2^j}\right\}} d\mu(y)$$

$$\leq \frac{2^\beta}{M^\beta} \sum_{j=0}^{\infty} 2^{j\beta} \mu\left(B\left(x, \frac{2M}{2^j}\right)\right) \leq C_{M,a,\beta} \sum_{j=0}^{\infty} 2^{j(\beta-a)} < \infty$$

where $C_{M,a,\beta}$ is a positive constant depending only on M, a, β. Thus

$$\mathbb{I}_\beta(\mu) = \int_{\mathbb{R}^d} \mathbb{V}_\mu^\beta(x) d\mu(x) < \infty$$

(ii) Let $a > 0$ and $F = \{x : \mathbb{V}_\mu^a(x) \leq 2\mathbb{I}_a(\mu)\}$.Then

$$\mathbb{I}_a(\mu) = \int_F \mathbb{V}_\mu^a(x)d\mu(x) + \int_{F^c} \mathbb{V}_\mu^a(x)d\mu(x) \geq \int_{F^c} \mathbb{V}_\mu^a(x)d\mu(x) \geq 2\mathbb{I}_a(\mu)(1-\mu(F))$$

Thus $\mu(F) \geq \frac{1}{2}$. Let $\nu(X) = \frac{\mu(X \cap F)}{\mu(F)},\ X \subseteq \mathbb{R}^d$

Let $x \in F$. If $r > 0$ then

$$\mathbb{V}_\nu^a(x) = \int_{\{|x-y|<r\}} \frac{1}{|x-y|^a} d\nu(y) + \int_{\{|x-y|\geq r\}} \frac{1}{|x-y|^a} d\nu(y)$$

$$\geq \int_{\{|x-y|<r\}} \frac{1}{|x-y|^a} d\nu(y) \geq r^{-a}\nu(B(x,r))$$

Also

$$\mathbb{V}_\nu^a(x) \leq \frac{1}{\mu(F)} \int_{\mathbb{R}^d} \frac{1}{|x-y|^a} 1_F(y) d\mu(y) \leq 2\mathbb{V}_\mu^a(x) \leq 4\mathbb{I}_a(\mu)$$

Thus

$$\nu(B(x,r) \leq 4\mathbb{I}_a(\mu)r^a,\ \forall x \in F, \forall r > 0$$

For $x \in \mathbb{R}^d \setminus F$ we consider two cases:

If $r > 0$ such that $B(x,r) \cap F = \emptyset$ then $\nu(B(x,r)) = 0$

If $r > 0$ such that $B(x,r) \cap F \neq \emptyset$, let $y \in F \cap B(x,r)$. Then

$$\nu(B(x,r)) \leq \nu(B(y,2r)) \leq 2^{a+2}\mathbb{I}_a(\mu)r^a$$

Combining all the above, we conclude that ν is a probability $a-$Frostman measure with the property: $\nu(X) \leq 2\mu(X)$ for all sets $X \subseteq R^d$. ■

If $E \subseteq \mathbb{R}^d$ we denote $P(E)$ the set of probability Borel measures supported on E.

Theorem 5.2. *If $E \subseteq \mathbb{R}^d$ compact,then the Hausdorff dimension of E coincides with the number*

$$\sup\{a : \exists \mu \in P(E) \ \textit{with}\ \mathbb{I}_a(\mu) < \infty\}$$

Proof Let $A = \{a : \exists \mu \in P(E) \text{ with } \mathbb{I}_a(\mu) < \infty\}$ and $s = \sup A$.

If $\beta \in A$ then from Lemma 5.1 exists a $\beta-$Frostman measure $\mu \in P(E)$, so by Frostman's Lemma $H_\beta(E) > 0 \implies \beta \leq \mathbf{dim}(E)$. Since this is true $\forall \beta \in A$,we have $s \leq \mathbf{dim}(E)$

Conversely if $\beta < \mathbf{dim}(E)$ then $\exists \epsilon > 0$ such that

$$\beta + \epsilon < \mathbf{dim}(E) \implies H_{\beta+\epsilon}(E) > 0$$

By Frostman's Lemma exists a $(\beta+\epsilon)-$Frostman measure $\mu \in P(E)$ and by Lemma 5.1 we have $\mathbb{I}_\beta(\mu) < \infty$ since $\beta < \beta + \epsilon$.

