The atomic structure of $H^1(\mathbb{R}^d)$ and the duality between $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$

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Chapter 1

Introduction

We shall define the spaces $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$ and prove that the dual space of the first is isomorphic to the second. The first proof of this result was given by Charles Fefferman and Elias Stein in 1972.

In the second chapter we shall define certain maximal functions which will play an important role in the sequel. In the third chapter we shall prove two theorems: the Whitney decomposition and (a generalization of) the Calderón-Zygmund decomposition. In the fourth chapter we shall define the spaces $H^1_{at}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ and prove that they are isomorphic. Finally, in the fifth chapter we shall define the space $BMO(\mathbb{R}^d)$ and give the proof of the main result:

$$(H^1(\mathbb{R}^d))^* \cong BMO(\mathbb{R}^d).$$

We denote $B(x,r)$ the open ball with center x and radius $r > 0$. The Euclidean norm and the Euclidean inner product are given by

$$|x| = \sqrt{x_1^2 + \cdots + x_d^2}, \qquad x \cdot y = x_1 y_1 + \cdots + x_d y_d$$

when $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

If $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_d, \qquad \partial^\beta = \partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}$$

and we write $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for every $j = 1, \ldots, d$.

The Lebesgue measure of a Lebesgue measurable $A \subseteq \mathbb{R}^d$ is denoted $|A|$.

The Schartz space $\mathcal{S}(\mathbb{R}^d)$ contains all functions $\phi \in C^\infty(\mathbb{R}^d)$ which satisfy

$$\sup_x (1 + |x|^2)^{m/2} |\partial^\beta \phi(x)| < +\infty$$

for every $m \in \mathbb{N}_0$ and $\beta \in \mathbb{N}_0^d$. For each $m \in \mathbb{N}_0$ we define the following norm on $\mathcal{S}(\mathbb{R}^d)$:

$$p_m(\phi) = \sup_{x, |\beta| \leq m} (1 + |x|^2)^{m/2} |\partial^\beta \phi(x)|, \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

The norms p_m, $m \in \mathbb{N}_0$, induce a translation invariant metric on $\mathcal{S}(\mathbb{R}^d)$ defined by

$$d(\phi, \psi) = \sum_{m=0}^{+\infty} \frac{1}{2^m} \frac{p_m(\phi - \psi)}{1 + p_m(\phi - \psi)}, \qquad \phi, \psi \in \mathcal{S}(\mathbb{R}^d).$$

For every $\alpha, \beta \in \mathbb{N}_0^d$ we define the following seminorm on the Schwartz space:

$$\|\phi\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \phi(x)|, \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

There are positive constants $c_{m,d}$ and $c'_{m,d}$, depending only on m and d such that

$$c_{m,d}\, p_m(\phi) \le \max_{|\alpha|,|\beta|\le m} \|\phi\|_{\alpha,\beta} \le c'_{m,d}\, p_m(\phi), \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Thus, the collection of all seminorms $\|\cdot\|_{\alpha,\beta}$ defines the same locally convex topology on the Schwartz space as the metric topology defined by the family of all p_m. The Schwartz space is complete and hence a Fréchet space.

When we say that $\mathcal{F}$ is a family of seminorms on $\mathcal{S}(\mathbb{R}^d)$ we mean that $\mathcal{F}$ consists of $\|\cdot\|_{\alpha,\beta}$ for a certain collection of pairs $\alpha, \beta \in \mathbb{N}_0^d$.

If $\phi \in \mathcal{S}(\mathbb{R}^d)$, then the functions

$$\tau_y\phi(x) = \phi(x-y), \qquad \widetilde{\phi}(x) = \phi(-x), \qquad \phi_t(x) = t^{-d}\phi(t^{-1}x) \quad (t>0)$$

also belong to $\mathcal{S}(\mathbb{R}^d)$.

We denote $\mathcal{S}'(\mathbb{R}^d)$ the space of all continuous linear functionals of $\mathcal{S}(\mathbb{R}^d)$. The elements of $\mathcal{S}'(\mathbb{R}^d)$ are called *tempered distributions*. A linear functional $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ is continuous and hence belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if there is an $m \in \mathbb{N}_0$ and a constant c so that

$$|f(\phi)| \le c\, p_m(\phi), \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

Now let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. We define the function $f * \phi : \mathbb{R}^d \to \mathbb{C}$ by

$$f * \phi(x) = f(\tau_x \widetilde{\phi}).$$

The convolution $f * \phi$ is a smooth function of polynomial growth. To prove this we observe that $\tau_x\widetilde{\phi}(y) = \phi(x-y)$ and we write

$$\begin{aligned} p_m(\tau_x\widetilde{\phi}) &= \sup_{y,\,|\beta|\le m} (1+|y|^2)^{m/2} |\partial^\beta \phi(x-y)| = \sup_{y,\,|\beta|\le m} (1+|x-y|^2)^{m/2} |\partial^\beta \phi(y)| \\ &\le 2^{m/2}(1+|x|^2)^{m/2} \sup_{y,\,|\beta|\le m} (1+|y|^2)^{m/2}|\partial^\beta\phi(y)| = 2^{m/2}(1+|x|^2)^{m/2}\, p_m(\phi), \end{aligned}$$

where for the first inequality we used the $(1+|x-y|^2) \le 2(1+|x|^2)(1+|y|^2)$. Since $f \in \mathcal{S}'(\mathbb{R}^d)$, there is an $m \in \mathbb{N}_0$ and a constant c so that $|f(\phi)| \le c\, p_m(\phi)$ and thus

$$|f * \phi(x)| = |f(\tau_x\widetilde{\phi})| \le c\, p_m(\tau_x\widetilde{\phi}) \le c 2^{m/2} p_m(\phi)(1+|x|^2)^{m/2}. \tag{1.1}$$

Therefore the function $f*\phi$ induces a tempered distribution. If we denote this tempered distribution with the same symbol $f * \phi$, we have

$$\begin{aligned} f * \phi(\psi) &= \int_{\mathbb{R}^d} \psi(x)(f*\phi)(x)\, dx = \int_{\mathbb{R}^d} \psi(x) f(\tau_x\widetilde{\phi})\, dx \\ &= f\Big(\int_{\mathbb{R}^d} \psi(x)\, \tau_x\widetilde{\phi}\, dx\Big) = f(\psi * \widetilde{\phi}) \end{aligned} \tag{1.2}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. Of course, the third equality above needs careful justification and this can be done by approximating $\int_{\mathbb{R}^d} \psi(x)\, \tau_x\widetilde{\phi}\, dx$ with appropriate Riemann sums.

The Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i\, x\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^d.$$

The function $\widehat{f}$ is continuous on $\mathbb{R}^d$ and $\widehat{f}(\xi) \to 0$ as $|\xi| \to +\infty$. Moreover, $\|\widehat{f}\|_\infty \le \|f\|_1$. We also have the *Fourier inversion formula* which says that, if also $\widehat{f} \in L^1(\mathbb{R}^d)$, then

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{2\pi i\, x\cdot\xi}\, d\xi, \qquad \text{a.e. } x \in \mathbb{R}^d.$$

If, besides $f \in L^1(\mathbb{R}^d)$, the functions $x^\alpha f(x)$, $|\alpha| \le M$, also belong to $L^1(\mathbb{R}^d)$, then

$$\partial^\alpha \widehat{f}(\xi) = \int_{\mathbb{R}^d} (-2\pi i x)^\alpha f(x) e^{-2\pi i\, x\cdot\xi}\, dx = \big((-2\pi i x)^\alpha f(x)\big)\widehat{\ }(\xi), \qquad \xi \in \mathbb{R}^d,$$

for all α with $|\alpha| \le M$.

If, besides $f \in L^1(\mathbb{R}^d)$, the functions $\partial^\alpha f$, $|\alpha| \le M$, also belong to $L^1(\mathbb{R}^d)$, then

$$(2\pi i\xi)^\alpha\, \widehat{f}(\xi) = (-1)^{|\alpha|} \int_{\mathbb{R}^d} f(x) \partial_x^\alpha e^{-2\pi i\, x\cdot\xi}\, dx = \int_{\mathbb{R}^d} \partial^\alpha f(x)\, e^{-2\pi i\, x\cdot\xi}\, dx = \widehat{\partial^\alpha f}(\xi), \ \ \xi \in \mathbb{R}^d,$$

for all α with $|\alpha| \le M$.

Based on these properties of the Fourier transform, we can see that it is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto itself.

By $c_{p,q,\dots}$ we denote a constant which depends upon the quantities $p, q, \dots$. The constant $c_{p,q,\dots}$ may depend upon other quantities besides those which are explicitly written, but these extra quantities must be mentioned. *We may use the same symbol $c_{p,q,\dots}$ for different constants even if they occur in the same argument.*

Chapter 2

Maximal functions

The well-known Hardy-Littlewood maximal function for functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

It is a fact that the operator M is *strong*-(p,p) for all p with $1 < p \leq +\infty$:

$$\|Mf\|_p \leq c_{p,d}\, \|f\|_p, \qquad f \in L^p(\mathbb{R}^d).$$

Also M is *weak*-$(1,1)$:

$$|\{x \mid Mf(x) > \lambda\}| \leq \frac{c}{\lambda}\, \|f\|_1, \qquad \lambda > 0,\ f \in L^1(\mathbb{R}^d).$$

Now let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and consider a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$. We define the following maximal functions:

$$\begin{aligned} M_\Phi f(x) &= \sup_{t>0} |f * \Phi_t(x)|, \\ M^*_\Phi f(x) &= \sup_{t>0,\, |y-x|<t} |f * \Phi_t(y)|, \\ M_{\Phi,d} f(x) &= \sup_{t>0,\, y} |f * \Phi_t(y)| \Big(1 + \frac{|y-x|}{t}\Big)^{-(d+1)}. \end{aligned}$$

Now, $|f * \Phi_t(x)| \leq \sup_{|y-x|<t} |f * \Phi_t(y)|$ and hence

$$M_\Phi f(x) \leq M^*_\Phi f(x).$$

Moreover, if $|y-x| < t$, then $|f * \Phi_t(y)| \leq 2^{d+1} |f * \Phi_t(y)| (1 + \frac{|y-x|}{t})^{-(d+1)}$. Therefore,

$$M^*_\Phi f(x) \leq 2^{d+1} M_{\Phi,d} f(x).$$

We just proved the inequality

$$M_\Phi f \leq M^*_\Phi f \leq 2^{d+1} M_{\Phi,d} f, \qquad f \in \mathcal{S}'(\mathbb{R}^d)$$

which implies, for example, $\|M_\Phi f\|_1 \leq \|M^*_\Phi f\|_1 \leq 2^{d+1}\, \|M_{\Phi,d} f\|_1$ or other similar inequalities related to magnitude.

Now, if $\mathcal{F}$ is a finite collection of seminorms on the Schwartz space, we define a new maximal function by

$$M_{\mathcal{F}} f(x) = \sup_{\Phi \in \mathcal{S}_{\mathcal{F}}} M_\Phi f(x),$$

where $\mathcal{S}_{\mathcal{F}}$ is the set defined by

$$\mathcal{S}_{\mathcal{F}} = \{\Phi \in \mathcal{S}(\mathbb{R}^d), \textstyle\int_{\mathbb{R}^d} \Phi \neq 0 \mid \|\Phi\|_{\alpha,\beta} \leq 1 \text{ for every } \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}.$$

We state our first theorem.

Theorem 2.1. *(i) Let $\mathcal{F}$ be any finite collection of seminorms. Then for every $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ there is a constant $c_{\Phi,\mathcal{F}} > 0$ so that*

$$\|M_\Phi f\|_1 \leq c_{\Phi,\mathcal{F}} \|M_\mathcal{F} f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

(ii) There is a fixed finite collection $\mathcal{F}_d$ of seminorms with the property: for every $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ there is a constant $c_{\Phi,d} > 0$ so that

$$\|M_{\mathcal{F}_d} f\|_1 \leq c_{\Phi,d} \|M_\Phi f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Proof. (i) Since the collection $\mathcal{F}$ is finite, there is a constant $c_{\Phi,\mathcal{F}} > 0$ depending only on $\mathcal{F}$ and Φ such that $\|\Phi\|_{\alpha,\beta} \leq c_{\Phi,\mathcal{F}}$ for every seminorm $\|\cdot\|_{\alpha,\beta}$ in $\mathcal{F}$. Thus, $\frac{\Phi}{c_{\Phi,\mathcal{F}}} \in \mathcal{S}_\mathcal{F}$ and hence

$$\|M_\Phi f\|_1 \leq c_{\Phi,\mathcal{F}} \|M_\mathcal{F} f\|_1.$$

(ii) This proof will be a consequence of certain lemmas and their corrolaries. □

Lemma 2.1. *let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then there is a constant c_d so that*

$$\int_{\mathbb{R}^d} \sup_{t>0,\,|y-x|<at} |f * \Phi_t(y)|\,dx \leq c_d\,(1+a)^d \int_{\mathbb{R}^d} \sup_{t>0,\,|y-x|<t} |f * \Phi_t(y)|\,dx$$

for every $a > 0$.

Proof. We shall prove

$$\left|\left\{x \,\middle|\, \sup_{t>0,\,|y-x|<at} |f * \Phi_t(y)| > \lambda\right\}\right| \leq c_d\,(1+a)^d \left|\left\{x \,\middle|\, \sup_{t>0,\,|y-x|<t} |f * \Phi_t(y)| > \lambda\right\}\right| \tag{2.1}$$

for every $\lambda > 0$. This is enough to finish the proof of the lemma: we just integrate (2.1) with respect to λ using the well-known identity $\int_{\mathbb{R}^d} |g(x)|\,dx = \int_0^{+\infty} |\{x \mid |g(x)| > \lambda\}|\,d\lambda$.
It is clearly enough to prove (2.1) when $a > 1$.
We define the sets:

$$C = \left\{x \,\middle|\, \sup_{t>0,\,|y-x|<at} |f * \Phi_t(y)| > \lambda\right\}, \quad O = \left\{x \,\middle|\, \sup_{t>0,\,|y-x|<t} |f * \Phi_t(y)| > \lambda\right\}.$$

When $0 < \gamma < 1$ we define

$$O_\gamma = \left\{x \,\middle|\, \frac{|O \cap B(x,r)|}{|B(x,r)|} > \gamma \text{ for some } r > 0\right\}. \tag{2.2}$$

Now let $x \in C$. Then there are $\bar{y}$ and $\bar{t} > 0$ with $|\bar{y} - x| < a\bar{t}$ such that $|f * \Phi_{\bar{t}}(\bar{y})| > \lambda$.
If $|\bar{y} - z| < \bar{t}$, then $\lambda < |f * \Phi_{\bar{t}}(\bar{y})| \leq \sup_{t>0,\,|y-z|<t} |f * \Phi_t(y)|$ and thus $z \in O$. Hence $B(\bar{y},\bar{t}) \subseteq O$. Moreover, it is clear that $B(\bar{y},\bar{t}) \subseteq B(x,(1+a)\bar{t})$ and we get $B(\bar{y},\bar{t}) \subseteq O \cap B(x,(1+a)\bar{t})$. Therefore, $|O \cap B(x,(1+a)\bar{t})| \geq |B(\bar{y},\bar{t})|$ and we find

$$\frac{|O \cap B(x,(1+a)\bar{t})|}{|B(x,(1+a)\bar{t})|} \geq \frac{1}{(1+a)^d}.$$

Now if we fix any γ with $0 < \gamma < \frac{1}{(1+a)^d}$, then we have $x \in O_\gamma$. Since $x \in C$ is arbitrary,

$$C \subseteq O_\gamma. \tag{2.3}$$

By the definition of the Hardy-Littlewood maximal function, we have

$$M(\chi_O)(x) = \sup_{r>0} \frac{|O \cap B(x,r)|}{|B(x,r)|}.$$

Now (2.2) says

$$O_\gamma = \{x \mid M(\chi_O)(x) > \gamma\}$$

and, since the Hardy-Littlewood maximal operator is weak-$(1,1)$, we get

$$|O_\gamma| \le \frac{c_d}{\gamma} \int_{\mathbb{R}^d} \chi_O(x)\, dx = \frac{c_d}{\gamma}\, |O|.$$

Therefore, (2.3) implies

$$|C| \le \frac{c_d}{\gamma}\, |O|.$$

Finally, we take the limit as $\gamma \to \frac{1}{(1+a)^d}-$ and we get (2.1). □

Corollary 2.1. *Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$. Then there is a constant c_d so that*

$$\|M_{\Phi,d} f\|_1 \le c_d\, \|M^*_\Phi f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Proof. For every $s > 0$ and x, z we have

$$|f * \Phi_s(z)| \Big(1 + \frac{|z-x|}{s}\Big)^{-(d+1)} \le \sum_{k=0}^{+\infty} 2^{(1-k)(d+1)} \sup_{t>0,\, |y-x|<2^k t} |f * \Phi_t(y)|. \tag{2.4}$$

Indeed, if $|z - x| < s$, then the term corresponding to $k = 0$ of the sum on the right side is not smaller than the left side of (2.4). Also, if $2^{k-1}s \le |z - x| < 2^k s$ for some $k \ge 1$, then $(1 + \frac{|z-x|}{s})^{-(d+1)} \le 2^{(1-k)(d+1)}$ and $|f * \Phi_s(z)| \le \sup_{t>0,\, |y-x|<2^k t} |f * \Phi_t(y)|$.
Now we take the supremum with respect to $s > 0$ and z of the left side of (2.4) and we integrate with respect to x:

$$\|M_{\Phi,d} f\|_1 \le \sum_{k=0}^{+\infty} 2^{(1-k)(d+1)} \int_{\mathbb{R}^d} \sup_{t>0,\, |y-x|<2^k t} |f * \Phi_t(y)|\, dx. \tag{2.5}$$

The result of Lemma 2.1 with $a = 2^k$ gives

$$\int_{\mathbb{R}^d} \sup_{t>0,\, |y-x|<2^k t} |f * \Phi_t(y)|\, dx \le c_d\, (1+2^k)^d \int_{\mathbb{R}^d} \sup_{t>0,\, |y-x|<t} |f * \Phi_t(y)|\, dx = \|M^*_\Phi f\|_1.$$

Therefore (2.5) implies

$$\|M_{\Phi,d} f\|_1 \le c_d \sum_{k=0}^{+\infty} 2^{(1-k)(d+1)} (1+2^k)^d \|M^*_\Phi f\|_1 = c_d\, \|M^*_\Phi f\|_1$$

and the proof is complete. □

Lemma 2.2. *Take $\alpha, \beta \in \mathbb{N}_0^d$, $M \ge 0$ and $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and let $\mathcal{F} = \mathcal{F}_{\alpha,\beta,M,d}$ be the collection of seminorms $\|\cdot\|_{\alpha',\beta'}$ with $|\alpha'| \le |\alpha| + d + 1$ and $|\beta'| \le |\beta| + [M] + d + 1$. Then for every $\Psi \in \mathcal{S}_\mathcal{F}$ there is a sequence of functions $\eta_k \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\Psi = \sum_{k=0}^{+\infty} \eta_k * \Phi_{2^{-k}}.$$

Moreover,

$$\|\eta_k\|_{\alpha,\beta} = \sup_x |x^\alpha \partial^\beta \eta_k(x)| \le c_{\alpha,\beta,M,\Phi,d}\, 2^{-kM}$$

for all k.

