PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 103, Number 3, July 1988

BEST UNIFORM APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS

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(Communicated by Irwin Kra)

ABSTRACT. This paper gives a counterexample to the conjecture that the continuity of the conjugate $\tilde{f}$ of an $f \in C(T)$ implies the continuity of the best uniform approximation $g \in H^\infty(T)$ of f. It also states two conditions which imply the continuity of g.

Let $L^\infty(T)$ the space of bounded measurable functions on the unit circle T, $H^\infty(T)$ the subalgebra of $L^\infty(T)$ consisting of nontangential limits of bounded analytic functions in the unit disk and write $\|f\|_\infty$ for the (essential supremum) norm of $f \in L^\infty(T)$. Also, let $C(T)$ be the space of all continuous functions on T.

It is known that any $f \in L^\infty(T)$ has at least one best approximation $g \in H^\infty(T)$, in the sense that

$$d = \|f - g\|_\infty = \inf_{h \in H^\infty} \|f - h\|_\infty$$

and that, by duality

$$(*) \qquad d = \sup\left\{\left|\int_0^{2\pi} f(\theta)F(\theta)\frac{d\theta}{2\pi}\right| : F \in H^1(T), F(0) = 0, \|F\|_1 \leq 1\right\}$$

where $H^p(T)$ $(0 < p < \infty)$ is the Hardy space of all nontangential limits of functions F analytic in the unit disc such that

$$\|F\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta})| \frac{d\theta}{2\pi} < +\infty.$$

Moreover, if f is continuous, then the best approximation g of f is unique and there is at least one F, for which the supremum $(*)$ is attained. Also f, g and any of those maximizing F's are connected by

$$(1) \qquad f(\theta) - g(\theta) = \|f - g\|_\infty \frac{\overline{F(\theta)}}{|F(\theta)|} \quad \text{a.e. } (d\theta)$$

which implies

$$|f(\theta) - g(\theta)| = \|f - g\|_\infty = d \quad \text{a.e. } (d\theta).$$

We need the following result (see [**1** or **2**]):

THEOREM 1 (CARLESON-JACOBS). *If $f \in C(T)$, $g \in H^\infty(T)$, $F \in H^1(T)$ are connected by* (1), *then*

(a) $F \in H^p(T)$, *for all* $p < +\infty$,

Received by the editors May 19, 1987.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30D55; Secondary 42A50.
Partially supported by NSF Grant # DMS85-03780.

(b) *if* $\tau \in [0, 2\pi]$ *and if*

$$f_\tau(\theta) = f(\theta) - f(\tau), \qquad g_\tau(\theta) = g(\theta) - f(\tau)$$

then there is $\delta > 0$ *and* $r_0 > 0$ *such that*

$$|g_\tau(z)| \geq \tfrac{1}{2} \cdot \|f - g\|_\infty \quad \text{on } W_\tau = \{z = re^{i\theta} : |\theta - \tau| < \delta, r_0 < r < 1\}$$

where δ *and* r_0 *can be independent of* τ.

We consider the problem of how the regularity of f affects the regularity of g. In [**1**] the following is proved.

THEOREM 2. *If f is Dini-continuous, i.e. if $\int_0 (\omega(t)/t)\,dt < +\infty$, where $\omega(t) = \sup_{|x-y|\leq t} |f(x) - f(y)|$ is the modulus of continuity of f, then its best approximation g is also continuous.*

In [**1**] a function f is constructed, continuous but not Dini-continuous, whose best approximation g is not continuous.

Because the Dini-continuity of f implies the continuity of its conjugate $\tilde{f}$ and because of the proof in [**1**], it was conjectured that, for $f \in C(T)$, the continuity of $\tilde{f}$ and the continuity of g are equivalent.

It was proved by Sarason that the continuity of g does not imply the continuity of $\tilde{f}$. See [**2**, p. 177].

