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Preface

The 15th Panhellenic Conference on Mathematical Analysis (http://fourier.math.uoc.gr/pcma2016/) was held at the Department of Mathematics and Applied Mathematics of the University of Crete, Greece, from 27 to 29 May, 2016. This is the central conference of Mathematical Analysis in Greece and takes place every couple of years. The 16th Conference will be held at the University of the Aegean (Samos).

The topics of the conference included:

1. Harmonic Analysis, Geometric Analysis, Complex Analysis, Real Analysis, Ergodic Theory.

2. Functional Analysis, Operator Theory, Convex Analysis.

3. Differential Equations, Integral Equations, Stochastic Differential Equations, Dynamical Systems, Probability.

4. Numerical Analysis, Optimization, Control Theory, Special Functions.

In the Conference Proceedings (this volume) we have included the contributions of those participants who chose to submit a paper or a detailed description of their presentation. We thank those participants for their contribution.

The Organizing Committee,

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The Operations on $\mathbb{B}$-convex Sets and $\mathbb{B}$-convex Functions

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Abstract

$\mathbb{B}$-convex sets and functions were introduced and studied in [3, 4, 11, 12, 13]. In this paper, we examine some operations on $\mathbb{B}$-convex functions and $\mathbb{B}$-convex sets.

AMS Subject Classification: 52A30, 52A41
Key Words: Abstract convexity, $\mathbb{B}$-convexity, $\mathbb{B}$-convex sets, $\mathbb{B}$-convex functions

1 Introduction

$\mathbb{B}$-convexity, which is examined in this article, is one of the generalizations of convexity ([3, 4, 11, 12, 13]). $\mathbb{B}$-convexity concept is determined in [5], properties of $\mathbb{B}$-convex sets and functions are given in [1, 2, 5, 7], separation properties are investigated in [6, 7].

$\mathbb{B}$-convex functions are also studied in [8]. In this work, the operations on $\mathbb{B}$-convex sets and $\mathbb{B}$-convex functions is examined.

For all $r \in \mathbb{N}$ the map $x \mapsto \varphi_r(x) = x^{2r+1}$ is a homeomorphism from $\mathbb{R}$ to itself; $x = (x_1, ..., x_n) \mapsto \Phi_r(x) = (\varphi_r(x_1), ..., \varphi_r(x_n))$ is a homeomorphism from $\mathbb{R}^n$ to itself. For a finite nonempty set $A = \{x_1, ..., x_m\} \subset \mathbb{R}^n$, the r-convex hull of A, denoted as $Co^r(A)$, is given by

$$Co^r(A) = \left\{ \Phi_r^{-1}\left(\sum_{i=1}^{m} t_i \Phi_r(x_i)\right) : t_i \geq 0, \sum_{i=1}^{m} t_i = 1 \right\} .$$

The structure of $\mathbb{B}$-convex sets, which has not yet defined, will involve the order structure, with respect to the positive cone of $\mathbb{R}^n$; denoted by $\overset{m}{\underset{i=1}{\vee}} x_i$ the least upper bound of $x_1, ..., x_m \in \mathbb{R}^n$, that is:

$$\overset{m}{\underset{i=1}{\vee}} x_i = (\max\{x_{1,1}, ..., x_{m,1}\}, ..., \max\{x_{1,n}, ..., x_{m,n}\})$$

where, $x_{i,j}$ denotes jth coordinate of the ith point.

The limit hull of a finite set A is defined as the Kuratowski-Painleve upper limit of the sequence of sets $\{Co^r(A)\}_{r\in\mathbb{N}}$ (The Kuratowski-Painleve upper limit of the sequence of sets $\{A_n\}$ is $\bigcap_n \overline{\bigcup_k A_{n+k}}$; it is also the set of points p for which there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}}$ and points $p_{n_k} \in A_{n_k}$ such that $p = \lim_{k\to\infty} p_{n_k}$.)[9].

Definition 1.1 *[5] The Kuratowski-Painleve upper limit of the sequence of sets $(Co^r(A))_{r\in\mathbb{N}}$, denoted by $Co^\infty(A)$ where A is finite set, is called $\mathbb{B}$-polytope of A.*

It can be shown that in $\mathbb{R}^n_+$ the upper-limit is in fact a limit and that elements of $Co^\infty(A)$ have a simple analytic description:

Theorem 1.1 *[5] For all nonempty finite subsets $A = \{x_1, ..., x_m\} \subset \mathbb{R}^n_+$ we have*

$$Co^\infty(A) = Lim_{r\to\infty} Co^r(A) = \left\{ \bigvee_{i=1}^{m} t_i x_i : t_i \in [0,1], \max_{1\le i\le m} \{t_i\} = 1 \right\} .$$

Definition 1.2 *A subset U of $\mathbb{R}^n$ is $\mathbb{B}$-convex if for all finite subset $A \subset U$ the $\mathbb{B}$-polytope $Co^\infty(A)$ is contained in U.*

In $\mathbb{R}^n_+$, $\mathbb{B}$-convex set is defined in a different way [5]:
A subset U of $\mathbb{R}^n_+$ is $\mathbb{B}$-convex if and only if for all $x_1, x_2 \in U$ and all $\lambda \in [0,1]$ one has $\lambda x_1 \vee x_2 \in U$.
Now, we mention the following definitions which will be necessary in the sequel.

Definition 1.3 *[10] Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$. The set*

$$\{(x,\mu) \,|x \in U, \mu \in \mathbb{R}, \mu \ge f(x)\}$$

is called the epigraph of f and is denoted by $epi(f)$.

Definition 1.4 *[11] Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$. The set*

$$\{(x,\mu) \,|x \in U, \mu \in \mathbb{R}, \mu \le f(x)\}$$

is called the hypograph of f and is denoted by $hyp(f)$.

Thus, we can define $\mathbb{B}$-convex functions.

Definition 1.5 *[8] Let $U \subset \mathbb{R}^n$. A function $f : U \to \mathbb{R} \bigcup \{\pm\infty\}$ is called a $\mathbb{B}$-convex function if $epi(f)$ is a $\mathbb{B}$-convex set.*

The following theorem provides a sufficient and necessary condition for $\mathbb{B}$-convex functions in $\mathbb{R}^n_+$ [5, 8].

Theorem 1.2 *Let* $U \subset \mathbb{R}_+^n, f : U \to \mathbb{R}_+ \cup \{+\infty\}$*. The function* f *is* $\mathbb{B}$*-convex if and only if* U *is a* $\mathbb{B}$*-convex set and for all* $x, y \in U$ *and all* $\lambda \in [0,1]$ *the following inequality holds:*

$$f(\lambda x \vee y) \leq \lambda f(x) \vee f(y) \ . \tag{1}$$

For $\mathbb{B}$-convex functions, we can give a large number of examples. For instance, $f : \mathbb{R}_{++}^2 \to \mathbb{R}_+ \cup \{+\infty\}$, $f(z_1, z_2) = \ln \frac{1}{z_1 z_2}$ is a $\mathbb{B}$-convex function.

Definition 1.6 *Let* $U \subset \mathbb{R}_+^n$*. A function* $f : U \to \mathbb{R}_+ \cup \{+\infty\}$ *is called a* $\mathbb{B}$*-concave function if* $hyp^+(f) = \{(x, \mu) : x \in U, 0 \leq \mu \leq f(x)\}$ *is a* $\mathbb{B}$*-convex set.*

The following theorem holds (see also [5]).

Theorem 1.3 *Let* $f : U \subset \mathbb{R}_+^n \to \mathbb{R}_+ \cup \{+\infty\}$*. The function* f *is* $\mathbb{B}$*-concave if and only if* U *is a* $\mathbb{B}$*-convex set and for all* $x, y \in U$ *and all* $\lambda \in [0,1]$ *the following inequality holds:*

$$f(\lambda x \vee y) \geq \lambda f(x) \vee f(y) \ . \tag{2}$$

2 Operation on $\mathbb{B}$-convex Sets

Many properties, which are true for classic convexity, are also true for $\mathbb{B}$-convexity [5].

Theorem 2.1 *[5]*
(a) The empty set, $\mathbb{R}^n$ *, as well as all the singletons are* $\mathbb{B}$*-convex;*
(b) if $\{S_\lambda : \lambda \in \Lambda\}$ *is an arbitrary family of* $\mathbb{B}$*-convex sets then* $\bigcap_\lambda S_\lambda$ *is* $\mathbb{B}$*-convex;*
(c) if $\{S_\lambda : \lambda \in \Lambda\}$ *is a family of* $\mathbb{B}$*-convex sets such that* $\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda_3 \in \Lambda$ *such that* $S_{\lambda_1} \bigcup S_{\lambda_2} \subset S_{\lambda_3}$ *then* $\bigcup_\lambda S_\lambda$ *is* $\mathbb{B}$*-convex.*

Given a set S there is, according to (a) above, a $\mathbb{B}$-convex set which contains S; by (b) the intersection of all such $\mathbb{B}$-convex sets is $\mathbb{B}$-convex; we call it the $\mathbb{B}$-convex hull of S and we write $\mathbb{B}[S]$ for the $\mathbb{B}$-convex hull of S.

Theorem 2.2 *[5] The following properties hold:*
(a) $\mathbb{B}[\emptyset] = \emptyset, \mathbb{B}[\mathbb{R}^n] = \mathbb{R}^n$*, for all* $x \in \mathbb{R}^n, \mathbb{B}[\{x\}] = \{x\}$*;*
(b) For all $S \subset \mathbb{R}^n, S \subset \mathbb{B}[S]$ *and* $\mathbb{B}[\mathbb{B}[S]] = \mathbb{B}[S]$*;*
(c) For all $S_1, S_2 \subset \mathbb{R}^n$*, if* $S_1 \subset S_2$ *then* $\mathbb{B}[S_1] \subset \mathbb{B}[S_2]$*;*
(d) For all $S \subset \mathbb{R}^n, \mathbb{B}[S] = \bigcup \{\mathbb{B}[A] : A \text{ is a finite subset of } S\}$*;*
(e) A subset $S \subset \mathbb{R}^n$ *is* $\mathbb{B}$*-convex if and only if for all finite subset* A *of* S*,* $\mathbb{B}[A] \subset S$*.*

3 Operations on $\mathbb{B}$-convex Functions

Theorem 3.1 *(i) Let $U \subset \mathbb{R}^n_+, f : U \to \mathbb{R}_+ \cup \{+\infty\}$ be a $\mathbb{B}$-convex function. Multiplying f by a positive scalar, we obtain also $\mathbb{B}$-convex function.*
(ii) Let $f : U \subset \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ be a $\mathbb{B}$-convex function. The restriction of f to a $\mathbb{B}$-convex subset of U is also a $\mathbb{B}$-convex function.
(iii) If $f : U \subset \mathbb{R}^n_+ \to V \subset \mathbb{R}_+ \cup \{+\infty\}$ is a $\mathbb{B}$-convex function and $g : V \subset \mathbb{R}_+ \cup \{+\infty\} \to \mathbb{R}_+ \cup \{+\infty\}$ is a non-decreasing $\mathbb{B}$-convex function, then $g \circ f$ is $\mathbb{B}$-convex.
(iv) Suppose that $f : U \subset \mathbb{R}_+ \to V \subset \mathbb{R}_+$ is a bijection. If f is increasing, then f is $\mathbb{B}$-convex if and only if f^{-1} is $\mathbb{B}$-concave. If f is a decreasing bijection, then f and f^{-1} are of the same type of $\mathbb{B}$-convexity.

Proof. i) Let f be a $\mathbb{B}$-convex function. From Theorem 1.2, f satisfies the inequality (1). For $c > 0$, we have

$$cf(\lambda x \vee y) \leq c\left[\lambda f(x) \vee f(y)\right] = \lambda (cf)(x) \vee (cf)(y) \ .$$

Hence, cf is $\mathbb{B}$-convex.

ii) Let f be a $\mathbb{B}$-convex function. From Theorem 1.2, the inequality (1) holds for all $x, y \in U$. Therefore, for every $\mathbb{B}$-convex subset of U, the restriction of f to this subset also satisfies (1), consequently it is $\mathbb{B}$-convex function.

iii) When we use that f is $\mathbb{B}$-convex and g is non-decreasing $\mathbb{B}$-convex, we deduce the following inequality

$$g(f(\lambda x \vee y)) \leq g(\lambda f(x) \vee f(y)) \leq \lambda g(f(x)) \vee g(f(y)) \ .$$

Namely, $g \circ f$ is a $\mathbb{B}$-convex function.

iv) Let the bijection f be increasing and suppose that f is a $\mathbb{B}$-convex function. In this case, for all $x, y \in U$ and all $\lambda \in [0, 1]$ we have the inequality (1). Let I_V and I_U be the identity functions on V and U, respectively. From $f \circ f^{-1} = I_V, f^{-1} \circ f = I_U$ and the inequality (1), for $s, t \in V$ we obtain that

$$f\left(f^{-1}(\lambda s \vee t)\right) = \lambda s \vee t = \lambda f\left(f^{-1}(s)\right) \vee f\left(f^{-1}(t)\right) \geq f\left(\lambda f^{-1}(s) \vee f^{-1}(t)\right) \ .$$

Since f is increasing, f^{-1} is increasing; hence we get $f^{-1}(\lambda s \vee t) \geq \lambda f^{-1}(s) \vee f^{-1}(t)$. So, f^{-1} is $\mathbb{B}$-concave.

To prove the sufficiency, suppose that f^{-1} is a $\mathbb{B}$-concave function. Thus for all $s, t \in V$ and all $\lambda \in [0, 1]$, the inequality (2) for f^{-1} holds. If we assume that $f^{-1}(s) = x, f^{-1}(t) = y$, we have $f(x) = s, f(y) = t$. From increasing of f and the inequality (2)

$$\begin{aligned} f^{-1}(\lambda f(x) \vee f(y)) \geq \lambda f^{-1}(f(x)) \vee f^{-1}(f(y)) &= \lambda x \vee y \\ f\left(f^{-1}(\lambda f(x) \vee f(y))\right) &\geq f(\lambda x \vee y) \\ \lambda f(x) \vee f(y) &\geq f(\lambda x \vee y) \ . \end{aligned}$$

Thereby, f is $\mathbb{B}$-convex function.
Other hypothesis in iv) are proved similarly. ■

Theorem 3.2 *Let* $\left\{ f_\alpha \left| f_\alpha : U \subset \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\} \right. \right\}$ *be a family of* $\mathbb{B}$*-convex functions. Then the function* f *defined by* $f(x) = \sup\{f_\alpha(x) | \alpha\}$ *is* $\mathbb{B}$*-convex.*

Proof. $epi\,(f) = \bigcap\limits_\alpha epi\,(f_\alpha)$, so $epi\,(f)$, being the intersection of $\mathbb{B}$-convex sets, is $\mathbb{B}$-convex [5]. Therefore, f is $\mathbb{B}$-convex. ∎

Theorem 3.3 *If* $f : U \subset \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ *and* $g : U \subset \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{+\infty\}$ *are both decreasing(increasing) and* $\mathbb{B}$*-convex then* $h(x) = f(x)\,g(x)$ *also exhibits these two properties.*

Proof. Since, decreasing(increasing) of h is clear, let us examine the $\mathbb{B}$-convexity of it. For $x, y \in U$, we begin by taking $x \preccurlyeq y$ ($x \preccurlyeq y \Leftrightarrow x_i \leq y_i$ *for all* $i \in \{1, ..., n\}$). From $\mathbb{B}$-convexity of f and g, for $\lambda \in [0, 1]$ we obtain

$$h(\lambda x \vee y) = f(\lambda x \vee y)\,g(\lambda x \vee y) \leq [\lambda f(x) \vee f(y)]\,[\lambda g(x) \vee g(y)] \;.$$

We should analyse the following four cases:

i) if $\lambda f(x) \vee f(y) = \lambda f(x)$, $\lambda g(x) \vee g(y) = \lambda g(x)$, then $h(\lambda x \vee y) \leq \lambda^2 f(x)\,g(x) \leq \lambda f(x)\,g(x) = \lambda h(x) \leq \lambda h(x) \vee h(y)$.

ii) if $\lambda f(x) \vee f(y) = \lambda f(x)$, $\lambda g(x) \vee g(y) = g(y)$, then since g is decreasing, we have $h(\lambda x \vee y) \leq \lambda f(x)\,g(y) \leq \lambda f(x)\,g(x)$. Thus, from i), h satisfies the required inequality.

iii) if $\lambda f(x) \vee f(y) = f(y)$, $\lambda g(x) \vee g(y) = \lambda g(x)$, then using decreasing of f, we obtain that $h(\lambda x \vee y) \leq f(y)\,\lambda g(x) \leq \lambda f(x)\,g(x) = \lambda h(x) \leq \lambda h(x) \vee h(y)$.

iv) if $\lambda f(x) \vee f(y) = f(y)$, $\lambda g(x) \vee g(y) = g(y)$, then $h(\lambda x \vee y) \leq f(y)\,g(y) = h(y) \leq \lambda h(x) \vee h(y)$.

Thus h is a $\mathbb{B}$-convex function. The case of $x \not\succcurlyeq y$ can be proven more easily.

Similarly, we can prove the $\mathbb{B}$-convexity of h when f and g are increasing. ∎

Remark 3.1 *In the above theorem, the condition of decreasing(increasing) of functions* f *and* g *at the same time is required. On the contrary, for example; let* $f : \mathbb{R}_+ \to \mathbb{R}_+$, $f(x) = x^{\frac{3}{2}}$ *and* $g : \mathbb{R}_{++} \to \mathbb{R}_+$, $g(x) = \frac{1}{x}$. *The function* f *is increasing,* $\mathbb{B}$*-convex and the function* g *is decreasing,* $\mathbb{B}$*-convex. In this case, if we examine the function* $h(x) = f(x)\,g(x) = x^{\frac{1}{2}}$, h *isn't a* $\mathbb{B}$*-convex function.*

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SOME SPECTRAL PROPERTIES OF ONE CLASS DISCONTINOUS STURM-LIOUVILLE OPERATOR

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1.INTRODUCTION

The Sturm-Liouville theory plays and important role in solving many problems in mathematical physics. It is an active area of research in pure and applied mathematics. From past to present, there has been a growing interest in Sturm-Louville problems with eigenparameter dependent boundary conditions (Walter 1973; Fulton 1977; Mukhtarov and Demir 1999; Mukhtarov et al. 2010; Altınışık et al 2012; Zhang et al.2013; Aydemir 2014;Mukhtarov and Aydemir 2014), i.e.,

The eigenparameter appears not only in the differential equations of the Sturm- Liouville problems but also in the boundary conditions.

In this study we investigate the Stum-Liouville equation

$$\ell y := -y''(x) + q(x)y(x) = \lambda y(x), \tag{1}$$

to hold in finite interval $(-1, 1)$ except one inner point $c \in (-1, 1)$, subject to the eigenparameter-dependent boundary conditions

$$L_1(y) := \lambda(\alpha_1' y(-1) - \alpha_2' y'(-1)) - (\alpha_1 y(-1) - \alpha_2 y'(-1)) = 0 \tag{2}$$

$$L_2(y) := \lambda(\beta_1' y(1) - \beta_2' y'(1)) - (\beta_1 y(1) - \beta_2 y'(1)) = 0 \tag{3}$$

and the eigenparamete dependent transmission conditions at the point of discontinuity

$$L_3(y) := \gamma_3 y(c + 0) - \gamma_4 y(c - 0) = 0 \tag{4}$$

$$L_4(y) := \gamma_2 y'(c + 0) - \gamma_1 y'(c - 0) + (\lambda\delta_1 + \delta_2)y(c) = 0, \tag{5}$$

Here $q(x)$ is real-valued continous function on $I = [-1, c) \cup (c, 1]$ and has finite limits $q(c \pm 0) := \lim_{x\to c\pm 0} q(x); \alpha_1, \alpha_1', \beta_1, \beta_1'$ and $\delta_i (i = 1, 2)$ are real numbers.

Boundary-value problems with transmission condiitons arise in the theory of heat ans mass transferAdjoint and self-adjoint boundary value problems with transmission conditions have been studied by Zettl (1968). Sturm-Liouville problems with transmission conditions at one interior point have been studied many authors (Yang 2013; Aydemir 2014; Mukhtarov and Aydemir 2014; Zhang 2014).

It must be noted that some special cases of the considered problem $(1) - (5)$ arise after an application of the method of seperation of variables to the varied assortment of physical problems. For example, some boundary-value problems with transmission conditions arise in heat and mass transfer problems (see for example, [4] Likov, A.V. and Mikhailov, Yu. A.: The Theory of Heat and Mass Transfer, Gosenergoizdat,1963 (Russian)),vibrating string problems when the string loaded additionally with point masses (see for example. [15]).Also, some problems with transmission conditions which arise in mechanics(thermal conduction problem for a hin laminated plate) were studied in the article [14]. Similar problems for differential equations with discontinuous coefficients were investigated by Raulov in monographs.[9,10] But, the considered discontinuous problems in these works do not contain transmission conditions.

2.AN OPERATOR FORMULATION OF THE PROBLEM(1)-(5) IN THE ADEQUATE HILBERT SPACE

In this section we introduce the special inner product in the Hilbert space $H := L_2[-1,1]\oplus C^3$ and such a way that the problem $(1)-(5)$ can be considered as the eigenvalue problem of this operator.