Proof. We take any $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. Then $\chi \in \mathcal{S}(\mathbb{R}^d)$ and, since the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$, there is a $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\widehat{\phi}(\xi) = \chi(\xi)$$

for all ξ. We also define the functions $\widehat{\psi}_k \in \mathcal{S}(\mathbb{R}^d)$ by

$$\widehat{\psi_0}(\xi) = \widehat{\phi}(\xi), \qquad \widehat{\psi_k}(\xi) = \widehat{\phi}(2^{-k}\xi) - \widehat{\phi}(2^{-(k-1)}\xi), \quad k \geq 1,$$

for all ξ. We observe that $\mathrm{supp}(\widehat{\psi_0}) \subseteq \{\xi \,|\, |\xi| \leq 2\}$ and $\mathrm{supp}(\widehat{\psi_k}) \subseteq \{\xi \,|\, 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ for $k \geq 1$. Also:

$$\sup_{\xi} |\partial^\beta \widehat{\psi_k}(\xi)| \leq c_\beta\, 2^{-k|\beta|}. \tag{2.6}$$

Now $\sum_{k=0}^n \widehat{\psi_k}(\xi) = \widehat{\phi}(2^{-n}\xi) \to \widehat{\phi}(0) = \chi(0) = 1$ as $n \to +\infty$ and hence

$$\sum_{k=0}^{+\infty} \widehat{\psi_k}(\xi) = 1 \tag{2.7}$$

for all ξ. We may assume $\widehat{\Phi}(0) = \int_{\mathbb{R}^d} \Phi = 1$. Hence there is $k_0 \in \mathbb{N}$ such that

$$|\widehat{\Phi}(\xi)| \geq 1/2, \qquad |\xi| \leq 2^{-(k_0-1)}.$$

And now we define the functions η_k by

$$\widehat{\eta_k}(\xi) = 0, \quad 1 \leq k \leq k_0 - 1, \qquad \widehat{\eta_k}(\xi) = \frac{\widehat{\psi_{k-k_0}}(\xi)}{\widehat{\Phi}(2^{-k}\xi)}\, \widehat{\Psi}(\xi), \quad k \geq k_0,$$

for all ξ. We have $\widehat{\eta_k} \in \mathcal{S}(\mathbb{R}^d)$, hence $\eta_k \in \mathcal{S}(\mathbb{R}^d)$. Finally, from (2.7),

$$\widehat{\Psi}(\xi) = \sum_{k=k_0}^{+\infty} \widehat{\eta_k}(\xi)\widehat{\Phi}(2^{-k}\xi) = \sum_{k=0}^{+\infty} \widehat{\eta_k}(\xi)\widehat{\Phi}(2^{-k}\xi)$$

for all ξ, and this implies the representation of Ψ in the statement of the lemma. Now, by the Fourier inversion formula,

$$\begin{aligned}
|x^\alpha \partial^\beta \eta_k(x)| &= \Big| \int_{\mathbb{R}^d} \big(x^\alpha \partial^\beta \eta_k(x)\big)\widehat{\ }(\xi) e^{2\pi i \xi \cdot x}\, d\xi \Big| \\
&= (2\pi)^{|\beta|-|\alpha|} \Big| \int_{\mathbb{R}^d} \partial^\alpha(\xi^\beta \widehat{\eta_k}(\xi)) e^{2\pi i \xi\cdot x}\, d\xi \Big| \\
&\leq (2\pi)^{|\beta|-|\alpha|} \int_{\mathbb{R}^d} |\partial^\alpha(\xi^\beta \widehat{\eta_k}(\xi))|\, d\xi \\
&= (2\pi)^{|\beta|-|\alpha|} \int_{2^{k-k_0-1} \leq |\xi| \leq 2^{k-k_0+1}} |\partial^\alpha(\xi^\beta \widehat{\eta_k}(\xi))|\, d\xi.
\end{aligned} \tag{2.8}$$

for all x. We temporarily set

$$\widehat{\eta_k}(\xi) = g(2^{-k}\xi)\widehat{\Psi}(\xi), \qquad k \geq k_0,$$

for all ξ, where

$$g(\xi) = \frac{\widehat{\phi}(2^{k_0}\xi) - \widehat{\phi}(2^{k_0+1}\xi)}{\widehat{\Phi}(\xi)},$$

and we shall estimate the quantity inside the last integral of (2.8). We have

$$\partial^\alpha(\xi^\beta \widehat{\eta_k}(\xi)) = \sum_{\delta \leq \alpha,\beta} c_{\alpha,\delta}\, \partial^\delta \xi^\beta\, \partial^{\alpha-\delta} \widehat{\eta_k}(\xi) = \sum_{\delta \leq \alpha,\beta} c_{\alpha,\beta,\delta} \xi^{\beta-\delta}\, \partial^{\alpha-\delta} \widehat{\eta_k}(\xi) \tag{2.9}$$

for all ξ. Also,

$$\begin{aligned}\partial^{\alpha-\delta}\widehat{\eta_k}(\xi) &= \partial^{\alpha-\delta}\big(g(2^{-k}\xi)\widehat{\Psi}(\xi)\big) = \sum_{\varepsilon\le\alpha-\delta} c_{\alpha,\delta,\varepsilon}\,\partial^{\varepsilon}(g(2^{-k}\xi))\,\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)\\ &= \sum_{\varepsilon\le\alpha-\delta} c_{\alpha,\delta,\varepsilon}2^{-k|\varepsilon|}\,\partial^{\varepsilon}g(2^{-k}\xi)\,\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)\end{aligned}\tag{2.10}$$

for all ξ. By the definition of g we get $\sup_\xi |\partial^\varepsilon g(\xi)| \le c_{\varepsilon,\Phi}$ and (2.9) and (2.10) imply

$$|\partial^\alpha(\xi^\beta\widehat{\eta_k}(\xi))| \le \sum_{\delta\le\alpha,\beta}\sum_{\varepsilon\le\alpha-\delta} c_{\alpha,\beta,\delta,\varepsilon,\Phi}|\xi|^{|\beta-\delta|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|$$

for all ξ. Thus,

$$\begin{aligned}&\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}}|\partial^\alpha(\xi^\beta\widehat{\eta_k}(\xi))|\,d\xi\\ &\le \sum_{\delta\le\alpha,\beta}\sum_{\varepsilon\le\alpha-\delta} c_{\alpha,\beta,\delta,\varepsilon,\Phi}\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}}|\xi|^{|\beta-\delta|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|\,d\xi.\end{aligned}\tag{2.11}$$

To estimate the last integral in (2.11) we consider any $\gamma\in\mathbb{N}_0^d$ with $|\gamma| = [M]+1+d$ and write

$$\begin{aligned}&\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}}|\xi|^{|\beta-\delta|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|\,d\xi\\ &\qquad = \int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}}\frac{|\xi|^{|\beta-\delta|+|\gamma|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|}{|\xi|^{|\gamma|}}\,d\xi.\end{aligned}\tag{2.12}$$

Now

$$\begin{aligned}|\xi|^{|\beta-\delta|+|\gamma|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)| &\le (|\xi_1|+\cdots+|\xi_d|)^{|\beta-\delta|+|\gamma|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|\\ &\le \sum_{|\zeta|=|\beta-\delta|+|\gamma|} c_{\zeta,\beta,\gamma,\delta,d}|\xi^\zeta\,\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|\\ &\le \sum_{|\zeta|=|\beta-\delta|+|\gamma|} c_{\zeta,\beta,\gamma,\delta,\alpha,\varepsilon,d}\big|\big(\partial^\zeta(x^{\alpha-\delta-\varepsilon}\Psi(x))\big)\widehat{\ }(\xi)\big|\end{aligned}\tag{2.13}$$

for all ξ. For the summands in (2.13) we have

$$\begin{aligned}&\big|\big(\partial^\zeta(x^{\alpha-\delta-\varepsilon}\Psi(x))\big)\widehat{\ }(\xi)\big| \le \int_{\mathbb{R}^d}\big|\partial^\zeta(x^{\alpha-\delta-\varepsilon}\Psi(x))\big|\,dx\\ &\le \int_{\mathbb{R}^d}\sum_{\theta\le\zeta,\alpha-\delta-\varepsilon} c_{\zeta,\theta}\,|x^{\alpha-\delta-\varepsilon-\theta}\,\partial^{\zeta-\theta}\Psi(x)|\,dx\\ &= \int_{\mathbb{R}^d}\frac{\sum_{\theta\le\zeta,\alpha-\delta-\varepsilon} c_{\zeta,\theta}\,(1+|x|)^{d+1}|x^{\alpha-\delta-\varepsilon-\theta}\,\partial^{\zeta-\theta}\Psi(x)|}{(1+|x|)^{d+1}}\,dx\\ &\le \int_{\mathbb{R}^d}\frac{\sum_{\theta\le\zeta,\alpha-\delta-\varepsilon}\sum_{|\lambda|\le d+1} c_{\zeta,\theta,\lambda}\,|x^{\lambda+\alpha-\delta-\varepsilon-\theta}\,\partial^{\zeta-\theta}\Psi(x)|}{(1+|x|)^{d+1}}\,dx\end{aligned}\tag{2.14}$$

for all ξ. We observe that, due to the restrictions imposed on $\lambda,\delta,\varepsilon,\zeta,\theta$, we have

$$|\lambda+\alpha-\delta-\varepsilon-\theta| \le |\alpha|+d+1, \qquad |\zeta-\theta| \le |\beta|+[M]+d+1.$$

Since $\mathcal{F}$ is the collection of seminorms $\|\cdot\|_{\alpha',\beta'}$ with $|\alpha'|\le|\alpha|+d+1$ and $|\beta'|\le|\beta|+[M]+d+1$, we conclude that, if $\Psi\in\mathcal{S}_{\mathcal{F}}$, then

$$\sup_x |x^{\lambda+\alpha-\delta-\varepsilon-\theta}\,\partial^{\zeta-\theta}\Psi(x)| \le \|\Psi\|_{\lambda+\alpha-\delta-\varepsilon-\theta,\zeta-\theta} \le 1.$$

Thus, (2.14) implies

$$\left|\left(\partial^{\zeta}(x^{\alpha-\delta-\varepsilon}\Psi(x))\right)\widehat{}(\xi)\right| \le c_{\alpha,\delta,\varepsilon,\zeta,d}$$

and from (2.13) we get

$$|\xi|^{|\beta-\delta|+|\gamma|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)| \le c_{\alpha,\beta,\gamma,\delta,\varepsilon,d}$$

for all ξ. Now, from (2.12) we have

$$\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}} |\xi|^{|\beta-\delta|}\,|\partial^{\alpha-\delta-\varepsilon}\widehat{\Psi}(\xi)|\,d\xi \\ \le c_{\alpha,\beta,\gamma,\delta,\varepsilon,d}\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}} \frac{1}{|\xi|^{|\gamma|}}\,d\xi = c_{\alpha,\beta,\gamma,\delta,\varepsilon,M,k_0,d}\,2^{-kM}.$$

Finally, (2.11) implies

$$\int_{2^{k-k_0-1}\le|\xi|\le 2^{k-k_0+1}} |\partial^{\alpha}(\xi^{\beta}\widehat{\eta_k}(\xi))|\,d\xi \le c_{\alpha,\beta,M,\Phi,d}\,2^{-kM}$$

and from (2.8) we get

$$\sup_x |x^{\alpha}\partial^{\beta}\eta_k(x)| \le c_{\alpha,\beta,M,\Phi,d}\,2^{-kM}$$

and the proof is complete. □

Corollary 2.2. *There is a fixed finite collection $\mathcal{F}_d$ of seminorms with the property: for every $\Phi\in\mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d}\Phi\neq 0$ there is a constant $c_{\Phi,d}>0$ so that*

$$M_{\mathcal{F}_d}f \le c_{\Phi,d}\,M_{\Phi,d}f, \qquad f\in\mathcal{S}'(\mathbb{R}^d).$$

Proof. We take $\alpha\in\mathbb{N}_0^d$ and we consider $\mathcal{F}_{\alpha,d}$ to be the collection of seminorms $\mathcal{F}_{\alpha,\beta,M,d}$ encountered in Lemma 2.2 with $\beta=(0,\ldots,0)$ and $M=2d+2$.
Now let $\Psi\in\mathcal{S}_{\mathcal{F}_{\alpha,d}}$. Then with the functions η_k as in Lemma 2.2 we have

$$\begin{aligned} M_{\Psi}f(x) &= \sup_{t>0}|f*\Psi_t(x)| \le \sup_{t>0}\sum_{k=0}^{+\infty}|f*(\eta_k*\Phi_{2^{-k}})_t(x)| \\ &= \sup_{t>0}\sum_{k=0}^{+\infty}|(f*\Phi_{2^{-k}t})*(\eta_k)_t(x)| \\ &= \sup_{t>0}\sum_{k=0}^{+\infty}\int_{\mathbb{R}^d}|(f*\Phi_{2^{-k}t})(x-y)|\,\frac{1}{t^d}\left|\eta_k\Big(\frac{y}{t}\Big)\right|dy \\ &\le \sup_{t>0}\sum_{k=0}^{+\infty}\int_{\mathbb{R}^d}M_{\Phi,d}f(x)\cdot\Big(1+\frac{|y|}{2^{-k}t}\Big)^{d+1}\frac{1}{t^d}\left|\eta_k\Big(\frac{y}{t}\Big)\right|dy \\ &= M_{\Phi,d}f(x)\sum_{k=0}^{+\infty}\int_{\mathbb{R}^d}(1+2^k|y|)^{d+1}|\eta_k(y)|\,dy \\ &\le M_{\Phi,d}f(x)\sum_{k=0}^{+\infty}2^{k(d+1)}\int_{\mathbb{R}^d}(1+|y|)^{d+1}|\eta_k(y)|\,dy \end{aligned} \tag{2.15}$$

for all x. If $|y|\le 1$,

$$(1+|y|)^{2d+2}|\eta_k(y)| \le 2^{2d+2}|\eta_k(y)|.$$

If $|y|\ge 1$,

$$(1+|y|)^{2d+2}|\eta_k(y)| \le 2^{2d+2}|y|^{2d+2}|\eta_k(y)| \le 2^{2d+2}\sum_{|\alpha|\le 2d+2}c_{\alpha}|y^{\alpha}\eta_k(y)|. \tag{2.16}$$

Since $\Psi \in \mathcal{S}_{\mathcal{F}_{\alpha,d}}$, Lemma 2.2 implies

$$\sup_y |y^\alpha \eta_k(y)| \le c_{\alpha,\Phi,d}\, 2^{-k(2d+2)}. \tag{2.17}$$

Now we consider the finite collection of seminorms

$$\mathcal{F}_d = \bigcup_{|\alpha|\le 2d+2} \mathcal{F}_{\alpha,d}$$

and take $c_{\Phi,d} = \max_{|\alpha|\le 2d+2} c_{\alpha,\Phi,d}$.
If $\Psi \in \mathcal{S}_{\mathcal{F}_d}$, then $\Psi \in \mathcal{S}_{\mathcal{F}_{\alpha,d}}$ for all α with $|\alpha| \le 2d+2$ and hence (2.16) and (2.17) imply

$$(1+|y|)^{2d+2}|\eta_k(y)| \le c_{\Phi,d}\, 2^{-k(2d+2)}$$

when $|y| \ge 1$ and from (2.15) we have

$$M_\Psi f(x) \le c_{\Phi,d}\, M_{\Phi,d} f(x) \sum_{k=0}^{+\infty} 2^{-k(d+1)} \int_{\mathbb{R}^d} \frac{1}{(1+|y|)^{d+1}}\, dy = c_{\Phi,d}\, M_{\Phi,d} f(x)$$

for all x. We conclude the proof by taking the supremum of the left side over all $\Psi \in \mathcal{S}_{\mathcal{F}_d}$. □

The collection of seminorms of Corollary 2.2 is

$$\mathcal{F}_d = \{\|\cdot\|_{\alpha,\beta} \mid |\alpha|, |\beta| \le 3d+3\}. \tag{2.18}$$

Lemma 2.3. *For every $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ there is a constant $c_{\Phi,d} > 0$ so that*

$$\|M^*_\Phi f\|_1 \le c_{\Phi,d} \|M_\Phi f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Proof. (i) We initially assume that $\|M^*_\Phi f\|_1 < +\infty$.
We consider the collection $\mathcal{F}_d$ given by (2.18), and which appears in Corollary 2.2, and for each $\lambda > 0$ we define the set:

$$E_\lambda = \{x \mid M_{\mathcal{F}_d} f(x) \le \lambda M^*_\Phi f(x)\}.$$

Then

$$\int_{E^c_\lambda} M^*_\Phi f(x)\, dx \le \frac{1}{\lambda} \int_{E^c_\lambda} M_{\mathcal{F}_d} f(x)\, dx \le \frac{1}{\lambda} \|M_{\mathcal{F}_d} f\|_1 \le \frac{c_{\Phi,d}}{\lambda} \|M_{\Phi,d} f\|_1 \le \frac{c_{\Phi,d}}{\lambda} \|M^*_\Phi f\|_1$$

where the last two inequalities come from Corollary 2.2 and Corollary 2.1, respectively. If we choose $\lambda = 2c_{\Phi,d}$, then $\int_{E^c_\lambda} M^*_\Phi f(x)\, dx \le \frac{1}{2} \|M^*_\Phi f\|_1$ and hence

$$\|M^*_\Phi f\|_1 \le 2 \int_{E_\lambda} M^*_\Phi f(x)\, dx. \tag{2.19}$$

We fix the λ we have chosen and for simplicity we write $E = E_\lambda$.
Now we take any x and then there are $\bar{y}, \bar{t}$ such that $|\bar{y} - x| < \bar{t}$ and

$$|f * \Phi_{\bar{t}}(\bar{y})| \ge \frac{1}{2} M^*_\Phi f(x). \tag{2.20}$$

We consider a small $r > 0$, to be made precise later on, and the ball $B(\bar{y}, r\bar{t})$.
For every $x' \in B(\bar{y}, r\bar{t})$ there is $x'' = (1-c)\bar{y} + cx'$ for some $c \in [0,1]$ such that

$$|f * \Phi_{\bar{t}}(x') - f * \Phi_{\bar{t}}(\bar{y})| \le |x' - \bar{y}|\, |\nabla(f * \Phi_{\bar{t}})(x'')|.$$