This paper provides a counterexample for the other half of the conjecture. It constructs a continuous function f, whose conjugate $\tilde{f}$ is continuous, but whose best approximation g is not. We also give two further conditions on f which imply g is continuous.

In the following $\overline{f}$ is the complex conjugate of f.

THEOREM 3. *If $\overline{f} \in A(T) = H^\infty(T) \cap C(T)$ and $\int_0 (\omega^2(t)/t)\,dt < +\infty$, then g, the best approximation of f, is continuous.*

THEOREM 4. *If $\overline{f} \in A(T)$ and $\widetilde{|f|^2} \in C(T)$ and $\int_0 (\omega^3(t)/t)\,dt < +\infty$, then g is continuous.*

THEOREM 5. *There exists a function f, such that $\overline{f} \in A(T)$, but such that its best approximation g is not continuous.*

Since $\tilde{f} = -if$ when $\overline{f} \in A(T)$, the function in Theorem 5 has a continuous conjugate.

PROOF OF THEOREM 3. Suppose $\|f - g\|_\infty = 1$. Fix $\tau \in [0, 2\pi]$. Then, from Theorem 1(b), $g_\tau(z)$ has a well-defined logarithm on W_τ, which is given by

$$\log g_\tau(z) = \frac{1}{2\pi} \int_{|\theta-\tau|\leq\delta} \log|g_\tau(\theta)| \cdot \frac{e^{i\theta} + z}{e^{i\theta} - z}\,d\theta + R_\tau(z), \qquad z \in W_\tau,$$

where $R_\tau(z)$ is the integral over $|\theta - \tau| > \delta$ plus the logarithm of the inner factor of g_τ. Since $|g_\tau| \geq \frac{1}{2}$ on W_τ, this inner factor is analytic across $|\theta - \tau| < \delta$. So $R_\tau(z)$ and its derivative are bounded on $|z - e^{i\tau}| < \delta_1$, for some $\delta_1 < \delta$, independent of τ. This implies

$$(2) \qquad |R_\tau(z) - R_\tau(w)| \leq c|z - w| \quad \text{for } |z - e^{i\tau}| < \delta_1,\ |w - e^{i\tau}| < \delta_1.$$

We also have

$$|f(\theta) - g(\theta)| = 1 \quad \text{a.e. } (d\theta)$$

from which

$$|f_\tau(\theta) - g_\tau(\theta)| = 1 \quad \text{a.e. } (d\theta)$$

and

$$|g_\tau|^2 = 1 + 2 \cdot \mathrm{Re}(\overline{f}_\tau \cdot g_\tau) - |f_\tau|^2.$$

Therefore

$$\begin{aligned}\log|g_\tau| = \tfrac{1}{2}\log|g_\tau|^2 &= \tfrac{1}{2}\left[2 \cdot \mathrm{Re}(\overline{f}_\tau \cdot g_\tau) - |f_\tau|^2 + O(|f_\tau|^2)\right] \\ &= \mathrm{Re}(\overline{f}_\tau g_\tau) + O(|f_\tau|^2),\end{aligned}$$

and

$$\begin{aligned}\log g_\tau(z) = \frac{1}{2\pi}\int_{|\theta-\tau|\le\delta} &\mathrm{Re}(\overline{f}_\tau g_\tau)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta \\ &+ \frac{1}{2\pi}\int_{|\theta-\tau|\le\delta} O(|f_\tau|^2)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta + R_\tau(z).\end{aligned}$$

Since $\overline{f}_\tau$ is analytic, $\overline{f}_\tau g_\tau$ is also analytic, which implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(\overline{f}_\tau g_\tau)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta = \overline{f}_\tau(z) \cdot g_\tau(z).$$

