CLOSED RANGE COMPOSITION OPERATORS ON B^1

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ABSTRACT. We give a necessary condition on ψ for $C_\psi : B^1 \to B^1$ to have closed range.

1. The Besov-1 space

$\mathbb{D}$ is the unit disc in $\mathbb{C}$ and the Möbius functions

$$\mu\phi_a(z) = \mu\frac{a-z}{1-\overline{a}z}\,, \qquad a \in \mathbb{D}, |\mu| = 1$$

are the conformal mappings of $\mathbb{D}$ onto itself. In case $a \in \partial\mathbb{D}$, then $\mu\phi_a$ is a constant function.

The Besov-1 space, B^1 , consists of all functions f analytic on $\mathbb{D}$ which can be written as

$$f = \sum_{n=1}^{+\infty} \lambda_n \phi_{a_n}\,, \qquad a_n \in \overline{\mathbb{D}}, \sum_{n=1}^{+\infty} |\lambda_n| < +\infty.$$

The

$$\|f\|_{B^1} = \inf \sum_{n=1}^{+\infty} |\lambda_n|$$

is a norm on B^1 under which we have $\|\phi_a\|_{B^1} = 1$ for all $a \in \overline{\mathbb{D}}$.

It is easily proved that, under this norm, B^1 is a Banach space *invariant* under right-composition by Möbius functions and that it is the *minimal* space with this property.

Equivalently, B^1 consists of all f analytic on $\mathbb{D}$ for which $\iint_{\mathbb{D}} |f''(z)|\,dA(z) < +\infty$, where dA is the normalized Lebesgue measure on $\mathbb{D}$. We have the following equivalence between norms:

$$\|f\|_{B^1} \asymp |f(0)| + |f'(0)| + \iint_{\mathbb{D}} |f''(z)|\,dA(z).$$

One direction of this equivalence comes from the formulae

$$\phi_a{}'(z) = \frac{|a|^2 - 1}{(1-\overline{a}z)^2}\,, \qquad \phi_a{}''(z) = 2\overline{a}\frac{|a|^2 - 1}{(1-\overline{a}z)^3}$$

and the resulting

$$\iint_{\mathbb{D}} |\phi_a{}''(z)|\,dA(z) \lesssim 1.$$

The other direction comes from an appropriate discretization of the formula

$$f(z) = f(0) + f'(0)z - \iint_{\mathbb{D}} \frac{1}{\overline{a}}\, f''(a)\phi_a(z)\,dA(a).$$

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2. Other Möbius invariant spaces

B^p $(1 < p < +\infty)$, the Besov-p space, consists of all f analytic in $\mathbb{D}$ with the norm

$$\|f\|_{B^p} = |f(0)| + \iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z)$$
$$\asymp_p |f(0)| + |f'(0)| + \iint_{\mathbb{D}} |f''(z)|^p (1-|z|^2)^{p-1}\, dA(z).$$

B is the Bloch space, consisting of all b analytic on $\mathbb{D}$ with

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|b'(z)| < +\infty.$$

B_0 is the little-Bloch space, consisting of all b analytic on $\mathbb{D}$ with

$$\lim_{z\in\mathbb{D}, |z|\to 1} (1-|z|^2)|b'(z)| = 0.$$

It is true that

$$B^1 \subseteq B^{p_1} \subseteq B^{p_2} \subseteq B_0 \subseteq B$$

for $1 < p_1 < p_2 < +\infty$ and B^1 is the "limit" of B^p as $p \to 1$.

Other Möbius invariant spaces are H^∞, the disc algebra A, the $BMOA$ and $VMOA$, all containing B^1.

3. Duality

For all $f \in B^1$ and all $b \in B$ we define the sesquilinear product

$$\langle f, b\rangle = f(0)\overline{b(0)} + f'(0)\overline{b'(0)} + \iint_{\mathbb{D}} \frac{1}{z} f''(z)(1-|z|^2)\overline{b'(z)}\, dA(z).$$

Through this product one may easily prove the dualities

$$B^{1*} \cong B, \qquad {B_0}^* \cong B^1.$$

4. Composition operators: boundedness

For analytic $\psi : \mathbb{D} \to \mathbb{D}$ we define the *composition operator*

$$C_\psi f = f \circ \psi.$$

It is easy to prove that $C_\psi : B^1 \to B^1$ is *bounded* if and only if

$$\|C_\psi\| = \sup_{a\in\mathbb{D}} \|C_\psi \phi_a\|_{B^1} < +\infty.$$

We have the following sequence of equivalent condtitions:

$$\sup_{a\in\mathbb{D}} (1-|a|^2) \iint_{\mathbb{D}} \Big|2\overline{a}\frac{\psi'(z)^2}{(1-\overline{a}\psi(z))^3} + \frac{\psi''(z)}{(1-\overline{a}\psi(z))^2}\Big|\, dA(z) < +\infty,$$

$$\sup_{a\in\mathbb{D}} (1-|a|^2) \left(\iint_{\mathbb{D}} \frac{|\psi'(z)|^2}{|1-\overline{a}\psi(z)|^3}\, dA(z) + \iint_{\mathbb{D}} \frac{|\psi''(z)|}{|1-\overline{a}\psi(z)|^2}\, dA(z) \right) < +\infty,$$

$$\sup_I \left(\frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \right) < +\infty,$$

where I is the typical arc of the unit circle and $S = S(I)$ is the Carleson square with I as its base.