Since $|x'-\bar{y}|\le r\bar{t}$, we have $|x''-\bar{y}|\le r\bar{t}$ and hence

$$|f*\Phi_{\bar{t}}(x')-f*\Phi_{\bar{t}}(\bar{y})|\le r\bar{t}\sup_{|z-\bar{y}|\le r\bar{t}}|\nabla(f*\Phi_{\bar{t}})(z)|.$$

From $|\bar{y}-x|\le\bar{t}$ we get

$$\begin{aligned}|f*\Phi_{\bar{t}}(x')-f*\Phi_{\bar{t}}(\bar{y})|&\le r\bar{t}\sup_{|z-x|\le(r+1)\bar{t}}|\nabla(f*\Phi_{\bar{t}})(z)|\\&=r\bar{t}\sup_{|z|\le(r+1)\bar{t}}|\nabla(f*\Phi_{\bar{t}})(x+z)|.\end{aligned}\tag{2.21}$$

We also have

$$\partial_{z_j}(f*\Phi_{\bar{t}})(y)=\frac{1}{\bar{t}}\,f*(\partial_{z_j}\Phi)_{\bar{t}}(y)\tag{2.22}$$

for all y. We consider any $j=1,\dots,d$ and the subset

$$A_j=\{\partial_{z_j}\Phi(\cdot+h)\,|\,|h|\le r+1\}$$

of $\mathcal{S}(\mathbb{R}^d)$. We take a sequence $\partial_{z_j}\Phi(\cdot+h_n)$ with $|h_n|\le r+1$. Then there are h_{n_k} such that $h_{n_k}\to h$ for some h with $|h|\le r+1$. Hence $\partial_{z_j}\Phi(\cdot+h_{n_k})\to\partial_{z_j}\Phi(\cdot+h)$ in $\mathcal{S}(\mathbb{R}^d)$ and $\partial_{z_j}\Phi(\cdot+h)\in A_j$. We have proved that each A_j is a compact subset of $\mathcal{S}(\mathbb{R}^d)$. If we assume $r\le 1$, then the compact set A_j depends only on Φ and j. Since every seminorm $\|\cdot\|_{\alpha,\beta}$ of the collection $\mathcal{F}_d$ is continuous, there are finite constants $c_{\alpha,\beta,j,\Phi}$ so that

$$\|\partial_{z_j}\Phi(\cdot+h)\|_{\alpha,\beta}\le c_{\alpha,\beta,j,\Phi}$$

for $|h|\le r+1\le 2$. But $\mathcal{F}_d$ is finite and we may consider the finite constant $c_{\Phi,d}=\max c_{\alpha,\beta,j,\Phi}$ for all $\|\cdot\|_{\alpha,\beta}$ of $\mathcal{F}_d$ and all $j=1,\dots,d$ and then we have

$$\frac{1}{c_{\Phi,d}}\,\partial_{z_j}\Phi(\cdot+h)\in\mathcal{S}_{\mathcal{F}_d},\qquad |h|\le r+1,\ j=1,\dots,d.\tag{2.23}$$

Now

$$\begin{aligned}f*(\partial_{z_j}\Phi)_{\bar{t}}(x+h\bar{t})&=f\big(\tau_{x+h\bar{t}}\widetilde{(\partial_{z_j}\Phi)_{\bar{t}}}\big)=f\big(\tau_x\big(\tau_{h\bar{t}}\widetilde{(\partial_{z_j}\Phi)_{\bar{t}}}\big)\big)=f\big(\tau_x\big(\tau_{-h\bar{t}}\widetilde{(\partial_{z_j}\Phi)_{\bar{t}}}\big)\big)\\&=f\big(\tau_x\big(\widetilde{(\tau_{-h}(\partial_{z_j}\Phi))_{\bar{t}}}\big)\big)=f\big(\tau_x\widetilde{\Psi_{\bar{t}}}\big)=f*\Psi_{\bar{t}}(x)\end{aligned}$$

where $\Psi(y)=\tau_{-h}\partial_{z_j}\Phi(y)=\partial_{z_j}\Phi(y+h)$ for all y. Now (2.23) says that $\frac{\Psi}{c_{\Phi,d}}\in\mathcal{S}_{\mathcal{F}_d}$ if $|h|\le r+1$ and hence

$$|f*(\partial_{z_j}\Phi)_{\bar{t}}(x+h\bar{t})|=|f*\Psi_{\bar{t}}(x)|\le c_{\Phi,d}\,M_{\mathcal{F}_d}f(x),\qquad |h|\le r+1,\ j=1,\dots,d.$$

Combining with (2.22),

$$|\nabla(f*\Phi_{\bar{t}})(x+h\bar{t}|=\frac{1}{\bar{t}}\Big(\sum_{j=1}^{d}|f*(\partial_{z_j}\Phi)_{\bar{t}}(x+h\bar{t})|^2\Big)^{1/2}\le\frac{c_{\Phi,d}\sqrt{d}}{\bar{t}}\,M_{\mathcal{F}_d}f(x)$$

for $|h|\le r+1$. Thus

$$\sup_{|z|\le(r+1)\bar{t}}|\nabla(f*\Phi_{\bar{t}})(x+z)|\le\frac{c_{\Phi,d}\sqrt{d}}{\bar{t}}\,M_{\mathcal{F}_d}f(x)$$

and (2.21) implies

$$|f*\Phi_{\bar{t}}(x')-f*\Phi_{\bar{t}}(\bar{y})|\le rc_{\Phi,d}\,\sqrt{d}\,M_{\mathcal{F}_d}f(x).$$

Now, if we consider $x \in E$, then we have

$$|f * \Phi_{\bar t}(x') - f * \Phi_{\bar t}(\bar y)| \le r c_{\Phi,d} \sqrt{d}\, M_\Phi^* f(x).$$

Therefore, if we choose $r = \min\{1, 1/(4c_{\Phi,d}\sqrt{d})\}$, which depends only on d and Φ, then

$$\big|\, |f * \Phi_{\bar t}(x')| - |f * \Phi_{\bar t}(\bar y)|\, \big| \le |f * \Phi_{\bar t}(x') - f * \Phi_{\bar t}(\bar y)| \le \frac{1}{4}\, M_\Phi^* f(x)$$

and due to (2.20) we finally get

$$|f * \Phi_{\bar t}(x')| \ge \frac{1}{4}\, M_\Phi^* f(x) \tag{2.24}$$

for $x \in E$ and $x' \in B(\bar y, r\bar t)$.
Now we see that $B(\bar y, r\bar t) \subseteq B(x, (r+1)\bar t)$.
We fix q so that $0 < q < 1$ and, because of (2.24), we get

$$\begin{aligned} M_\Phi^* f(x)^q &\le \frac{4^q}{|B(\bar y, r\bar t)|} \int_{B(\bar y, r\bar t)} |f * \Phi_{\bar t}(x')|^q\, dx' \\ &\le 4^q \Big(\frac{r+1}{r}\Big)^d \frac{1}{|B(x,(r+1)\bar t)|} \int_{B(x,(r+1)\bar t)} |f * \Phi_{\bar t}(x')|^q\, dx \\ &\le c_{\Phi,d}\, \frac{1}{|B(x,(r+1)\bar t)|} \int_{B(x,(r+1)\bar t)} M_\Phi f(x')^q\, dx \\ &\le c_{\Phi,d}\, M\big((M_\Phi f)^q\big)(x), \end{aligned} \tag{2.25}$$

for $x \in E$, where M is the Hardy-Littlewood maximal operator.
Now (2.19) and (2.25) imply

$$\|M_\Phi^* f\|_1 \le 2c_{\Phi,d}^{1/q} \int_E M\big((M_\Phi f)^q\big)(x)^{1/q}\, dx \le 2c_{\Phi,d}^{1/q} \int_{\mathbb{R}^d} M\big((M_\Phi f)^q\big)(x)^{1/q}\, dx.$$

Since the Hardy-Littlewood maximal operator is strong-$(1/q, 1/q)$, we get

$$\|M_\Phi^* f\|_1 \le c_{q,\Phi,d} \int_{\mathbb{R}^d} \big((M_\Phi f)^q(x)\big)^{1/q}\, dx = c_{q,\Phi,d} \|M_\Phi f\|_1,$$

where the constant $c_{q,\Phi,d}$ depends on q, Φ and d. But the constant $c_{q,\Phi,d}$ can be considered to depend only on Φ and d since we may take $q = \frac{1}{2}$.
(ii) Now we continue with the case $\|M_\Phi^* f\|_1 = +\infty$ and we consider a modification of M_Φ^* as follows:

$$M_\Phi^{*,\varepsilon,L} f(x) = \sup_{t < \frac{1}{\varepsilon},\, |y-x|<t} |f * \Phi_t(y)|\, \frac{t^L}{(\varepsilon + t + \varepsilon |y|)^L}$$

for all x, with $0 < \varepsilon \le 1$ and $L > 0$ to be specified shortly.
Since f is a tempered distribution, we apply (1.1) for appropriate $m \ge 0$ and $c > 0$ depending on f and get

$$\begin{aligned} |f * \Phi_t(y)| &\le c\, 2^{m/2}\, (1+|y|^2)^{m/2}\, p_m(\Phi_t) \\ &= c\, 2^{m/2}\, (1+|y|^2)^{m/2} \sup_{z,\, |\beta| \le m} (1+|z|^2)^{m/2}\, |\partial^\beta \Phi_t(z)| \\ &= c\, 2^{m/2}\, \frac{1}{t^d}\, (1+|y|^2)^{m/2} \sup_{z,\, |\beta| \le m} (1+|z|^2)^{m/2}\, \frac{1}{t^{|\beta|}} \Big|\partial^\beta \Phi\Big(\frac{z}{t}\Big)\Big| \\ &= c\, 2^{m/2}\, \frac{1}{t^d}\, (1+|y|^2)^{m/2} \sup_{z,\, |\beta| \le m} (1+t^2|z|^2)^{m/2}\, \frac{1}{t^{|\beta|}}\, |\partial^\beta \Phi(z)| \\ &= c\, 2^{m/2}\, \frac{\max\{t^m, t^{-m}\}}{t^d}\, (1+|y|^2)^{m/2} \sup_{z,\, |\beta| \le m} (1+|z|^2)^{m/2}\, |\partial^\beta \Phi(z)| \\ &= c\, 2^{m/2}\, \frac{\max\{t^m, t^{-m}\}}{t^d}\, (1+|y|^2)^{m/2}\, p_m(\Phi). \end{aligned}$$

Now, if we take $L > m + d$, then for $t < \frac{1}{\varepsilon}$ and $|y - x| < t$ we have

$$\begin{aligned}
|f * \Phi_t(y)| \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} &\le c\, 2^{m/2} \frac{\max\{t^{L+m-d}, t^{L-m-d}\}}{(\varepsilon + t + \varepsilon|y|)^L} (1 + |y|^2)^{m/2}\, p_m(\Phi) \\
&\le c\, 2^{m/2} \frac{\max\{t^{L+m-d}, t^{L-m-d}\}}{\varepsilon^L (1 + |y|)^L} (1 + |y|^2)^{m/2}\, p_m(\Phi) \\
&\le c\, 2^{m/2} \frac{\max\{t^{L+m-d}, t^{L-m-d}\}}{\varepsilon^L} \frac{1}{(1 + |y|^2)^{(L-m)/2}}\, p_m(\Phi) \\
&\le c\, 2^{m/2} \frac{\max\{t^{L+m-d}, t^{L-m-d}\}}{\varepsilon^L} \frac{2^{(L-m)/2}(1 + |y - x|^2)^{(L-m)/2}}{(1 + |x|^2)^{(L-m)/2}}\, p_m(\Phi) \\
&\le c\, 2^{L/2} \frac{\max\{t^{L+m-d}, t^{L-m-d}\}}{\varepsilon^L} \frac{(1 + t^2)^{(L-m)/2}}{(1 + |x|^2)^{(L-m)/2}}\, p_m(\Phi) \\
&\le c\, 2^{L-(m/2)} \frac{\max\{t^{2L-d}, t^{L-m-d}\}\, p_m(\Phi)}{\varepsilon^L} \frac{1}{(1 + |x|^2)^{(L-m)/2}} \\
&\le c\, \frac{2^{L-(m/2)}\, p_m(\Phi)}{\varepsilon^{3L-d}} \frac{1}{(1 + |x|^2)^{(L-m)/2}}.
\end{aligned}$$

Thus,

$$M_\Phi^{*,\varepsilon,L} f(x) \le c\, \frac{2^{L-(m/2)}\, p_m(\Phi)}{\varepsilon^{3L-d}} \frac{1}{(1 + |x|^2)^{(L-m)/2}}$$

for all x. Since $L > m + d$, we have that $M_\Phi^{*,\varepsilon,L} f \in L^1(\mathbb{R}^d)$.
The second step is to define

$$M_{\Phi,d}^{\varepsilon,L} f(x) = \sup_{y,\, 0<t<\frac{1}{\varepsilon}} |f * \Phi_t(y)| \left(1 + \frac{|y - x|}{t}\right)^{-(d+1)} \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L}$$

and prove a variant of the result of Corollary 2.2. We start by modifying (2.15). We consider any x and let $t < \frac{1}{\varepsilon}$ and $|y - x| < t$. Then

$$\begin{aligned}
|f * \Psi_t(y)| \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} &\le \sum_{k=0}^{+\infty} |f * (\eta_k * \Phi_{2^{-k}})_t(y)| \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \\
&= \sum_{k=0}^{+\infty} |(f * \Phi_{2^{-k}t}) * (\eta_k)_t(y)| \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \\
&= \sum_{k=0}^{+\infty} \int_{\mathbb{R}^d} |(f * \Phi_{2^{-k}t})(y - z)| \frac{1}{t^d} \left|\eta_k\left(\frac{z}{t}\right)\right| dz\, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \\
&\le \sum_{k=0}^{+\infty} \int_{\mathbb{R}^d} M_{\Phi,d}^{\varepsilon,L} f(x) \left(1 + \frac{|y - z - x|}{2^{-k}t}\right)^{d+1} \qquad (2.26) \\
&\quad \frac{(\varepsilon + 2^{-k}t + \varepsilon|y - z|)^L}{(2^{-k}t)^L} \frac{1}{t^d} \left|\eta_k\left(\frac{z}{t}\right)\right| dz\, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \\
&\le M_{\Phi,d}^{\varepsilon,L} f(x) \sum_{k=0}^{+\infty} 2^{k(L+d+1)} \int_{\mathbb{R}^d} \left(2 + \frac{|z|}{t}\right)^{L+d+1} \frac{1}{t^d} \left|\eta_k\left(\frac{z}{t}\right)\right| dz \\
&= M_{\Phi,d}^{\varepsilon,L} f(x) \sum_{k=0}^{+\infty} 2^{k(L+d+1)} \int_{\mathbb{R}^d} (2 + |z|)^{L+d+1} |\eta_k(z)|\, dz.
\end{aligned}$$

Now, exactly as in the proof of Corollary 2.2 we see that, if $\mathcal{F}_d^L = \{\|\cdot\|_{\alpha,\beta} \mid |\alpha|, |\beta| \le L + 3d + 3\}$

and $\Psi \in \mathcal{S}_{\mathcal{F}_d^L}$, then (2.26) implies

$$|f * \Psi_t(y)| \, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \leq c_{\Phi,d,L} \, M_{\Phi,d}^{\varepsilon,L} f(x)$$

when $t < \frac{1}{\varepsilon}$ and $|y - x| < t$. Thus

$$M_{\Psi}^{*,\varepsilon,L} f(x) \leq c_{\Phi,d,L} \, M_{\Phi,d}^{\varepsilon,L} f(x)$$

for all x. Now we define

$$M_{\mathcal{F}_d^L}^{\varepsilon,L} f = \sup_{\Psi \in \mathcal{S}_{\mathcal{F}_d^L}} M_{\Psi}^{*,\varepsilon,L} f$$

and our result is:

$$M_{\mathcal{F}_d^L}^{\varepsilon,L} f \leq c_{\Phi,d,L} \, M_{\Phi,d}^{\varepsilon,L} f. \tag{2.27}$$

Before we continue, we must remark that all these estimates as well as the collection of seminorms $\mathcal{F}_d^L$ depend on L, which in turn depends on f, but *they do not depend on* ε.
Examining the proof of Lemma 2.1, we see that it goes without change, resulting to

$$\begin{aligned} &\int_{\mathbb{R}^d} \sup_{t < \frac{1}{\varepsilon},\, |y-x| < at} |f * \Phi_t(y)| \, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \, dx \\ &\qquad \leq c_d \, (1 + a)^d \int_{\mathbb{R}^d} \sup_{t < \frac{1}{\varepsilon},\, |y-x| < t} |f * \Phi_t(y)| \, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L} \, dx \end{aligned} \tag{2.28}$$

for every $a > 0$. The next step is to take (2.4) and write it as

$$\begin{aligned} &|f * \Phi_s(z)| \Big(1 + \frac{|z - x|}{s}\Big)^{-(d+1)} \frac{s^L}{(\varepsilon + s + \varepsilon|z|)^L} \\ &\qquad \leq \sum_{k=0}^{+\infty} 2^{(1-k)(d+1)} \sup_{t < \frac{1}{\varepsilon},\, |y-x| < 2^k t} |f * \Phi_t(y)| \, \frac{t^L}{(\varepsilon + t + \varepsilon|y|)^L}. \end{aligned} \tag{2.29}$$

when $0 < s < \frac{1}{\varepsilon}$. The proof is the same. Now, taking the supremum of the left side of (2.29) over z and $s < \frac{1}{\varepsilon}$, integrating the resulting inequality and using (2.28) with $a = 2^k$, we end up with a variation of the result of Corollary 2.1:

$$\|M_{\Phi,d}^{\varepsilon,L} f\|_1 \leq c_d \, \|M_{\Phi}^{*,\varepsilon,L} f\|_1.$$

This together with (2.27) imply

$$\|M_{\mathcal{F}_d^L}^{\varepsilon,L} f\|_1 \leq c_{\Phi,d,L} \, \|M_{\Phi}^{*,\varepsilon,L} f\|_1. \tag{2.30}$$