Thus $\beta \leq s$ and since this is true for all $\beta < \mathbf{dim}(E)$ then $\mathbf{dim}(E) \leq s$. ∎

For $f \in L^1_{loc}(\mathbb{R}^d)$, we denote the **Hardy-Littlewood** maximal function

$$\mathcal{M}(f)(x) := \sup_{r>0} \frac{1}{m_d(B(x,r))} \int_{B(x,r)} |f(y)|dy$$

Lemma 5.3. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be any non-negative radial decreasing Schwartz function and let $0 < a < d$, and $x \in \mathbb{R}^d$. Then*

$$\int_{\mathbb{R}^d} |x-y|^{-a}|\phi(y)|dy \leq C_{a,d}\frac{||\phi||_1}{|x|^a}$$

where the constant $C_{a,d}$ depends only on a, d,

Proof Let $f(x) = \frac{1}{|x|^a}$, then it is easy to see that $f \in L^1_{loc}(\mathbb{R}^d)$ since $a < d$. Firstly we'll prove that $|(\phi * f)(x)| \leq ||\phi||_1 \mathcal{M}(f)(x), \forall x \neq 0$.

Since ϕ is radial we have that $\phi(re) = g(r), \forall e \in \mathbb{S}^{d-1}, \forall r > 0$ where $g : [0, +\infty) \to \mathbb{R}$ is decreasing. We define

$$F(r) = \int_{\mathbb{S}^{d-1}} |f(x - re)| d\sigma(e)$$

By polar coordinates

$$|(\phi * f)(x)| \leq \int_0^{+\infty} F(r) g(r) r^{d-1} dr$$

There exists an increasing sequence $\{h_n : n \in \mathbb{N}\}$ of step- functions defined on $[0 + \infty)$ where each h_n has bounded support, $h_n \to g$ pointwise and

$$h_n = \sum_{j=1}^{n} a_j 1_{[0,r_j]}, \ a_j > 0 \ \forall j \in \{1, 2, ..., n\}$$

Again by polar coordinates

$$\int_{\mathbb{R}^d} h_n(|x|) dx = \sum_{j=1}^{n} \sigma(\mathbb{S}^{d-1}) a_j \int_0^{r_j} r^{d-1} dr = \sum_{j=1}^{n} a_j m_d(B(0, r_j))$$

Thus we have

$$\int_0^{+\infty} F(r) h_n(r) r^{d-1} dr$$

$$= \sum_{j=1}^{n} a_j \int_0^{r_j} F(r) r^{d-1} dr$$

$$= \sum_{j=1}^{n} a_j \int_0^{r_j} \int_{\mathbb{S}^{d-1}} |f(x - re)| d\sigma(e) r^{d-1} dr$$

$$= \sum_{j=1}^{n} a_j \int_{B(0,r_j)} |f(x - y)| dy$$

$$= \sum_{j=1}^{n} a_j \int_{B(x,r_j)} |f(y)| dy$$

$$= \sum_{j=1}^{n} a_j m_d(B(x, r_j)) \frac{1}{m_d(B(x, r_j))} \int_{B(0,r_j)} |f(y)| dy$$

$$\leq \sum_{j=1}^{n} a_j m_d(B(x,r_j))\mathcal{M}(f)(x)$$

$$= \sum_{j=1}^{n} a_j m_d(B(0,r_j))\mathcal{M}(f)(x)$$

$$= \int_{\mathbb{R}^d} h_n(|y|)dy\mathcal{M}(f)(x)$$

$$\leq \int_{\mathbb{R}^d} \phi(x)dx\mathcal{M}(f)(x) = ||\phi||_1\mathcal{M}(f)(x), \forall n \in \mathbb{N}$$

By Fatou's Lemma we have

$$|(f * \phi)(x)| \leq ||\phi||_1\mathcal{M}(f)(x) \ \mathbf{(1)}$$

In order to finish the proof of the lemma, it remains to show that

$$\mathcal{M}(f)(x) \leq C_{d,a}\frac{1}{|x|^a}$$

Let $x \in \mathbb{R}^d \setminus \{0\}$ and $R > 0$.

