CONTINUITY OF THE OPERATOR OF BEST UNIFORM APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS

M. PAPADIMITRAKIS

Let $T = \{e^{i\theta} : 0 \leqslant \theta < 2\pi\}$ and $L^\infty = L^\infty(T)$, the space of all measurable essentially bounded complex-valued functions defined on T equipped with the sup-norm. $C(T)$ is the closed subspace of L^∞ of all continuous functions. Also, $H^\infty(T)$ is the closed subspace of L^∞ of all functions which are boundary values of bounded holomorphic functions defined in $D = \{z \in \mathbb{C} : |z| < 1\}$.

It is well known that for every $f \in C(T)$, there exists a unique $g \in H^\infty(T)$ such that $\|f-g\|_\infty = \operatorname{dist}(f, H^\infty) = \min\{\|f-h\|_\infty : h \in H^\infty\}$; see **[2]**. Hence the transformation of best approximation is defined as

$$S : C(T) \longrightarrow H^\infty, \quad S(f) = g.$$

The problem considered in this work is to find all continuity points of S. The following theorem will be proved.

THEOREM 1. *$f \in C(T)$ is a continuity point for S if and only if $f \in C(T) \cap H^\infty$.*

This was a conjecture of V. V. Peller; see **[4, 5]**. Also see **[3]**, for another proof of the same and related results.

To prove this theorem, we shall make use of two lemmas.

LEMMA 1. *Let $f_n, f \in C(T)$ $(n = 1, 2, 3, \ldots)$ and $d_n = \|f_n - g_n\|_\infty = \operatorname{dist}(f_n, H^\infty)$, $d = \|f-g\|_\infty = \operatorname{dist}(f, H^\infty)$. If $\|f_n - f\|_\infty \to 0$, then $d_n \to d$.*

Proof. $d_n = \|f_n - g_n\|_\infty \leqslant \|f_n - g\|_\infty \leqslant \|f_n - f\|_\infty + \|f-g\|_\infty$. Hence $\limsup d_n \leqslant d$. $d = \|f-g\|_\infty \leqslant \|f-g_n\|_\infty \leqslant \|f-f_n\|_\infty + \|f_n - g_n\|_\infty$. Hence $d \leqslant \liminf d_n$.

The second lemma is an elaboration of a construction used in **[1]**.

Let $w : [0, \delta] \to \mathbb{R}$, $\delta > 0$, be a continuous function with $w(0) = 0$, $w(x) > 0$, for every $x \in (0, \delta]$, and

$$\int_0^\delta \frac{w(x)}{x}\, dx = +\infty.$$

Consider $f : T \to \mathbb{C}$ defined as

$$f(e^{ix}) = \begin{cases} w(x), & 0 \leqslant x \leqslant \delta, \\ 0, & -\delta \leqslant x \leqslant 0, \end{cases}$$

and arbitrarily defined on $T \backslash [e^{-i\delta}, e^{i\delta}]$ but continuous on T.

Received 8 July 1991; revised 20 January 1992.

1991 *Mathematics Subject Classification* 30D55.

Bull. London Math. Soc. 25 (1993) 44–48

Let $g = S(f)$. Then $d = \|f-g\|_\infty > 0$ and, as is well known, $|f(e^{it})-g(e^{it})| = d$ for a.e. t.

LEMMA 2. *Let f, g be as above.*
Either (a) *there is a sequence $e^{it_k} \to 1$ such that*

$$|\mathrm{Re}\, g(e^{it_k})| < d\sin\frac{\pi}{8}, \quad \textit{for all } k,$$

$$|g(e^{it_k})| \to 1,$$

or (b) *there are two sequences, $e^{it'_k} \to 1$, $e^{it''_k} \to 1$, such that*

$$|g(e^{it'_k}) - g(e^{it''_k})| \geqslant 2d\sin\frac{\pi}{8}.$$

Proof. Suppose that there exists δ_1, $0 < \delta_1 \leqslant \delta$, such that

$$\mathrm{Re}\, g(e^{ix}) \geqslant d\sin\frac{\pi}{8} \quad \text{for a.e. } x \in [-\delta_1, \delta_1]. \tag{1}$$

Assume $f(e^{ix}) < \frac{1}{2}d$ for a.e. $x \in [-\delta_1, \delta_1]$. This implies, since $|f-g| = d$ a.e., that there exists $c > 0$ so that

$$\log|g(e^{ix})| > cw(x) \quad \text{for a.e. } x \in (0, \delta_1]. \tag{2}$$

On the other hand, (1) implies that $\mathrm{Re}\, g(z) \geqslant \frac{1}{2}d\sin\pi/8$ for all $z \in D$ which are close to the interval $(e^{-i\delta_1}, e^{i\delta_1})$. Hence $\arg g(z)$ is well-defined and stays bounded as $0 < z = r \to 1$.