The last step is to rework the proof of part (i) using $\|M_{\Phi}^{*,\varepsilon,L} f\|_1 < +\infty$ instead of $\|M_{\Phi}^{*} f\|_1 < +\infty$. For each $\lambda > 0$ we define the set:

$$E_\lambda = \{x \,|\, M_{\mathcal{F}_d^L}^{\varepsilon,L} f(x) \leq \lambda M_{\Phi}^{*,\varepsilon,L} f(x)\}.$$

Then, due to (2.30),

$$\int_{E_\lambda^c} M_{\Phi}^{*,\varepsilon,L} f(x) \, dx \leq \frac{1}{\lambda} \int_{E_\lambda^c} M_{\mathcal{F}_d^L}^{\varepsilon,L} f(x) \, dx \leq \frac{1}{\lambda} \, \|M_{\mathcal{F}_d^L}^{\varepsilon,L} f\|_1 \leq \frac{c_{\Phi,d,L}}{\lambda} \|M_{\Phi}^{*,\varepsilon,L} f\|_1.$$

We choose $\lambda = 2c_{\Phi,d,L}$, and we find

$$\|M_{\Phi}^{*,\varepsilon,L} f\|_1 \leq 2 \int_{E_\lambda} M_{\Phi}^{*,\varepsilon,L} f(x) \, dx. \tag{2.31}$$

We fix the λ we have chosen and for simplicity we write $E = E_\lambda$.
Now we take any x and then there are $\bar{y}, \bar{t}$ such that $\bar{t} < \frac{1}{\varepsilon}$, $|\bar{y} - x| < \bar{t}$ and

$$|f * \Phi_{\bar{t}}(\bar{y})| \geq |f * \Phi_{\bar{t}}(\bar{y})| \, \frac{\bar{t}^L}{(\varepsilon + \bar{t} + \varepsilon|\bar{y}|)^L} \geq \frac{1}{2} \, M_\Phi^{*,\varepsilon,L} f(x).$$

From this point we repeat the proof of part (i) without change (except that we work with $\mathcal{F}_d^L$ instead of $\mathcal{F}_d$) and we find a small $r > 0$, depending on d, Φ and L (but not on ε), so that

$$|f * \Phi_{\bar{t}}(x')| \geq \frac{1}{4} \, M_\Phi^{*,\varepsilon,L} f(x)$$

for $x \in E$ and $x' \in B(\bar{y}, r\bar{t})$. Therefore

$$M_\Phi f(x') \geq \frac{1}{4} M_\Phi^{*,\varepsilon,L} f(x)$$

for $x \in E$ and all $x' \in B(\bar{y}, r\bar{t}) \subseteq B(x, (r+1)\bar{t})$. The rest of the proof is the same.
We fix q so that $0 < q < 1$ and we get

$$\begin{aligned} M_\Phi^{*,\varepsilon,L} f(x)^q &\leq \frac{4^q}{|B(\bar{y}, r\bar{t})|} \int_{B(\bar{y}, r\bar{t})} M_\Phi f(x')^q \, dx' \\ &\leq 4^q \Big(\frac{r+1}{r}\Big)^d \frac{1}{|B(x, (r+1)\bar{t})|} \int_{B(x,(r+1)\bar{t})} M_\Phi f(x')^q \, dx' \\ &\leq c_{\Phi,d,L} \, M\big((M_\Phi f)^q\big)(x), \end{aligned} \tag{2.32}$$

for $x \in E$, where M is the Hardy-Littlewood maximal operator. Now (2.31) and (2.32) imply

$$\begin{aligned} \|M_\Phi^{*,\varepsilon,L} f\|_1 &\leq 2 c_{\Phi,d,L}^{1/q} \int_E M\big((M_\Phi f)^q\big)(x)^{1/q} \, dx \leq c_{q,\Phi,d,L} \int_{\mathbb{R}^d} M\big((M_\Phi f)^q\big)(x)^{1/q} \, dx \\ &\leq c_{q,\Phi,d,L} \int_{\mathbb{R}^d} \big((M_\Phi f)^q(x)\big)^{1/q} \, dx = c_{q,\Phi,d,L} \|M_\Phi f\|_1. \end{aligned}$$

Now we observe that $M_\Phi^{*,\varepsilon,L} f \uparrow M_\Phi^* f$ as $\varepsilon \downarrow 0$. Taking the limit as $\varepsilon \to 0+$ in the last inequality, we find that, if $M_\Phi f \in L^1(\mathbb{R}^d)$, then $\|M_\Phi^* f\|_1 \leq c_{q,\Phi,d,L} \, \|M_\Phi f\|_1 < +\infty$. Hence, if $M_\Phi f \in L^1(\mathbb{R}^d)$, then $M_\Phi^* f \in L^1(\mathbb{R}^d)$ and from part (i) we get

$$\|M_\Phi^* f\|_1 \leq c_{\Phi,d} \|M_\Phi f\|_1.$$

On the other hand, if $M_\Phi f \notin L^1(\mathbb{R}^d)$, then the last inequality is trivially true. □

We may now finish the proof of **Theorem 2.1**.

Proof. (ii) We consider the collection of seminorms $\mathcal{F}_d$ of (2.18), which appears in Corollary 2.2 and Lemma 2.3. Then Corollary 2.2 implies $\|M_{\mathcal{F}_d} f\|_1 \leq c_{\Phi,d} \, \|M_{\Phi,d} f\|_1$, Corollary 2.1 says that $\|M_{\Phi,d} f\|_1 \leq c_d \, \|M_\Phi^* f\|_1$ and Lemma 2.3 says that $\|M_\Phi^* f\|_1 \leq c_{\Phi,d} \, \|M_\Phi f\|_1$. □

Remark. Let $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and $\int_{\mathbb{R}^d} \Psi \neq 0$. Theorem 2.1 says that there are constants $c_{\Phi,d}$, $c'_{\Phi,d}$, $c_{\Psi,d}$ and $c'_{\Psi,d}$ so that

$$c_{\Phi,d} \, \|M_{\mathcal{F}_d} f\|_1 \leq \|M_\Phi f\|_1 \leq c'_{\Phi,d} \, \|M_{\mathcal{F}_d} f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d)$$

and

$$c_{\Psi,d} \, \|M_{\mathcal{F}_d} f\|_1 \leq \|M_\Psi f\|_1 \leq c'_{\Psi,d} \, \|M_{\mathcal{F}_d} f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Thus, there are constants $c_{\Phi,\Psi,d}$ and $c'_{\Phi,\Psi,d}$ so that

$$c_{\Phi,\Psi,d} \, \|M_\Psi f\|_1 \leq \|M_\Phi f\|_1 \leq c'_{\Phi,\Psi,d} \, \|M_\Psi f\|_1, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Corollary 2.3. *(i) Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$. If $M_\Phi f \in L^1(\mathbb{R}^d)$, then $M_\Phi f \in L^\infty(\mathbb{R}^d)$.*
(ii) Let $\mathcal{F}$ be a finite collection of seminorms and assume $M_{\mathcal{F}} f \in L^1(\mathbb{R}^d)$. Then for every $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ we have $M_\Phi f \in L^\infty(\mathbb{R}^d)$.

Proof. (i) We fix x and let $|y-x|<1$. Then

$$|f*\Phi_t(x)| \le \sup_{|z-y|<1} |f*\Phi_t(z)| \le \sup_{t'>0,\,|z-y|<t'} |f*(\Phi_t)_{t'}(z)|$$
$$= \sup_{t'>0,\,|z-y|<t'} |f*\Phi_{tt'}(z)| = M_\Phi^* f(y).$$

Hence

$$|f*\Phi_t(x)| \le \frac{1}{|B(x,1)|}\int_{B(x,1)} M_\Phi^* f(y)\,dy \le c_d\,\|M_\Phi^* f\|_1.$$

Now Lemma 2.3 implies

$$|f*\Phi_t(x)| \le c_{\Phi,d}\,\|M_\Phi f\|_1$$

and thus $M_\Phi f \in L^\infty(\mathbb{R}^d)$.
(ii) A consequence of (i) and Theorem 2.1. ☐

Remark. If $\mathcal{F}$ is a finite collection of seminorms $\|\cdot\|_{\alpha,\beta}$, we consider a new collection $\mathcal{F}_M$ defined as follows:

$$\mathcal{F}_M = \{\|\cdot\|_{\alpha,\beta} \,|\, |\alpha| \le M, |\beta| \le M\},$$

where $M = \max\{|\alpha|,|\beta| \,|\, \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}$.
Clearly, $\mathcal{F} \subseteq \mathcal{F}_M$ and hence $\mathcal{S}_{\mathcal{F}_M} \subseteq \mathcal{S}_{\mathcal{F}}$. Therefore,

$$M_{\mathcal{F}_M} f(x) \le M_{\mathcal{F}} f(x)$$

and hence

$$\|M_{\mathcal{F}_M} f\|_1 \le \|M_{\mathcal{F}} f\|_1.$$

Lemma 2.4. *Let $\mathcal{F}$ be a finite collection of seminorms and $M \in \mathbb{N}$ such that $\mathcal{F} \subseteq \mathcal{F}_M$ as in the last remark. Let $\phi \in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\phi) \subseteq B$, where B is an open ball of radius r, and let $f \in L^1(\mathbb{R}^d)$. If $\sup_x |\partial^\beta \phi(x)| \le \frac{c_M}{r^{d+|\beta|}}$ for every $\beta \in \mathbb{N}_0^d$ with $|\beta| \le M$, then*

$$\Big|\int_{\mathbb{R}^d} f(x)\phi(x)\,dx\Big| \le c_M\, M_{\mathcal{F}_M} f(\bar{x}) \le c_M\, M_{\mathcal{F}} f(\bar{x})$$

for every $\bar{x} \in B$.

Proof. The second inequality is included in the last remark. Now if $\bar{x} \in B$, we define

$$\psi(x) = r^d \phi(\bar{x} - rx) \tag{2.33}$$

for all x. Then

$$\phi(x) = \psi_r(\bar{x} - x)$$

for all x and hence

$$\Big|\int_{\mathbb{R}^d} f(x)\phi(x)\,dx\Big| = \Big|\int_{\mathbb{R}^d} f(x)\psi_r(\bar{x}-x)\,dx\Big|. \tag{2.34}$$

Now, (2.26) and $\mathrm{supp}(\phi) \subseteq B$ imply $\mathrm{supp}(\psi) \subseteq B(0,2)$ and

$$\sup_x |\partial^\beta \psi(x)| \le c_M$$

when $|\beta| \le M$. Therefore,

$$\sup_x |x^\alpha \partial^\beta \psi(x)| \le 2^M c_M = c_M$$

if $|\alpha|, |\beta| \le M$ and hence $\frac{\psi}{c_M} \in \mathcal{S}_{\mathcal{F}_M}$. Thus,

$$\Big|\int_{\mathbb{R}^d} f(x)\psi_r(\bar{x}-x)\,dx\Big| = |f*\psi_r(\bar{x})| \le M_\psi f(\bar{x}) \le c_M\, M_{\mathcal{F}_M} f(\bar{x})$$

and (2.27) finishes the proof. ☐

Chapter 3

Whitney and Calderón–Zygmund decompositions

We shall first discuss the theorem describing the well-known *Whitney decomposition* of an open set.

Theorem 3.1. *Let O be an open proper subset of $\mathbb{R}^d$. Then there is a countable collection $\mathcal{Q}$ of closed cubes with edges parallel to the coordinate axes and with the following properties:*
(i) $\bigcup_{Q\in\mathcal{Q}} Q = O$.
(ii) Different cubes in $\mathcal{Q}$ have disjoint interiors.
(iii) diam$(Q) \leq$ dist$(Q, O^c) \leq 4\,$diam(Q) for every $Q \in \mathcal{Q}$.

Proof. For each $k \in \mathbb{Z}$ let $\mathcal{M}_k$ be the collection of closed cubes with edge-length equal to 2^{-k} and vertices from the points

$$(a_1 2^{-k}, \dots, a_d 2^{-k}), \qquad a_1, \dots, a_d \in \mathbb{Z}.$$

If $Q \in \mathcal{M}_k$ then $\text{diam}(Q) = \sqrt{d}\, 2^{-k}$.
We observe that every cube of the collection $\mathcal{M}_k$ contains exactly 2^d cubes of the collection $\mathcal{M}_{k+1}$ and is contained in exactly one cube of the collection $\mathcal{M}_{k-1}$.
We consider a fixed constant $c > 0$, which will be specified in a moment, and for each $k \in \mathbb{Z}$ we define

$$O_k = \{x \in O \,|\, c\, 2^{-k} < \text{dist}(x, O^c) \leq c\, 2^{-(k-1)}\}.$$

It is obvious that $\bigcup_{k\in\mathbb{Z}} O_k = O$ and that the sets O_k, $k \in \mathbb{Z}$, are pairwise disjoint.
Now we shall make an initial choice of a collection $\mathcal{Q}_0$ of cubes and the final choice $\mathcal{Q}$ will result from $\mathcal{Q}_0$ by choosing certain of its cubes in a particular way. We define the collection of cubes

$$\mathcal{Q}_0 = \bigcup_{k\in\mathbb{Z}} \{Q \in \mathcal{M}_k \,|\, Q \cap O_k \neq \emptyset\}.$$

Now taking $c = 2\sqrt{d}$, we easily see that the cubes of the collection $\mathcal{Q}_0$ satisfy (i) and (iii). In fact, if $Q \in \mathcal{Q}_0$, then for some $k \in \mathbb{Z}$ we have $Q \in \mathcal{M}_k$ and $Q \cap O_k \neq \emptyset$. If $x \in Q \cap O_k$, then

$$\text{dist}(Q, O^c) \leq \text{dist}(x, O^c) \leq 4\sqrt{d}\, 2^{-k} = 4\,\text{diam}(Q)$$

and

$$\begin{aligned} \text{dist}(Q, O^c) &\geq \text{dist}(x, O^c) - \text{diam}(Q) > 2\sqrt{d}\, 2^{-k} - \text{diam}(Q) \\ &= 2\,\text{diam}(Q) - \text{diam}(Q) = \text{diam}(Q). \end{aligned}$$

Also, if $Q \in \mathcal{Q}_0$, then from $\text{dist}(Q, O^c) \geq \text{diam}(Q) > 0$ we get $Q \subseteq O$. Thus $\bigcup_{Q\in\mathcal{Q}_0} Q \subseteq O$. On the other hand, for every $x \in O$ there is some k so that $x \in O_k$ and also some $Q \in \mathcal{M}_k$ so

that $x \in Q$. Then $Q \cap O_k \neq \emptyset$ and hence $Q \in \mathcal{Q}_0$. So every $x \in O$ belongs to some $Q \in \mathcal{Q}_0$ and hence $\bigcup_{Q \in \mathcal{Q}_0} Q = O$. Although the cubes of $\mathcal{Q}_0$ satisfy (i) and (iii), they may not have pairwise disjoint interiors.
Now take any $Q \in \mathcal{Q}_0$. If $Q' \in \mathcal{Q}_0$ *contains* Q, then

$$\text{diam}(Q') \leq \text{dist}(Q', O^c) \leq \text{dist}(Q, O^c) \leq 4\,\text{diam}(Q).$$

Hence, if $Q \in \mathcal{M}_k$ for some k, then either $Q' = Q$, or Q' is the unique cube in $\mathcal{M}_{k-1}$ which contains Q, or Q' is the unique cube in $\mathcal{M}_{k-2}$ which contains Q. We conclude that for every $Q \in \mathcal{Q}_0$ there is a unique maximal cube $Q' \in \mathcal{Q}_0$ which contains Q.
Now we collect those maximal cubes of the collection $\mathcal{Q}_0$ and form the collection $\mathcal{Q}$:

$$\mathcal{Q} = \{Q' \in \mathcal{Q}_0 \,|\, Q' \text{ is maximal}\}.$$

Since for every $Q \in \mathcal{Q}_0$ there is a $Q' \in \mathcal{Q}$ so that $Q \subseteq Q'$, we have that $\bigcup_{Q' \in \mathcal{Q}} Q' = O$.
Since every $Q' \in \mathcal{Q}$ belongs to $\mathcal{Q}_0$, we have $\text{diam}(Q') \leq \text{dist}(Q', O^c) \leq 4\,\text{diam}(Q')$ for every $Q' \in \mathcal{Q}$.
If the interiors of $Q'_1, Q'_2 \in \mathcal{Q}$ intersect, then one of them contains the other and, since both are maximal, they are equal.
Therefore $\mathcal{Q}$ satisfies (i), (ii) and (iii). □

The collection of closed cubes $\mathcal{Q}$ described in Theorem 3.1 is called the **Whitney decomposition** of the open set O.

If Q_1, Q_2 are cubes of the collection $\mathcal{Q}$, we say that they *touch* if $\partial Q_1 \cap \partial Q_2 \neq \emptyset$.

Corollary 3.1. *Let O be a proper open subset of $\mathbb{R}^d$ and let $\mathcal{Q}$ be the collection of closed cubes considered in Theorem 3.1. Then:*
(i) If $Q_1, Q_2 \in \mathcal{Q}$ touch, then

$$4^{-1}\,\textit{diam}(Q_2) \leq \textit{diam}(Q_1) \leq 4\,\textit{diam}(Q_2).$$

(ii) If $Q \in \mathcal{Q}$, there are at most $N = 12^d$ cubes of $\mathcal{Q}$ which touch Q.
(iii) Let $1 < \theta < \frac{5}{4}$. From the collection $\mathcal{Q}$ of cubes Q we produce a new collection $\mathcal{Q}^*$ of corresponding closed cubes Q^* as follows: if $Q \in \mathcal{Q}$ has center x and edge-length l, then Q^* has the same center x and edge-length θl. Then $O = \bigcup_{Q^* \in \mathcal{Q}^*} Q^*$ and for every $x \in O$ there is a small open neighborhood W of x which intersects at most $N = 12^d$ cubes $Q^* \in \mathcal{Q}^*$.