(A) If $R < \frac{|x|}{2}$, then for $y \in B(x,R)$ we have that

$$|y| \geq |x| - |y-x| \geq \frac{|x|}{2}$$

thus

$$\frac{1}{m_d((B(x,R))}\int_{B(x,R)} \frac{1}{|y|^a}dy \leq 2^a\frac{1}{|x|^a}$$

(B) If $R \geq \frac{|x|}{2}$ then

$$\frac{1}{m_d((B(x,R))}\int_{B(x,R)} \frac{1}{|y|^a}dy$$

$$\leq \frac{1}{m_d((B(x,R))}\int_{B(0,3R)} \frac{1}{|y|^a}dy$$

$$= \frac{1}{R^d}\int_0^{3R} r^{d-1-a}dr = \frac{3^{d-a}}{d-a}\frac{1}{R^a} \leq \frac{2^a 3^{d-a}}{(d-a)|x|^a}$$

So **(A)**, **(B)**, **(1)** give the desired conclusion. ■

Theorem 5.4. *Let μ be a positive measure with compact support and $0 < a < d$. Then*

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x-y|^{-a} d\mu(x) d\mu(y) = c_a \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{-(d-a)} d\xi$$

where $c_a = \frac{\Gamma(\frac{d-a}{2})\pi^{a-\frac{d}{2}}}{\Gamma(\frac{a}{2})}$

<u>Proof</u> Suppose first that $f \in L^1(\mathbb{R}^d)$ is real and even, and that $d\mu(x) = \phi(x)dx$ with $0 \le \phi \in$ S. Then by Fubini's theorem and inversion formula we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y) d\mu(x) d\mu(y)$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\phi(x)\phi(y) dx dy$$

$$= \int_{\mathbb{R}^d} (f * \phi)(x)\phi(x) dx$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{(f * \phi)}(\xi) e^{2\pi i \xi \cdot x} \phi(x) d\xi dx$$

$$= \int_{\mathbb{R}^d} \widehat{(f * \phi)}(\xi) \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \phi(x) dx d\xi$$

$$= \int_{\mathbb{R}^d} \widehat{(f * \phi)}(\xi) \int_{\mathbb{R}^d} \overline{e^{-2\pi i \xi \cdot x}} \phi(x) dx d\xi$$

$$= \int_{\mathbb{R}^d} \widehat{(f * \phi)}(\xi) \overline{\int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} \phi(x) dx} d\xi$$

$$= \int_{\mathbb{R}^d} \widehat{(f * \phi)}(\xi) \overline{\hat{\phi}(\xi)} d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi)\hat{\phi}(\xi)\overline{\hat{\phi}(\xi)} d\xi$$

$$= \int_{\mathbb{R}^d} \hat{f}(\xi)|\hat{\phi}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} \hat{f}(\xi)|\hat{\mu}(\xi)|^2 d\xi$$

Now if we fix $0 \leq \phi \in$ S,then it is easy to see that the linear operators $S_\phi, T_\phi : L^2(\mathbb{R}^d) \to \mathbb{R}$

$$S_\phi(f) := \int_{\mathbb{R}^d} \hat{f}(\xi)|\hat{\mu}(\xi)|^2 d\xi$$

$$T_\phi(f) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\phi(x)\phi(y)dxdy$$

are bounded.Indeed by Plancherel's theorem

$$|S_\phi(f)| = \left|\int_{\mathbb{R}^d} \hat{f}(\xi)|\hat{\mu}(\xi)|^2 d\xi\right| = \left|\int_{\mathbb{R}^d} \hat{f}(\xi)|\hat{\phi}(\xi)|^2 d\xi\right|$$

$$\leq ||\phi||_1||\hat{\phi}||_2||\hat{f}||_2 = ||\phi||_1||\phi||_2||f||_2$$

and by Tonelli's theorem

$$|T_\phi(f)| \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)||\phi(x)||\phi(y)|dxdy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)||\phi(y)|dy|\phi(x)|dx$$

$$\leq ||f||_2||\phi||_2||\phi||_1$$

The operators agree on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, so by density they agree on $L^2(\mathbb{R}^d)$.

Thus the two operators agree for $f \in L^1(\mathbb{R}^d) + L^2(\mathbb{R}^d)$ and hence,by lemma 3.4 for $f(x) = \frac{1}{|x|^a}$, $\frac{d}{2} < a < d$.