5. Composition operators: compactness

It is also known that $C_\psi : B^1 \to B^1$ is *compact* if and only if

$$\lim_{a\in\mathbb{D},|a|\to 1} \|C_\psi \phi_a\|_{B^1} = 0.$$

We also have the corresponding sequence of equivalent condtitions, with the $\sup_{a\in\mathbb{D}}$ replaced by $\lim_{a\in\mathbb{D},|a|\to 1}$ and with $\sup_I$ replaced by $\lim_{|I|\to 0}$.

It is somewhat strange that all these conditions are equivalent to

$$\sup_{z\in\mathbb{D}} |\psi(z)| < 1.$$

6. Composition operators: closed range

As a general fact, $C_\psi : B^1 \to B^1$ has *closed range* if and only if

$$\|C_\psi f\|_{B^1} \ge c\|f\|_{B^1}, \qquad f \in B^1,$$

for some constant $c > 0$ independent of $f \in B^1$.

We shall prove that the condition

$$\liminf_{|I|\to 0} \sup_{a\in T(S)} (1-|a|^2) \iint_{\psi^{-1}(S)} \Big|2\overline{a}\frac{\psi'(z)^2}{(1-\overline{a}\psi(z))^3} + \frac{\psi''(z)}{(1-\overline{a}\psi(z))^2}\Big|\, dA(z) > 0,$$

where $T(S)$ is the inner half of the Carleson square S, is *necessary* for C_ψ to have closed range.

The last condition is equivalent to

$$\liminf_{|I|\to 0} \left(\frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \right) > 0.$$

Proof of the equivalence of the two conditions.

If $a \in T(S)$ and $\psi(z) \in S$, then $1-|a|^2 \asymp |I|$ and $|1-\overline{a}\psi(z)| \asymp |I|$. Therefore, the first condition trivially implies the second.

Now, suppose that for some sequence of I's with $|I| \to 0$ we have

$$\sup_{a\in T(S)} (1-|a|^2) \iint_{\psi^{-1}(S)} \Big|2\overline{a}\frac{\psi'(z)^2}{(1-\overline{a}\psi(z))^3} + \frac{\psi''(z)}{(1-\overline{a}\psi(z))^2}\Big|\, dA(z) \to 0.$$

This implies

$$\sup_{a\in T(S)} \frac{1}{|I|} \iint_{\psi^{-1}(S)} \Big|2\overline{a}\frac{\psi'(z)^2}{1-\overline{a}\psi(z)} + \psi''(z)\Big|\, dA(z) \to 0.$$

Applying this for any $a_1, a_2 \in T(S)$, we get

$$\frac{1}{|I|} \iint_{\psi^{-1}(S)} \Big|\frac{\overline{a_1}}{1-\overline{a_1}\psi(z)} - \frac{\overline{a_2}}{1-\overline{a_2}\psi(z)}\Big||\psi'(z)|^2\, dA(z) \to 0$$

and, if moreover $|a_1 - a_2| \asymp |I|$, then we get

$$\tfrac{1}{|S|} \textstyle\iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) \to 0, \qquad \tfrac{1}{|I|} \textstyle\iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \to 0.$$

Proof of the necessity of the second condition.

Suppose that for some sequence of I's with $|I| \to 0$ we have

$$\delta_I := \frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \to 0.$$

For each I (in our sequence) we consider a *much smaller* (to be determined) arc I' so that I and I' have the same center. We consider the corresponding $S = S(I)$ and $S' = S(I')$ and, also, two

$a_1, a_2 \in T(S')$ such that $|a_1 - a_2| \asymp |I'|$ and $1 - |a_1|^2 = 1 - |a_2|^2 \asymp |I'|$.