Throughout this study, we shall assume that the coefficients γ_1, γ_2 and δ_1 have the same sign (without losing the generality we shall assume that γ_1, γ_2 and δ_1 are positive)

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0 \text{ and } \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0$$

Let us introduce a new equivalent inner product on $L_2[-1,1]\oplus C^3$ by

$$(F,G) := \gamma_1 \int_{-1}^{c} f(x)\overline{g(x)}dx + \gamma_2 \int_{c}^{1} f(x)\overline{g(x)}dx + \frac{\gamma_1}{\rho_1} f_1\overline{g_1} + \frac{\gamma_2}{\rho_2} f_2\overline{g_2} + \frac{1}{\delta_1} f_3\overline{g_3}$$

for

$$F := \begin{pmatrix} f(x) \\ f_1 \\ f_2 \\ f_3 \end{pmatrix} \in H, G := \begin{pmatrix} g(x) \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \in H,$$

which is connected with coefficients of our problem. For a short exposition we shall use the following notations:

$$\begin{aligned}
B_{-1}(y) &:= \alpha_1 y(-1) - \alpha_2 y'(-1), \\
B'_{-1}(y) &:= \alpha_1' y(-1) - \alpha_2' y'(-1), \\
B_1(y) &:= \beta_1 y(1) - \beta_2 y'(1), \\
B_1'(y) &:= \beta_1' y(1) - \beta_2' y'(1), \\
T_c(y) &:= \gamma_2 y'(c+0) - \gamma_1 y'(c-0) + \delta_2 y(c), \\
T_c'(y) &:= -\delta_1 y(c).
\end{aligned}$$

In the Hilbert space H we define a linear operator A

$$\Omega(A) := \left\{ \begin{array}{c} F = (f(x), f_1, f_2, f_3) : \ f \ \ \textit{is absolutely continuous in} \ [-1,1], \\ f' \ \textit{is absolutely continuous on both} \ [-1,c) \cup (c,1] \ \textit{and has a finite} \text{ limits}, \\ f'(c \pm 0) = \lim_{x \to c \pm 0} f'(x), \ell f \in L_2[-1,1], \\ f_1 = B'_{-1}(f), \ f_2 = B'_1(f), \ f_3 = T'_c(f) \end{array} \right\}$$

and

$$AF = (\ell f, B_{-1}(f), -B_1(f), T_c(f)$$

Therefore, we can study the problem $(1)-(5)$ in H by considering the operator equation

$$AF = \lambda F,$$

where $F = (f(x), B_{-1}(f), B'_1(f), T'_c(f)) \in \Omega(A)$.

Naturally, by eigenvalues and eigenfunctions of the problem $(1)-(5)$ coincide with eigenvalues and first components of corresponding eigenelements of the operator A, respectively.

Let $F, G \in \Omega(A)$. By two partial integration we obtain

$$\begin{aligned} (AF, G) = (F, AG) + \gamma_1 W(f, \overline{g}; c) - \gamma_1 W(f, \overline{g}; -1) + \\ + \gamma_2 W(f, \overline{g}; 1) - \gamma_2 W(f, \overline{g}; c) + \frac{\gamma_1}{\rho_1}(B_{-1}(f)B'_{-1}(f) - B'_{-1}(f)B_1(\overline{g})) + \\ + \frac{\gamma_2}{\rho_2}(B'_1(f)B_1(\overline{g}) - B_1(f)B'_1(\overline{g}) + \frac{1}{\delta_1}(T_c(f)T'_c(\overline{g}) - T'_c(f)T_c(\overline{g})), \end{aligned} \tag{6}$$

where, $W(f, g; x)$ is denoted the Wronskians $f(x)g'(x) - f'(x)g(x)$.

It is easy to show that

$$\begin{aligned} B_{-1}(f)B'_{-1}(f) - B'_{-1}(f)B_1(\overline{g}) &= \rho_1 W(f, \overline{g}; -1), \\ B'_1(f)B_1(\overline{g}) - B_1(f)B'_1(\overline{g}) &= -\rho_2 W(f, \overline{g}; 1), \\ T_c(f)T'_c(\overline{g}) - T'_c(f)T_c(\overline{g}) &= -\delta_1\gamma_1 W(f, \overline{g}; c) + \delta_1\gamma_2 W(f, \overline{g}, c). \end{aligned}$$

Substituting into (6) we have

$$(AF, G) = (F, AG) \quad (F, G \in \Omega(A)), \tag{7}$$

Corollary 1 *The linear operator A is symmetric.*

Corollary 2 *All eigenvalues of the problem $(1)-(5)$ are real.*

We can now assume that all eigenfunctions of the problem $(1)-(5)$ are real-valued.

Corollary 3 *Let λ_1 and λ_2 be two different eigenvalues of the problem $(1)-(5)$. Then the corresponding eigenfunctions $y_1(x)$ and $y_2(x)$ of this problem is orthogonal in the sense of*

$$\int_{-1}^{c} y_1(x)y_2(x)dx + \int_{c}^{1} y_1(x)y_2(x)dx + \frac{1}{\rho_1}B'_{-1}(y_1)B'_{-1}(y_2) + \frac{1}{\rho_2}B'_{-1}(y_1)B'_1(y_2) + \frac{1}{\delta_1}T'_c(y_1)T'_c(y_2) = 0$$

3.FUNDAMENTAL SOLUTIONS

In this section we defined the fundemental solutions of the For next consideration, we need to give the following Lemma.

Lemma 4 *Let the real valued function $q(x)$ be continious in $[a,b]$ and $f(\lambda), g(\lambda)$ are given entire functions. Then for any $\lambda \in \mathbb{C}$, the equation*

$$\ell y := -y''(x) + q(x)y(x) = \lambda y(x), x \in [a,b]$$

has a unique solution $y = y(x,\lambda)$ satisfying the initial condiitons,

$$y(a) = f(\lambda), y'(a) = g(\lambda) \quad (\text{or } y(b) = f(\lambda), y'(b) = g(\lambda))$$

for each fixed $x \in [a,b]$, $y(x,\lambda)$ is an entire function of λ.

Let $\Phi_{1\lambda}(x) := \Phi_1(x,\lambda)$ be the solution of equation (1) in the interval $[-1,c)$ satisfying the initial conditions

$$y(-1) = -\alpha_2 + \lambda\alpha_2',\ y'(-1) = -\alpha_1 + \lambda\alpha_1' \tag{8}$$

Now we can define the solution $\Phi_{2\lambda}(x) := \Phi_2(x,\lambda)$ of equation (1) on the interval $(c,1]$ in terms of $\Phi_1(c-0,\lambda)$ and $\Phi_1'(c-0,\lambda)$ by the initial conditions

$$\begin{aligned} y(c) &= \gamma_4 y(c-0), \qquad (9)\\ y'(c) &= \frac{\gamma_1}{\gamma_2}\Phi_1'(c-0,\lambda) - \left(\frac{\lambda\delta_1+\delta_2}{\gamma_2}\right)\Phi_1(c-0,\lambda) \end{aligned}$$

Analogically, we can define the solutions $\chi_{1\lambda}(x)$ and $\chi_{2\lambda}(x)$ by initial conditions

$$\chi_{2\lambda}(1) = \beta_2 + \lambda\beta_2', \chi_{2\lambda}'(1) = \beta_1 + \lambda\beta_1' \tag{10}$$

and

$$\begin{aligned} \chi_{1\lambda}(c) &= \gamma_3 y(c+0), \qquad (11)\\ \chi_{1\lambda}'(c) &= \frac{\gamma_2}{\gamma_1}\chi_2'(c+0,\lambda) + \left(\frac{\lambda\delta_1+\delta_2}{\gamma_1}\right)\chi_2(c+0,\lambda) \end{aligned}$$

respectively.Finally let us define two "fundamental solutions" of equation (1) on whole $(-1,c) \cup (0,1]$ as

$$\Phi(x,\lambda) = \left.\begin{array}{c} \Phi_1(x,\lambda),\ \text{for } x\in[-1,c)\,, \\ \Phi_2(x,\lambda),\ \text{for } x\in(c,1]\,, \end{array}\right\} \tag{12}$$

$$\chi(x,\lambda) = \left.\begin{array}{c} \chi_1(x,\lambda)\ \text{,for } x\in[-1,c) \\ \chi_2(x,\lambda),\ \text{for } x\in(c,1] \end{array}\right\} \tag{13}$$

It must note that each of these solutions satisfy both transmission conditions (4) and (5). Moreover, $\Phi(x,\lambda)$ satisfies one of the boundary conditions (namely the condition (2), but $\chi(x,\lambda)$ the other boundary condition (3).Let us consider the Wronskians

$$\begin{aligned} \omega_i(\lambda) \quad &: \quad = W_\lambda(\Phi_i,\chi_i;x) \\ &: \quad = \Phi_i(x,\lambda)\chi_i'(x,\lambda) - \Phi_i'(x,\lambda)\chi_i(x,\lambda)\ (i=1,2) \end{aligned}$$

which are independent of x and are entire functions. With a short calcula-

tions give us $\gamma_1\omega_1(\lambda)=\gamma_2\omega_2(\lambda)$ and now we may introduce to the consideration the characteristic function $\omega(\lambda)$ as

$$\omega(\lambda) := \gamma_1\omega_1(\lambda) = \gamma_2\omega_2(\lambda)$$

Theorem 5 *The eigenvalues of the problem (1)-(5) are consist of the zeros of the functions*

$$\omega(\lambda)\ \text{and}\ \Delta(\lambda) := \frac{\gamma_2}{\gamma_1}\omega(\lambda) + (\lambda\delta_1+\delta_2)\chi_{2\lambda}(c)\Phi_{2\lambda}(c)$$

Lemma 6 *Let $\lambda = s^2$.Then the following integral equations hold for $k=0$ and $k=1$,*

$$\begin{aligned} \frac{d^k}{dx^k}\Phi_{1\lambda}(x) \quad &= \quad (-\alpha_2+s^2\alpha_2')\frac{d^k}{dx^k}\cos[s(x+1)] - \frac{1}{\gamma_4 s}(-\alpha_1+s^2\alpha_1')\frac{d^k}{dx^k}\sin[s(x+1)] + \\ &\quad + \frac{1}{s}\int_c^x \frac{d^k}{dx^k}\sin(s(x-y))q(y)\Phi_{1\lambda}(y)dy \end{aligned}$$

$$\begin{aligned} \frac{d^k}{dx^k}\Phi_{2\lambda}(x) \quad &= \quad \Phi_{1\lambda}(c)\frac{d^k}{dx^k}\cos[s(x-c)] + \frac{1}{\gamma_3 s}[\Phi_{1\lambda}'(c) - s^2\delta_1\Phi_{1\lambda}(c)]\frac{d^k}{dx^k}\sin[s(x-c)] + \\ &\quad + \frac{1}{s}\int_c^x \frac{d^k}{dx^k}\sin[s(x-y)]q(y)\Phi_{2\lambda}(y)dy] \end{aligned}$$

Similarly that lemma can establish for $\chi_i(x,\lambda)$ $(i=1,2)$.

4. RESOLVENT OPERATOR AND GREEN FUNCTION OF THE PROBLEM (1)-(5)

Corollary 7 *Let us assume that $\lambda \in \mathbb{C}$ is not an eigenvalue of the problem $(1)-(5)$. Then the functions $\Phi_1(x,\lambda), \chi_1(x,\lambda)$ are linearly independent in the interval $[-1,c]$, the functions $\Phi_2(x,\lambda), \chi_2(x,\lambda)$ are linearly independent in the interval $(c,1]$.*

Corollary implies that for all $\lambda \in \mathbb{C}$ which is not an eigenvalue of the problem $(1)-(5)$ we can write thegeneral solution of the differential equaiton (1) as

$$y(x,\lambda) = \begin{cases} C_1\Phi_1(x,\lambda) + C_2\Phi_2(x,\lambda),\ x \in [-1,c) \\ C_3\Phi_1(x,\lambda) + C_4\Phi_2(x,\lambda), x \in (c,1] \end{cases}$$

where C_i $(i = \overline{1,4})$ are arbitrary constants. Then applying the method of variation of constants, the following formula is obtained for $y(x,\lambda)$.

$$y(x,\lambda) = \frac{\Phi_{1\lambda}(x)}{\omega\left(\lambda\right)}\int_c^x f(y)\chi_{1\lambda}(y)dy + \frac{\chi_{1\lambda}(x)}{\omega\left(\lambda\right)}\int_{-1}^x f(y)\Phi_{1\lambda}(y)dy + \frac{\Phi_{1\lambda}(x)}{\gamma_3\omega\left(\lambda\right)}\int_c^1 f(y)\chi_{2\lambda}(y)dy, x \in [-1,c),$$

$$\frac{\Phi_{2\lambda}(x)}{\omega\left(\lambda\right)}\int_x^1 f(y)\chi_{2\lambda}(y)dy + \frac{\chi_{2\lambda}(x)}{\omega\left(\lambda\right)}\int_c^x f(y)\Phi_{2\lambda}(y)dy + + \frac{\chi_{2\lambda}(x)}{\gamma_4\omega\left(\lambda\right)}\int_1^c f(y)\Phi_{1\lambda}(y)dy, x \in (c,1]$$

Let

$$\Phi(x,\lambda) = \left.\begin{matrix} \Phi_1(x,\lambda),\ \text{for } x \in [-1,c)\,, \\ \Phi_2(x,\lambda),\ \text{for } x \in (c,1]\,, \end{matrix}\right\}, \chi(x,\lambda) = \left.\begin{matrix} \chi_1(x,\lambda)\ ,\text{for } x \in [-1,c) \\ \chi_2(x,\lambda),\ \text{for } x \in (c,1] \end{matrix}\right\}$$

Then we can rewrite the resolvent

$$y(x,\lambda) = \frac{\Phi(x,\lambda)}{\omega_i\left(\lambda\right)}\int_x^1 f(y)\chi_\lambda(y)dy + \frac{\chi_\lambda(x)}{\omega_i\left(\lambda\right)}\int_{-1}^x f(y)\Phi(y)dy,\ i = 1,2 \tag{14}$$

Therefore the resolvent of the boundary value transmission problem is obtained.We can find the Green function from the resolvent denoting below the following;

$$G(x,y;\lambda) = \begin{cases} \frac{\Phi_{i\lambda}(x,\lambda)\chi(x)}{\omega_i(\lambda)}, -1 \le y \le x \le 1, x \ne c, y \ne c, \\ \frac{\Phi(x)\chi_{i\lambda}(x)}{\omega_i(\lambda)}, -1 \le y \le x \le 1, x \ne c, y \ne c, i = 1,2. \end{cases} \tag{15}$$

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COMPACT MULTIPLICATION OPERATORS ON NEST ALGEBRAS

G. ANDREOLAS AND M. ANOUSSIS

Abstract. Let $\mathcal{N}$ be a nest on a Hilbert space H and $\operatorname{Alg}\mathcal{N}$ the corresponding nest algebra. We obtain a characterization of the compact and weakly compact multiplication operators defined on nest algebras. This characterization leads to a description of the closed ideal generated by the compact elements of $\operatorname{Alg}\mathcal{N}$. We also see that there is no non-zero weakly compact multiplication operator on $\operatorname{Alg}\mathcal{N} / \operatorname{Alg}\mathcal{N} \cap \mathcal{K}(H)$.

1. INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra. A *multiplication operator* $M_{a,b} : \mathcal{A} \to \mathcal{A}$ corresponding to $a, b \in \mathcal{A}$ is given by $M_{a,b}(x) = axb$. An operator $\Phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if $\Phi = \sum_{i=1}^{m} M_{a_i,b_i}$ for some $a_i, b_i \in \mathcal{A}$, $i = 1, \ldots, m$. Properties of compact multiplication operators have been investigated since 1964 when Vala published his work "On compact sets of compact operators" [12]. If $\mathcal{X}$ is a Banach space, we shall denote by $\mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded operators on $\mathcal{X}$ and by $\mathcal{K}(\mathcal{X})$ the Banach algebra of all compact operators on $\mathcal{X}$. Vala proved that a non-zero multiplication operator $M_{a,b} : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is compact if and only if the operators $a \in \mathcal{B}(\mathcal{X})$ and $b \in \mathcal{B}(\mathcal{X})$ are both compact.

This concept was further investigated by Ylinen in [13] who proved a similar result for abstract C*-algebras. An element a of a Banach algebra $\mathcal{A}$ is called *compact* if the multiplication operator $M_{a,a} : \mathcal{A} \to \mathcal{A}$ is compact. Ylinen characterized the compact elements of a C*-algebra. In the sequel, these results were generalized to various directions by several authors, such as Akemann and Wright [2], Fong and Sourour [5], Mathieu [8] and Timoney [11]. From the description of the compact elementary operators by Fong and Sourour, the following conjecture arose: *If* Φ *is a compact elementary operator on the Calkin algebra on a separable Hilbert space, then* $\Phi = 0$. This conjecture was confirmed in [3] by Apostol and Fialkow and by Magajna in [7]. In [8] Mathieu proves that if Φ is weakly compact, then $\Phi = 0$ as well. The weak compactness of multiplication operators has been studied in a Banach space setting by Saskmann - Tylli and Johnson - Schechtman in [10] and [6] respectively.

In this work we characterize the compact and weakly compact multiplication operators on nest algebras and show that there is not any non-zero weakly compact multiplication operator on $\operatorname{Alg}\mathcal{N} / \operatorname{Alg}\mathcal{N} \cap \mathcal{K}(H)$. Complete proofs of the following results will appear in [1].

Let us introduce some notation and definitions that will be used throughout. Nest algebras form a class of non-selfadjoint operator algebras that generalize the block upper triangular matrices to an infinite dimensional Hilbert space context.

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They were introduced by Ringrose in [9] and since then, they have been studied by many authors. The monograph of Davidson [4] is recommended as a reference. A nest $\mathcal{N}$ is a totally ordered family of closed subspaces of a Hilbert space H containing $\{0\}$ and H, which is closed under intersection and closed span. If H is a Hilbert space and $\mathcal{N}$ a nest on H, then the nest algebra $\mathrm{Alg}\,\mathcal{N}$ is the algebra of all operators T such that $T(N) \subseteq N$ for all $N \in \mathcal{N}$. We shall usually denote both the subspaces belonging to a nest and their corresponding orthogonal projections by the same symbol. If $(N_\lambda)_{\lambda\in\Lambda}$ is a family of subspaces of a Hilbert space, we denote by $\vee\{N_\lambda : \lambda \in \Lambda\}$ their closed linear span and by $\wedge\{N_\lambda : \lambda \in \Lambda\}$ their intersection. If $\mathcal{N}$ is a nest and $N \in \mathcal{N}$, then $N_- = \vee\{N' \in \mathcal{N} : N' < N\}$. Similarly we define $N_+ = \wedge\{N' \in \mathcal{N} : N' > N\}$. If e, f are elements of a Hilbert space H, we denote by $e \otimes f$ the rank one operator on H defined by $(e \otimes f)(h) = \langle h, e\rangle f$. We shall frequently use the fact that a rank one operator $e \otimes f$ belongs to a nest algebra, $\mathrm{Alg}\,\mathcal{N}$, if and only if the exist an element N of $\mathcal{N}$ such that $e \in N_-^\perp$ and $f \in N$, [4, Lemmas 2.8 and 3.7]. Throughout, we denote by $\mathcal{N}$ a nest acting on a Hilbert space H and by $\mathcal{K}(\mathcal{N})$ the ideal of compact operators of $\mathrm{Alg}\,\mathcal{N}$.

2. COMPACT MULTIPLICATION OPERATORS

Let H be a Hilbert space and a, b elements of $\mathcal{B}(H)$. Vala proved in [12] that if $a, b \in \mathcal{B}(H) - \{0\}$, then the map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$, $x \mapsto axb$ is compact if and only if the operators a and b are both compact. However, such a result does not hold for nest algebras. Let $\mathcal{N}$ be a nest containing a projection P such that $\dim(P) = \dim(P^\perp) = \infty$ and $a \in \mathrm{Alg}\,\mathcal{N}$ be a non-compact operator such that $a = PaP^\perp$. Then, the multiplication operator

$$\begin{aligned} M_{a,a} : \mathrm{Alg}\,\mathcal{N} &\to \mathrm{Alg}\,\mathcal{N}, \\ x &\mapsto axa \end{aligned}$$

coincides with the multiplication operator $M_{0,0}$, since

$$M_{a,a}(x) = axa = PaP^\perp xPaP^\perp = 0,$$

for $P^\perp xP = 0$.

Let $a, b \in \mathrm{Alg}\,\mathcal{N}$. We introduce the following projections:

$$R_a = \vee\{P \in \mathcal{N} : aP = 0\}$$

and

$$Q_b = \wedge\{P \in \mathcal{N} : P^\perp b = 0\}.$$

Proposition 2.1. *Let $a, b \in \mathrm{Alg}\,\mathcal{N}$. Then, $M_{a,b} = 0$ if and only if $Q_b \leq R_a$.*

The next theorem gives a necessary and sufficient condition for a non-zero multiplication operator $M_{a,b} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$, $M_{a,b}(x) = axb$ to be compact.

Theorem 2.2. *Let $a, b \in \mathrm{Alg}\,\mathcal{N}$ such that $M_{a,b} \neq 0$. The multiplication operator $M_{a,b} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is compact if and only if the operators P_+aP_+ and $P_-^\perp bP_-^\perp$ are both compact for all $P \in \mathcal{N}$, $R_a < P < Q_b$ in the case that $R_{a+} \neq Q_b$ or the operators Q_baQ_b and $R_a^\perp bR_a^\perp$ are both compact in the case that $R_{a+} = Q_b$.*

Remark 2.3. *Consider the nest $\mathcal{N} = \{\{0\}, H\}$ and let $a, b \in \mathrm{Alg}\,\mathcal{N} = \mathcal{B}(H)$ with $a, b \neq 0$. From Theorem 2.2 it follows that the multiplication operator $M_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ is compact if and only if the operators a and b are both compact. In that case the result coincides with Vala's Theorem.*

Let $\mathcal{A}$ be a C*-algebra and Φ an elementary operator on $\mathcal{A}$. Timoney proved in [11, Theorem 3.1] that Φ is compact if and only if Φ can be expressed as $\Phi(x) = \sum_{i=1}^{m} a_i x b_i$ for a_i and b_i compact elements of $\mathcal{A}$ $(1 \leq i \leq m)$. The following example shows that this fact does not hold in the class of nest algebras.