Thus:

$$\log g_\tau(z) - \overline{f}_\tau(z) g_\tau(z) = \frac{1}{2\pi}\int_{|\theta-\tau|\le\delta} O(|f_\tau|^2)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta + R_\tau^*(z)$$

where

$$R_\tau^*(z) = R_\tau(z) - \frac{1}{2\pi}\int_{|\theta-\tau|>\delta} \mathrm{Re}(\overline{f}_\tau g_\tau)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$$

and so, by (2),

(3) $$|R_\tau^*(z) - R_\tau^*(w)| \le c|z-w| \quad \text{for } |z-e^{i\tau}| < \delta_1,\ |w-e^{i\tau}| < \delta_1.$$

If z is in a truncated cone $\Gamma(\tau)$, which is inside $|z-e^{i\tau}| < \delta_1$ and has vertex $e^{i\tau}$, then

$$\left|\frac{e^{i\theta}+z}{e^{i\theta}-z}\right| < \frac{c}{|\theta-\tau|},$$

and so

$$|\log g_\tau(z) - \overline{f}_\tau(z) g_\tau(z) - R_\tau^*(z)| \le c \cdot \int_0^\delta \frac{\omega^2(t)}{t}\,dt.$$

Since $\overline{f}_\tau(z) \to 0$ as $z \to e^{i\tau}$,

$$|\log g_\tau(z) - R_\tau^*(z)| \le c\int_0^\delta \frac{\omega^2(t)}{t}\,dt + \eta(\delta)$$

where $\eta(\delta) \to 0$ as $\delta \to 0$. Hence, by (3),

$$|g_\tau(z) - g_\tau(w)| \le c|z-w| + \eta_1(\delta), \qquad z, w \in \Gamma(\tau),$$

where $\eta_1(\delta) \to 0$ as $\delta \to 0$.

Now, if σ and τ are close to each other and $z \in \Gamma(\tau) \cap \Gamma(\sigma)$ then

$$\begin{aligned}|g(e^{i\tau}) - g(e^{i\theta})| &\le |g(e^{i\tau}) - g(z)| + |g(z) - g(e^{i\sigma})| \\ &= |g_\tau(e^{i\tau}) - g_\tau(z)| + |g_\sigma(z) - g_\sigma(e^{i\sigma})| \\ &\le c|e^{i\tau} - z| + c|e^{i\sigma} - z| + 2\eta_1(\delta) \le c|\tau - \sigma| + 2\eta_1(\delta),\end{aligned}$$

and

$$\overline{\lim_{\sigma \to \tau}} |g(e^{i\tau}) - g(e^{i\sigma})| \le 2\eta_1(\delta)$$

so that

$$\lim_{\sigma \to \tau} g(e^{i\sigma}) = g(e^{i\tau})$$

and g is continuous.

PROOF OF THEOREM 4. Now we carry the expansion of $\log|g_\tau|$ one step further:

$$\begin{aligned}\log|g_\tau| &= \frac{1}{2}\left[2\operatorname{Re}(\overline{f}_\tau g_\tau) - |f_\tau|^2 - \frac{(2\operatorname{Re}(\overline{f}_\tau g_\tau) - |f_\tau|^2)^2}{2} + O(|f_\tau|^3)\right] \\ &= \operatorname{Re}(\overline{f}_\tau g_\tau) - \tfrac{1}{2}|f_\tau|^2 - (\operatorname{Re}(\overline{f}_\tau g_\tau))^2 + O(|f_\tau|^3) \\ &= \operatorname{Re}(\overline{f}_\tau g_\tau) - \tfrac{1}{2}|f_\tau|^2 - \tfrac{1}{2}|\overline{f}_\tau g_\tau|^2 - \tfrac{1}{2}\operatorname{Re}(\overline{f}_\tau g_\tau)^2 + O(|f_\tau|^3) \\ &= \operatorname{Re}(\overline{f}_\tau g_\tau) - \tfrac{1}{2}\operatorname{Re}(\overline{f}_\tau g_\tau)^2 - \tfrac{1}{2}|f_\tau|^2 \\ &\quad - \tfrac{1}{2}|f_\tau|^2(1 + 2\operatorname{Re}(\overline{f}_\tau g_\tau) - |f_\tau|^2) + O(|f_\tau|^3) \\ &= \operatorname{Re}(\overline{f}_\tau g_\tau) - \tfrac{1}{2}\operatorname{Re}(\overline{f}_\tau g_\tau)^2 - |f_\tau|^2 + O(|f_\tau|^3).\end{aligned}$$