This means that

$$\left|\lim_{r\to 1} \arg g(r)\right| \geqslant \text{constant} + \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<\delta_1} \frac{\log|g(e^{ix})|}{x}\,dx$$
$$= \text{constant} + \lim_{\varepsilon\to 0} \int_{\varepsilon<x<\delta_1} \frac{\log|g(e^{ix})|}{x}\,dx.$$

But, by (2), the last limit is $+\infty$ and we obtain a contradiction.

Similarly, the existence of a δ_1 such that $\mathrm{Re}\, g(e^{ix}) \leqslant -d\sin\pi/8$ for a.e. $x \in [-\delta_1, \delta_1]$ gives a contradiction.

Hence there are two cases.

(a) For all δ_1 with $0 < \delta_1 \leqslant \delta$, the set

$$\left\{-\delta_1 < x < \delta_1 : |\mathrm{Re}\, g(e^{ix})| < d\sin\frac{\pi}{8}\right\}$$

has positive measure.

Since $|f(e^{ix}) - g(e^{ix})| = d$ a.e. and $f(1) = 0$, we can choose a sequence $e^{it_k} \to 1$ such that

$$|\mathrm{Re}\, g(e^{it_k})| < d\sin\frac{\pi}{8}, \quad |g(e^{it_k})| \to 1.$$

This is conclusion (a) of Lemma 2.

(b) For all small enough δ_1, the two sets

$$\left\{-\delta_1 < x < \delta_1 : \operatorname{Re} g(e^{ix}) \geqslant d \sin\frac{\pi}{8}\right\}$$

and

$$\left\{-\delta_1 < x < \delta_1 : \operatorname{Re} g(e^{ix}) \leqslant -d \sin\frac{\pi}{8}\right\}$$

both have positive measure.

This clearly implies conclusion (b) of Lemma 2.

Proof of Theorem 1. A trivial general metric space argument shows that if $f \in C(T) \cap H^\infty$ then f is a continuity point of S. This is based on the inequality

$$\|S(f_n) - S(f)\|_\infty = \|S(f_n) - f\|_\infty \leqslant \|f_n - S(f_n)\|_\infty + \|f_n - f\|_\infty \leqslant 2\|f_n - f\|_\infty,$$

where we use that $S(f) = f$.

Suppose that $f \in C(T) \backslash H^\infty$ and $S(f) = g$.

If $g \notin C(T)$, take f_n continuously differentiable on T with $\|f_n - f\|_\infty \to 0$. By a well-known theorem (see [1]), $S(f_n) \in C(T)$ and f is not a continuity point of S.

So suppose $g \in C(T)$, and without loss of generality $f(1) = 0$.

Consider the function w of Lemma 2.

Take intervals $[-\delta_n, \delta_n]$ with $\delta_n \downarrow 0$, and functions $\phi_n \in C(T)$ with the following properties:

$$\|\phi_n\|_\infty \leqslant \frac{1}{n}, \tag{3}$$

$$\phi_{2n}(e^{ix}) + f(e^{ix}) = \begin{cases} w(x), & 0 \leqslant x \leqslant \delta_{2n}, \\ 0, & -\delta_{2n} \leqslant x \leqslant 0, \end{cases} \tag{4}$$

$$\phi_{2n+1}(e^{ix}) + f(e^{ix}) = \begin{cases} iw(x), & 0 \leqslant x \leqslant \delta_{2n+1}, \\ 0, & -\delta_{2n+1} \leqslant x \leqslant 0. \end{cases} \tag{5}$$

Let $f_n = f + \phi_n, g_n = S(f_n), d = \|f - g\|_\infty, d_n = \|f_n - g_n\|_\infty$. From (3), $\|f_n - f\|_\infty \to 0$.

Now we apply Lemma 2 to the functions f_{2n} and $-if_{2n+1}$. If conclusion (b) holds for infinitely many $2n$, then for these $2n$ we have two sequences $t'_{k,2n}, t''_{k,2n} \to 0$ such that

$$|g_{2n}(e^{it'_{k,2n}}) - g_{2n}(e^{it''_{k,2n}})| \geqslant 2d_{2n} \sin\frac{\pi}{8}.$$

If we define $B(x;f) = \lim_{\varepsilon \to 0} \operatorname{ess\,sup} \{|f(y)| : x - \varepsilon < y < x + \varepsilon\}$ then, by continuity of g at 1, and by $d_{2n} \to d$, we have $\lim_{n \to \infty} B(1; g_{2n} - g) \geqslant d \sin \pi/8$.