Example 2.4. *Let H be a Hilbert space, $\{e_i\}_{i\in\mathbb{N}}$ an orthonormal sequence of H, $\mathcal{N} = \{[\{e_i : i \in \mathbb{N}, i \leq n\}] : n \in \mathbb{N}\} \cup \{\{0\}, H\}$ and $b = \sum_{n\in\mathbb{N}} \frac{1}{n} e_n \otimes e_n$ a compact operator of $\mathrm{Alg}\,\mathcal{N}$. Then, the multiplication operator $M_{I,b}$ is compact (Theorem 2.2). We suppose that there exist compact operators $c_i, d_i \in \mathcal{B}(H)$, $i = 1, \ldots, l$ such that $M_{I,b} = \sum_{i=1}^{l} M_{c_i,d_i}$ and we shall arrive at a contradiction. We consider the following family of rank one operators,*

$$\{x_{r,s}\}_{\substack{r\in\mathbb{N}\\ s\in\mathbb{N}\cup\{0\}\\ s<r}} = \{e_r \otimes e_{r-s}\}_{\substack{r\in\mathbb{N}\\ s\in\mathbb{N}\cup\{0\}\\ s<r}} \in \mathrm{Alg}\,\mathcal{N}.$$

Then,

$$M_{I,b}(x_{r,s}) = \sum_{i=1}^{l} M_{c_i,d_i}(x_{r,s})$$

or

$$\frac{1}{r} e_r \otimes e_{r-s} = \sum_{i=1}^{l} d_i^*(e_r) \otimes c_i(e_{r-s}). \tag{2.1}$$

The relation (2.1) implies that

$$\frac{1}{r} = \sum_{i=1}^{l} \langle e_r, d_i^*(e_r)\rangle \langle e_{r-s}, c_i(e_{r-s})\rangle, \tag{2.2}$$

by taking the evaluations on e_r and then the inner product by e_{r-s} on each side of (2.1). For all $r \in \mathbb{N}$ and $i \in \{1, \ldots, l\}$ we set $D_{r,i} = \langle e_r, d_i^*(e_r)\rangle$ and $C_{r,i} = \langle e_r, c_i(e_r)\rangle$. We denote the vectors $(D_{r,1}, \ldots, D_{r,l}) \in \mathbb{C}^l$ and $(C_{r,1}, \ldots, C_{r,l}) \in \mathbb{C}^l$ by D_r and C_r respectively for all $r \in \mathbb{N}$. Now, we can write equation (2.2) in the form

$$\frac{1}{r} = \sum_{i=1}^{l} D_{r,i} C_{r-s,i}. \tag{2.3}$$

This implies

$$0 = \sum_{i=1}^{l} D_{r,i} \left(C_{r-s,i} - C_{1,i}\right) \tag{2.4}$$

The sequence $(\mathcal{V}_n)_{n\in\mathbb{N}} = (\mathrm{span}\{C_2 - C_1, \ldots, C_n - C_1\})_{n\in\mathbb{N}}$ of subspaces of $\mathbb{C}^l$ is increasing and therefore there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{V}_{n_0} = \mathcal{V}_n$ for all $n \geq n_0$. Therefore, the following holds for all $n \in \mathbb{N}$.

$$0 = \sum_{i=1}^{l} D_{n_0,i}(C_{n,i} - C_{1,i}). \tag{2.5}$$

Since the operators c_i, $i = 1, \ldots, l$ are compact, the sequence $(C_n)_{n\in\mathbb{N}}$ converges to 0. Taking limits in equation (2.5) as $n \to \infty$ we obtain $0 = -\frac{1}{n_0}$ which is a contradiction.

The set of compact elements of a nest algebra does not form an ideal in general. However, we characterize the ideal generated by the compact elements.

Theorem 2.5. *The ideal $\mathcal{J}_c$, generated by the compact elements of the nest algebra* $\mathrm{Alg}\mathcal{N}$*, is equal to* $\mathcal{K}(\mathcal{N})+\mathrm{Rad}(\mathcal{N})$*, where* $\mathrm{Rad}(\mathcal{N})$ *is an the Jacobson radical of* $\mathrm{Alg}\mathcal{N}$*.*

3. WEAKLY COMPACT MULTIPLICATION OPERATORS

The following lemma is instrumental in the proof of the main theorem (3.2) of this section.

Lemma 3.1. *Let $a, b \in \mathrm{Alg}\mathcal{N}$ and $(e_n)_{\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ orthonormal sequences in H such that $e_n \otimes f_n \in \mathrm{Alg}\mathcal{N}$ for all $n \in \mathbb{N}$. If there exists an $\varepsilon > 0$ such that $\|a(f_n)\| \geq \varepsilon$ and $\|b^*(e_n)\| \geq \varepsilon$ for all $n \in \mathbb{N}$, then there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that the operator $a\left(\sum_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}\right) b = \sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n}) \in \mathrm{Alg}\mathcal{N}$ is not compact and for any subsequence $(k_{n_m})_{m\in\mathbb{N}}$ the operator $\sum_{n\in\mathbb{N}} b^*(e_{k_{n_m}}) \otimes a(f_{k_{n_m}}) \in \mathrm{Alg}\mathcal{N}$ is non-compact as well.*

Now, we proceed to the main theorem of this section. To do so, we introduce the following projections:

$$U_a = \vee\{P \in \mathcal{N} : PaP \text{ is a compact operator}\}$$

and

$$L_b = \wedge\{P \in \mathcal{N} : P^{\perp}bP^{\perp} \text{ is a compact operator}\},$$

where $a, b \in \mathrm{Alg}\mathcal{N}$.

Theorem 3.2. *Let $a, b \in \mathrm{Alg}\mathcal{N}$. The multiplication operator $M_{a,b} : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact if and only if one of the following conditions is satisfied:*

(i) $U_a > L_b$.
(ii) *$U_a = L_b = S$ and the operators SaS and $S^{\perp}bS^{\perp}$ are both compact.*
(iii) *$U_a = L_b = S$, the operator SaS is compact, the operator $S^{\perp}bS^{\perp}$ is non-compact and for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{N}$, $P > S$ such that $\|a(P-S)\| < \varepsilon$.*
(iv) *$U_a = L_b = S$, the operator $S^{\perp}bS^{\perp}$ is compact, the operator SaS is non-compact and for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{N}$, $P < S$ such that $\|(S-P)b\| < \varepsilon$.*

The next theorem provides an other characterization of weakly compact multiplication operators.

Theorem 3.3. *Let $a, b \in \mathrm{Alg}\mathcal{N}$. The multiplication operator $M_{a,b} : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$ is weakly compact if and only if for all $\varepsilon > 0$ there exist two projections $P_1, P_2 \in \mathcal{N}$, with $P_1 \leq P_2$, such that the operators P_1aP_1 and $P_2^{\perp}bP_2^{\perp}$ are both compact and $\|a(P_2-P_1)\| < \varepsilon$ or $\|(P_2-P_1)b\| < \varepsilon$.*

Corollary 3.4. *Let $\mathcal{N} = \{P_n\}_{n\in\mathbb{N}} \cup \{\{0\}, H\}$ be a nest consisting of a sequence of finite rank projections that increase to the identity, and let $a, b \in \mathrm{Alg}\mathcal{N}$. The multiplication operator $M_{a,b} : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact if and only if either the operator a is compact or the operator b is compact.*

Remark 3.5. *Let $\mathcal{N}$ be a nest as in Corollary 3.4 and $a, b \in \operatorname{Alg}\mathcal{N}$. From Theorem 2.2 and Corollary 3.4 it follows that the multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is weakly compact while being non-compact if and only if the operator a is compact and the operator b is non-compact.*

4. MULTIPLICATION OPERATORS ON $\operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$

In this section, we see that there is not any non-zero weakly compact multiplication operator on $\operatorname{Alg}\mathcal{N}$.

Theorem 4.1. *Let $a, b \in \operatorname{Alg}\mathcal{N}$ and $\pi : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ be the quotient map. The multiplication operator $M_{\pi(a),\pi(b)} : \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N}) \to \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ is weakly compact if and only if $M_{\pi(a),\pi(b)} = 0$.*

Remark 4.2. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. Then, the following are equivalent:*

(i) *The multiplication operator $M_{\pi(a),\pi(b)} : \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N}) \to \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ is compact.*
(ii) *The multiplication operator $M_{\pi(a),\pi(b)} : \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N}) \to \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ is weakly compact.*
(iii) $M_{\pi(a),\pi(b)} = 0$.
(iv) $M_{a,b}(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(H)$.
(v) *The multiplication operator $M_{a,b}$ is weakly compact.*

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TOPOLOGICAL RADICALS IN NEST ALGEBRAS

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Abstract. Let $\mathcal{N}$ be a nest on a Hilbert space H and Alg$\mathcal{N}$ the corresponding nest algebra. We determine the hypocompact radical of Alg$\mathcal{N}$. Other topological radicals are also characterized.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. The *Jacobson radical* of $\mathcal{A}$ is defined as the intersection of the kernels of the algebraically irreducible representations of $\mathcal{A}$. A topologically irreducible representation of $\mathcal{A}$ is a continuous homomorphism of $\mathcal{A}$ into the Banach algebra of bounded linear operators on a Banach space X for which no nontrivial, closed subspace of X is invariant. It has been shown in [5] that the intersection of the kernels of these representations is in a reasonable sense a new radical that can be strictly smaller than the Jacobson radical.

The theory of topological radicals of Banach algebras originated with Dixon [5] in order to study this new radical as well as other radicals associated with various types of representations.

Shulman and Turovskii have further developed the theory of topological radicals in a series of papers [8, 9, 10, 11, 12, 13] and applied it to the study of various problems of Operator Theory and Banach algebras. They introduced many new topological radicals. Among them there are the hypocompact radical, the hypofinte radical and the scattered radical. These radicals are closely related to the theory of elementary operators on Banach algebras [3, 10].

Let us recall Dixon's definition of topological radicals.

Definition 1.1. A *topological radical* is a map $\mathcal{R}$ associating with each Banach algebra $\mathcal{A}$ a closed ideal $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{A}$ such that the following hold.

(1) $\mathcal{R}(\mathcal{R}(\mathcal{A})) = \mathcal{R}(\mathcal{A})$.
(2) $\mathcal{R}(\mathcal{A}/\mathcal{R}(\mathcal{A})) = \{0\}$, where $\{0\}$ denotes the zero coset in $\mathcal{A}/\mathcal{R}(\mathcal{A})$.
(3) If $\mathcal{A}$, $\mathcal{B}$ are Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism, then $\phi(\mathcal{R}(\mathcal{A})) \subseteq \mathcal{R}(\mathcal{B})$.
(4) If $\mathcal{I}$ is a closed ideal of $\mathcal{A}$, then $\mathcal{R}(\mathcal{I})$ is a closed ideal of $\mathcal{A}$ and $\mathcal{R}(\mathcal{I}) \subseteq \mathcal{R}(\mathcal{A}) \cap \mathcal{I}$.

An element a of a Banach algebra $\mathcal{A}$ is said to be *compact* if the map $M_{a,a} : \mathcal{A} \to \mathcal{A}$, $x \mapsto axa$ is compact. Following Shulman and Turovskii [12] we will call a Banach algebra $\mathcal{A}$ *hypocompact* if any nonzero quotient $\mathcal{A}/\mathcal{J}$ by a closed ideal $\mathcal{J}$ contains a nonzero compact element. Shulman and Turovskii have proved that any Banach algebra $\mathcal{A}$ has a largest hypocompact ideal [12, Corollary 3.10] which is denoted by $\mathcal{R}_{hc}(\mathcal{A})$ and that the map $\mathcal{A} \to \mathcal{R}_{hc}(\mathcal{A})$ is a topological radical [12, Theorem 3.13]. The ideal $\mathcal{R}_{hc}(\mathcal{A})$ is called the hypocompact radical of $\mathcal{A}$.

If X is a Banach space, we shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded operators on X and by $\mathcal{K}(X)$ the Banach algebra of all compact operators on X. Vala has shown in [14] that if X is a Banach space, an element $a \in \mathcal{B}(X)$ is a compact element if and only if $a \in \mathcal{K}(X)$. Since by [3, Lemma 8.2] the compact elements are always contained in the hypocompact radical, we obtain $\mathcal{K}(X) \subseteq \mathcal{R}_{hc}(\mathcal{B}(X))$. It follows that if H is a separable Hilbert space, the hypocompact radical of $\mathcal{B}(H)$ is $\mathcal{K}(H)$. Indeed, the ideal $\mathcal{K}(H)$ is the only proper ideal of $\mathcal{B}(H)$ while the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ does not have any non-zero compact element [6, section 5].

Shulman and Turovskii observe in [12, p. 298] that there exist Banach spaces X, such that the hypocompact radical $\mathcal{R}_{hc}(\mathcal{B}(X))$ of $\mathcal{B}(X))$ contains all the weakly compact operators and contains strictly the ideal of compact operators $\mathcal{K}(X)$.

Argyros and Haydon construct in [2] a Banach space X such that every operator in $\mathcal{B}(X)$ is a scalar multiple of the identity plus a compact operator. In that case, it follows that $\mathcal{B}(X)/\mathcal{K}(X)$ is finite-dimensional and hence $\mathcal{R}_{hc}((\mathcal{B}(X)) = (\mathcal{B}(X)$.

In this work we characterize the hypocompact radical of a nest algebra. The detailed proofs of the results presented in this note, may be found in [1].

Nest algebras form a class of non-selfadjoint operator algebras that generalize the block upper triangular matrices to an infinite dimensional Hilbert space context. They were introduced by Ringrose in [7] and since then, they have been studied by many authors. The monograph of Davidson [4] is recommended as a reference.

Ringrose characterized the Jacobson radical of a nest algebra in [7, Theorem 5.3]. Moreover, it follows from [7, Theorem 4.9 and 5.3] that the intersection of the kernels of the topologically irreducible representations of a nest algebra coincides with the Jacobson radical.

We introduce now some definitions and notations that we will use in the sequel. A nest $\mathcal{N}$ is a totally ordered family of closed subspaces of a Hilbert space H containing $\{0\}$ and H, which is closed under intersection and closed span. If H is a Hilbert space and $\mathcal{N}$ a nest on H, then the nest algebra Alg$\mathcal{N}$ is the algebra of all operators $T \in \mathcal{B}(H)$ such that $T(N) \subseteq N$ for all $N \in \mathcal{N}$. We shall usually denote both the subspaces belonging to a nest and their corresponding orthogonal projections by the same symbol.

Throughout we denote by $\mathcal{N}$ a nest acting on a Hilbert space H. In addition, all ideals are considered to be closed. The Jacobson radical of the nest algebra Alg$\mathcal{N}$ will be denoted by Rad(Alg$\mathcal{N}$). The following is [7, Theorem 5.4].

Theorem 1.2 (Ringrose's Theorem)**.** *Let $\mathcal{N}$ be a nest on a Hilbert space H and $a \in$ Alg$\mathcal{N}$. Then $a \in$ Rad(Alg$\mathcal{N}$) if and only if the following condition is satisfied: for every $\epsilon > 0$, there exist $m \in \mathbb{N}$ and $P_0, P_1, ..., P_m \in \mathcal{N}$ such that*

$$\{0\} = P_0 < P_1 < P_2 < ...P_m = H$$

and

$$\|(P_i - P_{i-1})a(P_i - P_{i-1})\| < \epsilon$$

$\forall i = 1, 2, ...m.$

2. Main Result

Proposition 2.1. $(\mathrm{Alg}\mathcal{N} \cap \mathcal{K}(H)) + \mathrm{Rad}(\mathrm{Alg}\mathcal{N}) \subseteq \mathcal{R}_{hc}(\mathrm{Alg}\mathcal{N})$.

Proof. If $a \in \mathrm{Alg}\mathcal{N} \cap \mathcal{K}(H)$ then it follows from the result of Vala that $M_{a,a}$ is compact, hence a is a compact element of Alg$\mathcal{N}$. Since by [3, Lemma 8.2] the compact elements are always contained in the hypocompact radical, we obtain that a is in $\mathcal{R}_{hc}(\mathcal{A})$.

Let $a \in$ Rad(Alg$\mathcal{N}$). Let $\epsilon > 0$, $m \in \mathbb{N}$ and $P_0, P_1, ..., P_m \in \mathcal{N}$ such that

$$\{0\} = P_0 < P_1 < P_2 < ...P_m = H$$

and

$$\|(P_i - P_{i-1})a(P_i - P_{i-1})\| < \epsilon$$

$\forall i = 1, 2, ...m.$

Write

$$a = \sum_{i=1}^{m}(P_i - P_{i-1})a(P_i - P_{i-1}) + \sum_{i=1}^{m}(P_i - P_{i-1})aP_i^{\perp}.$$

We have

$$(P_i - P_{i-1})aP_i^{\perp} = P_i(P_i - P_{i-1})aP_i^{\perp}.$$

We show that if $b \in$ Alg$\mathcal{N}$ and $P \in \mathcal{N}$, then $PbP^{\perp}$ is a compact element of Alg$\mathcal{N}$.

The multiplication operator

$$M_{PbP^{\perp},PbP^{\perp}} : \mathrm{Alg}\mathcal{N} \to \mathrm{Alg}\mathcal{N},$$

satisfies

$$M_{PbP^{\perp},PbP^{\perp}}(x) = axa = PbP^{\perp}xPbP^{\perp} = 0,$$

since $P^{\perp}xP = 0$, $\forall x \in \mathrm{Alg}\mathcal{N}$ and hence $PbP^{\perp}$ is a compact element of $\mathrm{Alg}\mathcal{N}$.

We have

$$\|\sum_{i=1}^{m}(P_i - P_{i-1})a(P_i - P_{i-1})\| = \sup_{i=1,2,\ldots,m} \|(P_i - P_{i-1})a(P_i - P_{i-1})\| < \epsilon.$$

Hence a is approximated by elements of $\mathcal{R}_{hc}(\mathcal{A})$ and since $\mathcal{R}_{hc}(\mathcal{A})$ is closed it follows that $a \in \mathcal{R}_{hc}(\mathcal{A})$.

□

The characterization of the hypocompact radical of a nest algebra is given in the following theorem:

Theorem 2.2. *The hypocompact radical of* $\mathrm{Alg}\mathcal{N}$ *is the ideal*

$$\mathrm{Alg}\mathcal{N} \cap \mathcal{K}(H) + \mathrm{Rad}(\mathrm{Alg}\mathcal{N}).$$

The following definitions and results are taken from [13]. An element a of a Banach algebra $\mathcal{A}$ is said to be finite rank if the map $M_{a,a} : \mathcal{A} \to \mathcal{A}$, $x \mapsto axa$ is finite rank. A Banach algebra $\mathcal{A}$ is called hypofinite if any nonzero quotient $\mathcal{A}/\mathcal{J}$ by a closed ideal $\mathcal{J}$ contains a nonzero finite rank element. A Banach algebra $\mathcal{A}$ has a largest hypofinite ideal which is denoted by $\mathcal{R}_{hf}(\mathcal{A})$ and the map $\mathcal{A} \to \mathcal{R}_{hf}(\mathcal{A})$ is a topological radical [13, 2.3.6]. The ideal $\mathcal{R}_{hf}(\mathcal{A})$ is called the hypofinite radical of $\mathcal{A}$. A Banach algebra is called *scattered* if the spectrum of every element $a \in \mathcal{A}$ is finite or countable. A Banach algebra $\mathcal{A}$ has a largest scattered ideal denoted by $\mathcal{R}_{sc}(\mathcal{A})$ and the map $\mathcal{A} \to \mathcal{R}_{sc}(\mathcal{A})$ is a topological radical [13, Theorems 8.10, 8.11]. The ideal $\mathcal{R}_{sc}(\mathcal{A})$ is called the scattered radical of $\mathcal{A}$.

Corollary 2.3. $\mathcal{R}_{hf}(\mathrm{Alg}\mathcal{N}) = \mathcal{R}_{hc}(\mathrm{Alg}\mathcal{N}) = \mathcal{R}_{sc}(\mathrm{Alg}\mathcal{N})$.

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SPACES OF DISTRIBUTIONS WITH MIXED LEBESGUE NORMS

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Abstract. We consider smoothness spaces of distributions on $\mathbb{R}^n$ with mixed Lebesgue norms, where different level of integrability is used for every coordinate. In this note we will state our recent results in this area and we will present some new properties of mixed-norm Besov and Triebel-Lizorkin spaces.

1. Introduction

The theory of spaces of functions and distributions forms an integral part of functional analysis. Here we aim to present some recent and some new properties of smoothness spaces. Besov and Triebel-Lizorkin spaces form two closely related families of smoothness spaces with numerous applications in approximation theory and functional analysis, see [12, 27, 30]. The construction of the above mentioned spaces is based on a dyadic decomposition of the frequency space, and their proven usefulness for applications relies to a large degree on the fact that universal and stable discrete decomposition systems exist for the two families of spaces.

The significance of these spaces can be partially understood by the fact that several spaces of functional analysis, with their own history, are recovered for specific values of the parameters in the definitions of Besov and Triebel-Lizorkin spaces. Some examples are Lebesgue, Hardy, Sobolev and Lipschitz spaces.

The study of Besov and Triebel-Lizorkin spaces has been expanded significantly since the introduction of the so called φ-transform by Frazier and Jawerth in their seminal papers [10–12]. As solid bases for introduction in the study of these spaces we refer the reader to the books of Peetre [27], Triebel [30] and the booklet of Frazier, Jawerth and Weiss [13].

The influence of [10–12] on mathematical analysis has been impressive. Any citation database will show a huge number of citations to the above papers. Moreover these papers have guided researchers with specialities in distribution spaces, wavelets, and approximation theory. Some related works on $\mathbb{R}^n$ are [4–6, 22, 24]. For decompositions on other settings such as on the ball, on the sphere and the interval, see for example [20, 21, 23, 26, 28].

In this paper we present some recent and some new results for Besov and Triebel-Lizorkin spaces in a mixed-norm setting. The content of the article has been presented by the first named author during the fifteenth Panhellenic conference of

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Recently, there has been significant interest in the study of inhomogeneous Besov and Triebel-Lizorkin spaces with mixed Lebesgue norms, see [15–19].

In [7] we introduced and studied homogeneous mixed-norm Besov spaces $\dot{B}^s_{\vec{p}q}$, for $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$ and $q \in (0,\infty]$. The homogeneous spaces are defined over the class $\mathcal{S}'/\mathcal{P}$ of tempered distributions modulo the polynomials. Homogeneous mixed-norm Triebel-Lizorkin spaces $\dot{F}^s_{\vec{p}q}$, are introduced in the recent preprint [14].

Here we present some first properties on $\dot{B}^s_{\vec{p}q}$ spaces proven in [7] and we offer some new results as well. Namely we will prove the connection between inhomogeneous and homogeneous mixed-norm Besov and Triebel-Lizorkin spaces.

Notation: Through the article, positive constants will denoted by c and they may vary at every occurrence. The Fourier transform of a (proper) function f will be stated by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$. The set of positive integers will be denote by $\mathbb{N} := \{1, 2, \dots\}$. For two quasi-normed spaces X, Y we will denoted by $X \hookrightarrow Y$ a continuous embedding.

2. Preliminaries

In this section we present some background needed for the development of mixed norm Besov and Triebel-Lizorkin spaces.