Observe that if $f, g \in$ S then $\widehat{fg} = \hat{f} * \hat{g}$. Indeed by inversion formula and Fubini's theorem we have

$$\widehat{fg}(\xi) = \int_{\mathbb{R}^d} f(x)g(x)e^{-2\pi i\xi\cdot x}dx$$

$$= \int_{\mathbb{R}^d} f(x)\bigg(\int_{\mathbb{R}^d} \hat{g}(z)e^{2\pi iz\cdot x}dz\bigg)e^{-2\pi i\xi\cdot x}dx$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\hat{g}(z)e^{-2\pi i(\xi-z)\cdot x}dzdx$$

$$= \int_{\mathbb{R}^d} \hat{g}(z)\hat{f}(\xi-z)dz = (\hat{f} * \hat{g})(\xi)$$

So for $d\mu = \phi(x)dx, 0 \leq \phi \in \mathrm{S}$

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^a}\phi(x)\phi(y)dxdy$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|z|^a}\phi(y-z)\phi(y)dzdy$$

$$= \int_{\mathbb{R}^d} \frac{1}{|z|^a}(\tilde{\phi} * \phi)(z)dz$$

We have that $\tilde{\phi} * \phi \in \mathrm{S}$ and also $(\tilde{\phi} * \phi)$ is the Fourier transform of $\hat{\phi}\bar{\hat{\phi}} = |\hat{\phi}|^2$. Indeed by the previous observation and inversion formula

$$\widehat{\hat{\phi}\bar{\hat{\phi}}}(\xi) = (\hat{\hat{\phi}} * \hat{\bar{\hat{\phi}}})(\xi) = (\tilde{\phi} * \phi)(\xi)$$

So for $0 < a < \frac{d}{2}$ we have by lemma 3.4

$$\mathbb{I}_a(\mu) = \int_{\mathbb{R}^d} \frac{1}{|z|^a}(\tilde{\phi} * \phi)(z)dz$$

$$= c_a \int_{\mathbb{R}^d} \frac{1}{|\xi|^{d-a}}|\hat{\phi}(\xi)|^2 d\xi$$

$$c_a \int_{\mathbb{R}^d} \frac{1}{|\xi|^{d-a}}|\hat{\mu}(\xi)|^2 d\xi$$

Now for the general case, let μ a positive measure with compact support. We denote $G_t(x) = \frac{1}{t^d}e^{-\pi\frac{|x|^2}{t^2}} = \frac{1}{t^d}G(\frac{x}{t})$, where G is the Gaussian

$$G(x) = e^{-\pi|x|^2} \in \mathrm{S}$$

The Gaussian has the properties:

1. $\hat{G}(\xi) = G(\xi)$

2. $G_t * G_s = G_{t+s}$

We can easily see by definition of a Schwartz function that $G_t * \mu \in \mathrm{S}$
Define

$$I_t(z,w) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{|x-y|^a} G_t(x-z)G_t(y-w)dxdy$$

By Fubini's theorem and applying the previous case to the measure $d\nu = G_t * \mu(x)dx$ we have

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} I_t(z,w)d\mu(z)d\mu(w) = c_a \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \big(\hat{G}(t\xi)\big)^2 \frac{1}{|\xi|^{d-a}} d\xi \ (1)$$

If we apply the change of variables:

$$u = x - z, \;\; s = y - w$$

we have that

$$I_t(z,w) = \Big[(G_t * G_t) * |\cdot|^{-a}\Big](z-w) = (G_{2t} * |\cdot|^{-a})(z-w) \to^{t\to 0} \frac{1}{|z-w|^a} \;(2)$$

Also by lemma 5.3 and change of variables

$$I_t(z,w) = \int_{\mathbb{R}^d} \frac{1}{|s-(z-w)|^a} G_{2t}(s)ds \leq C_{a.d} \frac{1}{|z-w|^a} \;(3)$$

(A) If $\mathbb{I}_a(\mu) < +\infty$ then by $(2),(3)$ and the dominated convergence theorem we have that the left hand side of (1) converges to $\mathbb{I}_a(\mu)$ as $t \to 0$.
If $\mathbb{I}_a(\mu) = +\infty$,then by Fatou's lemma we have the same convergence as before when $t \to 0$.

(B) On the right hand side of (1) if $\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \frac{1}{|\xi|^{d-a}} d\xi < +\infty$ then by dominated convergence we have that

$$c_a \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \big(\hat{G}(t\xi)\big)^2 \frac{1}{|\xi|^{d-a}} d\xi \to^{t\to 0} \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \frac{1}{|\xi|^{d-a}} d\xi$$

since $\big(\hat{G}(t\xi)\big)^2$ is bounded by 1.
If $\int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 \frac{1}{|\xi|^{d-a}} d\xi = +\infty$ then by Fatou's lemma we have the same convergence as before when $t \to 0$.