Proof. (i) If $Q_1, Q_2 \in \mathcal{Q}$ touch, then there is some $x \in Q_1 \cap Q_2$. Hence

$$\text{diam}(Q_1) \leq \text{dist}(Q_1, O^c) \leq \text{dist}(x, O^c) \leq \text{dist}(Q_2, O^c) + \text{diam}(Q_2) \leq 5\,\text{diam}(Q_2).$$

Since $\text{diam}(Q_2)/\text{diam}(Q_1)$ must be a power of 2, we get that $\text{diam}(Q_1) \leq 4\,\text{diam}(Q_2)$ and the symmetric relation gives $\text{diam}(Q_1) \geq 4^{-1}\,\text{diam}(Q_2)$.
(ii) Take $Q \in \mathcal{Q}$. Then $Q \in \mathcal{M}_k$ for some $k \in \mathbb{Z}$, where the families $\mathcal{M}_k$ are those defined in the proof of Theorem 3.1. It is clear that there are exactly 3^d cubes in $\mathcal{M}_k$ touching Q (Q itself and its "neighbors"). Now each of these 3^d cubes either is contained in exactly one cube in $\mathcal{Q}$ which touches Q or (because of (i)) contains at most 4^d cubes in $\mathcal{Q}$ which touch Q. Therefore, there are at most $3^d 4^d = 12^d$ cubes in $\mathcal{Q}$ which touch Q.
(iii) Take $Q_0 \in \mathcal{Q}$ and another $Q \in \mathcal{Q}$ with center x and edge-length l. Then, because of (i), all cubes in $\mathcal{Q}$ which touch Q must have diameters $\geq 4^{-1}\,\text{diam}(Q)$. Hence the union of all cubes in $\mathcal{Q}$ which touch Q contains the cube Q^{**} with center x and edge-length $\frac{3}{2}\,l$. Therefore, if Q_0 does not touch Q, then $Q_0 \cap Q^{**} = \emptyset$ and hence there is an *open* cube Q_1, slightly larger than Q_0, which does not intersect Q^*. In other words, if Q^* intersects Q_1 then Q touches Q_0.
Now take any $x \in O$. Then x belongs to at least one $Q_0 \in \mathcal{Q}$. We take $W = Q_1$, which is an open neighborhood of x, and then the cubes Q^* which intersect W must come from the cubes Q which touch Q_0 and hence their number is not more than $N = 12^d$. □

We say that a collection of sets $\mathcal{A}$ has the **bounded intersection property** if there is an $N \in \mathbb{N}$ such that every x has an open neighborhood which intersects at most N sets of $\mathcal{A}$.

Now we go on with a generalization of the well-known **Calderón–Zygmund decomposition**.

Theorem 3.2. *Consider $f \in L^1(\mathbb{R}^d)$, $\lambda > 0$ and a finite collection $\mathcal{F}$ of seminorms on $\mathcal{S}(\mathbb{R}^d)$ such that $M_{\mathcal{F}} f \in L^1(\mathbb{R}^d)$. Then there is a decomposition of f as a sum*

$$f = g + \sum_{k=1}^{+\infty} b_k,$$

and a collection of closed cubes $\mathcal{Q}^* = \{Q_1^*, Q_2^*, \dots\}$ so that:
(1) g is essentially bounded, with $\|g\|_\infty \leq c_{\mathcal{F},d}\,\lambda$.
(2) For every k we have $b_k \in L^1(\mathbb{R}^d)$, $\mathrm{supp}(b_k) \subseteq Q_k^*$ and $\int_{Q_k^*} b_k = 0$ and

$$\int_{\mathbb{R}^d} M_\Phi b_k(x)\,dx \leq c_{\Phi,\mathcal{F},d} \int_{Q_k^*} M_{\mathcal{F}} f(x)\,dx$$

for every $\Phi \in C^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \neq 0$.
(3) The collection $\mathcal{Q}^*$ has the bounded intersection property and

$$\bigcup_{k=1}^{+\infty} Q_k^* = \{x \,|\, M_{\mathcal{F}} f(x) > \lambda\}.$$

Proof. We consider the open set

$$O = \{x \,|\, M_{\mathcal{F}} f(x) > \lambda\}$$

and its Whitney decomposition, which is a collection of closed cubes $\{Q_1, Q_2, \dots\}$ such that:
(i) $\bigcup_{k=1}^{+\infty} Q_k = O$.
(ii) If Q_k, Q_l are different then they have disjoint interiors.
(iii) $\mathrm{diam}(Q_k) \leq \mathrm{dist}(Q_k, O^c) \leq 4\,\mathrm{diam}(Q_k)$ for every k.
We take $1 < \widetilde{\theta} < \theta < \frac{5}{4}$ and, as in Corollary 3.1, if x_k is the center and l_k is the edge-length of Q_k, we consider the closed cubes $\widetilde{Q}_k^*$ and Q_k^* with center x_k and edge-lengths $\widetilde{\theta}\,l_k$ and θl_k, respectively. Then

$$\bigcup_{k=1}^{+\infty} \widetilde{Q}_k^* = \bigcup_{k=1}^{+\infty} Q_k^* = O.$$

Moreover, each of the collections $\{\widetilde{Q}_1^*, \widetilde{Q}_2^*, \dots\}$ and $\{Q_1^*, Q_2^*, \dots\}$ has the bounded intersection property.
Now we consider a fixed function $\zeta \in C^\infty(\mathbb{R}^d)$ such that $0 \leq \zeta \leq 1$ in $\mathbb{R}^d$, $\zeta = 1$ in the cube with center 0 and edge-length 1 and $\zeta = 0$ outside the cube with center 0 and edge-length $\widetilde{\theta}$. We then define the functions $\zeta_k \in C^\infty(\mathbb{R}^d)$ by

$$\zeta_k(x) = \zeta\Big(\frac{x - x_k}{l_k}\Big)$$

for every x and every $k \in \mathbb{N}$. Then for every k we have that $0 \leq \zeta_k \leq 1$ in $\mathbb{R}^d$, $\zeta_k = 1$ in Q_k and $\mathrm{supp}(\zeta_k) \subseteq \widetilde{Q}_k^*$. Finally, we define the functions η_k by

$$\eta_k(x) = \begin{cases} \dfrac{\zeta_k(x)}{\sum_{j=1}^{+\infty} \zeta_j(x)}, & x \in O \\ 0, & x \in O^c \end{cases}$$

for every $k \in \mathbb{N}$. For every $x \in O$ we have $\sum_{j=1}^{+\infty} \zeta_j(x) \geq 1$ since x belongs to at least one Q_j. On the other hand, for every x there are at most N cubes $\widetilde{Q}_j^*$ which contain x. Hence,

$$1 \leq \sum_{j=1}^{+\infty} \zeta_j(x) \leq N \tag{3.1}$$

for every $x \in O$. Of course, if $x \in O^c$, then $\sum_{j=1}^{+\infty} \zeta_j(x) = 0$. Now, if we take any $x \in O$, then there is an open neighborhood W of x so that at most $N = 12^d$ of the cubes $\widetilde{Q}_j^*$ intersect W and hence there is some M so that $\sum_{j=1}^{+\infty} \zeta_j(y) = \sum_{j=1}^{M} \zeta_j(y)$ for all $y \in W$. Therefore, every function η_k is in $C^\infty(W)$. Also, for every $x \in O^c$ there is an open neighborhood W of x so that $\eta_k = 0$ in W. We conclude that $\eta_k \in C^\infty(\mathbb{R}^d)$ for every k.
We also observe that supp$(\eta_k) \subseteq \widetilde{Q}_k^*$ and that $\eta_k \geq 0$ in $\mathbb{R}^d$ for every k and that

$$\sum_{k=1}^{+\infty} \eta_k(x) = 1 \tag{3.2}$$

for every $x \in O$. Thus, the functions η_k, $k \in \mathbb{N}$, form a *partition of unity* for the set O with respect to the collection $\{\widetilde{Q}_1^*, \widetilde{Q}_2^*, \ldots\}$. From the definition of ζ_k we have

$$\sup_x |\partial^\beta \zeta_k(x)| \leq c_{\beta,d}\, l_k^{-|\beta|}$$

for all $\beta \in \mathbb{N}_0^d$ and from this and (3.1) we get

$$\sup_x |\partial^\beta \eta_k(x)| \leq c_{\beta,d}\, l_k^{-|\beta|} \tag{3.3}$$

for all $\beta \in \mathbb{N}_0^d$. The constants $c_{\beta,d}$ depend on β and on the function ζ and hence on β and d. We also see that (3.1) implies

$$\int_{\mathbb{R}^d} \eta_k = \int_{\widetilde{Q}_k^*} \eta_k \geq N^{-1} l_k^d \tag{3.4}$$

for every k. Next we define constants ρ_k by

$$\rho_k = \frac{1}{\int_{\mathbb{R}^d} \eta_k} \int_{\mathbb{R}^d} f(x)\eta_k(x)\, dx = \frac{1}{\int_{\widetilde{Q}_k^*} \eta_k} \int_{\widetilde{Q}_k^*} f(x)\eta_k(x)\, dx$$

and the functions b_k by

$$b_k(x) = (f(x) - \rho_k)\, \eta_k(x)$$

for all x. Then $b_k = 0$ outside $\widetilde{Q}_k^*$ and

$$\int_{\mathbb{R}^d} b_k = \int_{\widetilde{Q}_k^*} b_k = 0$$

for every k. Finally, we define

$$g(x) = \begin{cases} \sum_{k=1}^{+\infty} \rho_k\, \eta_k(x), & x \in O \\ f(x), & x \in O^c \end{cases}$$

Then (3.2) implies

$$f(x) = g(x) + \sum_{k=1}^{+\infty} b_k(x)$$

for every x.
From (iii) of Theorem 3.1 we get that for every k there is some $\bar{x} \in O^c$ such that $\text{diam}(Q_k) \leq \text{dist}(\bar{x}, Q_k) \leq 4\,\text{diam}(Q_k)$, i.e.

$$\sqrt{d}\, l_k \leq \text{dist}(\bar{x}, Q_k) \leq 4\sqrt{d}\, l_k.$$

Now we consider the ball B_k with center x_k and radius $5\sqrt{d}\, l_k$. Then

$$Q_k^* \subseteq B_k, \qquad \bar{x} \in B_k. \tag{3.5}$$

We consider $M = \max\{|\alpha|, |\beta| \mid \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}$, where $\mathcal{F}$ is the given finite collection of seminorms, and we apply Lemma 2.4 with our ball B_k and the function $\phi = \frac{1}{\int_{\tilde{Q}_k^*} \eta_k} \eta_k$, which appears in the definition of ρ_k. Using (3.3), (3.4) and (3.5) to verify the assumptions of Lemma 2.4, we get

$$|\rho_k| = \Big| \frac{1}{\int_{\mathbb{R}^d} \eta_k} \int_{\mathbb{R}^d} f(x)\eta_k(x)\, dx \Big| \leq N c_{M,d}\, M_{\mathcal{F}} f(\bar{x}) \leq N c_{M,d}\, \lambda = c_{\mathcal{F},d}\, \lambda, \tag{3.6}$$

where the last inequality is justified by $\bar{x} \in O^c$. In exactly the same manner we prove that for every $x \in Q_k^*$ we have

$$|\rho_k| \leq c_{\mathcal{F},d}\, M_{\mathcal{F}} f(x). \tag{3.7}$$

Now from the definition of g and from (3.2) and (3.6) we get

$$|g(x)| \leq c_{\mathcal{F},d}\, \lambda, \qquad x \in O. \tag{3.8}$$

We take any particular $\Psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Psi \neq 0$, and we remember from the proof of part (i) of Theorem 2.1 that there is a constant $c_{\Psi,\mathcal{F}} = c_{\mathcal{F},d}$ so that $\frac{\Psi}{c_{\mathcal{F},d}} \in \mathcal{S}_{\mathcal{F}}$. Now, since $f \in L^1(\mathbb{R}^d)$, we have $\lim_{t\to 0+} |f * \Psi_t(x)| = |f(x)|$ for a.e. x. Therefore, for a.e. $x \in O^c$ we have

$$|g(x)| = |f(x)| \leq \sup_{t>0} |f * \Psi_t(x)| = M_\Psi f(x) \leq c_{\mathcal{F},d}\, M_{\mathcal{F}} f(x) \leq c_{\mathcal{F},d}\, \lambda.$$

This together with (3.8) imply

$$|g(x)| \leq c_{\mathcal{F},d}\, \lambda$$

for a.e. x.
The only thing which remains to be proved is the integral inequality of part (2) and we consider any $\Phi \in C^\infty(\mathbb{R}^d)$ with $\text{supp}(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \neq 0$.
At first we shall prove:

$$\begin{aligned} M_\Phi b_k(x) &\leq c_{\Phi,\mathcal{F},d}\, M_{\mathcal{F}} f(x), \qquad x \in Q_k^* \\ M_\Phi b_k(x) &\leq c_{\Phi,\mathcal{F},d}\, \lambda\, \frac{l_k^{d+1}}{|x - x_k|^{d+1}}, \qquad x \notin Q_k^* \end{aligned} \tag{3.9}$$

To prove the first inequality in (3.9) we observe that

$$M_\Phi b_k(x) = M_\Phi(f\eta_k - \rho_k\eta_k)(x) \leq M_\Phi(f\eta_k)(x) + |\rho_k|\, M_\Phi \eta_k(x). \tag{3.10}$$

Since $0 \leq \eta_k \leq 1$, we have

$$|\eta_k * \Phi_t(x)| = \Big| \int_{\mathbb{R}^d} \eta_k(x-y)\Phi_t(y)\, dy \Big| \leq \int_{\mathbb{R}^d} |\Phi_t(y)|\, dy = \int_{B(0,1)} |\Phi(y)|\, dy = c_\Phi.$$

This together with (3.7) imply

$$|\rho_k|\, M_\Phi \eta_k(x) \leq c_{\Phi,\mathcal{F},d}\, M_{\mathcal{F}} f(x), \qquad x \in Q_k^*. \tag{3.11}$$

Now we write

$$(f\eta_k) * \Phi_t(x) = \int_{\mathbb{R}^d} f(y)\eta_k(y)\Phi_t(x-y)\,dy. \tag{3.12}$$

For the function $\phi(y) = \eta_k(y)\Phi_t(x-y)$ we have supp$(\phi) \subseteq B(x,t)$. If $t \leq l_k$, then, using (3.3), we get

$$\sup_y |\partial^\beta \phi(y)| \leq \frac{c_{\beta,\Phi,d}}{t^{|\beta|+d}}.$$

Then, considering again the quantity $M = \max\{|\alpha|, |\beta| \mid \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}$, we apply Lemma 2.4 to (3.12) and find

$$|(f\eta_k) * \Phi_t(x)| \leq c_{\Phi,\mathcal{F},d}\, M_{\mathcal{F}} f(x), \qquad x \in Q_k^*, \tag{3.13}$$

if $t \leq l_k$. If $t \geq l_k$, we observe that for the function $\phi(y) = \eta_k(y)\Phi_t(x-y)$ we have supp$(\phi) \subseteq B(x_k, \sqrt{d}\, l_k)$. Also, since $t \geq l_k$, we have

$$\sup_y |\partial^\beta \phi(y)| \leq \frac{c_{\beta,\Phi,d}}{(\sqrt{d}\, l_k)^{|\beta|+d}}.$$

We apply Lemma 2.4 again to (3.12) and get

$$|(f\eta_k) * \Phi_t(x)| \leq c_{\Phi,\mathcal{F},d}\, M_{\mathcal{F}} f(x), \qquad x \in Q_k^*,$$

if $t \geq l_k$. Combining with (3.13) we get

$$M_\Phi(f\eta_k)(x) \leq c_{\Phi,\mathcal{F},d}\, M_{\mathcal{F}} f(x), \qquad x \in Q_k^*.$$

This together with (3.10) and (3.11) imply the first inequality in (3.9) and we continue with the second inequality in (3.9).
If $x \notin Q_k^*$, then

$$b_k * \Phi_t(x) = \int_{\widetilde{Q}_k^*} b_k(y)\Phi_t(x-y)\,dy = \int_{\widetilde{Q}_k^*} b_k(y)(\Phi_t(x-y) - \Phi_t(x-x_k))\,dy,$$

where we use that $\int_{\widetilde{Q}_k^*} b_k = 0$. Hence

$$\begin{aligned} b_k * \Phi_t(x) = &\int_{\widetilde{Q}_k^*} f(y)\eta_k(y)(\Phi_t(x-y) - \Phi_t(x-x_k))\,dy \\ &- \rho_k \int_{\widetilde{Q}_k^*} \eta_k(y)(\Phi_t(x-y) - \Phi_t(x-x_k))\,dy. \end{aligned} \tag{3.14}$$

We consider the function $\phi(y) = \eta_k(y)(\Phi_t(x-y) - \Phi_t(x-x_k))$ with supp$(\phi) \subseteq \widetilde{Q}_k^*$ and we take $x \notin Q_k^*$. To estimate the integrands in (3.14) we may of course assume that $y \in \widetilde{Q}_k^*$.
If $|\beta| \geq 1$, then

$$|\partial_y^\beta(\Phi_t(x-y) - \Phi_t(x-x_k))| = |\partial_y^\beta(\Phi_t(x-y))| \leq \frac{c_{\beta,\Phi}}{t^{d+|\beta|}}.$$

For the case $|\beta| = 0$ we have that there is $x' = (1-c)y + cx_k$ for some $c \in [0,1]$ such that

$$|\Phi_t(x-y) - \Phi_t(x-x_k)| \leq |y-x_k||\nabla(\Phi_t)(x-x')| \leq \frac{\sqrt{d}\, c_\Phi\, |y-x_k|}{t^{d+1}} \leq \frac{c_{\Phi,d}\, l_k}{t^{d+1}}.$$

Combining these last two estimates with (3.3) and assuming as we may that $\Phi_t(x-y) \neq 0$ for at least one $y \in \widetilde{Q}_k^*$ and hence that $t \geq cl_k$, where c depends on the parameters $\widetilde{\theta}, \theta$, we get

$$\sup_y |\partial^\beta \phi(y)| \leq c_{\beta,\Phi,d}\, \frac{l_k^{d+1}}{|x-x_k|^{d+1}}\, \frac{1}{l_k^{|\beta|+d}}. \tag{3.15}$$

Considering as before the ball B_k with center x_k and radius $5\sqrt{d}\,l_k$, which contains some $\bar{x} \in O^c$, and using Lemma 2.4, we get

$$\begin{aligned}\Big|\int_{\widetilde{Q}_k^*} f(y)\eta_k(y)(\Phi_t(x-y)-\Phi_t(x-x_k))\,dy\Big| &\leq c_{\Phi,\mathcal{F},d}\,\frac{l_k^{d+1}}{|x-x_k|^{d+1}}\,M_{\mathcal{F}}f(\bar{x}) \\ &\leq c_{\Phi,\mathcal{F},d}\,\frac{l_k^{d+1}}{|x-x_k|^{d+1}}\,\lambda.\end{aligned} \tag{3.16}$$