2.1. **Schwartz functions and distributions.** Let us recall some basic facts about Schwartz functions and distributions. We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. A function $\varphi \in \mathcal{C}^\infty$ belongs to $\mathcal{S}$, when for every $k \in \mathbb{N} \cup \{0\}$ and every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$\mathcal{P}_{k,\alpha}(\varphi) := \sup_{x\in\mathbb{R}^n} (1 + |x|)^k |D^\alpha \varphi(x)| < \infty. \tag{2.1}$$

The dual $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}$ is the space of tempered distributions.

We will further denote

$$\mathcal{S}_\infty := \mathcal{S}_\infty(\mathbb{R}^n) = \Big\{\psi \in \mathcal{S} : \int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0,\ \forall \alpha \in (\mathbb{N} \cup \{0\})^n\Big\}.$$

We note that $\mathcal{S}_\infty$ is a Fréchet space, because it is closed in $\mathcal{S}$ and its dual is $\mathcal{S}'_\infty = \mathcal{S}'/\mathcal{P}$, where $\mathcal{P}$ the family of polynomials on $\mathbb{R}^n$.

We will define inhomogeneous mixed-norm Besov spaces for elements of $\mathcal{S}'$ and the homogeneous ones for tempered distributions modulo polynomials $\mathcal{S}'/\mathcal{P}$.

2.2. **Mixed norm Lebesgue spaces.** In our setting, the integrability will be measured in terms of the mixed Lebesgue norms which we present immediately.

Let $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$ and $f : \mathbb{R}^n \to \mathbb{C}$. We say that $f \in L_{\vec{p}} = L_{\vec{p}}(\mathbb{R}^n)$ if

$$\|f\|_{\vec{p}} := \|f\|_{L_{\vec{p}}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}} \cdots dx_n\right)^{\frac{1}{p_n}} < \infty, \tag{2.2}$$

The quasi-norm $\|\cdot\|_{\vec{p}}$, is actually a norm when $\min(p_1, \dots, p_n) \geq 1$ and turns $(L_{\vec{p}}, \|\cdot\|_{\vec{p}})$ into a Banach space. Note that when $\vec{p} = (p, \dots, p)$, then $L_{\vec{p}}$ coincides with L_p. More properties of $L_{\vec{p}}$, can be found for example in [1–3, 9, 25, 29]. For smoothness spaces with mixed Lebesgue norms we refer the reader to [15–17, 25] and their references.

3. Inhomogeneous mixed-norm Besov and Triebel-Lizorkin spaces

Inhomogeneous mixed-norm Besov and Triebel-Lizorkin spaces have been extensively studied the last years, see for example [15, 18, 19] and the references therein. Let us recall their definitions.

Let a function $\phi_0 \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp} \widehat{\phi_0} \subseteq \{\xi \in \mathbb{R}^n : \ |\xi| \leq 2\}, \tag{3.3}$$

and

$$|\widehat{\phi_0}(\xi)| \geq c > 0 \quad \text{if } \ |\xi| \leq 2^{3/4}. \tag{3.4}$$

Let also $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp} \widehat{\phi} \subseteq \{\xi \in \mathbb{R}^n : \ 1/2 \leq |\xi| \leq 2\}, \tag{3.5}$$

and

$$|\widehat{\phi}(\xi)| \geq c > 0 \quad \text{if } \ 2^{-3/4} \leq |\xi| \leq 2^{3/4}. \tag{3.6}$$

We set $\phi_\nu(x) := 2^{\nu n}\phi(2^\nu x), \ \ \forall \nu \in \mathbb{N}$.

Definition 3.1. *Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$ and ϕ_0, ϕ as above.*

(i) The inhomogeneous mixed-norm Besov space $B^s_{\vec{p}q}$, is the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{B^s_{\vec{p}q}} := \Big(\sum_{\nu=0}^{\infty}(2^{\nu s}\|\phi_\nu * f\|_{\vec{p}})^q\Big)^{1/q} < \infty, \tag{3.7}$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

(ii) The inhomogeneous mixed-norm Triebel-Lizorkin space $F^s_{\vec{p}q}$, is the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{F^s_{\vec{p}q}} := \Big\|\Big(\sum_{\nu=0}^{\infty}(2^{\nu s}|\phi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}} < \infty, \tag{3.8}$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

4. Homogeneous mixed-norm Besov spaces

In this section we present the extension of the classical homogeneous Besov spaces (see Triebel [30], Peetre [27] and Frazier-Jawerth [10]), which we developed in [7] using mixed-norms.

We will say that a test function $\varphi \in \mathcal{S}$ is admissible when it satisfies (3.5) and (3.6). Furthermore, we set $\varphi_\nu(x) := 2^{\nu n}\varphi(2^\nu x)$, $\forall \nu \in \mathbb{Z}$. We present the following:

Definition 4.1. *[7] For $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0, \infty)^n$, $q \in (0, \infty]$ and φ admissible, we define the homogeneous mixed-norm Besov space $\dot{B}^s_{\vec{p}q}$, as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that*

$$\|f\|_{\dot{B}^s_{\vec{p}q}} := \Big(\sum_{\nu\in\mathbb{Z}}(2^{\nu s}\|\varphi_\nu * f\|_{\vec{p}})^q\Big)^{1/q} < \infty, \tag{4.9}$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

Remark 4.2. *Several remarks regarding the homogeneous mixed-norm Besov spaces defined above and some results proven in [7] are in order.*

(α) *By (3.6) we have that $\|f\|_{\dot{B}^s_{\vec{p}q}} = 0 \Leftrightarrow f \in \mathcal{P}$, which is why we work over the quotient $\mathcal{S}'/\mathcal{P}$.*

(β) *When $\vec{p} = (p, \ldots, p)$, then $\dot{B}^s_{\vec{p}q}$, coincides with $\dot{B}^s_{pq}$, the standard homogeneous Besov space.*

(γ) *Homogeneous mixed-norm Besov space $\dot{B}^s_{\vec{p}q}$ is quasi-Banach for all $s \in \mathbb{R}$, $\vec{p} = (p_1, \ldots, p_n) \in (0,\infty)^n$ and $q \in (0,\infty]$. The triangle inequality does not hold in general in $\dot{B}^s_{\vec{p}q}$. Instead we have the sub-additivity*

$$\|f+g\|^r_{\dot{B}^s_{\vec{p}q}} \le \|f\|^r_{\dot{B}^s_{\vec{p}q}} + \|g\|^r_{\dot{B}^s_{\vec{p}q}}, \quad \text{where } r := \min(1, p_1, \ldots, p_n, q).$$

Furthermore $\dot{B}^s_{\vec{p}q}$ is a Banach space when $\vec{p} \in [1,\infty)^n$, $q \in [1,\infty]$.

(δ) *The quasi-norm in the definition of $\dot{B}^s_{\vec{p}q}$ depends on the choice of the admissible function φ, but for different admissible functions, we get equivalent quasi-norms. Therefore $\dot{B}^s_{\vec{p}q}$ space is independent of the admissible function φ.*

(ε) *All the construction has been based on the dyadic decomposition of the frequency space. We can use instead, any other number $\beta > 1$ in all the procedure of Subsection 2.2, as well as in the Definition 4.1 of Besov spaces (replace $2^{\nu s}$ by $\beta^{\nu s}$) and get the same spaces with equivalent norms.*

($\sigma\tau$) *Some embeddings between homogeneous mixed-norm Besov spaces, provided in [7] are presented below:*

($\sigma\tau$1) *Let $s \in \mathbb{R}$, $\vec{p} \in (0,\infty)^n$ and $0 < q < r \le \infty$. Then we have the embedding*

$$\dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^s_{\vec{p}r},$$

coming from the well known embedding between the sequence spaces; $\ell_q \hookrightarrow \ell_r$.

($\sigma\tau$2) *Homogeneous mixed-norm Besov spaces and the classes $\mathcal{S}_\infty, \mathcal{S}'_\infty$ are connected in the following way:*

Proposition 4.3. *Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n) \in (0,\infty)^n$ and $q \in (0,\infty]$. Then*

$$\mathcal{S}_\infty \hookrightarrow \dot{B}^s_{\vec{p}q} \quad \text{and} \quad \dot{B}^s_{\vec{p}q} \hookrightarrow \mathcal{S}'_\infty.$$

($\sigma\tau$3) *Spaces of different smoothness levels are connected as below:*

Proposition 4.4. *Let $s, t \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n)$, $\vec{r} = (r_1, \ldots, r_n) \in (0,\infty)^n$ and $q \in (0,\infty]$ be such that*

$$t < s, \quad p_1 \le r_1, \ldots, p_n \le r_n, \quad \text{and} \quad s - \frac{1}{p_1} - \cdots - \frac{1}{p_n} = t - \frac{1}{r_1} - \cdots - \frac{1}{r_n},$$

then

$$\dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^t_{\vec{r}q}.$$

Specifically we have the following relation between mixed and unmixed spaces:

Let $s \in \mathbb{R}$, $\vec{p} = (p_1, \cdots, p_n) \in (0,\infty)^n$ and $q \in (0,\infty]$. We set $p_m := \min(p_1, \ldots, p_n)$ and $p_M := \max(p_1, \ldots, p_n)$, then

$$\dot{B}^t_{p_m q} \hookrightarrow \dot{B}^s_{\vec{p}q} \hookrightarrow \dot{B}^\tau_{p_M q},$$

where

$$t = s - \Big(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\Big) + \frac{n}{p_m} \quad \text{and} \quad \tau = s - \Big(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\Big) + \frac{n}{p_M}.$$

4.1. **Homogeneous mixed-norm Triebel-Lizorkin spaces.** The development of homogeneous mixed-norm Triebel-Lizorkin spaces has been obtained in [14]. Let us present here only the definition of these spaces.

Definition 4.5. *[14] For $s \in \mathbb{R}$, $\vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n$, $q \in (0,\infty]$ and φ admissible, we define the homogeneous mixed-norm Triebel-Lizorkin space $\dot{F}^s_{\vec{p}q}$, as the set of all $f \in \mathcal{S}'/\mathcal{P}$ such that*

$$\|f\|_{\dot{F}^s_{\vec{p}q}} := \Big\| \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} |\varphi_\nu * f(\cdot)|)^q \Big)^{1/q} \Big\|_{\vec{p}} < \infty, \tag{4.10}$$

with the ℓ_q-norm replaced by the $\sup_\nu$ if $q = \infty$.

Note that the remarks we presented for the case of homogeneous mixed-norm Besov spaces, apply for $\dot{F}^s_{\vec{p}q}$ spaces too.

5. Comparison of inhomogeneous and homogeneous spaces

In this section we give some new results, inspired by the unmixed case presented in [13]. We give the relation connecting the inhomogeneous and homogeneous mixed-norm Besov and Triebel-Lizorkin spaces, but let us first justify the title "homogeneous" which we use for some of our spaces.

Let $f \in \mathcal{S}'$. We set $f_\mu(x) := 2^{\mu n} f(2^\mu x)$ for every $\mu \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. We will show that

$$\|f_\mu\|_{\dot{B}^s_{\vec{p}q}} = 2^{\mu N} \|f\|_{\dot{B}^s_{\vec{p}q}}, \ \forall s \in \mathbb{R},\ \vec{p} = (p_1, \dots, p_n) \in (0,\infty)^n,\ q \in (0,\infty], \tag{5.11}$$

where N is an exponent depending only on the parameters s, $\vec{p}$, q.

Indeed, let $\nu, \mu \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. By changing variables we obtain that

$$\varphi_\nu * f_\mu(x) = 2^{\mu n} \big(\varphi_{\nu-\mu} * f\big)(2^\mu x). \tag{5.12}$$

Now the mixed Lebesgue norm of $\big(\varphi_{\nu-\mu} * f\big)(2^\mu x)$, by changing the variables $2^\mu x_j =: y_j$, for every direction $j = 1, \dots, n$, equals to

$$\|(\varphi_{\nu-\mu} * f)(2^\mu \cdot)\|_{\vec{p}} = 2^{-\mu\left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)} \|\varphi_{\nu-\mu} * f\|_{\vec{p}}. \tag{5.13}$$

From (5.12) and (5.13), it follows that

$$\begin{aligned}
\|f_\mu\|_{\dot{B}^s_{\vec{p}q}} &= \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} \|\varphi_\nu * f_\mu\|_{\vec{p}})^q \Big)^{1/q} \\
&= \Big(\sum_{\nu \in \mathbb{Z}} \Big(2^{\nu s} 2^{\mu n} 2^{-\mu\left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)} \|\varphi_{\nu-\mu} * f\|_{\vec{p}} \Big)^q \Big)^{1/q} \\
&= 2^{\mu\left(s + n - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)\right)} \Big(\sum_{\nu \in \mathbb{Z}} \big(2^{(\nu-\mu)s} \|\varphi_{\nu-\mu} * f\|_{\vec{p}} \big)^q \Big)^{1/q} \\
&= 2^{\mu\left(s + n - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\right)\right)} \|f\|_{\dot{B}^s_{\vec{p}q}}.
\end{aligned}$$

So (5.11) holds true for $N := s + n - \big(\frac{1}{p_1} + \cdots + \frac{1}{p_n}\big)$. Note that (5.11) remains true for the homogeneous mixed-norm Triebel-Lizorkin spaces as well (with the same N) and does not hold for the inhomogeneous spaces.

The exponent N is called the *homogeneous dimension* of $\dot{B}^s_{\vec{p}q}$ (or $\dot{F}^s_{\vec{p}q}$) space. Note that for the unmixed case the homogeneous dimension we derived turns to $N = s + n\big(1 - \frac{1}{p}\big)$ as in [13].

Now let us present the relation connecting the inhomogeneous and homogeneous spaces with mixed-norms, inspired by the classical, unmixed, situation, see [13].

Theorem 5.1. *Let* $s > 0$, $\vec{p} = (p_1, \dots, p_n)$ *with* $\min(p_1, \dots, p_n) \geq 1$ *and* $0 < q \leq \infty$. *Then*

$$\text{(i) } B^s_{\vec{p}q} = L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q} \ \textit{ and } \ \text{(ii) } F^s_{\vec{p}q} = L_{\vec{p}} \cap \dot{F}^s_{\vec{p}q}.$$

Proof. (i) Let $f \in B^s_{\vec{p}q}$. Let also $\phi_0, \phi \in \mathcal{S}$ satisfying (3.3)-(3.6) be such that

$$\sum_{\nu \geq 0} \widehat{\phi_\nu}(\xi) = 1, \quad \text{for every } \xi \in \mathbb{R}^n.$$

Then

$$f = \sum_{\nu \geq 0} \phi_\nu * f \quad (\text{convergence in } \mathcal{S}').$$

Using the fact that $\min(p_1, \dots, p_n) \geq 1$ and hence $\| \cdot \|_{\vec{p}}$ turns to a norm, it follows that

$$\begin{aligned}
\|f\|_{L_{\vec{p}}} &= \Big\| \sum_{\nu \geq 0} \phi_\nu * f \Big\|_{\vec{p}} \leq \sum_{\nu \geq 0} \|\phi_\nu * f\|_{\vec{p}} \\
&\leq \sum_{\nu \geq 0} 2^{-\nu s} \sup_{\mu \geq 0} 2^{\mu s} \|\phi_\mu * f\|_{\vec{p}} \\
&= c_s \sup_{\mu \geq 0} 2^{\mu s} \|\phi_\mu * f\|_{\vec{p}} \leq c \|f\|_{B^s_{\vec{p}q}}, \qquad (5.14)
\end{aligned}$$

where for the last equality, we used the assumption $s > 0$.

Let now $\varphi \in \mathcal{S}$ satisfying (3.5) and (3.6). By Penedek-Panzone [3, Theorem 1.b, p. 319] and by the fact that $\min(p_1, \dots, p_n) \geq 1$, we have the following behaviour for the mixed-norms of convolution operators:

$$\|\varphi_\nu * f\|_{\vec{p}} \leq \|\varphi_\nu\|_1 \|f\|_{\vec{p}} = c\|f\|_{\vec{p}}, \quad \text{for every } \nu \in \mathbb{Z}, \qquad (5.15)$$

since we can easily observe that $\|\varphi_\nu\|_1 = \|\varphi\|_1$, for every $\nu \in \mathbb{Z}$ and hence we get immediately

$$\begin{aligned}
\|f\|_{\dot{B}^s_{\vec{p}q}} &= \Big(\sum_{\nu \in \mathbb{Z}} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} \\
&\leq c \Big(\sum_{\nu \leq 0} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} + c \Big(\sum_{\nu > 0} (2^{\nu s} \|\varphi_\nu * f\|_{\vec{p}})^q \Big)^{1/q} \\
&\leq \Big(\sum_{\nu \leq 0} 2^{\nu s q} \Big)^{1/q} \|f\|_{\vec{p}} + c \|f\|_{B^s_{\vec{p}q}} \\
&\leq c (\|f\|_{\vec{p}} + \|f\|_{B^s_{\vec{p}q}}), \qquad (5.16)
\end{aligned}$$

where we used again the fact that $s > 0$. Combining (5.14) and (5.16) we have the embedding

$$B^s_{\vec{p}q} \hookrightarrow L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q}.$$

For the other direction, note that (5.15) holds true for the functions $\phi_\nu,\ \nu \geq 0$ as well. Then,

$$\begin{aligned} \|f\|_{B^s_{\vec{p}q}} &= \Big(\sum_{\nu\geq 0}(2^{\nu s}\|\phi_\nu * f\|_{\vec{p}})^q\Big)^{1/q} \\ &\leq c\|\phi_0 * f\|_{\vec{p}} + c\Big(\sum_{\nu>0}(2^{\nu s}\|\phi_\nu * f\|_{\vec{p}})^q\Big)^{1/q} \\ &\leq c(\|f\|_{\vec{p}} + \|f\|_{\dot{B}^s_{\vec{p}q}}), \end{aligned}$$

which guarantees the embedding

$$L_{\vec{p}} \cap \dot{B}^s_{\vec{p}q} \hookrightarrow B^s_{\vec{p}q}.$$

(ii) We will follow [13]. Let $f \in \mathcal{S}'$ and $\phi_0, \phi \in \mathcal{S}$ satisfying (3.3)-(3.6) be such that $\{\widehat{\phi_\nu}\}_{\nu\geq 0}$ to be a partition of unity. Then

$$f = \sum_{\nu\geq 0} \phi_\nu * f \quad \text{(convergence in } \mathcal{S}'). \tag{5.17}$$

We turn to estimate

$$\sum_{\nu\geq 1} |\phi_\nu * f(x)|.$$

We distinguish the cases $q \geq 1$ and $q < 1$.

Case $\alpha:\ 1 \leq q \leq \infty$. By Hölder's inequality, denoting by q' the conjugate index of q, we obtain

$$\begin{aligned} \sum_{\nu\geq 1} |\phi_\nu * f(x)| &\leq \Big(\sum_{\nu\geq 1} 2^{-\nu s q'}\Big)^{1/q'} \Big(\sum_{\nu\geq 1} \big(2^{\nu s}|\phi_\nu * f(x)|\big)^q\Big)^{1/q} \\ &\leq c_{s,q}\Big(\sum_{\nu\geq 1} \big(2^{\nu s}|\phi_\nu * f(x)|\big)^q\Big)^{1/q}, \end{aligned}$$

thanks to the assumption $s > 0$.

Case $\beta:\ 0 < q < 1$. Using the q-triangle inequality and the fact that $s > 0$, we derive

$$\sum_{\nu\geq 1} |\phi_\nu * f(x)| \leq \sum_{\nu\geq 1} 2^{\nu s}|\phi_\nu * f(x)| \leq \Big(\sum_{\nu\geq 1} \big(2^{\nu s}|\phi_\nu * f(x)|\big)^q\Big)^{1/q}.$$

Since now $\min(p_1, \ldots, p_n) \geq 1$, by assumption, relation (5.17) and the bounds above lead us to

$$\begin{aligned} \|f\|_{\vec{p}} &\leq c\|\phi_0 * f\|_{\vec{p}} + c\Big\|\Big(\sum_{\nu\geq 1}(2^{\nu s}|\phi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}} \\ &\leq c\Big\|\Big(\sum_{\nu\geq 0}(2^{\nu s}|\phi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}} = c\|f\|_{F^s_{\vec{p}q}}. \end{aligned} \tag{5.18}$$

Let now $\varphi \in \mathcal{S}$ satisfying (3.5) and (3.6). Then,

$$\begin{aligned} \|f\|_{\dot{F}^s_{\vec{p}q}} &\leq c\Big\|\Big(\sum_{\nu\leq 0}(2^{\nu s}|\varphi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}} + c\Big\|\Big(\sum_{\nu>0}(2^{\nu s}|\varphi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}} \\ &=: c\big(\Sigma_1 + \Sigma_2\big). \end{aligned} \tag{5.19}$$

Of course it holds that

$$\Sigma_2 \leq \|f\|_{F^s_{\vec{p}q}} \tag{5.20}$$

and so we restrict our interest to Σ_1. We consider separately the cases when q is smaller than 1 or not.

Case $\alpha:$ $0 < q \leq 1$. By Hölder's inequality, denoting by $(1/q)'$ the conjugate index of $1/q$, we obtain

$$\begin{aligned}\sum_{\nu\leq 0}\big(2^{\nu s}|\varphi_\nu * f(\cdot)|\big)^q &\leq \Big(\sum_{\nu\leq 0}2^{(\nu sq/2)(1/q)'}\Big)^{1/(1/q)'}\Big(\sum_{\nu\leq 0}2^{\nu s/2}|\varphi_\nu * f(\cdot)|\Big)^q\\ &\leq c_{s,q}\Big(\sum_{\nu\leq 0}2^{\nu s/2}|\varphi_\nu * f(\cdot)|\Big)^q.\end{aligned}$$

The last inequality gives us

$$\begin{aligned}\Sigma_1 &\leq c\Big\|\sum_{\nu\leq 0}2^{\nu s/2}|\varphi_\nu * f(\cdot)|\Big\|_{\vec{p}} \leq c\sum_{\nu\leq 0}2^{\nu s/2}\|\varphi_\nu * f\|_{\vec{p}}\\ &\leq c\Big(\sum_{\nu\leq 0}2^{\nu s/2}\Big)\|f\|_{\vec{p}} \leq c\|f\|_{\vec{p}}, \qquad (5.21)\end{aligned}$$

where for the second inequality we used the fact that $\|\cdot\|_{\vec{p}}$ is a norm under our assumptions, for the third the inequality (5.15) and for the last the assumption $s > 0$.

