ON THE BEST CONSTANT FOR THE DISCRETE HILBERT TRANSFORM

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Abstract. We relate the best constant for the discrete Hilbert transform to the best constant for another, non-discrete, multiplier transform. All comes from two letters to prof. A. Gillespie in Edinbourgh sent on March 5 and 18 of 2004.

Let $1 \leq p < +\infty$.

Suppose $C_{d,p}$ is the smallest constant so that

$$\|H_d(a)\|_p \leq C_{d,p} \|a\|_p$$

for all $a = (a_n)_{n \in \mathbf{Z}}$, where H_d is defined by

$$H_d(a)_n = \frac{1}{\pi} \sum_{m \in \mathbf{Z}, m \neq n} \frac{1}{n-m} a_m$$

for all $n \in \mathbf{Z}$.

Suppose, also, that $C_{\phi,p}$ is the smallest constant so that

$$\|H_\phi(f)\|_p \leq C_{\phi,p} \|f\|_p$$

for all f, where H_ϕ is defined by $\widehat{H_\phi(f)} = \phi \widehat{f}$ on $\mathbf{R}$ and ϕ is 1-periodic with $\phi(\xi) = 1 - 2\xi$ for $0 < \xi < 1$.

We shall prove that $C_{d,p} = C_{\phi,p}$.

Here is something more general.

Consider a function ϕ on $\mathbf{R}$ periodic with period 1 and integrable on $[0,1]$ and an arbitrary ψ in the Schwartz class $\mathcal{S}(\mathbf{R})$ which is supported in the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Define $f(x) = \sum_n a_n \psi(x-n)$ where the sum is, for simplicity, a finite sum. Then $\widehat{f}(\xi) = \sum_n a_n e^{-2\pi i n \xi} \widehat{\psi}(\xi)$ and if we define

$$\widehat{H_\phi f}(\xi) = \phi(\xi) \widehat{f}(\xi)$$

Date: September 10, 2011.

we get

$$\begin{aligned} H_\phi f(x) &= \sum_n a_n \int_{-\infty}^{+\infty} \phi(\xi)\widehat{\psi}(\xi)e^{-2\pi i(n-x)\xi}d\xi \\ &= \sum_n a_n \sum_m \int_0^1 \phi(\xi)\widehat{\psi}(\xi+m)e^{-2\pi i(n-x)(\xi+m)}d\xi \\ &= \sum_n a_n \int_0^1 \phi(\xi)\sum_m \widehat{\psi}(\xi+m)e^{-2\pi i(n-x)(\xi+m)}d\xi \\ &= \sum_n a_n \int_0^1 \phi(\xi)\sum_m \psi(x+m-n)e^{-2\pi i m\xi}d\xi \\ &= \sum_n a_n \sum_m \widehat{\phi}(m)\psi(x+m-n) \\ &= \sum_n a_n \sum_m \widehat{\phi}(m+n)\psi(x+m) \\ &= \sum_m \big(\sum_n \widehat{\phi}(n-m)a_n\big)\psi(x-m), \end{aligned}$$

where for the fourth equality we use the Poisson summation formula.

We also define

$$H_d(\{a_n\}) = \{b_m\}$$

where $b_m = \sum_n \widehat{\phi}(n-m)a_n$ for all m.

Then the above says that

$$H_\phi(f) = g$$

where

$$g(x) = \sum_m b_m\psi(x-m) \qquad f(x) = \sum_n a_n\psi(x-n).$$

Now we use the assumption about the support of ψ to get

$$\begin{aligned} \sum_m |b_m|^p &= \int_{-\infty}^{+\infty} |g(x)|^p dx \\ &\le C_{\phi,p}^p \int_{-\infty}^{+\infty} |f(x)|^p dx \\ &= C_{\phi,p}^p \sum_n |a_n|^p. \end{aligned}$$

This gives $C_{d,p} \le C_{\phi,p}$.

For the opposite inequality we write (at least for $f \in \mathcal{S}(\mathbf{R})$ and for ϕ as above and of bounded variation on $[0,1]$):

$$\begin{aligned} H_\phi f(x) &= \int_{-\infty}^{+\infty} \phi(\xi)\widehat{f}(\xi)e^{2\pi i x\xi}d\xi \\ &= \sum_m \widehat{\phi}(m)\int_{-\infty}^{+\infty} \widehat{f}(\xi)e^{2\pi i(x+m)\xi}d\xi \\ &= \sum_m \widehat{\phi}(m)f(x+m). \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{-\infty}^{+\infty} |H_\phi f(x)|^p dx &= \int_0^1 \sum_k |H_\phi f(x+k)|^p dx \\
&= \int_0^1 \sum_k |\sum_m \widehat{\phi}(m) f(x+k+m)|^p dx \\
&= \int_0^1 \sum_k |\sum_m \widehat{\phi}(m-k) f(x+m)|^p dx \\
&\le C_{d,p}^p \int_0^1 \sum_m |f(x+m)|^p dx \\
&= C_{d,p}^p \int_{-\infty}^{+\infty} |f(x)|^p dx.
\end{aligned}$$

And finally, $C_{\phi,p} \le C_{d,p}$.

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