Moreover, using (3.15) with $\beta = (0,\ldots,0)$, we find

$$\sup_y |\phi(y)| \leq c_{\Phi,d}\,\frac{l_k}{|x-x_k|^{d+1}}$$

and (3.6) implies

$$|\rho_k|\,\Big|\int_{\widetilde{Q}_k^*} \eta_k(y)(\Phi_t(x-y)-\Phi_t(x-x_k))\,dy\Big| \leq c_{\Phi,\mathcal{F},d}\,\frac{l_k^{d+1}}{|x-x_k|^{d+1}}\,\lambda. \tag{3.17}$$

Finally, (3.14), (3.16) and (3.17) imply the second inequality in (3.9) and now we finish the proof by verifying the inequality of part (2) of our theorem.
Using (3.9) we get

$$\begin{aligned}\int_{\mathbb{R}^d} M_\Phi b_k(x)\,dx &= \int_{Q_k^*} M_\Phi b_k(x)\,dx + \int_{\mathbb{R}^d\setminus Q_k^*} M_\Phi b_k(x)\,dx \\ &\leq c_{\Phi,\mathcal{F},d}\int_{Q_k^*} M_{\mathcal{F}}f(x)\,dx + c_{\Phi,\mathcal{F},d}\,\lambda\,l_k^{d+1}\int_{\mathbb{R}^d\setminus Q_k^*}\frac{1}{|x-x_k|^{d+1}}\,dx.\end{aligned} \tag{3.18}$$

We observe that, if $|x-x_k| \leq \frac{l_k}{2}$, then $x \in Q_k^*$. Therefore

$$\int_{\mathbb{R}^d\setminus Q_k^*}\frac{1}{|x-x_k|^{d+1}}\,dx \leq \int_{\{x\,|\,|x|\geq l_k/2\}}\frac{1}{|x|^{d+1}}\,dx = \frac{c_d}{l_k}$$

and (3.18) implies

$$\int_{\mathbb{R}^d} M_\Phi b_k(x)\,dx \leq c_{\Phi,\mathcal{F},d}\int_{Q_k^*} M_{\mathcal{F}}f(x)\,dx + c_{\Phi,\mathcal{F},d}\,\lambda\,l_k^d. \tag{3.19}$$

But, for $x \in Q_k^*$ we have $M_{\mathcal{F}}f(x) > \lambda$ and hence

$$\int_{Q_k^*} M_{\mathcal{F}}f(x)\,dx \geq \lambda\,|Q_k^*| = c_d\,\lambda\,l_k^d.$$

Finally, (3.19) gives

$$\int_{\mathbb{R}^d} M_\Phi b_k(x)\,dx \leq c_{\Phi,\mathcal{F},d}\int_{Q_k^*} M_{\mathcal{F}}f(x)\,dx$$

and the proof is complete. □

The splitting

$$f = g + b = g + \sum_{k=1}^{+\infty} b_k$$

of f, described in Theorem 3.2, is called **Calderon-Zygmund type decomposition** of f. The function g is the “good” function, since it is bounded, and the function $b = \sum_{k=1}^{+\infty} b_k$ is the “bad” function.

It is easy to see that $b \in L^1(\mathbb{R}^d)$. In fact, by the definition of each ρ_k and each b_k, we have

$$\int_{\mathbb{R}^d} |b_k(x)|\,dx \le \int_{\widetilde{Q}_k^*} |f(x)|\eta_k(x)\,dx + |\rho_k| \int_{\widetilde{Q}_k^*} \eta_k(x)\,dx \le 2\int_{\widetilde{Q}_k^*} |f(x)|\eta_k(x)\,dx$$
$$\le 2\int_{\widetilde{Q}_k^*} |f(x)|\,dx.$$

Therefore

$$\int_{\mathbb{R}^d} |b(x)|\,dx \le \sum_{k=1}^{+\infty} \int_{\mathbb{R}^d} |b_k(x)|\,dx \le 2\sum_{k=1}^{+\infty} \int_{\widetilde{Q}_k^*} |f(x)|\,dx = 2\int_{\mathbb{R}^d} |f(x)| \sum_{k=1}^{+\infty} \chi_{\widetilde{Q}_k^*}(x)\,dx$$
$$\le 2N \int_{\mathbb{R}^d} |f(x)|\,dx < +\infty,$$

since every x belongs to at most $N = 12^d$ of the sets $\widetilde{Q}_k^*$.

Since both f and b belong to $L^1(\mathbb{R}^d)$, we have that $g \in L^1(\mathbb{R}^d)$. But also $g \in L^\infty(\mathbb{R}^d)$ and hence $g \in L^p(\mathbb{R}^d)$ for all p with $1 \le p \le +\infty$.

The “bad” function b, on the other hand, is not so bad. We can decompose b in “pieces” b_k, each of which is supported in a cube $\widetilde{Q}_k^*$ and its integral is 0. These cubes are almost mutually disjoint, in the sense that they have the bounded intersection property. Moreover, we can control the maximal function of each piece b_k by the maximal function of f over the corresponding $\widetilde{Q}_k^*$.

Chapter 4

The spaces $H^1_{at}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$

Definition. *A function $a : \mathbb{R}^d \to \mathbb{C}$ is called **atom** if it there is some ball B so that*
(i) $a = 0$ a.e outside B.
(ii) $\|a\|_\infty \leq \frac{1}{|B|}$.
(iii) $\int_{\mathbb{R}^d} a = \int_B a = 0$.

Lemma 4.1. *If a is an atom, then $a \in L^1(\mathbb{R}^d)$ and*

$$\|a\|_1 \leq 1.$$

Proof. Obvious. □

Lemma 4.2. *Let $\Phi \in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \neq 0$. If a is an atom then*

$$\int_{\mathbb{R}^d} M_\Phi a(x)\, dx \leq c_{\Phi,d},$$

where $c_{\Phi,d}$ is a constant independent of the atom a.

Proof. Let $B = B(\bar{x}, \bar{r})$ be the ball corresponding to the atom a (by the definition of the atom) and let $B^* = B(\bar{x}, 2\bar{r})$. Then, if $x \in B^*$,

$$\begin{aligned} M_\Phi a(x) &= \sup_{t>0} |a * \Phi_t(x)| \leq \sup_{t>0} \int_{\mathbb{R}^d} |a(y)|\, |\Phi_t(x-y)|\, dy \\ &\leq \frac{1}{|B|} \sup_{t>0} \int_{\mathbb{R}^d} |\Phi_t(x-y)|\, dy = \frac{1}{|B|} \int_{B(0,1)} |\Phi(y)|\, dy = \frac{c_\Phi}{|B|}. \end{aligned}$$

Therefore,

$$\int_{B^*} M_\Phi a(x)\, dx \leq c_{\Phi,d}. \tag{4.1}$$

If $x \notin B^*$, we write

$$\begin{aligned} a * \Phi_t(x) &= \int_{\mathbb{R}^d} a(y)\Phi_t(x-y)\, dy = \int_B a(y)\Phi_t(x-y)\, dy \\ &= \int_B a(y)(\Phi_t(x-y) - \Phi_t(x-\bar{x}))\, dy. \end{aligned}$$

Hence

$$|a * \Phi_t(x)| \leq \frac{1}{|B|} \int_B |\Phi_t(x-y) - \Phi_t(x-\bar{x})|\, dy. \tag{4.2}$$

Now, there is $x' = (1-c)y + c\bar{x}$ for some $c \in [0,1]$ so that

$$\begin{aligned} |\Phi_t(x-y) - \Phi_t(x-\bar{x})| &\leq |y-\bar{x}|\, |\nabla(\Phi_t)(x-x')| \leq \bar{r}\, |\nabla(\Phi_t)(x-x')| \\ &= \frac{\bar{r}}{t^{d+1}} \left| \nabla\Phi\Big(\frac{x-x'}{t}\Big) \right| \leq c_\Phi \frac{\bar{r}}{t^{d+1}}. \end{aligned} \tag{4.3}$$

We observe in (4.3) that, if $|\Phi_t(x-y)-\Phi_t(x-\bar{x})| \neq 0$, then $|x-x'| \leq t$. If $y \in B$, then this last inequality implies

$$|x-\bar{x}| \leq |x-x'|+|x'-\bar{x}| \leq t+|y-\bar{x}| \leq t+\bar{r} \leq t+\frac{|x-\bar{x}|}{2}$$

and hence $|x-\bar{x}| \leq 2t$. Now (4.3) implies

$$|\Phi_t(x-y)-\Phi_t(x-\bar{x})| \leq c_{\Phi,d}\,\frac{\bar{r}}{|x-\bar{x}|^{d+1}}$$

when $y \in B$ and from (4.2) we get

$$M_\Phi a(x) \leq c_{\Phi,d}\,\frac{\bar{r}}{|x-\bar{x}|^{d+1}}.$$

Thus

$$\int_{\mathbb{R}^d\setminus B^*} M_\Phi a(x)\,dx \leq c_{\Phi,d}\,\bar{r}\int_{\mathbb{R}^d\setminus B^*}\frac{1}{|x-\bar{x}|^{d+1}}\,dx = c_{\Phi,d}.$$

This, together with (4.1) imply $\int_{\mathbb{R}^d} M_\Phi a(x)\,dx \leq c_{\Phi,d}$. □

We continue with the definition of some of the main spaces of our work.

Definition. *We define*

$$H^1_{at}(\mathbb{R}^d) = \Big\{\sum_{j=1}^{+\infty}\lambda_j\,a_j\,\Big|\,a_j \text{ is an atom},\ \lambda_j\in\mathbb{C},\ \sum_{j=1}^{+\infty}|\lambda_j|<+\infty\Big\}$$

and

$$H^1(\mathbb{R}^d) = \{f\in\mathcal{S}'(\mathbb{R}^d)\,|\,\text{there is a } \Phi\in\mathcal{S}(\mathbb{R}^d) \text{ with } \textstyle\int_{\mathbb{R}^d}\Phi\neq 0 \text{ so that } M_\Phi f\in L^1(\mathbb{R}^d)\}.$$

Corollary 4.1. *If a is an atom, then $a \in H^1_{at}(\mathbb{R}^d)$ and $a\in H^1(\mathbb{R}^d)$*

Proof. The first is trivial and the second is a consequence of Lemma 4.2. □

Now we shall prove that both $H^1_{at}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ are subspaces of $L^1(\mathbb{R}^d)$.

Proposition 4.1. $H^1_{at}(\mathbb{R}^d)\subseteq L^1(\mathbb{R}^d)$.

Proof. Let $f\in H^1_{at}(\mathbb{R}^d)$. Then there are atoms a_j and $\lambda_j\in\mathbb{C}$ with $\sum_{j=1}^{+\infty}|\lambda_j|<+\infty$ so that

$$f=\sum_{j=1}^{+\infty}\lambda_j\,a_j.$$

Thus

$$\int_{\mathbb{R}^d}|f(x)|\,dx \leq \sum_{j=1}^{+\infty}|\lambda_j|\int_{\mathbb{R}^d}|a_j(x)|\,dx \leq \sum_{j=1}^{+\infty}|\lambda_j| < +\infty,$$

where the second inequality id due to Lemma 4.1 □

We observe that in the last proof we have got the following inequality:

$$\|f\|_1 \leq \inf\Big\{\sum_{j=1}^{+\infty}|\lambda_j|\,\Big|\,a_j \text{ is an atom},\ \lambda_j\in\mathbb{C},\ f=\sum_{j=1}^{+\infty}\lambda_j\,a_j\Big\}. \tag{4.4}$$

Proposition 4.2. $H^1(\mathbb{R}^d)\subseteq L^1(\mathbb{R}^d)$.

Proof. Let $f \in H^1(\mathbb{R}^d)$. Then $f \in \mathcal{S}'(\mathbb{R}^d)$ and there is some $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ so that $M_\Phi f \in L^1(\mathbb{R}^d)$. We may also assume that $\int_{\mathbb{R}^d} \Phi = 1$.
Now we define the functions

$$h_t(x) = f * \Phi_t(x)$$

for each $t > 0$. Then $h_t \in L^1(\mathbb{R}^d)$ and Corollary 2.3 implies that $h_t \in L^\infty(\mathbb{R}^d)$ and

$$\|h_t\|_\infty \leq \|M_\Phi f\|_\infty \leq c_{\Phi,d} \|M_\Phi f\|_1. \tag{4.5}$$

We know that h_t can also be equivalently defined as a tempered distribution:

$$h_t(\psi) = f * \Phi_t(\psi) = f(\psi * \widetilde{\Phi_t}), \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Since $\psi * \widetilde{\Phi_t} \to \psi$ in $\mathcal{S}(\mathbb{R}^d)$ as $t \to 0+$, we get

$$h_t(\psi) = f(\psi * \widetilde{\Phi_t}) \to f(\psi), \qquad \psi \in \mathcal{S}(\mathbb{R}^d),$$

as $t \to 0+$. Since $h_t \in L^1(\mathbb{R}^d)$, we have the following connection between the function h_t and the tempered distribution h_t:

$$h_t(\psi) = \int_{\mathbb{R}^d} h_t(x)\psi(x)\, dx \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Therefore,

$$\int_{\mathbb{R}^d} h_t(x)\psi(x)\, dx \to f(\psi), \qquad \psi \in \mathcal{S}(\mathbb{R}^d), \tag{4.6}$$

as $t \to 0+$.
Now every h_t is in $L^1(\mathbb{R}^d)$ and hence defines an absolutely continuous finite Borel measure μ_{h_t} on $\mathbb{R}^d$ by:

$$\mu_{h_t}(E) = \int_E h_t(x)\, dx, \qquad E \text{ Borel subset of } \mathbb{R}^d.$$

The total variation of μ_{h_t} is

$$\|\mu_{h_t}\| = \|h_t\|_1 \leq \|M_\Phi f\|_1 \tag{4.7}$$

and by the Banach-Alaoglu theorem, there is a sequence $t_m \to 0+$ and a finite Borel measure μ on $\mathbb{R}^d$ so that

$$\mu_{h_{t_m}} \xrightarrow{w^*} \mu. \tag{4.8}$$

As a consequence of (4.7) we get

$$\|\mu\| \leq \|M_\Phi f\|_1. \tag{4.9}$$

Now, (4.6) and (4.8) imply

$$f(\psi) = \int_{\mathbb{R}^d} \psi(x)\, d\mu(x), \qquad \psi \in \mathcal{S}(\mathbb{R}^d). \tag{4.10}$$

We shall prove that μ is absolutely continuous. Let E be a Borel set with $|E| = 0$. Then there is an open set $U \supseteq E$ so that $|U| < \epsilon$ and then

$$|\mu|(U) = \sup\Big\{\Big|\int_{\mathbb{R}^d} \psi(x)\, d\mu(x)\Big| \;\Big|\; \psi \in \mathcal{S}(\mathbb{R}^d),\ \operatorname{supp}(\psi) \subseteq U,\ \|\psi\|_\infty \leq 1\Big\}. \tag{4.11}$$

For every $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}(\psi) \subseteq U$ and $\|\psi\|_\infty \leq 1$ we have from (4.5) that

$$\Big|\int_{\mathbb{R}^d} \psi(x) h_{t_m}(x)\, dx\Big| \leq c_{\Phi,d} \|M_\Phi f\|_1 |U| \leq c_{\Phi,d} \|M_\Phi f\|_1\, \epsilon.$$

Now, (4.11) implies

$$|\mu|(E) \le |\mu|(U) \le c_{\Phi,d} \, \|M_\Phi f\|_1 \, \epsilon$$

and this gives $|\mu|(E) = 0$. Thus, μ is absolutely continuous and so there is a $g \in L^1(\mathbb{R}^d)$ so that

$$\mu(E) = \int_E g(x) \, dx, \qquad E \text{ Borel subset of } \mathbb{R}^d.$$

From this and (4.10) we have

$$f(\psi) = \int_{\mathbb{R}^d} \psi(x) g(x) \, dx, \qquad \psi \in \mathcal{S}(\mathbb{R}^d)$$

which says that the tempered distribution f is identified with the function $g \in L^1(\mathbb{R}^d)$. Therefore we consider f as a function in $L^1(\mathbb{R}^d)$. $\square$

A consequence of the last proof is that every $f \in H^1(\mathbb{R}^d)$ is in $L^1(\mathbb{R}^d)$ and moreover

$$\|f\|_1 \le \|M_\Phi f\|_1. \tag{4.12}$$

Indeed, f was identified with g and from (4.9) we get $\|f\|_1 = \|g\|_1 = \|\mu\| \le \|M_\Phi f\|_1$.

Because of Propositions 4.1 and 4.2 we may re-define the two spaces as follows.

$$H^1_{at}(\mathbb{R}^d) = \Big\{ f \in L^1(\mathbb{R}^d) \, \Big| \, f = \sum_{j=1}^{+\infty} \lambda_j a_j, \ a_j \text{ is an atom}, \ \lambda_j \in \mathbb{C}, \ \sum_{j=1}^{+\infty} |\lambda_j| < +\infty \Big\}$$

and for each $f \in H^1_{at}(\mathbb{R}^d)$ we set

$$\|f\|_{H^1_{at}} = \inf \Big\{ \sum_{j=1}^{+\infty} |\lambda_j| \, \Big| \, a_j \text{ is an atom}, \ \lambda_j \in \mathbb{C}, \ f = \sum_{j=1}^{+\infty} \lambda_j \, a_j \Big\}.$$

Also,

$$H^1(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d) \, | \, \text{there is a } \Phi \in \mathcal{S}(\mathbb{R}^d) \text{ with } \textstyle\int_{\mathbb{R}^d} \Phi \neq 0 \text{ so that } M_\Phi f \in L^1(\mathbb{R}^d) \}.$$

and for each $f \in H^1(\mathbb{R}^d)$ we set

$$\|f\|_{H^1} = \|M_\Phi f\|_1$$

for any $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and $M_\Phi f \in L^1(\mathbb{R}^d)$.

Remark. By the remark after the proof of Theorem 2.1 (just before Corollary 2.3) we have that the choice of Φ is irrelevant. Indeed, if we consider any two $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Phi \neq 0$ and $\int_{\mathbb{R}^d} \Psi \neq 0$, then $M_\Phi f \in L^1(\mathbb{R}^d)$ if and only if $M_\Psi f \in L^1(\mathbb{R}^d)$. Also, the quantities $\|M_\Phi f\|_1$ and $\|M_\Psi f\|_1$ are equivalent: their ratio is between two positive constants independent of f.