Case $\beta:$ $1 < q \leq \infty$. By the identity $|a+b|^{1/q} \leq |a|^{1/q} + |b|^{1/q}$, we derive

$$\sum_{\nu\leq 0}\big(2^{\nu s}|\varphi_\nu * f(\cdot)|\big)^q \leq \Big(\sum_{\nu\leq 0}2^{\nu s}|\varphi_\nu * f(\cdot)|\Big)^q.$$

So with the same steps as before we get for this case too

$$\Sigma_1 \leq c\|f\|_{\vec{p}}. \tag{5.22}$$

Combining (5.18)-(5.22) we have that

$$\|f\|_{\dot{F}^s_{\vec{p}q}} \leq c\|f\|_{F^s_{\vec{p}q}}$$

which together with (5.18) offers the inclusion

$$F^s_{\vec{p}q} \hookrightarrow L_{\vec{p}} \cap \dot{F}^s_{\vec{p}q}.$$

The converse embedding comes straight from the expression (5.17) and the estimation (5.15), indeed

$$\begin{aligned}\|f\|_{F^s_{\vec{p}q}} &\leq c\|\phi_0 * f\|_{\vec{p}} + c\Big\|\Big(\sum_{\nu>0}(2^{\nu s}|\phi_\nu * f(\cdot)|)^q\Big)^{1/q}\Big\|_{\vec{p}}\\ &\leq c\|f\|_{\vec{p}} + c\|f\|_{\dot{F}^s_{\vec{p}q}}\end{aligned}$$

and the proof is complete.

□

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The Main Equation for a Sturm-Liouville Operator with a Piecewise Continuous Coefficient

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Abstract. We consider a boundary value problem for a Sturm-Liouville operator with a piecewise continuous coefficient:

$$-y'' + q(x)y = \lambda^2 \rho(x) y, \qquad 0 \le x \le \pi, \tag{1}$$

$$y(0) = 0, \tag{2}$$

$$\lambda^2 \left[\beta_1 y'(\pi) + \beta_2 y(\pi)\right] + \alpha_1 y(\pi) + \alpha_2 y'(\pi) = 0 \tag{3}$$

where $q(x) \in L_2[0,\pi]$, λ is a complex parameter, α_i, β_i $(i = 1,2)$ are real numbers and

$$\rho(x) = \begin{cases} 1, & 0 \le x < a, \\ \alpha^2, & a < x \le \pi \end{cases}$$

as $0 < \alpha \ne 1$. We derive the Gelfand-Levitan-Marchenko type main equation for boundary value problem (1)-(3) and we prove the uniqueness of its solution. We also give the uniqueness theorem for the solution of the inverse problem. The direct and inverse problem with respect to the Weyl function for the boundary value problem (1)-(3) is examined in [1].

Keywords. Sturm-Liouville operator, inverse problem, main equation.

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Matrix transformations between some sequence space of generalized weighted means

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The study of the topological properties is fundamental in our study. We will determine the multipliers and duals of certain sequence spaces. The knowledge of the β–dual of a given sequence space X is essential for the characterization of linear operators from X into a sequence space Y. This is why we will focus on the β–dual of our sequence spaces to establish necessary and sufficient conditions on the entries of an infinite matrix A to be in the class (X, Y).

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Global asymptotic behavior of a system with exponential difference equations

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Abstract

In this paper we study the asymptotic behavior of the positive solutions of the systems of the two difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n},$$

where the constants a, b, c, d are positive real numbers, and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers.

Keywords: System of difference equations, equilibrium, boundedness, persistence, attractivity, global asymptotic stability.

1 Introduction

The mathematical modeling of a discrete problem very often leads to systems of nonlinear difference equations. So there are many applications of such systems in Economics, Biology, Biomathematics, Bioengineering, Population Dynamics, Genetics and other sciences. Thus, an extended literature has been developed regarding to difference equations and systems of difference equations as we can see in [1-11] and the references cited therein.

Now, in this paper we study a system of nonlinear difference equations which becomes from the following difference equation

$$x_{n+1} = a + bx_{n-1}e^{-x_n}$$

that has been studied in [6]. More detailed in this manuscript we investigate the boundedness and the persistence of the positive solutions, the existence of a unique positive equilibrium and the global asymptotic stability of the equilibrium of the following system of difference equations

$$x_{n+1} = a + by_{n-1}e^{-x_n}, \quad y_{n+1} = c + dx_{n-1}e^{-y_n}, \tag{1.1}$$

where the constants a, b, c, d are positive real numbers and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive real numbers.

2 Boundedness

Firstly, we study the boundedness and persistence of the solutions of system (1.1).

Proposition 2.1 *Let a, b, c, d be positive real numbers such that*

$$p = bde^{-a-c} < 1. \tag{2.1}$$

Then every solution of (1.1) is positive, bounded and persists.

Proof Since the initial x_{-1}, x_0, y_{-1}, y_0 of (1.1) are positive, every solution of (1.1) is positive.
Let (x_n, y_n) be an arbitrary solution of (1.1). From (1.1) it is obvious that

$$x_n \geq a, \quad y_n \geq c, \quad n = 1, \ 2, \ ... \tag{2.2}$$

Every solution of (1.1) persists.
Moreover from (1.1) and (2.2) it follows that for $n = 2, 3, ...$

$$\begin{aligned} x_{n+1} &= a + b(c + dx_{n-3}e^{-y_{n-2}})e^{-x_n} \leq a + bce^{-a} + px_{n-3}, \\ y_{n+1} &= c + d(a + by_{n-3}e^{-x_{n-2}})e^{-y_n} \leq c + dae^{-c} + py_{n-3} \end{aligned} \tag{2.3}$$

We consider the system of difference equations

$$u_{n+1} = a + bce^{-a} + pu_{n-3}, \quad v_{n+1} = c + dae^{-c} + pv_{n-3}, \quad n = 2, 3, ... \tag{2.4}$$

Let (u_n, v_n) be a solution of (2.4) such that

$$\begin{aligned} &u_{-1} = x_{-1}, \quad u_0 = x_0, \quad u_1 = x_1, \quad u_2 = x_2, \\ &v_{-1} = y_{-1}, \quad v_0 = y_0, \quad v_1 = y_1, \quad v_2 = y_2. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) we obtain

$$u_3 = a + bce^{-a} + px_{-1} > 0, \quad v_3 = c + dae^{-c} + py_{-1} > 0$$

and working inductively it follows that

$$u_n > 0, \quad v_n > 0, \quad n = 2, 3, ... \ .$$

Moreover, from (2.4) for $n = 3, 4, ...,$ we have

$$u_n = \lambda_1 p^{\frac{n}{4}} + \lambda_2(-p)^{\frac{n}{4}} + \lambda_3 p^{\frac{n}{4}} cos(\frac{n\pi}{2}) + \lambda_4 p^{\frac{n}{4}} sin(\frac{n\pi}{2}) + \frac{a + bce^{-a}}{1 - p}, \tag{2.6}$$

$$v_n = \mu_1 p^{\frac{n}{4}} + \mu_2(-p)^{\frac{n}{4}} + \mu_3 p^{\frac{n}{4}} cos(\frac{n\pi}{2}) + \mu_4 p^{\frac{n}{4}} sin(\frac{n\pi}{2}) + \frac{c + dae^{-c}}{1 - p}, \tag{2.7}$$

where $\lambda_1, \ \lambda_2, \ \lambda_3, \ \lambda_4$ (resp. $\mu_1, \ \mu_2, \ \mu_3, \ \mu_4$) are constants defined by $x_{-1}, \ x_0, \ x_1, \ x_2$ (resp. $y_{-1}, \ y_0, \ y_1, \ y_2$).
Using (2.3), (2.4) and (2.5) we can prove by induction that

$$x_n \leq u_n, \quad y_n \leq v_n, \quad n = -1, \ 0, \ ... \tag{2.8}$$

Then from (2.2), (2.6), (2.7) and (2.8) we obtain that every solution of (1.1) is bounded.
Hence, the proof is completed.

In the next proposition we prove the existence of invariant intervals for the System (1.1).

Proposition 2.2 *Let a, b, c, d be positive numbers such that (2.1) hold. Consider the intervals*

$$I_1 = \Big[a, \frac{a+bce^{-a}}{1-p}\Big], \quad I_2 = \Big[c, \frac{c+ade^{-c}}{1-p}\Big],$$

$$I_3 = \Big[a, \frac{a+bce^{-a}+\epsilon}{1-p}\Big], \quad I_4 = \Big[c, \frac{c+ade^{-c}+\epsilon}{1-p}\Big],$$

where p is defined in relation (2.1) and ϵ is an arbitrary positive number. Then, if (x_n, y_n) is a positive solution of (1.1) such that

$$x_{-1}, x_0 \in I_1, \quad y_{-1}, y_0 \in I_2, \tag{2.9}$$

we have

$$x_n \in I_1, \quad y_n \in I_2, \quad n = 1, 2, \dots .$$

Moreover, if (x_n, y_n) is an arbitrary positive solution of (1.1), then there exists an $m \in N$ such that

$$x_n \in I_3, \quad y_n \in I_4, \quad n \geq m. \tag{2.10}$$

Proof (i) Let (x_n, y_n) be a positive solution of (1.1), such that (2.9) hold. Then, from (1.1) we obtain

$$a \leq x_1 = a + by_{-1}e^{-x_0} \leq a + b\frac{c+ade^{-c}}{1-p}e^{-a} = \frac{a+bce^{-a}}{1-p}$$

$$c \leq y_1 = c + dx_{-1}e^{-y_0} \leq c + d\frac{a+bce^{-a}}{1-p}e^{-c} = \frac{c+ade^{-c}}{1-p}.$$

and working inductively we can prove that

$$a \leq x_n \leq \frac{a+bce^{-a}}{1-p}, \quad c \leq y_n \leq \frac{c+ade^{-c}}{1-p}, \quad , n = 2, 3, \dots .$$

This completes the proof of the first part of the proposition.

Let (x_n, y_n) be an arbitrary positive solution of (1.1). Then, from Proposition 2.1, we obtain

$$\limsup_{n\to\infty} x_n = M < \infty, \quad \limsup_{n\to\infty} y_n = L < \infty. \tag{2.11}$$

Thus from (2.3) and (2.11) we get

$$M \leq \frac{a+bce^{-a}}{1-p}, \quad L \leq \frac{c+ade^{-c}}{1-p},$$

and so there exists an $m \in I\!N$ such that (2.10) hold. This completes the proof of the proposition.

3 Attractivity

In this section we study the existence of a unique positive equilibrium for system (1.1) and the attractivity of the unique positive equilibrium. Arguing as in Theorem 1.6.5 of [1], in Theorems 1.11-1.16 of [2] and in Theorems 1.4.5-1.4.8 of [4] we state the following lemma.

Lemma 3.1 *Let f, g, $f : R^+ \times R^+ \to R^+$, $g : R^+ \times R^+ \to R^+$ be continuous functions, $R^+ = (0, \infty)$ and a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1$, $a_2 < b_2$.*

Suppose that

$$f : [a_1, b_1] \times [a_2, b_2] \to [a_1, b_1], \quad g : [a_1, b_1] \times [a_2, b_2] \to [a_2, b_2].$$

In addition, assume that $f(x, y)$ (resp. $g(x, y)$) is decreasing with respect to x (resp. y) for every y (resp. x) and increasing with respect to y (resp. x) for every x (resp. y). Finally suppose that, if the real numbers m, M, r, R satisfy the system

$$M = f(m, R), \quad m = f(M, r), \quad R = g(M, r), \quad r = g(m, R),$$

then $m = M$ and $r = R$. Then the following system of difference equations

$$x_{n+1} = f(x_n, y_{n-1}), \quad y_{n+1} = g(x_{n-1}, y_n) \tag{3.1}$$

has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution (x_n, y_n) of the system (3.1) which satisfies

$$x_{n_0} \in [a_1, b_1], \quad x_{n_0+1} \in [a_1, b_1], \quad y_{n_0} \in [a_2, b_2], \quad y_{n_0+1} \in [a_2, b_2], \; n_0 \in I\!N$$

tends to the unique positive equilibrium of (3.1).

Proposition 3.1 *Let a, b, c, d be positive numbers. Assume that*

$$\theta_1 = be^{-a} < 1, \quad \theta_2 = de^{-c} < 1. \tag{3.2}$$

Suppose also that

$$(1 + a)p + c\theta_1 < 1, \quad (1 + c)p + a\theta_2 < 1 \tag{3.3}$$

and

$$\lambda = \frac{p(1 - p)^2}{\Big[1 - (1 + a)p - c\theta_1\Big]\Big[1 - (1 + c)p - a\theta_2\Big]} < 1. \tag{3.4}$$

Then the system (1.1) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every solution of (1.1) tends to the unique positive equilibrium of (1.1) as $n \to \infty$.

Proof Let $f : R^+ \times R^+ \to R^+$, $g : R^+ \times R^+ \to R^+$ be continuous functions, such that

$$f(x,y) = a + bye^{-x}, \quad g(x,y) = c + dxe^{-y}.$$

Then, if $x \in I_3$, $y \in I_4$ from (3.2) we have

$$a \le f(x,y) \le a + b\frac{c + ade^{-c} + \epsilon}{1-p}e^{-a} = \frac{a + c\theta_1 + \epsilon\theta_1}{1-p} < \frac{a + c\theta_1 + \epsilon}{1-p}$$

$$c \le g(x,y) \le c + d\frac{a + bce^{-a} + \epsilon}{1-p}e^{-c} = \frac{c + a\theta_2 + \epsilon\theta_2}{1-p} < \frac{c + a\theta_2 + \epsilon}{1-p}.$$

Therefore f, g are continuous functions such that $f : I_3 \times I_4 \to I_3$, $g : I_3 \times I_4 \to I_4$.

Let now, $m, M \in I_3$, $r, R \in I_4$ be positive real numbers such that

$$M = a + bRe^{-m}, \quad m = a + bre^{-M}, \quad R = c + dMe^{-r}, \quad r = c + dme^{-R}. \tag{3.5}$$

Then, from (3.5), we have

$$m = a + bce^{-M} + bdme^{-R}e^{-M}, \quad r = c + dae^{-R} + bdre^{-M}e^{-R}$$

and so

$$m = \frac{a + bce^{-M}}{1 - bde^{-R-M}}, \quad r = \frac{c + ade^{-R}}{1 - bde^{-R-M}}. \tag{3.6}$$

Then since $M \ge a, R \ge c$ it holds

$$m \le \frac{a + bce^{-a}}{1-p} = \frac{a + c\theta_1}{1-p}, \quad r \le \frac{c + ade^{-c}}{1-p} = \frac{c + a\theta_2}{1-p}. \tag{3.7}$$

Furthermore, there exists a ξ, $\min\{m, M\} \le \xi \le \max\{m, M\}$ such that

$$e^M - e^m = e^\xi(M - m). \tag{3.8}$$

From (3.5) and (3.8) and since $M, m \ge a$ we get

$$\begin{aligned} M - m \quad &= b(Re^{-m} - re^{-M}) = be^{-m}(R - r) + bre^{-m-M}(e^M - e^m) = \\ &be^{-m}(R - r) + bre^{-m-M+\xi}(M - m) \le \\ &\theta_1(R - r) + r\theta_1(M - m). \end{aligned} \tag{3.9}$$

Hence from (3.7) and (3.9) it follows that

$$M - m \le \theta_1(R - r) + \frac{\theta_1(c + a\theta_2)}{1-p}(M - m). \tag{3.10}$$

Then since $p = \theta_1\theta_2$, from (3.10) we obtain

$$(M - m)\Big(\frac{1 - p - c\theta_1 - ap}{1 - p}\Big) \le \theta_1(R - r). \tag{3.11}$$

Therefore from (3.3) and (3.11) we have

$$M - m \le \frac{\theta_1(1 - p)}{1 - c\theta_1 - (a + 1)p}(R - r). \tag{3.12}$$

Similarly, we have

$$R - r \le \frac{\theta_2(1 - p)}{1 - a\theta_2 - (c + 1)p}(M - m). \tag{3.13}$$

Relations (3.12) and (3.13) imply that

$$M - m \le \lambda(M - m). \tag{3.14}$$

Therefore from (3.4) and (3.14) we have $M = m$ and so from (3.5) $r = R$. Consequently, from Lemma 3.1, System (1.1) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of System (1.1) tends to $(\overline{x}, \overline{y})$. This completes the proof of the Statement (i). (ii) We define the functions $f : R^+ \times R^+ \to R^+$, $g : R^+ \times R^+ \to R^+$ as follows

$$f(u, v) = a + bve^{-u}, \quad g(z, w) = c + dwe^{-z}.$$

Then, if $z, w \in J_3$, $u, v \in J_4$ and arguing as in Statement (i) we have

$$f(u, v) \in J_3, \quad g(z, w) \in J_4.$$

So f and g are continuous functions such that

$$f : J_4 \times J_4 \to J_3, \quad g : J_3 \times J_3 \to J_4.$$

Let now, $m, M \in J_3$, $r, R \in J_4$ be real numbers such that

$$M = a+bRe^{-r}, \quad m = a+bre^{-R}, \quad R = c+dMe^{-m}, \quad r = c+dme^{-M}. \tag{3.15}$$

Moreover, there exists a ξ, $\min\{r, R\} \le \xi \le \max\{r, R\}$ such that

$$Re^R - re^r = (1 + \xi)e^{\xi}(R - r). \tag{3.16}$$

Then from (3.15) and (3.16) and since $r, R \ge c$ we get

$$\begin{aligned} M - m \quad &= b(Re^{-r} - re^{-R}) = be^{-r-R}(Re^R - re^r) = \\ &be^{-r-R+\xi}(1 + \xi)(R - r) \le be^{-c}(1 + \xi)(R - r). \end{aligned} \tag{3.17}$$

Moreover, from (3.15), we obtain

$$r = c + dae^{-M} + bdre^{-R}e^{-M}, \quad R = c + dae^{-m} + bdRe^{-r}e^{-m}$$

which implies that

$$r = \frac{c + ade^{-M}}{1 - bde^{-R-M}} \leq \frac{c + a\zeta_2}{1-p}, \quad R = \frac{c + ade^{-m}}{1 - bde^{-r-m}} \leq \frac{c + a\zeta_2}{1-p}. \tag{3.18}$$

Furthermore since $\xi \leq \max\{r, R\}$ we have either $\xi \leq r$ or $\xi \leq R$. Then from (3.18) it follows that

$$\xi \leq \frac{c + a\zeta_2}{1-p}. \tag{3.19}$$

Thus, from (3.17) and (3.19), we get

$$M - m \leq \frac{\zeta_1(1 - p + c + a\zeta_2)}{1-p}(R - r). \tag{3.20}$$

Similarly, we obtain

$$R - r \leq \frac{\zeta_2(1 - p + a + c\zeta_1)}{1-p}(M - m). \tag{3.21}$$

So, from (3.20) and (3.21) we have

$$M - m \leq \mu(M - m). \tag{3.22}$$

Then, from (**??**), (3.15) and (3.22) it is obvious that $M = m$ and $R = r$. Therefore, from Lemma 3.1, System (**??**) has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of System (**??**) tends to $(\overline{x}, \overline{y})$. This completes the proof of the proposition.

Proposition 3.2 *Let a, b, c, d be positive numbers. Assume that (3.2), (3.3) and (3.4) hold. Suppose also that*

$$\kappa = \frac{c\theta_1 + a\theta_2 + (a+c)p}{1-p} + \frac{p(a + c\theta_1)(c + a\theta_2)}{(1-p)^2} + p < 1 \tag{3.23}$$

Then the unique positive equilibrium $(\bar{x}, \bar{y})$ of (1.1) is globally asymptotically stable.

Proof First we will prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable. The linearized system of (1.1) about $(\bar{x}, \bar{y})$ is the following:

$$x_{n+1} = -b\bar{y}e^{-\bar{x}}x_n + be^{-\bar{x}}y_{n-1}, \quad y_{n+1} = de^{-\bar{y}}x_{n-1} - d\bar{x}e^{-\bar{y}}y_n \ . \qquad (3.24)$$

which is equivalent to the system

$$w_{n+1} = Aw_n, \quad A = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \gamma & \delta & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$\alpha = -b\bar{y}e^{-\bar{x}}, \quad \beta = be^{-\bar{x}} \quad \gamma = -d\bar{x}e^{-\bar{y}}, \quad \delta = de^{-\bar{y}} \ .$$

Then the characteristic equation of A is

$$\lambda^4 - (\alpha + \gamma)\lambda^3 + \alpha\gamma\lambda^2 - \beta\delta = 0. \qquad (3.25)$$

Using Remark of 1.3.1 of [3] all the roots of Equation (3.25) are of modulus less than 1, if

$$|\alpha| + |\gamma| + |\alpha\gamma| + |\beta\delta| < 1. \qquad (3.26)$$

Since $(\bar{x}, \bar{y})$ is an equilibrium for (1.1) we have that

$$\bar{x} = a + b(c + d\bar{x}e^{-\bar{y}})e^{-\bar{x}}, \quad \bar{y} = c + d(a + b\bar{y}e^{-\bar{x}})e^{-\bar{y}}.$$

Hence

$$\bar{x} = \frac{a + bce^{-\bar{x}}}{1 - bde^{-\bar{x}-\bar{y}}} \leq \frac{a + c\theta_1}{1 - p}, \quad \bar{y} = \frac{c + ade^{-\bar{y}}}{1 - bde^{-\bar{x}-\bar{y}}} \leq \frac{c + a\theta_2}{1 - p}. \qquad (3.27)$$

Then, since $\bar{x} \geq a$, $\bar{y} \geq c$, from (3.23) and (3.27), we get

$$|\alpha| + |\gamma| + |\alpha\gamma| + |\beta\delta| =$$

$$b\bar{y}e^{-\bar{x}} + d\bar{x}e^{-\bar{y}} + bd\bar{x}\bar{y}e^{-\bar{x}-\bar{y}} + bde^{-\bar{x}-\bar{y}} \leq \kappa < 1$$

and so (3.26) is satisfied. Therefore $(\bar{x}, \bar{y})$ is locally asymptotically stable. So, since from Statement (i) of Proposition 3.1, every positive solution of (1.1) tends to the unique positive equilibrium of (1.1), the proof is completed.

4 Unbounded solutions

In this section we find unbounded solutions for the System (1.1).