Similarly, conclusion (a) of Lemma 2 for infinitely many $2n+1$ implies

$$\lim_{n \to \infty} B(1; g_{2n+1} - g) \geqslant d \sin\frac{\pi}{8}.$$

Hence suppose that conclusion (a) of Lemma 2 holds for eventually all n. Then for some $t_{k,2n} \to 0$ $(k \to \infty)$,

$$|g_{2n}(e^{it_{k,2n}})| \to d_{2n}, \tag{6}$$

$$|\operatorname{Re} g_{2n}(e^{it_{k,2n}})| < d_{2n} \sin\frac{\pi}{8}. \tag{7}$$

Also, for some $t_{k,2n+1} \to 0$ $(k \to \infty)$,

$$|g_{2n+1}(e^{it_{k,2n+1}})| \to d_{2n+1}, \tag{8}$$

$$|\operatorname{Im} g_{2n+1}(e^{it_{k,2n+1}})| < d_{2n+1} \sin\frac{\pi}{8}. \tag{9}$$

Now (6)–(9) imply

$$|g_{2n}(e^{it_{k,2n}}) - g_{2n+1}(e^{it_{k,2n+1}})| \geqslant |d_{2n} e^{i\pi/8} - d_{2n+1} e^{-i\pi/8}| + o(1)$$

as $k \to \infty$. Thus

$$\lim_{n\to\infty} \max\{B(1; g_{2n} - g), B(1; g_{2n+1} - g)\} \geqslant d \sin\frac{\pi}{8}.$$

In all cases, continuity of g at 1 implies the existence of a sequence f_n such that

$$\lim_{n\to\infty} B(1; g_n - g) \geqslant d \sin\frac{\pi}{8}.$$

This obviously contradicts $\|g_n - g\|_\infty \to 0$, and this is the end of the proof.

One can prove the following quantitative theorem.

THEOREM 2. *Let $f \in C(T)$ and $g = S(f) \in H^\infty$ with $d = \|f-g\|_\infty$. There is an absolute constant $c > 0$ such that there exists a sequence $f_n \in C(T)$ with $g_n = S(f_n)$ and $\|f_n - f\|_\infty \to 0$, $\lim \|g_n - g\|_\infty \geqslant cd$. The constant c can be taken as $c = \frac{1}{2}\sin \pi/8$.*

Proof. In case $g \in C(T)$, the proof of the previous theorem gives the result with $c = \sin \pi/8$.

So let $g \in H^\infty \setminus C(T)$, and take f_n continuously differentiable on T with $\|f_n - f\|_\infty \to 0$. Let $g_n = S(f_n)$. Then $g_n \in C(T)$. Hence there exist $f_{n,m} \in C(T)$ with

$$\|f_{n,m} - f_n\|_\infty \to 0 \quad (m \to \infty),$$
$$\lim \|g_{n,m} - g_n\|_\infty \geqslant d_n \sin\frac{\pi}{8}.$$

In case $\overline{\lim} \|g_n - g\|_\infty \geqslant \frac{1}{2} d \sin \pi/8$, a subsequence of f_n gives the desired result.

But if $\overline{\lim} \|g_n - g\|_\infty < \frac{1}{2} d \sin \pi/8$, then taking for each n a sufficiently large m (and using $d_n \to d$), the sequence $f_{n,m}$ gives the result.

REMARK. Let $A_m(T)$ be the space of boundary values of functions meromorphic in D with at most m poles and bounded close to T. A similar transformation S_m can be defined as

$$S_m(f) = g, \quad f \in C(T), \quad g \in A_m(T),$$
$$\|f - g\|_\infty = \operatorname{dist}(f, A_m(T)).$$

Exactly the same proof applies for the proof of the following.

THEOREM 3. *$f \in C(T)$ is a continuity point of S_m if and only if $f \in C(T) \cap A_m$.*

References

1. L. CARLESON and S. JACOBS, 'Best uniform approximation by analytic functions', *Ark. Mat.* 10 (1972) 219–229.
2. J. GARNETT, *Bounded analytic functions* (Academic Press, New York, 1981).
3. ORLANDO MERINO, 'Stability of qualitative properties and continuity of solutions to problems of optimization over spaces of analytic functions', preprint.
4. V. V. PELLER, 'Continuity properties of the operator of best approximation by analytic functions', LOMI Preprints, E-13-87, Leningrad, 1987.
5. V. V. PELLER, 'Hankel operators and continuity properties of best approximation operators', *Algebra i Analiz* 2 (1990) 162–190.

Department of Mathematics
Washington University
St Louis, MO 63130
USA