It is easy to prove that $\| \cdot \|_{H^1_{at}}$ is a norm on $H^1_{at}(\mathbb{R}^d)$ and that $\| \cdot \|_{H^1}$ is a norm on $H^1(\mathbb{R}^d)$. We also observe that (4.4) says

$$\|f\|_1 \le \|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mathbb{R}^d).$$

Similarly, (4.12) says

$$\|f\|_1 \le \|f\|_{H^1}, \qquad f \in H^1(\mathbb{R}^d).$$

Hence the embeddings of both $H^1_{at}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$ are bounded.

Now we shall prove that the spaces $H^1_{at}(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ are equal and their norms are equivalent.

Theorem 4.1. $H^1_{at}(\mathbb{R}^d) \subseteq H^1(\mathbb{R}^d)$ *and*

$$\|f\|_{H^1} \leq c_d \, \|f\|_{H^1_{at}}, \qquad f \in H^1_{at}(\mathbb{R}^d).$$

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with supp$(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \neq 0$.
We take any $f \in H^1_{at}(\mathbb{R}^d)$ and then there are atoms a_j and $\lambda_j \in \mathbb{C}$ with $\sum_{j=1}^{=\infty} |\lambda_j| < +\infty$ so that

$$f = \sum_{j=1}^{+\infty} \lambda_j a_j.$$

Then Lemma 4.2 implies

$$\|M_\Phi f\|_1 \leq \sum_{j=1}^{+\infty} |\lambda_j| \|M_\Phi a_j\|_1 \leq c_{\Phi,d} \sum_{j=1}^{+\infty} |\lambda_j| < +\infty. \tag{4.13}$$

Therefore $f \in H^1(\mathbb{R}^d)$ and, taking the infimum of the right side of (4.13), we get $\|f\|_{H^1} = \|M_\Phi f\|_1 \leq c_{\Phi,d} \, \|f\|_{H^1_{at}}$. □

The converse is more difficult and more substantial.

Theorem 4.2. $H^1(\mathbb{R}^d) \subseteq H^1_{at}(\mathbb{R}^d)$ *and*

$$\|f\|_{H^1_{at}} \leq c_d \, \|f\|_{H^1}, \qquad f \in H^1(\mathbb{R}^d).$$

Proof. Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with supp$(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \neq 0$.
We take any $f \in H^1(\mathbb{R}^d)$ and then $M_\Phi f \in L^1(\mathbb{R}^d)$ and $\|f\|_{H^1} = \|M_\Phi f\|_1$.
We consider the finite family of seminorms $\mathcal{F}_d$ appearing in Theorem 2.1 and we have $M_{\mathcal{F}_d} f \in L^1(\mathbb{R}^d)$ with

$$\|M_{\mathcal{F}_d} f\|_1 \leq c_{\Phi,d} \, \|M_\Phi f\|_1 = c_{\Phi,d} \|f\|_{H^1}.$$

Now we apply Theorem 3.2 with $\lambda = 2^n$ for each $n \in \mathbb{Z}$. Thus for every $n \in \mathbb{Z}$ there is a decomposition of f as a sum

$$f = g^{(n)} + b^{(n)} = g^{(n)} + \sum_{k=1}^{+\infty} b_k^{(n)},$$

and a collection of closed cubes $\mathcal{Q}^{n,*} = \{Q_1^{n,*}, Q_2^{n,*}, \ldots\}$ so that:
(1) $g^{(n)}$ is essentially bounded, with $\|g^{(n)}\|_\infty \leq c_d \, 2^n$.
(2) For every k we have $b_k^{(n)} \in L^1(\mathbb{R}^d)$, supp$(b_k^{(n)}) \subseteq Q_k^{n,*}$ and $\int_{Q_k^{n,*}} b_k^{(n)} = 0$ and

$$\int_{\mathbb{R}^d} M_\Phi b_k^{(n)}(x) \, dx \leq c_{\Phi,d} \int_{Q_k^{n,*}} M_{\mathcal{F}_d} f(x) \, dx.$$

(3) The collection $\mathcal{Q}^{n,*}$ has the property of bounded intersection and

$$\bigcup_{k=1}^{+\infty} Q_k^{n,*} = \{x \,|\, M_{\mathcal{F}_d} f(x) > 2^n\}.$$

We set $O^{(n)} = \{x \,|\, M_{\mathcal{F}_d} f(x) > 2^n\}$.
At first we shall prove

$$\|f - g^{(n)}\|_{H^1} \to 0 \tag{4.14}$$

as $n \to +\infty$. We have

$$\int_{\mathbb{R}^d} M_\Phi b^{(n)}(x)\,dx \le \sum_{k=1}^{+\infty} \int_{\mathbb{R}^d} M_\Phi b_k^{(n)}(x)\,dx \le c_{\Phi,d} \sum_{k=1}^{+\infty} \int_{Q_k^{n,*}} M_{\mathcal{F}_d} f(x)\,dx.$$

Since for each n every x belongs to at most $N = 12^d$ of the cubes $Q_k^{n,*}$ we get

$$\|b^{(n)}\|_{H^1} = \int_{\mathbb{R}^d} M_\Phi b^{(n)}(x)\,dx \le c_{\Phi,d}\, N \int_{O^{(n)}} M_{\mathcal{F}_d} f(x)\,dx.$$

Now, $O^{(n+1)} \subseteq O^{(n)}$ for every n and, since $M_{\mathcal{F}_d} f \in L^1(\mathbb{R}^d)$, we have $\bigcap_{n=1}^{+\infty} O^{(n)} = \emptyset$. Therefore, $\int_{O^{(n)}} M_{\mathcal{F}_d} f(x)\,dx \to 0$ as $n \to +\infty$. Hence $\|b^{(n)}\|_{H^1} \to 0$ as $n \to +\infty$ and this is (4.14).
From $\|g^{(n)}\|_\infty \le c_d\, 2^n$ we see that

$$g^{(n)} \to 0 \quad \text{uniformly as } n \to -\infty.$$

Now we shall construct atoms and corresponding coefficients for f. We recall some definitions and notation from the proof of Theorem 3.2, which form the basis of the construction of the decomposition $f = g^{(j)} + b^{(j)} = g^{(j)} + \sum_{k=1}^{+\infty} b_k^{(j)}$. We have

$$b_k^{(j)}(x) = (f(x) - \rho_k^{(j)})\eta_k^{(j)}(x), \qquad \rho_k^{(j)} = \frac{\int_{\mathbb{R}^d} f(x)\eta_k^{(j)}(x)\,dx}{\int_{\mathbb{R}^d} \eta_k^{(j)}(x)\,dx},$$

where $\eta_k^{(j)} \in C^\infty(\mathbb{R}^d)$, $\mathrm{supp}(\eta_k^{(j)}) \subseteq Q_k^{j,*}$ and also

$$\sum_{k=1}^{+\infty} \eta_k^{(j)}(x) = 1, \qquad x \in O^{(j)}.$$

Now we define

$$A_k^{(j)}(x) = b_k^{(j)}(x) - \eta_k^{(j)}(x) b^{(j+1)}(x) + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \eta_m^{(j+1)}(x),$$

where

$$\rho_{k,m}^{(j)} = \frac{\int_{\mathbb{R}^d} b_m^{(j+1)}(x)\eta_k^{(j)}(x)\,dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\,dx}.$$

We observe that

$$\sum_{k=1}^{+\infty} b_k^{(j)}(x) = b^{(j)}(x), \quad \sum_{k=1}^{+\infty} \rho_{k,m}^{(j)} = \frac{\int_{\mathbb{R}^d} \sum_{k=1}^{+\infty} b_m^{(j+1)}(x)\eta_k^{(j)}(x)\,dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\,dx} = \frac{\int_{\mathbb{R}^d} b_m^{(j+1)}(x)\,dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\,dx} = 0$$

and

$$\sum_{k=1}^{+\infty} \eta_k^{(j)}(x) b^{(j+1)}(x) = b^{(j+1)}(x), \qquad x \in O^{(j)}.$$

We used that $\sum_{k=1}^{+\infty} \eta_k^{(j)}(x) = 1$ when $x \in O^{(j)}$ and that $O^{(j+1)} \subseteq O^{(j)}$. We used also that every $Q_m^{j+1,*}$ intersects only finitely many $Q_k^{j,*}$, $k \in \mathbb{N}$.
Therefore, for every $x \in O^{(j)}$, we have $\sum_{k=1}^{+\infty} A_k^{(j)}(x) = b^{(j)}(x) - b^{(j+1)}(x) = g^{(j+1)}(x) - g^{(j)}(x)$. On the other hand, if $x \in (O^{(j)})^c$, then $\sum_{k=1}^{+\infty} A_k^{(j)}(x) = b^{(j)}(x) = f(x) - g^{(j)}(x) = g^{(j+1)}(x) - g^{(j)}(x)$. Thus,

$$\sum_{k=1}^{+\infty} A_k^{(j)}(x) = g^{(j+1)}(x) - g^{(j)}(x) \tag{4.15}$$

for all x.
We continue with the study of the support of $A_k^{(j)}$. The terms $b_k^{(j)}$ and $\eta_k^{(j)} b^{(j+1)}$ in the definition of $A_k^{(j)}$ are supported in $Q_k^{j,*}$. We also see that $\rho_{k,m}^{(j)} \neq 0$ implies that $Q_m^{j+1,*}$ intersects $Q_k^{j,*}$. Now, $Q_k^{j,*}$ interects at most 12^d cubes $Q_r^{j,*}$, $r \in \mathbb{N}$, and every $Q_r^{j,*}$ contains 2^d cubes $Q_m^{j+1,*}$, $m \in \mathbb{N}$. Hence, $Q_k^{j,*}$ intersects at most 24^d cubes $Q_m^{j+1,*}$, $m \in \mathbb{N}$. If l_j is the side-length of $Q_k^{j,*}$, then we have $l_{j+1} = \frac{1}{2}\, l_j$ and hence

$$\operatorname{supp}(A_k^{(j)}) \subseteq B_k^{(j)} = B(x_k, \sqrt{d}\, l_j). \tag{4.16}$$

Now we shall examine more carefully the definition of $A_k^{(j)}$:

$$\begin{aligned}
A_k^{(j)}(x) &= b_k^{(j)}(x) - \eta_k^{(j)}(x) \sum_{m=1}^{+\infty} b_m^{(j+1)}(x) + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \eta_m^{(j+1)}(x) \\
&= f(x)\eta_k^{(j)}(x) - \rho_k^{(j)} \eta_k^{(j)}(x) - f(x)\eta_k^{(j)}(x) \sum_{m=1}^{+\infty} \eta_m^{(j+1)}(x) \\
&\quad + \eta_k^{(j)}(x) \sum_{m=1}^{+\infty} \rho_m^{(j+1)} \eta_m^{(j+1)}(x) + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \eta_m^{(j+1)}(x) \\
&= f(x)\eta_k^{(j)}(x) - \rho_k^{(j)} \eta_k^{(j)}(x) - f(x)\eta_k^{(j)}(x) \chi_{O^{(j+1)}}(x) \\
&\quad + \eta_k^{(j)}(x) \sum_{m=1}^{+\infty} \rho_m^{(j+1)} \eta_m^{(j+1)}(x) + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \eta_m^{(j+1)}(x) \\
&= f(x)\eta_k^{(j)}(x) \chi_{(O^{(j+1)})^c}(x) - \rho_k^{(j)} \eta_k^{(j)}(x) \\
&\quad + \eta_k^{(j)}(x) \sum_{m=1}^{+\infty} \rho_m^{(j+1)} \eta_m^{(j+1)}(x) + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \eta_m^{(j+1)}(x).
\end{aligned} \tag{4.17}$$

We have

$$\rho_{k,m}^{(j)} = \frac{\int_{\mathbb{R}^d} f(x)\eta_m^{(j+1)}(x)\eta_k^{(j)}(x)\, dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\, dx} - \rho_m^{(j+1)} \frac{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\eta_k^{(j)}(x)\, dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\, dx}.$$

In the same manner that we verified (3.6) in the proof of Theorem 3.2, we see that

$$\left| \frac{\int_{\mathbb{R}^d} f(x)\eta_m^{(j+1)}(x)\eta_k^{(j)}(x)\, dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\, dx} \right| \leq c_d\, 2^{j+1}.$$

We also know that $|\rho_m^{(j+1)}| \leq c_d\, 2^{j+1}$ and it is easy to see that

$$\left| \frac{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\eta_k^{(j)}(x)\, dx}{\int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\, dx} \right| \leq c_d.$$

Therefore,

$$|\rho_{k,m}^{(j)}| \leq c_d\, 2^{j+1}.$$

In the proof of Theorem 3.2 we saw that

$$|f(x)\eta_k^{(j)}(x)\chi_{(O^{(j+1)})^c}(x)| \leq |f(x)| \leq c_d\, 2^{j+1}.$$

Now (4.17) implies

$$\begin{aligned} |A_k^{(j)}(x)| &\le |f(x)\eta_k^{(j)}(x)\chi_{(O^{(j+1)})^c}(x)| + |\rho_k^{(j)}|\eta_k^{(j)}(x) \\ &\quad + \eta_k^{(j)}(x)\sum_{m=1}^{+\infty} |\rho_m^{(j+1)}|\eta_m^{(j+1)}(x) + \sum_{m=1}^{+\infty} |\rho_{k,m}^{(j)}|\eta_m^{(j+1)}(x) \\ &\le c_d\, 2^{j+1} + c_d\, 2^j + c_d\, 2^{j+1}\sum_{m=1}^{+\infty} \eta_m^{(j+1)}(x) + c_d\, 2^{j+1}\sum_{m=1}^{+\infty} \eta_m^{(j+1)}(x) \\ &\le c_d\, 2^{j+1}. \end{aligned} \tag{4.18}$$

Also

$$\begin{aligned} \int_{\mathbb{R}^d} A_k^{(j)}(x)\,dx &= \int_{\mathbb{R}^d} b_k^{(j)}(x)\,dx - \int_{\mathbb{R}^d} \eta_k^{(j)}(x) b^{(j+1)}(x)\,dx + \sum_{m=1}^{+\infty} \rho_{k,m}^{(j)} \int_{\mathbb{R}^d} \eta_m^{(j+1)}(x)\,dx \\ &= -\int_{\mathbb{R}^d} \eta_k^{(j)}(x) b^{(j+1)}(x)\,dx + \sum_{m=1}^{+\infty} \int_{\mathbb{R}^d} b_m^{(j+1)}(x)\eta_k^{(j)}(x)\,dx \\ &= \int_{\mathbb{R}^d} \Big(\sum_{m=1}^{+\infty} b_m^{(j+1)}(x) - b_m^{(j+1)} \Big) \eta_k^{(j)}(x)\,dx = 0. \end{aligned}$$

Now we look at (4.18) and with the *same* constant c_d we set

$$\lambda_k^{(j)} = c_d\, 2^{j+1}\, |B_k^{(j)}|,$$

where $B_k^{(j)} = B(x_k, \sqrt{d}\, l_j)$ are the balls which appear in (4.16) and define

$$a_k^{(j)}(x) = \frac{A_k^{(j)}(x)}{\lambda_k^{(j)}}.$$

All functions $a_k^{(j)}$ are atoms and (4.15) says that

$$\sum_{k=1}^{+\infty} \lambda_k^{(j)}\, a_k^{(j)}(x) = g^{(j+1)}(x) - g^{(j)}(x)$$

for all x. Since $g^{(n)}(x) \to 0$ as $n \to -\infty$ for a.e. x, we get

$$g^{(n)}(x) = \sum_{j=-\infty}^{n-1} \big(g^{(j+1)}(x) - g^{(j)}(x) \big) = \sum_{j=-\infty}^{n-1} \sum_{k=1}^{+\infty} \lambda_k^{(j)}\, a_k^{(j)}(x)$$

for a.e. x. We also have

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} |\lambda_k^{(j)}| &= \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} c_d\, 2^{j+1}\, |B_k^{(j)}| \\ &= \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} c_d\, 2^j\, |Q_k^{(j)}| = c_d \sum_{j=-\infty}^{+\infty} 2^j\, |O^{(j)}| \end{aligned}$$

because the cubes $Q_k^{(j)}$, $k \in \mathbb{N}$, have disjoint interiors and $O^{(j)}$ is their union. Thus,

$$\begin{aligned} \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} |\lambda_k^{(j)}| &= c_d \sum_{j=-\infty}^{+\infty} 2^j \, |\{x \,|\, M_{\mathcal{F}_d} f(x) > 2^j\}| \\ &\le 2c_d \sum_{j=-\infty}^{+\infty} \int_{2^{j-1}}^{2^j} |\{x \,|\, M_{\mathcal{F}_d} f(x) > t\}| \, dt \\ &= 2c_d \int_0^{+\infty} |\{x \,|\, M_{\mathcal{F}_d} f(x) > t\}| \, dt \\ &= 2c_d \int_{\mathbb{R}^d} M_{\mathcal{F}_d} f(x) \, dx = 2c_d \, \|M_{\mathcal{F}_d} f\|_1 < +\infty. \end{aligned} \tag{4.19}$$

This says that the function

$$F = \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} \lambda_k^{(j)} \, a_k^{(j)}$$

is in $H^1_{at}(\mathbb{R}^d)$. Since $g^{(n)} \to f$ in $H^1(\mathbb{R}^d)$ we get that $g^{(n)} \to f$ in $L^1(\mathbb{R}^d)$ as $n \to +\infty$. On the other hand,

$$\|F - g^{(n)}\|_{H^1_{at}} \le \sum_{j=n}^{+\infty} \sum_{k=1}^{+\infty} |\lambda_k^{(j)}| \to 0$$

as $n \to +\infty$. Thus $g^{(n)} \to F$ in $L^1(\mathbb{R}^d)$ as $n \to +\infty$ and we conclude that

$$f = \sum_{j=-\infty}^{+\infty} \sum_{k=1}^{+\infty} \lambda_k^{(j)} \, a_k^{(j)}$$

a.e. in $\mathbb{R}^d$.
We proved that $f \in H^1_{at}(\mathbb{R}^d)$ and now (4.19) implies

$$\|f\|_{H^1_{at}} \le c_d \|M_{\mathcal{F}_d} f\|_1 \le c_{\Phi,d} \|M_\Phi f\|_1$$

and hence $\|f\|_{H^1_{at}} \le c_{\Phi,d} \|f\|_{H^1}$. □

Chapter 5

The space $BMO(\mathbb{R}^d)$ as the dual space of $H^1(\mathbb{R}^d)$

Definition. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. We define*

$$M_* f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy,$$

where the supremum is over the balls B which contain x and where we denote

$$f_B = \frac{1}{|B|} \int_B f(y) \, dy$$

the mean value of f over B.