Proposition 4.1 *Suppose that*

$$\theta_1 > 1, \quad \theta_2 > 1, \tag{4.1}$$

where θ_1, θ_2 are defined in (3.2). Then there exist unbounded solutions (x_n, y_n) of (1.1) such that one of the following relations hold:

$$\lim_{n\to\infty} x_{2n+1} = \infty, \quad \lim_{n\to\infty} x_{2n} = a, \quad \lim_{n\to\infty} y_{2n+1} = \infty, \quad \lim_{n\to\infty} y_{2n} = c \tag{4.2}$$

$$\lim_{n\to\infty} x_{2n+1} = a, \quad \lim_{n\to\infty} x_{2n} = \infty, \quad \lim_{n\to\infty} y_{2n+1} = c, \quad \lim_{n\to\infty} y_{2n} = \infty. \tag{4.3}$$

Proof First we find solutions of (1.1) such that (4.2) are satisfied. Let (x_n, y_n) be a solution of (**??**) with initial values x_{-1}, x_0, y_{-1}, y_0 which satisfy

$$x_0 < m_1, \; x_{-1} > M, \; y_0 < m_2, \; y_{-1} > M \tag{4.4}$$

where

$$m_1 = \ln b, \; m_2 = \ln d, \; M = \max\Big\{\ln\Big(\frac{dm_1}{m_2 - c}\Big), \;\; \ln\Big(\frac{bm_2}{m_1 - a}\Big)\Big\}.$$

Then using (1.1) and (4.4) we have

$$x_1 = a + by_{-1}e^{-x_0} > a + by_{-1}e^{-m_1} = a + y_{-1},$$

$$y_1 = c + dx_{-1}e^{-y_0} > c + dx_{-1}e^{-m_2} = c + x_{-1},$$

$$x_2 = a + by_0 e^{-x_1} < a + bm_2 e^{-y_{-1}} < a + bm_2\Big(\frac{m_1 - a}{bm_2}\Big) = m_1,$$

$$y_2 = c + dx_0 e^{-y_1} < c + dm_1 e^{-x_{-1}} < c + dm_1\Big(\frac{m_2 - c}{dm_1}\Big) = m_2,$$

and working inductively we obtain

$$x_{2n+1} > a + y_{2n-1}, \quad y_{2n+1} > c + x_{2n-1}, \quad x_{2n} < m_1, \quad y_{2n} < m_2, \quad n = 1, 2, \ldots \tag{4.5}$$

Using (1.1) and (4.5) we can prove that (4.2) hold.

Let now (x_n, y_n) be a solution such that

$$x_{-1} < m_1, \; x_0 > M, \; y_{-1} < m_2, \; y_0 > M.$$

Then arguing as above we can show that relations (4.3) are satisfied. This completes the proof of the proposition.

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A SURVEY ON THE OSCILLATION OF DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH SEVERAL OSCILLATING COEFFICIENTS

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Abstract. Consider the retarded difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0$$

and the (dual) advanced difference equation

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \, n \in \mathbb{N},$$

which represent the discrete analogues of the retarded dfifferential equation

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \geq 0,$$

and the (dual) advanced differential equation

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0, \quad t \geq 1,$$

A survey on the oscillation of all solutions to these equations is presented in the case of several oscillating coefficients

Keywords: Oscillating coefficients, retarded argument, advanced argument, oscillatory solutions, nonoscillatory solutions.

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1. INTRODUCTION

Consider the retarded difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E$_\mathrm{R}$}$$

and the (dual) advanced difference equation

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \tag{E$_\mathrm{A}$}$$

where $m \in \mathbb{N}$, $\{p_i(n)\}$, $1 \leq i \leq m$, are oscillating sequences of real numbers, $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$, $1 \leq i \leq m$, are sequences of integers such that

$$\tau_i(n) \leq n-1 \quad \forall n \geq 0, \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \leq i \leq m, \tag{1.1}$$

$\{\sigma_i(n)\}_{n\in\mathbb{N}}$, $1 \leq i \leq m$, are sequences of integers such that

$$\sigma_i(n) \geq n+1, \quad n \in \mathbb{N}, \quad 1 \leq i \leq m, \tag{1.2}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ denotes the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

In the last few decades, the oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study the oscillation of ($\mathrm{E_R}$) [($\mathrm{E_A}$)], since the difference $\Delta x(n)$ [$\nabla x(n)$] of any nonoscillatory solution of ($\mathrm{E_R}$) [($\mathrm{E_A}$)] is always oscillatory. Therefore, the results on oscillation of difference and differential equations with oscillating coefficients are relatively scarce. Thus, a small number of paper are dealing with this case. See, for example, [1−6, 8, 9, 11−17] and the references cited therein.

Set

$$w = -\min_{\substack{n\geq 0\\ 1\leq i\leq m}} \tau_i(n).$$

(Clearly, $w \in \mathbb{N}$.)

By a *solution* of the retarded difference equation ($\mathrm{E_R}$), we mean a sequence of real numbers $\{x(n)\}_{n\geq -w}$ which satisfies ($\mathrm{E_R}$) for all $n \in \mathbb{N}_0$. It is clear that, for each choice of real numbers c_{-w}, $c_{-w+1}, \ldots,$ c_{-1}, c_0, there exists a unique solution $\{x(n)\}_{n\geq -w}$ of ($\mathrm{E_R}$) which satisfies the initial conditions $x(-w) = c_{-w}$, $x(-w+1) = c_{-w+1}, \ldots,$ $x(-1) = c_{-1}$, $x(0) = c_0$. By a *solution* of the advanced difference equation ($\mathrm{E_A}$), we mean a sequence of real numbers $\{x(n)\}_{n\in\mathbb{N}_0}$ which satisfies ($\mathrm{E_A}$) for all $n \in \mathbb{N}$.

A solution $\{x(n)\}_{n\geq -w}$ [$\{x(n)\}_{n\in\mathbb{N}_0}$] of the difference equation ($\mathrm{E_R}$) [($\mathrm{E_A}$)] is called *oscillatory,* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory.*

Strong interest in Eq.($\mathrm{E_R}$) with several variable retarded arguments is motivated by the fact that it represents a discrete analogue of the differential equation with several variable retarded arguments (see [6] and the references cited therein)

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0, \quad t \geq 0, \tag{1.3}$$

where, for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[0, \infty)$, and τ_i is a continuous real-valued function on $[0, \infty)$ such that

$$\tau_i(t) \leq t, \quad t \geq 0, \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty,$$

while, Eq.($\mathrm{E_A}$) represents a discrete analogue of the advanced differential equation (see [6] and the references cited therein)

$$x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0, \quad t \geq 1, \tag{1.4}$$

where, for every $i \in \{1, \ldots, m\}$, p_i is an oscillating continuous real-valued function in the interval $[1, \infty)$, and σ_i is a continuous real-valued function on $[1, \infty)$ such that

$$\sigma_i(t) \geq t, \quad t \geq 1.$$

For $m = 1$, equations ($\mathrm{E_R}$) and ($\mathrm{E_A}$) take the forms

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n \in \mathbb{N}_0, \tag{$\mathrm{E_{R1}}$}$$

and

$$\nabla x(n) - p(n)x(\sigma(n)) = 0, \quad n \in \mathbb{N}, \tag{$\mathrm{E_{A1}}$}$$

respectively. These equations represent the discrete analogues of the differential equations (see [6] and the references cited therein)

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq 0, \tag{1.5}$$

and

$$x'(t) - p(t)x(\sigma(t)) = 0, \quad t \geq 1, \tag{1.6}$$

respectively, where $\tau(t) \leq t$, $\sigma(t) \geq t$, and the coefficient p is a continuous function which is allowed to oscillate.

If $\tau_i(n) = n - k_i$ and $\sigma_i(n) = n + k_i$, where $k_i \in \mathbb{N}$, $1 \leq i \leq m$, then equations ($\mathrm{E_R}$) and ($\mathrm{E_A}$) take the forms

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(n - k_i) = 0, \quad n \in \mathbb{N}_0, \tag{$\mathrm{E_R'}$}$$

and

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(n + k_i) = 0, \quad n \in \mathbb{N}, \tag{$\mathrm{E_A'}$}$$

respectively.

For $m = 1$, equations ($\mathrm{E_R'}$) and ($\mathrm{E_A'}$) take the forms

$$\Delta x(n) + p(n)x(n - k) = 0, \quad n \in \mathbb{N}_0, \tag{$\mathrm{E_{R1}'}$}$$

and

$$\nabla x(n) - p(n)x(n + k) = 0, \quad n \in \mathbb{N}, \tag{$\mathrm{E_{A1}'}$}$$

respectively. These equations represent the discrete analogues of the differential equations (see [9, 10] and the references cited therein)

$$x'(t) + p(t)x(t - \tau) = 0, \quad t \geq 0, \tag{1.7}$$

and

$$x'(t) - p(t)x(t + \sigma) = 0, \quad t \geq 1, \tag{1.8}$$

respectively, where τ, σ are positive constants and the coefficient p is a continuous function which is allowed to oscillate.

In this paper, a survey on the ocillation of all solutions to the above equations is presented especially in the case that the coefficients oscillate.

2. OSCILLATION CRITERIA FOR DIFFERENTIAL EQUATIONS

In 1982, Ladas, Sficas and Stavroulakis [9] established the following theorems.

Theorem 2.1 ([9, Theorem 2.1])**.** *Assume that $p(t) > 0$ (at least) on a sequence of disjoint intervals $\bigcup_{n\in\mathbb{N}} (\xi(n), t(n))$ with $t(n) - \xi(n) = 2\tau$. If*

$$\limsup_{n\to\infty} \int_{t(n)-\tau}^{t(n)} p(s)\, ds > 1,$$

then all solutions of (1.7) oscillate.

Theorem 2.2 ([9, Theorem 2.1]). *Assume that* $p(t) > 0$ *(at least) on a sequence of disjoint intervals* $\bigcup_{n\in\mathbb{N}} (\xi(n), t(n))$ *with* $t(n) - \xi(n) = 2\sigma$. *If*

$$\limsup_{n\to\infty} \int_{\xi(n)}^{\xi(n)+\sigma} p(s)\ ds > 1,$$

then all solutions of (1.8) *oscillate.*

In 1984, Fukagai and Kusano [6] extended the above results to the differential equations (1.5) and (1.6) as follows.

Theorem 2.3 ([6, Theorem 4(i)]). *Assume that* $\tau(t) \le t$ *for* $t \ge 0$. *If there exists a sequence of numbers* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [\tau(\tau(t(n))), t(n)]$ *are disjoint,*

$$p(t) \ge 0 \qquad \text{for } t \in \bigcup_{n\in\mathbb{N}} [\tau(\tau(t(n))), t(n)],$$

and

$$\int_{\tau(t(n))}^{t(n)} p(s)\ ds \ge 1,$$

then all solutions of (1.5) *oscillate.*

Theorem 2.4 ([6, Theorem 4(ii)]). *Assume that* $\sigma(t) \ge t$ *for* $t \ge 1$. *If there exists a sequence of numbers* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [t(n), \sigma(\sigma(t(n)))]$ *are disjoint,*

$$p(t) \ge 0 \qquad \text{for } t \in \bigcup_{n\in\mathbb{N}} [t(n), \sigma(\sigma(t(n)))],$$

and

$$\int_{t(n)}^{\sigma(t(n))} p(s)\ ds \ge 1,$$

then all solutions of (1.6) *oscillate.*

In the same paper [6], the authors also studied, the oscillating coefficients case and established the following theorems.

Theorem 2.5 ([6, Theorem 3′ (i)]). *Assume* (1.4) *and that there is a continuous nondecreasing function* $\tau^*(t)$ *such that* $\tau_i(t) \le \tau^*(t) \le t$ *for* $t \ge 0$, $1 \le i \le m$. *Supose moerover that there is a sequence* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [(\tau^*)^n (t(n)), t(n)]$ *are disjoint and*

$$p_i(t) \ge 0 \qquad \text{for all } t \in \bigcup_{n\in\mathbb{N}} [(\tau^*)^n (t(n)), t(n)], \qquad 1 \le i \le m.$$

If there is a constant c *such that*

$$\int_{\tau^*(t)}^{t} \sum_{i=1}^{m} p_i(s)\ ds > c > \frac{1}{e} \qquad \text{for all } t \in \bigcup_{n\in\mathbb{N}} \left[(\tau^*)^{n-1} (t(n)), t(n)\right],$$

then all solutions of (1.3) *oscillate.*

Theorem 2.6 ([6, Theorem 3′ (ii)]). *Assume* (1.6) *and that there is a continuous nondecreasing function* $\sigma_*(t)$ *such that* $t \leq \sigma_*(t) \leq \sigma_i(t)$ *for* $t \geq 0$, $1 \leq i \leq m$. *Supose moerover that there is a sequence* $\{t(n)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t(n) = \infty$, *the intervals* $\bigcup_{n\in\mathbb{N}} [t(n), (\sigma_*)^n (t(n))]$ *are disjoint and*

$$p_i(t) \geq 0 \qquad \text{for all } t \in \bigcup_{n\in\mathbb{N}} [t(n), (\sigma_*)^n (t(n))], \qquad 1 \leq i \leq m.$$

If there is a constant c *such that*

$$\int_t^{\sigma_*(t)} \sum_{i=1}^m p_i(s)\, ds > c > \frac{1}{e} \qquad \text{for all } t \in \bigcup_{n\in\mathbb{N}} \left[t(n), (\sigma_*)^{n-1} (t(n))\right],$$

then all solutions of (1.4) *oscillate.*

3. OSCILLATION CRITERIA FOR DIFFERENCE EQUATIONS

In 1992, Qian, Ladas and Yan [11] studied the difference equation $(\mathrm{E}'_{\mathrm{R}1})$ with constant retarded argument and established the following theorem.

Theorem 3.1 ([12, Theorem 1]). *Assume that there exist two sequences* $\{r(m)\}$ *and* $\{s(m)\}$ *of positive integers such that* $s(m) - r(m) \geq 2k$ *for* $m \in \mathbb{N}$. *If*

$$p(n) \geq 0 \quad \text{for} \quad n \in \bigcup_{m\in\mathbb{N}} \{r(m), r(m)+1, \ldots, s(m)\}$$

and

$$\limsup_{m\to\infty} \sum_{i=s(m)-k}^{s(m)} p^+(i) > 1,$$

where $p^+(n) = \max\{p(n), 0\}$, *then all solutions of* $(\mathrm{E}'_{\mathrm{R}1})$ *oscillate.*

For equations $(\mathrm{E}_{\mathrm{R}})$ and $(\mathrm{E}_{\mathrm{A}})$ with oscillating coefficients, in 2014 and in 2015, Bohner, Chatzarakis and Stavroulakis [2, 3] established the following theorems.

Theorem 3.2 ([2, Theorem 2.4]). *Assume* (1.1) *and that the sequences* $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$ *are increasing for all* $i \in \{1, \ldots, m\}$. *Suppose also that for each* $i \in \{1, \ldots, m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$ *and*

$$p_k(n) \geq 0, \;\; n \in A = \bigcap_{i=1}^m \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m$$

where

$$\tau(n) = \max_{1\leq i\leq m} \tau_i(n), \qquad n \in \mathbb{N}_0.$$

If, moreover

$$\limsup_{j\to\infty} \sum_{i=1}^m \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1,$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E}_{\mathrm{R}})$ *oscillate.*

Theorem 3.3 ([2, Theorem 3.4]). *Assume* (1.2) and *that the sequences* $\{\sigma_i(n)\}_{n\in\mathbb{N}}$ *are increasing for all* $i \in \{1,\ldots,m\}$. *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$ *and*

$$p_k(n) \geq 0, \;\; n \in B = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m,$$

where

$$\sigma(n) = \min_{1\leq i\leq m} \sigma_i(n), \qquad n \in \mathbb{N}.$$

If, moreover

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1,$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_A})$ *oscillate.*

Theorem 3.4 ([3, Theorem 2.1]). *Assume* (1.1) *and that the sequences* $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$ *are increasing for all* $i \in \{1,\ldots,m\}$. *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0, \;\; n \in C = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau_i(\tau_i(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0, \;\; \text{for all } n \in C.$$

If, moreover

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau_i(n_i(j))}^{n_i(j)-1} p_i(q) > \frac{1}{e},$$

then all solutions of $(\mathrm{E_R})$ *oscillate.*

Theorem 3.5 ([3, Theorem 3.1]). *Assume* (1.2) *and that the sequences* $\{\sigma_i(n)\}_{n\in\mathbb{N}}$ *are increasing for all* $i \in \{1,\ldots,m\}$. *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0, \;\; n \in D = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma_i(\sigma_i(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \;\; \text{for all } n \in D.$$

If, moreover

$$\liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n_i(j)+1}^{\sigma_i(n_i(j))} p_i(q) > \frac{1}{e},$$

then all solutions of $(\mathrm{E_A})$ *oscillate.*

In 2014, Berezansky et al. [1] and in 2015, Chatzarakis et al. [4] astablished the following theorems.

Theorem 3.6 ([1, Theorem 8 and 4, Theorem 2.1])**.** *Assume that* (1.1) *holds, the sequences* $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$ *are increasing for all* $i \in \{1,\ldots,m\}$ *and the sequence* τ *is defined by* (1.4)*. Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0, \;\; n \in A = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m.$$

Set

$$\alpha := \liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)-1} p_i(q),$$

where $n(j) = \min\{n_i(j) : 1 \leq i \leq m\}$.

If $0 < \alpha \leq 1/2$, *and*

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1 - \frac{\alpha^2}{4(1-\alpha)},$$

or

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)} p_i(q) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right),$$

then all solutions of $(\mathrm{E_R})$ *oscillate.*

Theorem 3.7 ([1, Theorem 9 and 4, Theorem 3.1])**.** *Assume* (1.2) *holds, the sequences* $\{\sigma_i(n)\}_{n\in\mathbb{N}}$ *are increasing for all* $i \in \{1,\ldots,m\}$ *and the swquence* σ *is defined by* (1.7)*. Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0, \;\; n \in B = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \, \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m.$$

Set

$$\alpha := \liminf_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n(j)+1}^{\sigma(n(j))} p_i(q),$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$.

If $0 < \alpha \leq 1/2$, *and*

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1 - \frac{\alpha^2}{4(1-\alpha)},$$

or

$$\limsup_{j\to\infty} \sum_{i=1}^{m} \sum_{q=n(j)}^{\sigma(n(j))} p_i(q) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1-2\alpha}\right),$$

then all solutions of $(\mathrm{E_A})$ *oscillate.*

In 2015, Chatzarakis et al [5], established the following theorems.

Theorem 3.8 ([5]) *Assume that* (1.1) *holds, the sequences* $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$ *are increasing for all* $i \in \{1,\ldots,m\}$ *and the sequence* τ *is defined by* (1.4). *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) > 0, \;\; n \in A = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \le k \le m$$

with

$$\liminf_{n\to\infty} \{p_k(n) : n \in A\} > 0, \quad 1 \le k \le m.$$

If, moreover

$$\left[\prod_{i=1}^{m} \left(\sum_{\ell=1}^{m} \liminf_{j\to\infty} \sum_{k=\tau_\ell(n(j))}^{n(j)-1} p_i(k) \right) \right]^{1/m} > \frac{1}{e},$$

where $n(j) = \min\{n_i(j) : 1 \le i \le m\}$, *then all solutions of* $(\mathrm{E_R})$ *oscillate.*

Theorem 3.9 ([5]) *Assume that* (1.2) *holds, the sequences* $\{\sigma_i(n)\}_{n\in\mathbb{N}}$ *are increasing for all* $i \in \{1,\ldots,m\}$ *and the sequence* σ *is defined by* (1.7). *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \ge 0, \;\; n \in B = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \le k \le m,$$

with

$$\liminf_{n\to\infty} \{p_k(n) : n \in B\} > 0, \quad 1 \le k \le m.$$

If, moreover

$$\left[\prod_{i=1}^{m} \left(\sum_{\ell=1}^{m} \liminf_{j\to\infty} \sum_{k=n(j)+1}^{\sigma_\ell(n(j))} p_i(k) \right) \right]^{1/m} > \frac{1}{e},$$

where $n(j) = \max\{n_i(j) : 1 \le i \le m\}$, *then all solutions of* $(\mathrm{E_A})$ *oscillate.*

Theorem 3.10 ([5]) *Assume that* (1.1) *holds, the sequences* $\{\tau_i(n)\}_{n\in\mathbb{N}_0}$ *are increasing for all* $i \in \{1,\ldots,m\}$ *and the sequence* τ *is defined by* (1.4). *Suppose also that for each* $i \in \{1,\ldots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) > 0, \;\; n \in A = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [\tau(\tau(n_i(j))), n_i(j)] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \le k \le m$$

with

$$\liminf_{n\to\infty} \{p_k(n) : n \in A\} > 0, \quad 1 \le k \le m.$$

If, moreover

$$\frac{1}{m} \sum_{i=1}^{m} \liminf_{j\to\infty} \sum_{k=\tau_i(n(j))}^{n(j)-1} p_i(k) + \frac{2}{m} \sum_{\substack{i<\ell \\ i,\ell=1}}^{m} \left(\liminf_{j\to\infty} \sum_{k=\tau_\ell(n(j))}^{n(j)-1} p_i(k) \times \liminf_{j\to\infty} \sum_{k=\tau_i(n(j))}^{n(j)-1} p_\ell(k) \right)^{1/2} > \frac{1}{e},$$

where $n(j) = \min\{n_i(j) : 1 \le i \le m\}$, *then all solutions of* $(\mathrm{E_R})$ *oscillate.*

Theorem 3.11 ([5]) *Assume that* (1.2) *holds, the sequences* $\{\sigma_i(n)\}_{n\in\mathbb{N}}$ *are increasing for all* $i \in \{1,\dots,m\}$ *and the sequence* σ *is defined by* (1.7). *Suppose also that for each* $i \in \{1,\dots,m\}$ *there exists a sequence* $\{n_i(j)\}_{j\in\mathbb{N}}$ *such that* $\lim_{j\to\infty} n_i(j) = \infty$,

$$p_k(n) \geq 0, \;\; n \in B = \bigcap_{i=1}^{m} \left\{ \bigcup_{j\in\mathbb{N}} [n_i(j), \sigma(\sigma(n_i(j)))] \cap \mathbb{N} \right\} \neq \emptyset, \; 1 \leq k \leq m,$$

with

$$\liminf_{n\to\infty} \{p_k(n) : n \in B\} > 0, \quad 1 \leq k \leq m.$$

If, moreover

$$\frac{1}{m}\sum_{i=1}^{m} \liminf_{j\to\infty} \sum_{k=n(j)+1}^{\sigma_i(n(j))} p_i(k) + \frac{2}{m} \sum_{\substack{i<\ell \\ i,\ell=1}}^{m} \left(\liminf_{j\to\infty} \sum_{k=n(j)+1}^{\sigma_\ell(n(j))} p_i(k) \times \liminf_{j\to\infty} \sum_{k=n(j)+1}^{\sigma_i(n(j))} p_\ell(k) \right)^{1/2} > \frac{1}{e},$$

where $n(j) = \max\{n_i(j) : 1 \leq i \leq m\}$, *then all solutions of* $(\mathrm{E_A})$ *oscillate.*

A slight modification in the proofs of Theorem 2.7, 2.8. 2.9 and 2.10 leads to the following results about difference inequalities.