Now we take

$$f = \phi_{a_1} - \phi_{a_2}\,.$$

It is easy to see that

$$\|f\|_{B^1} \asymp 1$$

and we shall prove that

$$\|C_\psi f\|_{B^1} \to 0.$$

Now

$$f'(z) = \frac{|a_1|^2 - 1}{(1 - \overline{a_1} z)^2} - \frac{|a_2|^2 - 1}{(1 - \overline{a_2} z)^2} = (|a_1|^2 - 1)(\overline{a_1} - \overline{a_2}) \frac{2 - (\overline{a_1} + \overline{a_2}) z}{(1 - \overline{a_1} z)^2 (1 - \overline{a_2} z)^2}$$

and, hence,

$$|f'(z)| \asymp \frac{|I'|^2}{|1 - \overline{a} z|^3}\,,$$

where a is any of a_1 or a_2. Similarly:

$$|f''(z)| \asymp \frac{|I'|^2}{|1 - \overline{a} z|^4}\,.$$

We have

$$\|C_\psi f\|_{B^1} \asymp |f(\psi(0))| + |f'(\psi(0))||\psi'(0)| + A + B,$$

where

$$A := \iint_{\psi^{-1}(S)} |f''(\psi(z))\psi'(z)^2 + f'(\psi(z))\psi''(z)|\, dA(z),$$

$$B := \iint_{\mathbb{D}\setminus\psi^{-1}(S)} |f''(\psi(z))\psi'(z)^2 + f'(\psi(z))\psi''(z)|\, dA(z).$$

Trivially:

$$|f(\psi(0))| + |f'(\psi(0))||\psi'(0)| \asymp |I'| + |I'|^2 \to 0.$$

Also:

$$\begin{aligned} A &\lesssim |I'|^2 \iint_{\psi^{-1}(S)} \frac{|\psi'(z)|^2}{|1 - \overline{a}\psi(z)|^4}\, dA(z) + |I'|^2 \iint_{\psi^{-1}(S)} \frac{|\psi''(z)|}{|1 - \overline{a}\psi(z)|^3}\, dA(z) \\ &\lesssim \frac{|I|^2}{|I'|^2} \frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2\, dA(z) + \frac{|I|}{|I'|} \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)|\, dA(z) \\ &\lesssim \frac{|I|^2}{|I'|^2}\, \delta_I \end{aligned}$$

and

$$\begin{aligned} B &\lesssim |I'|^2 \iint_{\mathbb{D}\setminus\psi^{-1}(S)} \frac{|\psi'(z)|^2}{|1 - \overline{a}\psi(z)|^4}\, dA(z) + |I'|^2 \iint_{\mathbb{D}\setminus\psi^{-1}(S)} \frac{|\psi''(z)|}{|1 - \overline{a}\psi(z)|^3}\, dA(z) \\ &\lesssim \frac{|I'|}{|I|} (1 - |a|^2) \left(\iint_{\mathbb{D}} \frac{|\psi'(z)|^2}{|1 - \overline{a}\psi(z)|^3}\, dA(z) + \iint_{\mathbb{D}} \frac{|\psi''(z)|}{|1 - \overline{a}\psi(z)|^2}\, dA(z) \right) \\ &\lesssim \frac{|I'|}{|I|}\,. \end{aligned}$$

Therefore,

$$A + B \lesssim \frac{|I|^2}{|I'|^2}\, \delta_I + \frac{|I'|}{|I|}\,.$$

Now let

$$|I'| = \sqrt[3]{\delta_I}\, |I|$$

and conclude that

$$A + B \lesssim \sqrt[3]{\delta_I} \to 0.$$

The conjecture is:

$C_\psi : B^1 \to B^1$ has closed range if and only if the condition

$$\liminf_{|I|\to 0} \left(\frac{1}{|S|} \iint_{\psi^{-1}(S)} |\psi'(z)|^2 \, dA(z) + \frac{1}{|I|} \iint_{\psi^{-1}(S)} |\psi''(z)| \, dA(z) \right) > 0$$

on ψ is true.

7. "Closed range" results for other spaces

Surprisingly little is known about conditions on ψ which are equivalent for $C_\psi : X \to X$ to have closed range, when X is a space of analytic functions. Here is what we know.

1. Zorboska in "Composition operators with closed range" (1994) solves the problem when X is any of the weighted Bergman spaces A^2_α defined by $\iint_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < +\infty$ and when X is the H^2 space ($\iint_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < +\infty$).

2. Jovovic and McCluer in "Composition operators on Dirichlet spaces" (1997) consider the weighted Dirichlet spaces $\mathcal{D}_\alpha$ defined by $\iint_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^\alpha \, dA(z) < +\infty$. ($\mathcal{D}_0 = \mathcal{D}, \mathcal{D}_1 = H^2, \mathcal{D}_2 = A^2$). When $X = \mathcal{D}$, they prove that $|\psi(\mathbb{D}) \cap S| \geq c|S|$ for all S is a sufficient (but not necessary) condition for C_ψ to have closed range. They also prove that $\frac{1}{|S|} \iint_S n_\psi(w) \, dA(w) \geq c$ for all S is a necessary condition. But Luecking has proved that it is not sufficient.

3. Akeroyd and Ghatage in "Closed range composition operators on A^2" (2008) solve the problem when $X = A^2$. Their condition is different from Zorboska's condition.

4. Akeroyd, Ghatage and Tjani in "Closed range composition operators on A^2 and the Bloch space" (2010) solve the problem when X is the Bloch space.

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