So **(A),(B)** combined, yield equality (1). $\blacksquare$

6 Marstrand's Projection Theorem

For $e \in \mathbb{S}^{d-1}, d \geq 2$, define the projection $P_e : \mathbb{R}^d \to \mathbb{R}$,

$$P_e(x) = e \cdot x$$

This is essentially the orthogonal projection onto the line $L = \{te : t \in \mathbb{R}\}$.

If μ is a measure supported on a compact set E and $e \in \mathbb{S}^{d-1}$ then we denote μ_e the projected measure $P_e\mu(B) = \mu(P_e^{-1}(B)),\ \ B \subseteq \mathbb{R}$.

With "a.e. $e \in \mathbb{S}^{d-1}$" we always mean, almost everywhere with respect to the surface measure σ on $\mathbb{S}^{d-1}$.

Theorem 6.1 (Marstrand)**.** *Assume that $E \subseteq \mathbb{R}^d$ is compact and $\boldsymbol{dim}(E) = a$. Then: (i) If $a \leq 1$ then for a.e. $e \in \mathbb{S}^{d-1}$ we have $\boldsymbol{dim}(P_e(E)) = a$.*
(ii) If $a > 1$ then for a.e. $e \in \mathbb{S}^{d-1}$ we have $m_1(P_e(E)) > 0$.

Proof (i) Let $a < \mathbf{dim}(E)$, and let μ be a probability measure supported on E with $\mathbb{I}_a(\mu) < +\infty$.
(Note that we can always find such measure by Theorem 5.2)
For $\xi \in \mathbb{R}$ and by lemma 2.1

$$\hat{\mu_e}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} d\mu_e(x) = \int_{\mathbb{R}^d} e^{-2\pi i \xi (x \cdot e)} d\mu(x) = \hat{\mu}(\xi e)$$

Now by theorem 5.4 and polar coordinates we have:

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}} \mathbb{I}_a(\mu_e) d\sigma(e) &= c_a \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \frac{|\hat{\mu_e}(\xi)|^2}{|\xi|^{1-a}} d\xi d\sigma(e) \\
&= 2c_a \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{|\hat{\mu_e}(\xi)|^2}{|\xi|^{1-a}} d\xi d\sigma(e) \\
&= 2c_a \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{|\hat{\mu}(\xi e)|^2}{|\xi|^{1-a}} d\xi d\sigma(e) \\
&= 2c_a \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{|\hat{\mu}(re)|^2}{r^{1-a}} dr d\sigma(e) \\
&= 2c_a \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{r^{d-1}|\hat{\mu}(re)|^2}{r^{d-a}} dr d\sigma(e)
\end{aligned}$$

$$= 2c_a \int_{\mathbb{R}^d} \frac{|\hat{\mu}(y)|^2}{|y|^{d-a}} dy = 2\mathbb{I}_a(\mu) < +\infty$$

So $\mathbb{I}_a(\mu_e) < +\infty$ a.e. $e \in \mathbb{S}^{d-1}$.Since $supp(\mu_e) \subseteq P_e(E)$, it follows by theorem 5.2 that $\mathbf{dim}(P_e(E)) \geq a$ a.e. $e \in \mathbb{S}^{d-1}$.

This is true for all $a < \mathbf{dim}(E)$ thus

$$\mathbf{dim}(P_e(E)) \geq \mathbf{dim}(E) \text{ a.e. } e \in \mathbb{S}^{d-1}$$

Also as P_e is Lipschitz, $\mathbf{dim}(P_e(E)) \leq \mathbf{dim}(E), \forall e \in \mathbb{S}^{d-1}$ so

$$\mathbf{dim}(P_e(E)) = \mathbf{dim}(E)$$

(ii) If $\mathbf{dim}(E) > 1$ then for $s = \frac{1+\mathbf{dim}(E)}{2}$ we have $1 < s < \mathbf{dim}(E)$ so $H_s(E) > 0$.
By Frostman's Lemma there exists a probability $s-$Frostman measure μ supported on E and by lemma 5.1 we have that $\mathbb{I}_1(\mu) < +\infty$.