Theorem 3.12 ([5]) *Assume that all conditions of Theorem* 2.1 *or* 2.9 *hold. Then*

(i) *the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \qquad n \in \mathbb{N}_0$$

has no eventually positive solutions;

(ii) *the difference inequality*

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \geq 0, \qquad n \in \mathbb{N}_0$$

has no eventually negative solutions.

Theorem 3.13 ([5]) *Assume that all conditions of Theorem* 2.8 *or* 2.10 *hold. Then*

(i) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \geq 0, \qquad n \in \mathbb{N}$$

has no eventually positive solutions;

(ii) *the difference inequality*

$$\nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) \leq 0, \qquad n \in \mathbb{N}$$

has no eventually negative solutions.

3.1. **Special cases.** In the case where p_i, $i = 1, 2, \ldots, m$, are oscillating real constants and τ_i are constant retarded arguments of the form $\tau_i(n) = n - k_i$, [σ_i are constant advanced arguments of the form $\sigma_i(n) = n + k_i$], $k_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$, equation ($\mathrm{E_R}$) [($\mathrm{E_A}$)] takes the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i x(n - k_i) = 0,\ n \in \mathbb{N}_0 \quad \left[\nabla x(n) - \sum_{i=1}^{m} p_i x(n + k_i) = 0,\ n \in \mathbb{N}\right]. \tag{E}$$

For this equation, as a consequence of Theorems 2.1 [2.8] and 2.9 [2.10], we have the following corollary:

Corollary 3.1 ([5] cf. [10]) *Assume that*

$$\left[\prod_{i=1}^{m} p_i\right]^{1/m} \left(\sum_{i=1}^{m} k_i\right) > \frac{1}{e},$$

or

$$\frac{1}{m}\left(\sum_{i=1}^{m} \sqrt{p_i k_i}\right)^2 > \frac{1}{e}.$$

Then all solutions of (E) *oscillate.*

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On the formation of freak waves in a wave train

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ABSTRACT

A novel theoretical approach is applied to investigate the process of the formation and transformation of freak waves in a wave train. A semi-analytical nonlinear solution is derived to describe the propagation and evolution of wave components in a wave train and to investigate the formation and physics of freak waves. The results show that the nonlinear transformation of wave components in a wave train leads to the formation of freak waves. The analysis indicates that the interaction of wave components in a wave train is one of the potential sources of the formation of freak waves.

KEY WORDS: Freak waves; evolution of nonlinear waves; wave instability; semi-analytical solution

INTRODUCTION

Freak waves are probably the most dangerous type of extreme waves. They are believed to be one of the causes of the failure of coastal and offshore structures as well as ship accidents. The mechanisms of the formation and physics of freak waves are still not fully recognised. A proper description of freak wave phenomenon is indispensable for the studies of the attack of extreme waves on maritime structures or their impact on ships.

In the present study, a novel theoretical approach is applied to investigate the process of the formation and evolution of freak waves in a wave train. A theoretical model is described in Section 2. Results are presented in Section 3. Finally, in Section 4 conclusions and recommendations are provided.

THEORY

The formation and transformation of freak waves in a wave train is considered. A right-hand Cartesian coordinate system is selected such that the x axis is horizontal and coincides with an undisturbed free surface and z points vertically upwards. It is assumed that:

- The fluid is inviscid and incompressible.
- The fluid motion is irrotational.
- The bottom is impervious

In accordance with the assumptions, a velocity vector, $\boldsymbol{V}(x, z, t)$, may be computed from a velocity potential Φ (x, z, t):

$$\mathbf{V} = \nabla\Phi(x,z,t) \tag{1}$$

where ∇ $(\cdot)$ is the two-dimensional vector differential operator.

The fluid motion is governed by the continuity equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{2a}$$

and the Bernoulli equation

$$\frac{\partial\Phi}{\partial t} + \frac{1}{\rho}P + gz + \frac{1}{2}\left(\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right) = 0 \tag{2b}$$

where ρ is the fluid mass density, P is the pressure and g is the acceleration due to gravity.
The velocity potential, Φ (x, z, t), satisfies the Laplace equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} = 0; \quad -h \le z \le \eta(x,t) \tag{3a}$$

with the following boundary conditions

$$\frac{\partial\eta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\Phi}{\partial z} = 0, \; z = \eta(x,t) \tag{3b}$$

$$\frac{\partial\Phi}{\partial t} + g\eta + \frac{1}{2}\left(\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right) = 0, \; z = \eta(x,t) \tag{3c}$$

$$\frac{\partial \Phi}{\partial z} = 0, \quad z = -h \tag{3d}$$

In addition, the velocity potential must satisfy the boundary condition at infinity and the initial condition (Wehausen 1960).

The boundary-value problem, (3), is solved by expanding the free-surface boundary conditions in a Taylor series, and then applying eigenfunction expansions and FFT (Sulisz and Paprota 2004, 2008). Accordingly, the velocity potential, Φ, and the free-surface elevation, η, are sought in the following form

$$\Phi = \Phi_0 + \sum_{n=1} \frac{\cosh \lambda_n (z+h)}{\cosh \lambda_n h} (A_n \cos \lambda_n x + B_n \sin \lambda_n x) \tag{5b}$$

$$\eta = \eta_0 + \sum_{n=1} (a_n \cos \lambda_n x + b_n \sin \lambda_n x) \tag{5a}$$

where

$$\lambda_n = \frac{2\pi(n-1)}{b} \tag{5c}$$

in which η_0 , Φ_0 are known functions related with imposed initial conditions, A_n, B_n and a_n, b_n are coefficients, b is the length of a sector over which the solution is assumed to be periodic.

A time-stepping procedure and FFT are applied to determine the unknown coefficients of the eigenfunction expansions (Sulisz and Paprota 2004, 2008). The derived solution is very efficient. The application of eigenfunction expansions and FFT allows to predict the process of wave propagation and transformation in very large domains.

RESULTS

The derived model was applied to investigate the evolution of wave components in a modulated wave train and the formation of freak waves. The model is applied for N=6 and 8 waves in a modulated wave train segment and carrier waves of amplitude A, wave number k and period T.

The results presented in Fig. 1-6 show that a train of basically sinusoidal waves may drastically change its form within a relatively short distance from its original position. Significant changes of wave profile leads to the formation of freak waves. This process is accompanied by drastic changes of wave spectrum which evolves from a very narrow-banded spectrum to multi-peak spectrum and often retrieves its original shape in a fairly short period of time.

The nonlinear transformation of wave components in a wave train leads to the formation of freak waves. The analysis shows that the interaction of wave components in a wave train is one of the potential sources of the formation of freak waves.

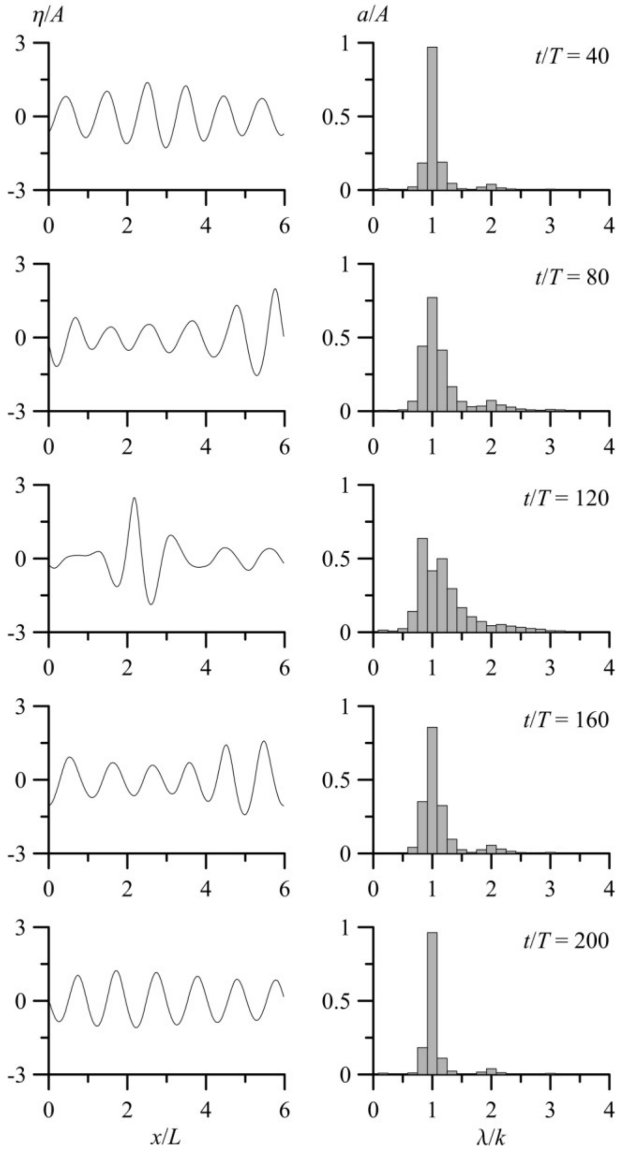

Fig.1a. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.1.

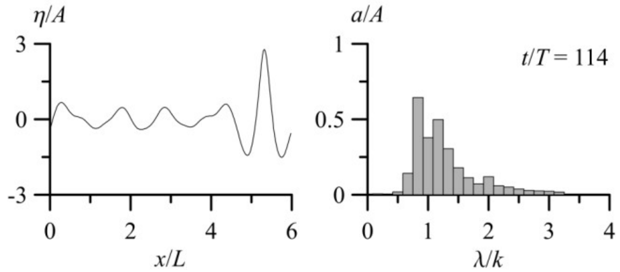

Fig.1b. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.1.

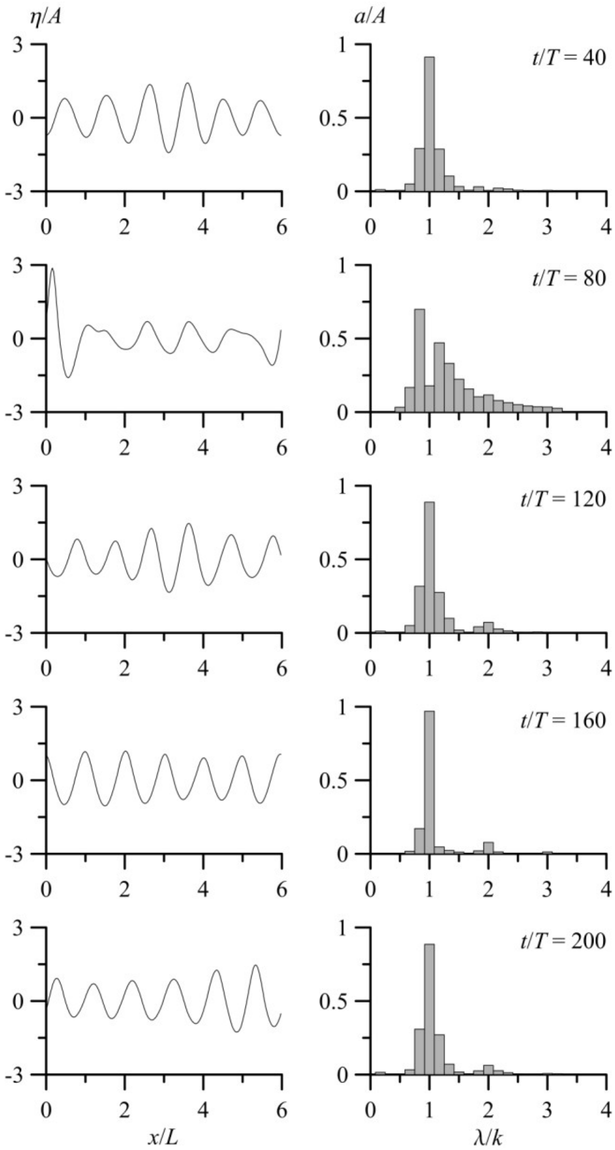

Fig.2a. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.12.

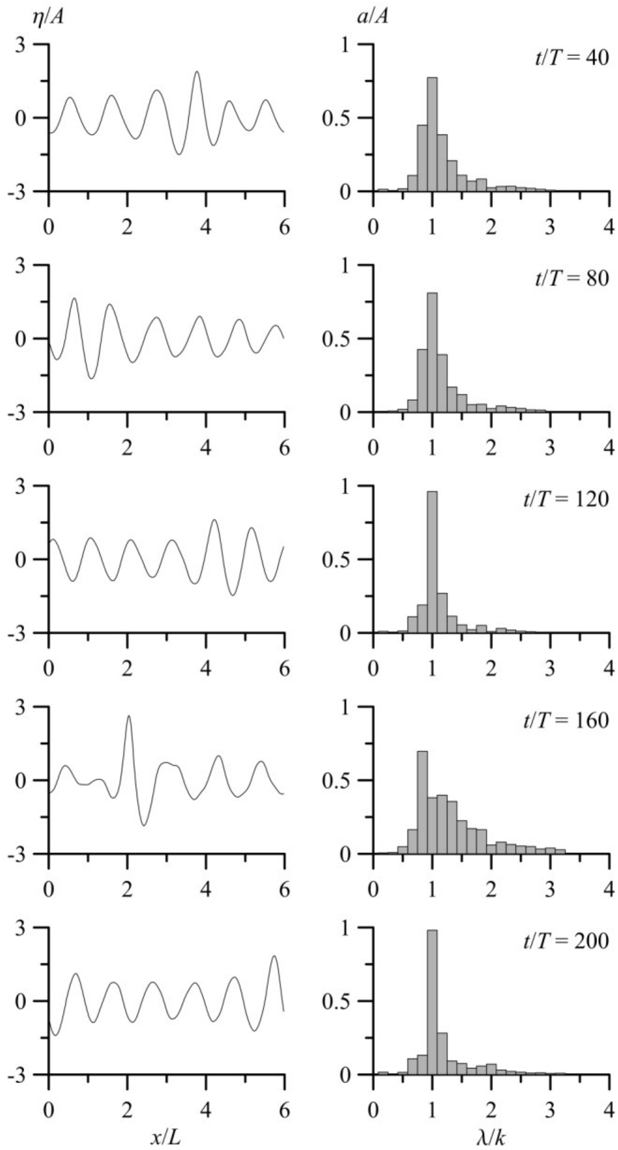

Fig.3a. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.14.

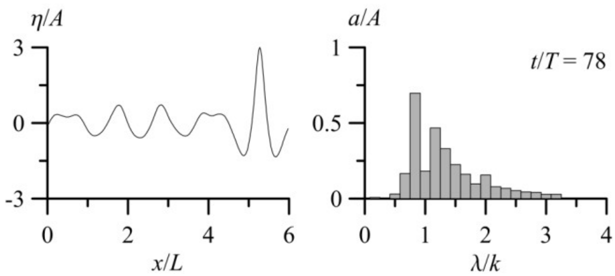

Fig.2b. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.12.

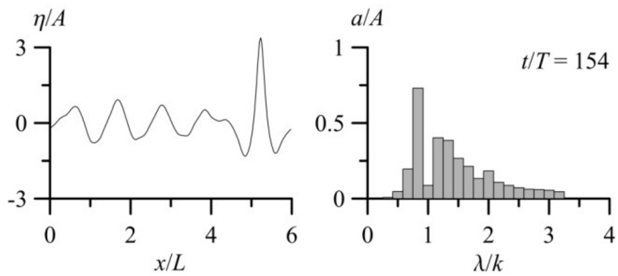

Fig.3. Free-surface elevation and corresponding Fourier amplitudes for N=6, Ak=0.14.

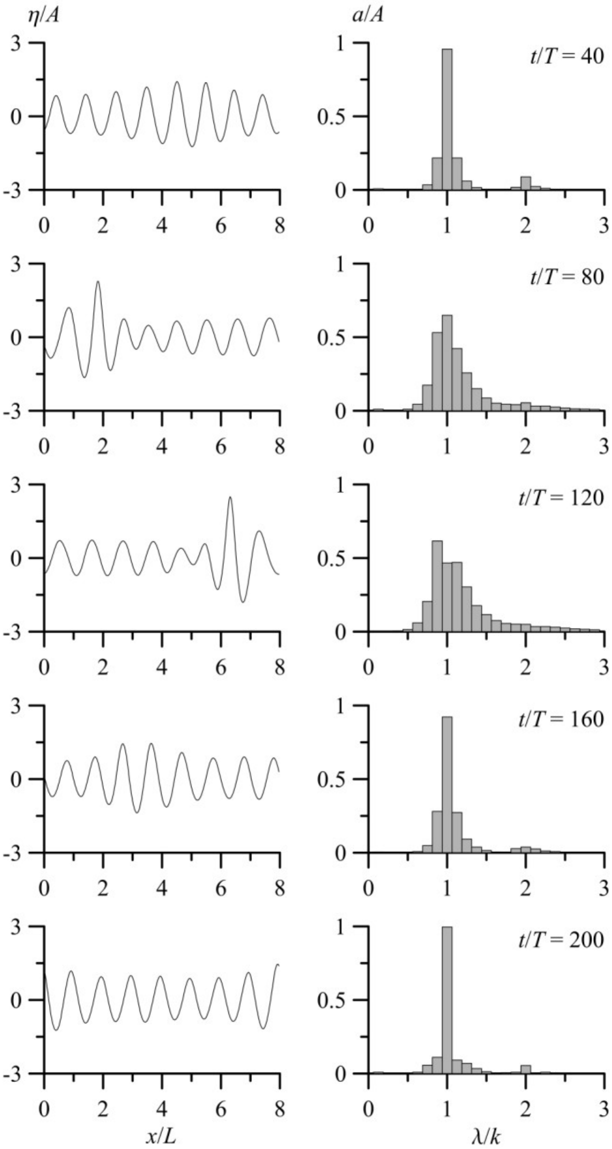

Fig.4a. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.1.

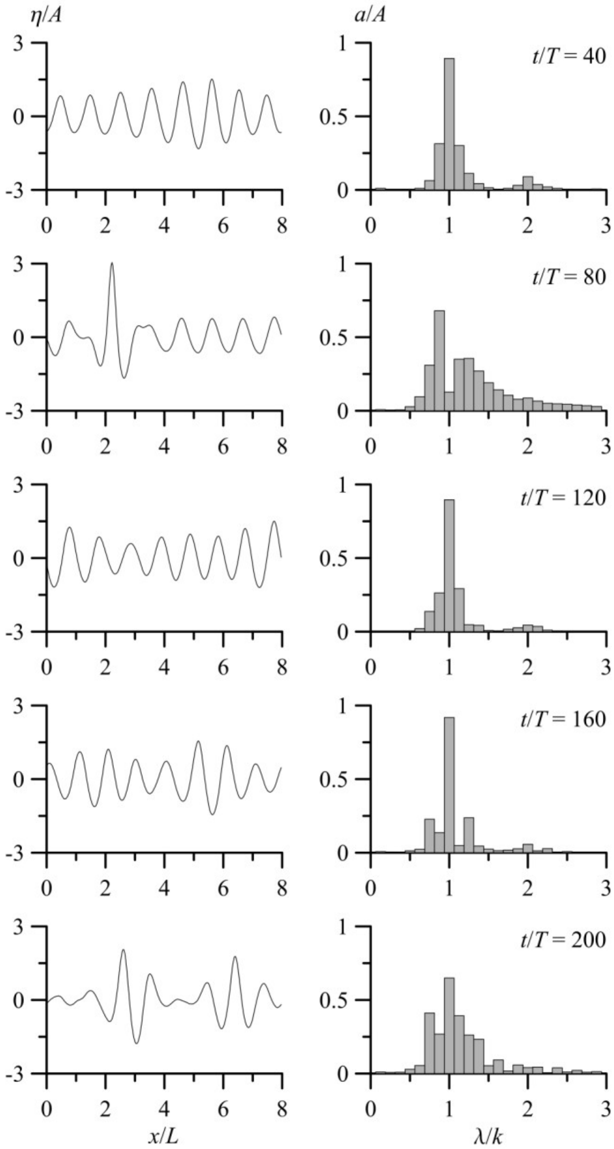

Fig.5a. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.12.

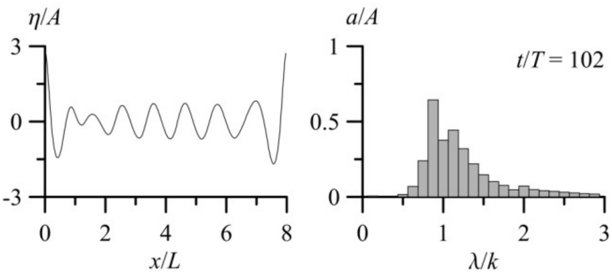

Fig.4b. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.1.

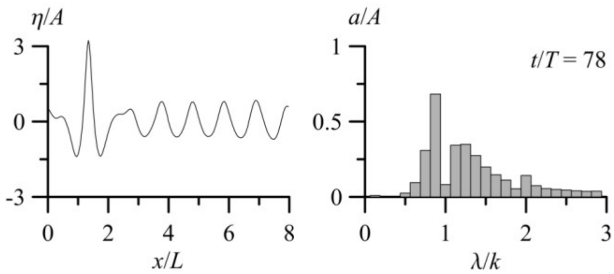

Fig.5b. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.12.