Now, because

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}(\overline{f}_\tau g_\tau)^2 \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta = (\overline{f}_\tau(z) g_\tau(z))^2$$

since $\overline{f}g \in H^\infty(T)$, we get

$$\begin{aligned}&\log g_\tau(z) - \overline{f}_\tau(z) g_\tau(z) + \tfrac{1}{2}(\overline{f}_\tau(z) g_\tau(z))^2 \\ &= -\frac{1}{2\pi}\int_{|\theta - \tau| \le \delta} |f_\tau(\theta)|^2 \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta + \frac{1}{2\pi}\int_{|\theta - \tau| \le \delta} O(|f_\tau|^3)\frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta + R_\tau^{**}(z)\end{aligned}$$

where

$$R_\tau^{**}(z) = R_\tau^*(z) + \frac{1}{2\pi}\int_{|\theta - \tau| > \delta} \operatorname{Re}(\overline{f}_\tau g_\tau)^2 \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta$$

and so

$$|R_\tau^{**}(z) - R_\tau^{**}(w)| \le c|z - w| \quad \text{for } |z - e^{i\tau}| \le \delta_1,\ |w - e^{i\tau}| \le \delta_1.$$

Now, the continuity of $\widetilde{|f|^2}$ implies the continuity of $\widetilde{|f_\tau|^2}$, and this implies the continuity of the first integral. The rest of the proof proceeds as in Theorem 3.

PROOF OF THEOREM 5. Consider the function

$$\begin{aligned}u(t) &= -\alpha_1 \log|\log t|, &&0 < t < \tfrac{1}{2}, \\ &= -\alpha_2 \log|\log|t||, &&-\tfrac{1}{2} < t < 0,\end{aligned}$$

extended to be smooth in $[-\pi, \pi] - \{0\}$, and consider the harmonic extension $u(z)$ of $u(t)$ inside the unit disk, its conjugate $\tilde{u}(z)$ and $f(z) = e^{u(z) - i\tilde{u}(z)}$.

Then, since $\tilde{u}(t)$ is continuous in $[-\pi, \pi] - \{0\}$, and $|f(z)| = e^{u(z)} \to 0$ as $z \to 1$, we see that $\overline{f} \in A(T)$.

If $\frac{1}{3} < \alpha_1 \le \frac{1}{2}$ and $\frac{1}{2} < \alpha_2$, then

$$\int_0^{1/2} \frac{|f(t)|^3}{t}\,dt < +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^3}{|t|}\,dt < +\infty$$

but

$$\int_0^{1/2} \frac{|f(t)|^2}{t}\,dt = +\infty \quad \text{and} \quad \int_{-1/2}^{0} \frac{|f(t)|^2}{|t|}\,dt < +\infty.$$

The last two imply that

$$\widetilde{|f|^2}(r) \to +\infty \quad \text{as } r \to 1-.$$

From

$$\begin{aligned}\log g(r) &- \overline{f}(r)g(r) + \tfrac{1}{2}(\overline{f}(r)\cdot g(r))^2 \\ &= |f|^2(r) + i\widetilde{|f|^2}(r) + \frac{1}{2\pi}\int_{|\theta-\tau|\le\delta} O(|f|^3)\frac{e^{i\theta}+r}{e^{i\theta}-r}\,d\theta + R^{**}(r)\end{aligned}$$

we get that

$$\arg g(r) \to +\infty \quad \text{as } r \to 1-.$$

Thus g is not continuous.

References

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