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |\hat{\mu_e}(\xi)|^2 d\xi d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} |\hat{\mu_e}(\xi)|^2 d\xi d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} |\hat{\mu}(re)|^2 dr d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{r^{d-1}|\hat{\mu}(re)|^2}{r^{d-1}} dr d\sigma(e)$$

$$= 2 \int_{\mathbb{R}^d} \frac{|\hat{\mu}(y)|^2}{|y|^{d-1}} dy = c\mathbb{I}_1(\mu) < +\infty, \ c > 0$$

Thus $\hat{\mu_e} \in L^2(\mathbb{R})$ a.e. $e \in \mathbb{S}^{d-1}$.

By theorem 3.2, μ_e is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ a.e. $e \in \mathbb{S}^{d-1}$.

If $m_1(P_e(E)) = 0$ for any $e \in \mathbb{S}^{d-1}$, then $\mu_e(P_e(E)) = 0$, which contradicts the fact that $supp(\mu_e) \subseteq P_e(E)$. ■

Theorem 6.2. *Let $E \subseteq \mathbb{R}^d$ compact with $\boldsymbol{dim}(E) > 2$. Then $P_e(E)$ has non-empty interior for a.e. $e \in \mathbb{S}^{d-1}$.*

Proof For $a = \frac{\mathbf{dim}(E)+2}{2}$ we take $c \in (a, \mathbf{dim}(E))$ so that $H_c(E) > 0$. By Frostman's Lemma there exists a probability $c-$Frostman measure supported on E and by lemma 5.1 we have that $\mathbb{I}_a(\mu) < +\infty$

By Schwartz's inequality,theorem 5.4 and polar coordinates,we obtain:

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |\hat{\mu_e}(r)| dr d\sigma(e)$$

$$= 2\int_{\mathbb{S}^{d-1}} \int_0^1 |\hat{\mu_e}(r)| dr d\sigma(e) + 2\int_{\mathbb{S}^{d-1}} \int_1^{\infty} |\hat{\mu_e}(r)| dr d\sigma(e)$$

$$\leq 2||\mu||\sigma(\mathbb{S}^{d-1}) + 2\int_{\mathbb{S}^{d-1}} \int_1^{\infty} |\hat{\mu_e}(r)| dr d\sigma(e)$$

$$= 2||\mu||\sigma(\mathbb{S}^{d-1}) + 2\int_{\mathbb{S}^{d-1}} \int_1^{\infty} |\hat{\mu_e}(r)| r^{\frac{a-d-1+d}{2}} r^{\frac{1-a}{2}} dr d\sigma(e)$$

$$\leq 2\sqrt{\int_{\mathbb{S}^{d-1}} \int_1^{\infty} |\hat{\mu_e}(r)|^2 r^{a-d-1+d} dr d\sigma(e)} \sqrt{\int_{\mathbb{S}^{d-1}} \int_1^{\infty} r^{1-a} dr d\sigma(e)} + 2||\mu||\sigma(\mathbb{S}^{d-1})$$

$$= 2\sqrt{\int_{\mathbb{S}^{d-1}} \int_1^{\infty} |\hat{\mu}(re)|^2 r^{a-d-1+d} dr d\sigma(e)} \sqrt{\int_{\mathbb{S}^{d-1}} \int_1^{\infty} r^{1-a} dr d\sigma(e)} + 2||\mu||\sigma(\mathbb{S}^{d-1})$$

$$\leq 2\sqrt{\frac{\sigma(\mathbb{S}^{d-1})}{a-2}} \sqrt{\int_{\mathbb{S}^{d-1}} \int_0^{\infty} \frac{|\hat{\mu}(re)|^2 r^{d-1}}{r^{d-a}} dr d\sigma(e)} + 2||\mu||\sigma(\mathbb{S}^{d-1})$$

$$= 2\sqrt{\frac{\sigma(\mathbb{S}^{d-1})}{a-2}} \sqrt{\int_{\mathbb{R}^d} \frac{|\hat{\mu}(y)|^2}{|y|^{d-a}} dy} + 2||\mu||\sigma(\mathbb{S}^{d-1})$$

$$= M_{a,d}\sqrt{\mathbb{I}_a(\mu)} + 2||\mu||\sigma(\mathbb{S}^{d-1}) < +\infty$$

where $M_{a,d}$ is a positive constant that depends only on a, d.
Hence $\hat{\mu_e} \in L^1(\mathbb{R})$ a.e. $e \in \mathbb{S}^{d-1}$ and by theorem 3.3 μ_e is a continuous function for such e. Since $supp(\mu_e) \subseteq P_e(E)$ we conclude that $P_e(E)$ has a non-empty interior. ■

Note that theorem 6.2 implies part *(ii)* o Marstrand's theorem because every Lebesgue measurable set with non-empty interior has positive Lebesgue measure.