Definition. *We define the space $BMO(\mathbb{R}^d)$ by*

$$BMO(\mathbb{R}^d) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^d) \mid M_* f \text{ is bounded in } \mathbb{R}^d \}.$$

The space $BMO(\mathbb{R}^d)$ is called the space of functions of **bounded mean oscillation** and we say that the $f, g \in BMO(\mathbb{R}^d)$ are equal if their difference $f - g$ is an a.e. constant function. Under this agreement, the $\| \cdot \|_*$ defined by

$$\|f\|_* = \sup_x M_* f(x)$$

is a norm on the linear space $BMO(\mathbb{R}^d)$.

Lemma 5.1. *If $f \in BMO(\mathbb{R}^d)$, then $|f| \in BMO(\mathbb{R}^d)$.*

Proof. Let $f \in BMO(\mathbb{R}^d)$ and take any x and any ball B containing x. Then

$$\big| |f(y)| - |f|_B \big| \le \big| |f(y)| - |f_B| \big| + \big| |f_B| - |f|_B \big| \le |f(y) - f_B| + \big| |f_B| - |f|_B \big|. \tag{5.1}$$

Also,

$$\begin{aligned} \big| |f_B| - |f|_B \big| &= \Big| |f_B| - \frac{1}{|B|} \int_B |f(y)| \, dy \Big| = \frac{1}{|B|} \Big| \int_B (|f(y)| - |f_B|) \, dy \Big| \\ &\le \frac{1}{|B|} \int_B \big| |f(y)| - |f_B| \big| \, dy \le \frac{1}{|B|} \int_B |f(y) - f_B| \, dy. \end{aligned} \tag{5.2}$$

Now, (5.1) and (5.2) imply

$$\frac{1}{|B|} \int_B \big| |f(y)| - |f|_B \big| \, dy \le \frac{2}{|B|} \int_B |f(y) - f_B| \, dy \le 2\|f\|_*.$$

Hence, $|f| \in BMO(\mathbb{R}^d)$ and $\| \, |f| \, \|_* \le 2\|f\|_*$. ◻

Lemma 5.2. $L^\infty(\mathbb{R}^d) \subseteq BMO(\mathbb{R}^d)$.

Proof. Let $f \in L^\infty(\mathbb{R}^d)$ and take any x and any ball B containing x. Then

$$\frac{1}{|B|}\int_B |f(y)-f_B|\,dy \le \frac{1}{|B|}\int_B |f(y)|\,dy + |f_B| \le \frac{2}{|B|}\int_B |f(y)|\,dy \le 2\|f\|_\infty.$$

Therefore, $f \in BMO(\mathbb{R}^d)$ and $\|f\|_* \le 2\|f\|_\infty$. □

Now we consider the following dense subspace of $H^1_{at}(\mathbb{R}^d)$:

$$H^1_a(\mathbb{R}^d) = \Big\{ \sum_{j\in J} \lambda_j\, a_j \,\Big|\, a_j \text{ is an atom}, \ \lambda_j \in \mathbb{C}, \ J \text{ is finite} \Big\}.$$

To see that $H^1_a(\mathbb{R}^d)$ is dense in $H^1_{at}(\mathbb{R}^d)$ we take any $f \in H^1_{at}(\mathbb{R}^d)$. Then $f = \sum_{j=1}^{+\infty} \lambda_j\, a_j$ for certain atoms a_j and $\lambda_j \in \mathbb{C}$ with $\sum_{j=1}^{+\infty} |\lambda_j| < +\infty$. Then we take $f_n = \sum_{j=1}^{n} \lambda_j\, a_j$ which belongs to $H^1_a(\mathbb{R}^d)$ and we have

$$\|f - f_n\|_{H^1_{at}} \le \sum_{j=n+1}^{+\infty} |\lambda_j| \to 0$$

as $n \to +\infty$.

Theorem 5.1. *Let $f \in BMO(\mathbb{R}^d)$. Then the linear functional $l_f : H^1_a(\mathbb{R}^d) \to \mathbb{C}$ defined by*

$$l_f(g) = \int_{\mathbb{R}^d} g(x) f(x)\,dx, \qquad g \in H^1_a(\mathbb{R}^d),$$

can be extended as a bounded linear functional $l_f : H^1_{at}(\mathbb{R}^d) \to \mathbb{C}$. Moreover

$$\|l_f\| \le \|f\|_*.$$

Proof. Let $g \in H^1_a(\mathbb{R}^d)$. Then

$$g = \sum_{j\in J} \lambda_j\, a_j$$

where each a_j is an atom, each λ_j is a number and J is finite. For each a_j there is a ball B_j so that $\mathrm{supp}(a_j) \subseteq B_j$, $\|a_j\|_\infty \le \frac{1}{|B_j|}$ and $\int_{B_j} a_j(x)\,dx = 0$. Then $l_f(g)$ is well defined, since $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, and we have

$$l_f(g) = \sum_{j\in J} \lambda_j \int_{B_j} a_j(x) f(x)\,dx = \sum_{j\in J} \lambda_j \int_{B_j} a_j(x)(f(x) - f_{B_j})\,dx.$$

Hence

$$|l_f(g)| \le \sum_{j\in J} |\lambda_j| \frac{1}{|B_j|} \int_{B_j} |f(x) - f_{B_j}|\,dx \le \|f\|_* \sum_{j\in J} |\lambda_j|.$$

Taking the infimum over all representations of g as sums $\sum_{j\in J} \lambda_j\, a_j$, we get

$$|l_f(g)| \le \|f\|_*\, \|g\|_{H^1_{at}}.$$

Thus, the linear functional l_f is bounded on the dense subspace $H^1_a(\mathbb{R}^d)$ of $H^1_{at}(\mathbb{R}^d)$ and its norm on $H^1_a(\mathbb{R}^d)$ satisfies $\|l_f\| \le \|f\|_*$. Therefore l_f can be extended as a bounded linear functional on $H^1_{at}(\mathbb{R}^d)$ with $\|l_f\| \le \|f\|_*$. □

Remark. We have proved the equality of spaces

$$H^1(\mathbb{R}^d) = H^1_{at}(\mathbb{R}^d)$$

and that this space has two equivalent norms: $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^1_{at}}$. Thus the functional l_f defined in Theorem 5.1 is a bounded linear functional on $H^1(\mathbb{R}^d)$. The inequality $\|l_f\| \le \|f\|_*$ has to change though since the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_{H^1_{at}}$ are *not* equal. Indeed, we have

$$|l_f(g)| \le \|f\|_* \, \|g\|_{H^1_{at}}$$

and hence

$$|l_f(g)| \le \|f\|_* \, \|g\|_{H^1_{at}} \le c_d \, \|f\|_* \, \|g\|_{H^1}.$$

Therefore, we have $\|l_f\| \le \|f\|_*$ if we consider the norm $\|\cdot\|_{H^1_{at}}$ on $H^1(\mathbb{R}^d) = H^1_{at}(\mathbb{R}^d)$, and we have $\|l_f\| \le c_d \, \|f\|_*$ if we consider the norm $\|\cdot\|_{H^1}$ on $H^1(\mathbb{R}^d) = H^1_{at}(\mathbb{R}^d)$.

Now comes the converse of Theorem 5.1.

Theorem 5.2. *Let l be any bounded linear functional on $H^1(\mathbb{R}^d) = H^1_{at}(\mathbb{R}^d)$. Then there is a unique $f \in BMO(\mathbb{R}^d)$ so that*

$$l(g) = l_f(g) = \int_{\mathbb{R}^d} g(x) f(x) \, dx, \qquad g \in H^1_a(\mathbb{R}^d).$$

Moreover,

$$\|f\|_* \le c_d \, \|l\|.$$

Proof. Take any ball B. We define the space

$$L^2_B(\mathbb{R}^d) = \{g \in L^2(\mathbb{R}^d) \mid g = 0 \text{ a.e. in } B^c\}.$$

We consider the norm $\|\cdot\|_{2,B}$ on $L^2_B(\mathbb{R}^d)$ to be the restriction of the usual norm $\|\cdot\|_2$ on B. Thus,

$$\|g\|_{2,B} = \Big(\int_B |g(x)|^2 \, dx \Big)^{1/2}, \qquad g \in L^2_B(\mathbb{R}^d).$$

Of course $L^2_B(\mathbb{R}^d)$ is a Hilbert space. We also define the closed linear subspace of $L^2_B(\mathbb{R}^d)$:

$$L^2_{B,0}(\mathbb{R}^d) = \Big\{ g \in L^2_B(\mathbb{R}^d) \, \Big| \, \int_B g(x) \, dx = 0 \Big\}.$$

Let $\Phi \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}(\Phi) \subseteq B(0,1)$ and $\int_{\mathbb{R}^d} \Phi \ne 0$. We also assume that Φ is a radial function and that $\Phi(|x|)$ is a decreasing function of $|x|$.
Now we consider any $g \in L^2_{B,0}(\mathbb{R}^d)$ and we look back at the proof of Lemma 4.2. If $B = B(\bar{x}, \bar{r})$, we take also the ball $B^* = B(\bar{x}, 2\bar{r})$.
We use the well-known inequality

$$|g * \Phi_t(x)| \le c_\Phi \, Mg(x),$$

where Mg is the Hardy-Littlewood maximal function of g, and we get $M_\Phi g(x) \le c_\Phi \, Mg(x)$. Since the Hardy-Littlewood maximal operator is strong-$(2,2)$, we have

$$\begin{aligned} \int_{B^*} M_\Phi g(x) \, dx &\le |B^*|^{1/2} \Big(\int_{B^*} M_\Phi g(x)^2 \, dx \Big)^{1/2} \le c_\Phi \, |B^*|^{1/2} \Big(\int_{\mathbb{R}^d} Mg(x)^2 \, dx \Big)^{1/2} \\ &\le c_{\Phi,d} \, |B^*|^{1/2} \Big(\int_{\mathbb{R}^d} |g(x)|^2 \, dx \Big)^{1/2} = c_{\Phi,d} \, |B|^{1/2} \, \|g\|_{2,B}. \end{aligned} \tag{5.3}$$

If $x \notin B^*$, then

$$\begin{aligned} |g * \Phi_t(x)| &= \Big| \int_{\mathbb{R}^d} g(y)(\Phi_t(x-y) - \Phi_t(x-\bar{x}))\, dy \Big| \\ &\le \|g\|_{2,B} \Big(\int_B |\Phi_t(x-y) - \Phi_t(x-\bar{x})|^2\, dy \Big)^{1/2}. \end{aligned} \tag{5.4}$$

We saw in the proof of Lemma 4.2 that

$$|\Phi_t(x-y) - \Phi_t(x-\bar{x})| \le c_{\Phi,d}\, \frac{\bar{r}}{|x-\bar{x}|^{d+1}}$$

for $y \in B$. Then (5.4) implies

$$M_\Phi g(x) \le c_{\Phi,d}\, \|g\|_{2,B}\, |B|^{1/2}\, \frac{\bar{r}}{|x-\bar{x}|^{d+1}}.$$

when $x \notin B^*$ and hence

$$\int_{\mathbb{R}^d \setminus B^*} M_\Phi g(x)\, dx \le c_{\Phi,d}\, |B|^{1/2}\, \|g\|_{2,B}.$$

Considering also (5.3), we find

$$\|g\|_{H^1} = \|M_\Phi g\|_1 \le c_{\Phi,d}\, |B|^{1/2}\, \|g\|_{2,B}. \tag{5.5}$$

We have thus proved that $L^2_{B,0}(\mathbb{R}^d) \subseteq H^1(\mathbb{R}^d)$ and we can restrict our linear functional on $L^2_{B,0}(\mathbb{R}^d)$. In fact (5.5) implies

$$|l(g)| \le \|l\|\, \|g\|_{H^1} \le c_{\Phi,d}\, |B|^{1/2}\, \|l\|\, \|g\|_{2,B}, \qquad g \in L^2_{B,0}(\mathbb{R}^d). \tag{5.6}$$

If we denote $l^B : L^2_{B,0}(\mathbb{R}^d) \to \mathbb{C}$ this restriction of l, then from (5.6) we have

$$\|l^B\| \le c_{\Phi,d}\, |B|^{1/2}\, \|l\|.$$

Since $L^2_{B,0}(\mathbb{R}^d)$ is a Hilbert space, there is a function $F^B \in L^2_{B,0}(\mathbb{R}^d)$ such that

$$l(g) = l^B(g) = \int_{\mathbb{R}^d} g(x) F^B(x)\, dx = \int_B g(x) F^B(x)\, dx, \qquad g \in L^2_{B,0}(\mathbb{R}^d). \tag{5.7}$$

and

$$\|F^B\|_{2,B} = \|l^B\| \le c_{\Phi,d}\, |B|^{1/2}\, \|l\|. \tag{5.8}$$

Up to now we have a function F^B corresponding to each ball B. From these functions we shall construct a function $f \in BMO(\mathbb{R}^d)$ so that for every ball B the difference $f - F^B$ is a.e. constant on B.

At first we observe that, if B and B' are balls with $B \subseteq B'$, then $L^2_{B,0}(\mathbb{R}^d) \subseteq L^2_{B',0}(\mathbb{R}^d)$ and hence (5.7) implies

$$\int_{\mathbb{R}^d} g(x)(F^B(x) - F^{B'}(x))\, dx = l^B(g) - l^{B'}(g) = l(g) - l(g) = 0, \qquad g \in L^2_{B,0}(\mathbb{R}^d).$$

Therefore, $F^B - F^{B'}$ is a.e. constant on B.

Now we consider the balls $B_n = B(0,n)$, $n \in \mathbb{N}$, and we define the constants c_n, $n \in \mathbb{N}$, so that:

$$c_n = F^{B_1}(x) - F^{B_n}(x), \qquad \text{a.e. } x \in B_1.$$

We also define the functions $f_n : B_n \to \mathbb{C}$, $n \in \mathbb{N}$, by

$$f_n(x) = F^{B_n}(x) + c_n, \qquad x \in B_n.$$

If $n < m$, then for a.e. $x \in B_n$ we have

$$f_n(x) - f_m(x) = F^{B_n}(x) + c_n - F^{B_m}(x) - c_m = c,$$

where c is a constant, because $F^{B_n} - F^{B_m}$ must be a.e. constant on B_n. To find the value of c we take $x \in B_1$ and we get

$$c = F^{B_n}(x) + c_n - F^{B_m}(x) - c_m = F^{B_1}(x) - F^{B_1}(x) = 0.$$

Therefore,

$$f_n(x) = f_m(x), \qquad \text{a.e. } x \in B_n,$$

and we may define the function $f : \mathbb{R}^d \to \mathbb{C}$ so that

$$f(x) = f_n(x) = F^{B_n}(x) + c_n, \qquad \text{a.e. } x \in B_n.$$

Now take any x and any ball B containing x. Then there is a B_n such that $B \subseteq B_n$. Hence there is a constant c so that $F^{B_n}(x) - F^B(x) = c$ for a.e. $x \in B$. Then

$$f(x) = F^B(x) + c + c_n, \qquad \text{a.e. } x \in B \tag{5.9}$$

and thus

$$f_B = \frac{1}{|B|} \int_B (F^B(x) + c + c_n)\, dx = c + c_n$$

since $\int_B F^B(x)\, dx = 0$. Therefore,

$$\frac{1}{|B|} \int_B |f(x) - f_B|\, dx \le \frac{1}{|B|^{1/2}} \Big(\int_B |f(x) - f_B|^2\, dx \Big)^{1/2} = \frac{1}{|B|^{1/2}} \Big(\int_B |F^B(x)|^2\, dx \Big)^{1/2}$$
$$\le c_{\Phi,d}\, \|l\|$$

from (5.8). We conclude that $f \in BMO(\mathbb{R}^d)$ and

$$\|f\|_* \le c_{\Phi,d}\, \|l\|.$$

For every ball B we have

$$l^B(g) = \int_B g(x) F^B(x)\, dx, \qquad g \in L^2_{B,0}(\mathbb{R}^d).$$

From (5.9) we get

$$l^B(g) = \int_B g(x) f(x)\, dx, \qquad g \in L^2_{B,0}(\mathbb{R}^d), \tag{5.10}$$

since $\int_B g(x)\, dx = 0$. Now let $g \in H^1_a(\mathbb{R}^d)$. Then there is a finite set J, atoms a_j and numbers λ_j so that

$$g = \sum_{j \in J} \lambda_j a_j.$$

If B_j is the ball corresponding to a_j, we take a ball B containing the union of all B_j, $j \in J$ and we get

$$\int_B |g(x)|^2\, dx \le |J| \int_B \sum_{j \in J} |\lambda_j|^2 |a_j(x)|^2\, dx = |J| \sum_{j \in J} |\lambda_j|^2 \int_{B_j} |a_j(x)|^2\, dx$$
$$\le |J| \sum_{j \in J} \frac{|\lambda_j|^2}{|B_j|} < +\infty.$$

Also

$$\int_B g(x)\,dx = \sum_{j\in J} \lambda_j \int_{B_j} a_j(x)\,dx = 0.$$

Hence $g \in L^2_{B,0}(\mathbb{R}^d)$ and now (5.7) and (5.10) imply

$$l(g) = \int_B g(x)f(x)\,dx = l_f(g), \qquad g \in H^1_a(\mathbb{R}^d).$$

We have already proved that $\|f\|_* \le c_{\Phi,d}\,\|l\|$, where the constant can be made to depend only on d since Φ is any particular function with properties described above.
To prove the uniqueness of f, we assume that $f \in BMO(\mathbb{R}^d)$ has the property:

$$\int_{\mathbb{R}^d} g(x)f(x)\,dx = 0, \qquad g \in H^1_a(\mathbb{R}^d).$$

Then for every atom a supported in a certain ball B we have

$$\int_B a(x)f(x)\,dx = 0$$

and hence

$$\int_B a(x)(f(x) - f_B)\,dx = 0.$$

Therefore, for any constant c and any atom supported in B we have

$$\int_B (a(x) + c)(f(x) - f_B)\,dx = 0$$

and, finally,

$$\int_B h(x)(f(x) - f_B)\,dx = 0$$

for every bounded function h in B. This implies that $f - f_B = 0$ and hence f is a.e. constant in B. Since the ball B is arbitrary, we conclude that f is a.e. constant and hence $f = 0$ in $BMO(\mathbb{R}^d)$. $\square$

Theorems 5.1 and 5.2, together, say that the linear operator

$$T : BMO(\mathbb{R}^d) \to \left(H^1(\mathbb{R}^d)\right)^*$$

defined by

$$T(f) = l_f, \qquad f \in BMO(\mathbb{R}^d),$$

is an isomorphism of $BMO(\mathbb{R}^d)$ onto $\left(H^1(\mathbb{R}^d)\right)^*$.

Basic Bibliography

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