Now we prove a quantitative result for the average length of projections of a compact set.

Theorem 6.3. *Let $E \subseteq \mathbb{R}^d$ compact and μ a probability Borel measure supported on E, with $\mathbb{I}_1(\mu) < +\infty$. Then*

$$\int_{\mathbb{S}^{d-1}} m_1(P_e(E))d\sigma(e) \geq \frac{c_1 \sigma(\mathbb{S}^{d-1})^2}{2\mathbb{I}_1(\mu)}$$

where c_1 is the constant c_a in Theorem 5.4 for $a = 1$

<u>Proof</u> If E is a finite union of dyadic cubes then $m_1(P_e(E))$ is continuous. If E is compact the $E = \bigcap_{n=1}^{\infty} E_n$ where $\{E_n : n \in \mathbb{N}\}$ is decreasing sequence of open sets and each E_n is a finite union of dyadic cubes. Thus $m_1(P_e(E))$ is a pointwise limit of continuous functions thus $\sigma-$measurable.

By theorem 5.4, we have

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |\hat{\mu_e}(\xi)|^2 d\xi d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{\infty} |\hat{\mu_e}(\xi)|^2 d\xi d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} |\hat{\mu}(re)|^2 dr d\sigma(e)$$

$$= 2 \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \frac{r^{d-1}|\hat{\mu}(re)|^2}{r^{d-1}} dr d\sigma(e)$$

$$= 2 \int_{\mathbb{R}^d} \frac{|\hat{\mu}(y)|^2}{|y|^{d-1}} dy = \frac{2}{c_1}\mathbb{I}_1(\mu) < +\infty$$

where μ_e is the projected measure.
So $\hat{\mu_e} \in L^2(\mathbb{R})$ a.e. $e \in \mathbb{S}^{d-1}$ and by theorem 3.2 μ_e is absolutely continuous

with respect to the Lebesgue measure on $\mathbb{R}$ a.e. $e \in \mathbb{S}^{d-1}$.
Also by Plancherel's theorem, $\mu_e \in L^2(\mathbb{R})$ a.e. $e \in \mathbb{S}^{d-1}$ and

$$\int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} |\hat{\mu_e}(x)|^2 dx d\sigma(e) = \frac{2}{c_1}\mathbb{I}_1(\mu)$$

By Schwartz's inequality and absolute continuity,

$$1 = \mu_e(\mathbb{R})^2 = \left(\int_{P_e(E)} \mu_e(x)dx \right)^2 \leq m_1(P_e(A)) \int_{\mathbb{R}} \mu_e^2(x)dx$$

$$\Longrightarrow \int_{\mathbb{S}^{d-1}} \frac{1}{m_1(P_e(A))} d\sigma(e) \leq \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mu_e^2(x) dx d\sigma(e) = \frac{2}{c_1}\mathbb{I}_1(\mu)$$

and finally, again by Schwartz's inequality

$$\sigma(\mathbb{S}^{d-1})^2 = \left(\int_{\mathbb{S}^{d-1}} 1 d\sigma(e) \right)^2 = \left(\int_{\mathbb{S}^{d-1}} \frac{1}{\sqrt{m_1(P_e(A))}} \sqrt{m_1(P_e(A))} d\sigma(e) \right)^2$$

$$\leq \left(\int_{\mathbb{S}^{d-1}} \frac{1}{m_1(P_e(A))} d\sigma(e) \right) \left(\int_{\mathbb{S}^{d-1}} m_1(P_e(A)) d\sigma(e) \right)$$

$$\Longrightarrow \int_{\mathbb{S}^{d-1}} m_1(P_e(A)) d\sigma(e) \geq \frac{\sigma(\mathbb{S}^{d-1})^2}{\int_{\mathbb{S}^{d-1}} \frac{1}{m_1(P_e(A))} d\sigma(e)} \geq \frac{c_1 \sigma(\mathbb{S}^{d-1})^2}{2\mathbb{I}_1(\mu)}. \blacksquare$$

7 References

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