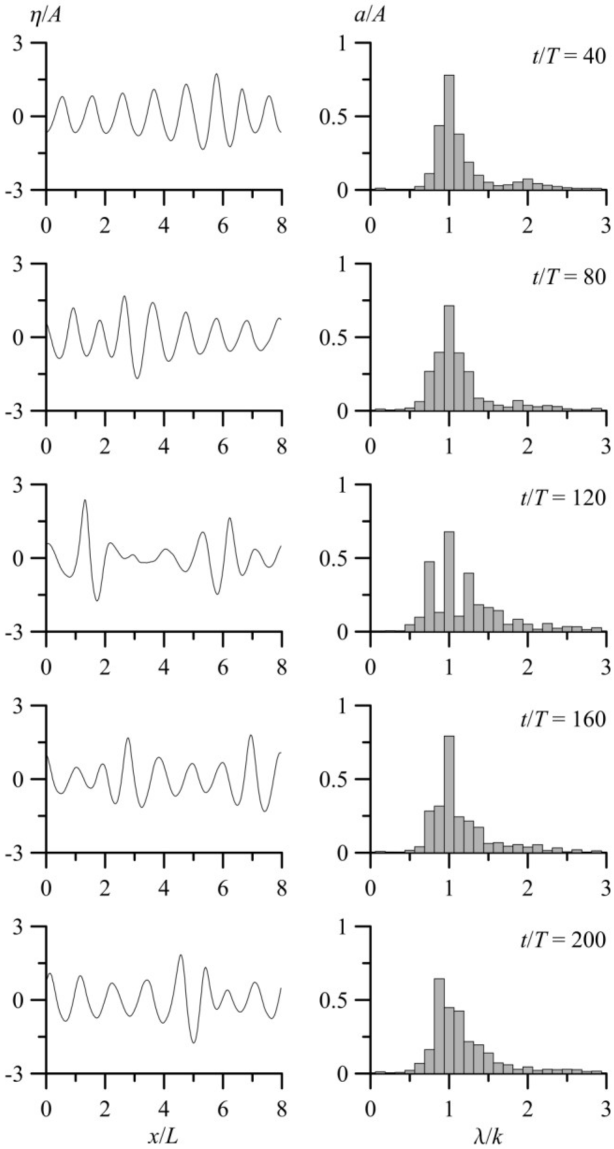

Fig.6a. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.14.

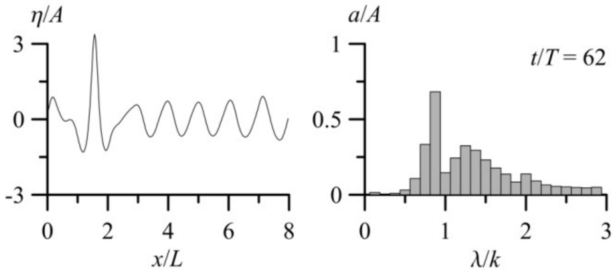

Fig.6b. Free-surface elevation and corresponding Fourier amplitudes for N=8, Ak=0.14.

CONCLUSIONS

A novel theoretical approach is applied to investigate the process of the formation and transformation of freak waves in a wave train. A semi-analytical nonlinear solution is derived to describe the propagation and evolution of wave components in a wave train and to investigate the formation and physics of freak waves.

The results show that a train of basically sinusoidal waves may drastically change its form within a relatively short distance from its original position. A significant evolution of wave profile leads to a formation of freak waves. This process is accompanied by considerable change of wave spectrum which evolves from a very narrow-banded spectrum to a broad-banded or even multi-peak spectrum and often retrieves its original shape in a fairly short period of time.

The analysis indicates that the nonlinear Schrödinger equation or its modifications cannot predict wave evolution with sufficient accuracy. The solution of the nonlinear Schrödinger equation provides insight into the instability of weakly nonlinear waves, however, its practical applicability range is very limited. An analysis shows that the wave evolution is a very sensitive process and solutions derived by applying the Schrödinger equation cannot describe this process with sufficient accuracy

This method has been shown to be an efficient technique in the modeling of the propagation and transformation of nonlinear waves. The derived model is very efficient and allows to obtain a solution for large spatial and time domains.

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Weighted trace Hardy inequalities with sharp remainder terms

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Abstract

This work concerns improvements of certain weighted trace Hardy inequalities, by adding Hardy-Sobolev type remainder terms. We obtain, as special limiting instances, critical Sobolev and Hardy type improvements. We also show that the improvement is optimal in the sense that the remainder terms involve weights with the best possible singularity for an L^p improvement to be valid.

Keywords: Hardy-Sobolev inequalities, Kato inequality, weighted trace Hardy inequality

1 Introduction

Sobolev spaces play a basic role in the study of differential equations especially for their embedding properties. Regarding in particular the Hilbert space $H^1(U)$, on a domain $U \subseteq \mathbb{R}^n$, most of the results assert its embedding into certain weighted or non weighted Lebesgue spaces $L^p(U)$ or $L^p(\partial U)$ for $p \geq 2$.

The standard Hardy inequality on the whole space $\mathbb{R}^n$, $n \geq 3$, asserts that

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \,\mathrm{d}x \leq \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x, \qquad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

It is well known that the constant $(n-2)^2/4$ is the best possible, but it is not achieved in the space of functions for which the right hand side is finite.

By standard reflection arguments we deduce that inequality (1) still holds with the same optimal constant on the upper half space

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, \, x_n > 0\},$$

without the restriction $u = 0$ on the boundary $\partial\mathbb{R}^n_+$, that is

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{|x|^2} \,\mathrm{d}x \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n).$$

Such an inequality does not give any information about the summability properties of the trace of the functions u which do not vanish on the boundary $\partial\mathbb{R}^n_+$. Such summability properties can be deduced from the following trace Hardy inequality (also known in the literature as Kato's inequality)

$$H_n \int_{\partial\mathbb{R}^n_+} \frac{u^2(x', 0)}{|x'|} \,\mathrm{d}x' \leq \int_{\mathbb{R}^n_+} |\nabla u|^2 \,\mathrm{d}x, \qquad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{2}$$

where the constant

$$H_n = 2\frac{\Gamma^2(\frac{n}{4})}{\Gamma^2(\frac{n-2}{4})}$$

is the best possible and is not attained in the space of functions for which the right hand side is finite.

Despite the lack of extremals, it is well known that no extra terms can added in the left hand side of inequalites (1), (2). Passing from the whole space to bounded domains containing the origin, inequalities (1), (2) are still valid with the same optimal constants, as is easily seen by the scaling invariance. In this case several remainder terms have been considered in the left hand side; see for instance [1], [3], [9] and the references therein.

Hardy and Kato inequalities are ones of the well known mathematical formulations of the uncertainty principle in Quantum Mechanics, in the relativistic and non relativistic case respectively. They are of fundamental importance in many branches of mathematical analysis, geometry and mathematical physics. These inequalities have been extensively studied and the relative literature is vast encompassing many generalizations and extensions in several directions. Although classical, are of special interest and many modifications are still forthcoming. In this work we are concerned with the following weighted version of Kato inequality [11]

$$H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x'\le\int\limits_{\mathbb{R}^n_+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\qquad\forall u\in C_0^\infty(\mathbb{R}^n).\tag{3}$$

It can be shown that no L^p improvement is possible (see Appendix), in the sense that there are no positive constant C, exponent $p>0$, and nontrivial potential $V\ge 0$ such that the following inequality is valid

$$C\left(\int\limits_{\mathbb{R}^n_+}V(x)\,|u(x)|^p\,dx\right)^{\frac{2}{p}}\le\int\limits_{\mathbb{R}^n_+}x_n^\alpha\,|\nabla u|^2\,dx-H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,dx',\qquad\forall u\in C_0^\infty(\mathbb{R}^n).\tag{4}$$

Let now U be a bounded domain containing, as an interior point, the origin. If we restrict our attention to the functions $u\in C_0^\infty(U)$, inequality (3), still holds in $\mathbb{R}^n_+\cap U$, with the same optimal constant,

$$H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x'\le\int\limits_{U^+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad\forall u\in C_0^\infty(U).\tag{5}$$

Contrary to the case of the half space $\mathbb{R}^n_+$, inequality (5) can be refined by adding L^p norms of u; see e.g. [6], [18]. In the present work we proceed with our investigation on such inequalities, considering lower bounds for the trace Hardy difference functional

$$\int\limits_{U^+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x-H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x',\qquad\forall u\in C_0^\infty(U).$$

Actually, we consider the best possible L^p remainder terms, involving superquadratic exponents of u with Hardy type potentials of optimal singularity, covering the critical Sobolev exponent as well. For $0<\rho\le 1$, we set $X(\rho)=(1-\ln\rho)^{-1}$. The result is stated as follows.

Theorem 1. *Let $\alpha\in(-1,1)$, $n+\alpha-2>0$, $0\le\theta\le 2-\alpha$, and U be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $C>0$, depending only on n, α and θ, such that*

$$H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+\cap U}\frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x'+C\left(\int\limits_{U^+}\frac{X^{p(\theta)}\,|u|^{2^*(\theta)}}{|x|^\theta}\,\mathrm{d}x\right)^{\frac{2}{2^*(\theta)}}\le\int\limits_{U^+}x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x,\quad\forall u\in C_0^\infty(U),\tag{6}$$

where $2^*(\theta)=\frac{2(n-\theta)}{n+\alpha-2}$, $p(\theta)=\frac{2n+\alpha-2-\theta}{n+\alpha-2}$ and $X=X(|x|/D)$, with $D=\sup_{x\in\mathbb{R}^n_+\cap U}|x|$. Moreover, the logarithmic correction $X^{p(\theta)}$ cannot be replaced by a smaller power of X.

Under a suitable transformation, inequality (6) will turn out to be equivalent with the following inequality (see [1], [9] for the endpoint cases $\theta=0$, $\theta=2$)

$$C_{n,\theta}\left(\int_U\frac{X^{p(\theta)}(|x|)}{|x|^n}\,|v|^{2^*(\theta)}\,\mathrm{d}x\right)^{\frac{2}{2^*(\theta)}}\le\int_U\frac{x_n^\alpha\,|\nabla v|^2}{|x|^{n-2+\alpha}}\,\mathrm{d}x,\quad\forall v\in C_0^\infty(U),\tag{7}$$

where the logarithmic weight $X^{p(\theta)}$ is again optimal, in the sense that it cannot be replaced by a smaller power of X. Notice also that in the unweighted case $\alpha=0$, (7) reads

$$C_{n,\theta}\left(\int_U\frac{X^{p(\theta)}(|x|)}{|x|^n}\,|v|^{2^*(\theta)}\,\mathrm{d}x\right)^{\frac{2}{2^*(\theta)}}\le\int_U\frac{|\nabla v|^2}{|x|^{n-2}}\,\mathrm{d}x,\quad\forall u\in C_0^\infty(U),\tag{8}$$

which may be seen as the limiting case, as $\gamma \to (n-2)/2$, of the Caffarelli-Konh-Nirenberg inequalities [4]

$$C_{n,\beta,\gamma}\left(\int_U \frac{|v|^p}{|x|^{\beta p}}\,\mathrm{d}x\right)^{\frac{2}{p}} \le \int_U \frac{|\nabla v|^2}{|x|^{2\gamma}}\,\mathrm{d}x, \quad \forall v \in C_0^\infty(U),$$

where $\gamma < \frac{n-2}{2}$, $\gamma \le \beta \le \gamma + 1$, and $p = \frac{2n}{n-2+2(\beta-\gamma)}$.

2 Preliminaries

We briefly outline here some known results, playing a fundamental role in our proof in the subsequent section. A proof of (5) was given in [18, Theorem 1], where it has been identified the energetic solution $\psi(x)$ of the associated Euler-Lagrange equations

$$\begin{cases} div(x_n^\alpha \nabla\psi) = 0, & \text{in } \mathbb{R}^n_+, \\ \lim\limits_{x_n \to 0^+} x_n^\alpha \frac{\partial \psi(x',x_n)}{\partial x_n} = -H(n,\alpha)\,\frac{\psi(x',0)}{|x'|^{1-\alpha}}, & \text{on } \partial\mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{9}$$

Although we consider the full parameter range $\alpha \in (-1,1)$, we suppress the dependence of ψ on α, for the sake of simplicity. Let us also recall the following uniform asymptotics for ψ ([18, Lemma 2]). Hereafter, for functions $f : \mathbb{R}^n_+ \to \mathbb{R}$, $g : \mathbb{R}^n_+ \to \mathbb{R}$, we will write $f \sim g$ in $\mathbb{R}^n_+$, when there exist constants $c_1 > 0$, $c_2 > 0$, such that $c_1 f(x) \le g(x) \le c_2 f(x)$, $\forall x \in \mathbb{R}^n_+$.

Lemma. *There holds*

$$\psi \sim |x|^{\frac{2-\alpha-n}{2}}, \ \textit{in } \mathbb{R}^n_+. \tag{10}$$

Moreover, for $\alpha \in (-1,0]$, there holds

$$|\nabla\psi| \sim |x|^{-\frac{\alpha+n}{2}}, \ \textit{in } \mathbb{R}^n_+.$$

If $\alpha \in (0,1)$, then there holds

$$|\nabla\psi| \sim |x|^{\frac{\alpha-n}{2}} x_n^{-\alpha}, \ \textit{in } \mathbb{R}^n_+.$$

3 Hardy-Sobolev type remainder terms

In this section we give the proof of Theorem 1. Let us first fix some notation, that will be used within the proof. We define the unit ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, the upper half ball $B_R^+ = \{x \in \mathbb{R}^n_+ : |x| < R\}$, the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and the upper half sphere $\mathbb{S}^{n-1}_+ = \{x \in \mathbb{R}^n_+ : |x| = 1\}$. We also abbreviate $U^+ = \mathbb{R}^n_+ \cap U$. Moreover, $\int_{\mathbb{S}^{n-1}_+} u\,\mathrm{d}S$ denote the $(n-1)-$dimensional Lebesgue integral of the function u over $\mathbb{S}^{n-1}_+$.

For any point $(x', x_n) \in \mathbb{S}^{n-1}_+$ we define $\varphi = \arccos x_n$, $\varphi \in [0, \pi/2]$, so that $\cos\varphi$ equals the distance of (x', x_n) to $\partial\mathbb{R}^n_+$. We also follow the usual convention of denoting by C a general positive constant, possibly varying from line to line. Relevant dependencies on parameters will be emphasized by using parentheses or subscripts. In particular, we denote $\mathcal{C}_{n,\alpha} = \int_{\mathbb{S}^{n-1}_+} x_n^\alpha\,\mathrm{d}S$ and $\gamma_n = \int_{\mathbb{S}^{n-1}} 1\,\mathrm{d}S$.

We are now ready to give the

proof of Theorem 1. An essential role, in both parts of the proof, will play the function ψ, introduced in Section (2). Notice first, that by standard approximation, it suffices to prove (6) for $u \in C_0^\infty(U \setminus \{0\})$. Indeed, let $\epsilon > 0$, and consider the functions $u_\epsilon = u\eta_\epsilon$, where $u \in C_0^\infty(U)$ and $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, $\eta_\epsilon(x) = 1$, for $|x| \ge \epsilon$ and $|\nabla\eta_\epsilon| \le c/\epsilon$. Then, by the Lebesgue dominated theorem, we have

$$\int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u_\epsilon^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \to \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2}{|x'|^{1-\alpha}}\,\mathrm{d}x' \quad \text{and} \quad \int\limits_{U^+} \frac{X^{p(\theta)}\,|u_\epsilon|^{2^*(\theta)}}{|x|^\theta}\,\mathrm{d}x \to \int\limits_{U^+} \frac{X^{p(\theta)}\,|u|^{2^*(\theta)}}{|x|^\theta}\,\mathrm{d}x, \ \text{ as } \epsilon \to 0. \tag{11}$$

Similarly we have

$$\int\limits_{U^+} x_n^\alpha\,\eta_\epsilon^2\,|\nabla u|^2\,\mathrm{d}x \to \int\limits_{U^+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y, \ \text{ as } \epsilon \to 0.$$

Moreover, taking into account $|\nabla \eta_\epsilon| \le c/\epsilon$, we get

$$\int_{U^+} x_n^\alpha \, |\nabla \eta_\epsilon|^2 \, u^2 \, \mathrm{d}x \le c\epsilon^{n-2+\alpha} \to 0, \ \text{ as } \ \epsilon \to 0,$$

hence

$$\int_{U^+} x_n^\alpha \, |\nabla u_\epsilon|^2 \, \mathrm{d}x \to \int_{U^+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x, \ \text{ as } \ \epsilon \to 0. \tag{12}$$

By (11), (12) we conclude that it suffices to prove the result for $u \in C_0^\infty(U \setminus \{0\})$. We expand now the square and integrate by parts to obtain

$$\int_{U^+} x_n^\alpha \, |\nabla u - \frac{\nabla \psi}{\psi} \, u|^2 \, \mathrm{d}x = \int_{U^+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x + \int_{U^+} x_n^\alpha \, |\nabla \psi|^2 \left(\frac{u}{\psi}\right)^2 \mathrm{d}x + \int_{U^+} u^2 \, div \left(x_n^\alpha \, \frac{\nabla \psi}{\psi} \right) \mathrm{d}x$$
$$+ \int_{\partial \mathbb{R}^n_+ \cap U} \lim_{x_n \to 0^+} x_n^\alpha \, \frac{\partial \psi(x', x_n)}{\partial x_n} \frac{u^2}{\psi} \, \mathrm{d}x' = \int_{U^+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x - H(n, \alpha) \int_{\partial \mathbb{R}^n_+ \cap U} \frac{u^2(x', 0)}{|x'|^{1-\alpha}} \, \mathrm{d}x'.$$

In the last equation we used equations (9). Notice also that on $supp\, u$, the function u/ψ is well defined. Actually $u/\psi \in C_c^\infty(\overline{\mathbb{R}^n_+} \cap U \setminus \{0\})$. Therefore we have

$$\int_{U^+} x_n^\alpha \, |\nabla u|^2 \, \mathrm{d}x - H(n, \alpha) \int_{\partial \mathbb{R}^n_+ \cap U} \frac{u^2(x', 0)}{|x'|^{1-\alpha}} \, \mathrm{d}x' = \int_{U^+} x_n^\alpha \, |\nabla (u/\psi)|^2 \, \psi^2 \, \mathrm{d}x, \tag{13}$$

hence, we have to show that there exists a constant $C > 0$, depending only on n, α, θ, such that for all $u \in C_0^\infty(U \setminus \{0\})$ there holds

$$C \left(\int_{U^+} \frac{X^{p(\theta)} \, |u|^{2^*(\theta)}}{|x|^\theta} \, \mathrm{d}x \right)^{\frac{2}{2^*(\theta)}} \le \int_{U^+} x_n^\alpha \, \psi^2 \, |\nabla (u/\psi)|^2 \, . \tag{14}$$

Now, making the substitution $u = \upsilon \, \psi$, taking into account (10), that is $\psi \sim |x|^{-\frac{n-2+\alpha}{2}}$ in $\mathbb{R}^n_+$, and noting that $U \subseteq B_D$, we conclude that (14) will follow on its turn after showing the existence of a positive constant $C = C(n, \alpha, \theta)$, independent of D, such that for all $\upsilon \in C_0^\infty(B_D)$, there holds

$$c \left(\int_{B_D^+} \frac{X^{p(\theta)}}{|x|^n} \, |\upsilon|^{2^*(\theta)} \, \mathrm{d}x \right)^{\frac{2}{2^*(\theta)}} \le \int_{B_D^+} \frac{x_n^\alpha}{|x|^{n-2+\alpha}} \, |\nabla \upsilon|^2 \, \mathrm{d}x. \tag{15}$$

To this aim we consider the minimization problem

$$c_{n,\alpha,\theta} = \inf_{\substack{\upsilon \in C_0^\infty(B_D) \\ \upsilon \not\equiv 0}} I[\upsilon], \quad \text{where} \quad I[\upsilon] = \frac{\int_{B_D^+} \frac{x_n^\alpha}{|x|^{n-2+\alpha}} \, |\nabla \upsilon|^2 \, \mathrm{d}x}{\left(\int_{B_D^+} \frac{X^{p(\theta)}\left(\frac{|x|}{D}\right)}{|x|^n} \, |\upsilon|^{2^*(\theta)} \, \mathrm{d}x \right)^{\frac{2}{2^*(\theta)}}} = \frac{I_1[\upsilon]}{I_2[\upsilon]}.$$

We point out that despite the presence of D, we do not incorporate the subscript D, in the notation of the above infimum, since it is independent D, due to the scaling invariance. Actually, inspired by an idea of Adimurthi, Filippas and Tertikas [1], we will relate the constant $c_{n,\alpha,\theta}$ with the weighted Hardy-Sobolev constant $S_{n,\alpha,\theta}$ defined by

$$S_{n,\alpha,\theta} = \inf_{\substack{\upsilon \in C_0^\infty(B_1) \\ \upsilon \not\equiv 0}} J[\upsilon], \quad \text{where} \quad J[\upsilon] = \frac{\int_{B_1^+} x_n^\alpha \, |\nabla \upsilon|^2 \, \mathrm{d}x}{\left(\int_{B_1^+} \frac{|\upsilon|^{2^*(\theta)}}{|x|^\theta} \, \mathrm{d}x \right)^{\frac{2}{2^*(\theta)}}} = \frac{J_1[\upsilon]}{J_2[\upsilon]}. \tag{16}$$

We express $J_1[\upsilon]$, $J_2[\upsilon]$ in terms of polar coordinates, writing $\upsilon(x) = \upsilon(r, \vartheta)$, where

$$r = |x|, \;\; \vartheta \;=\; \frac{x}{|x|} \in \mathbb{S}_+^{n-1}. \tag{17}$$

Then we make the change of $r-$variable, setting

$$t = r^{2-n-\alpha} \quad \text{and} \quad \upsilon(r, \vartheta) = h(t, \vartheta), \tag{18}$$

to obtain

$$\begin{aligned} J_1[\upsilon] \;&=\; \int_0^1 \int_{\mathbb{S}_+^{n-1}} r^{n-1+\alpha} \cos^\alpha \varphi \left(\upsilon_r^2 \,+\, \frac{1}{r^2}\, |\nabla_\vartheta \upsilon|^2 \right) \,\mathrm{d}S\,\mathrm{d}r \\ &=\; (n-2+\alpha) \int_1^\infty \int_{\mathbb{S}_+^{n-1}} \cos^\alpha \varphi \,\left(h_t^2 \,+\, (n-2+\alpha)^{-2}\, t^{-2}\, |\nabla_\vartheta h|^2 \right) \,\mathrm{d}S\,\mathrm{d}t. \end{aligned} \tag{19}$$

Transforming the denominator $J_2[\upsilon]$, we have

$$J_2[\upsilon] = \left(\int_0^1 \int_{\mathbb{S}_+^{n-1}} r^{n-1-\theta}\, |\upsilon|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}r \right)^{\frac{2}{2^*(\theta)}} \;=\; \frac{1}{(n-2+\alpha)^{\frac{2}{2^*(\theta)}}} \left(\int_1^\infty \int_{\mathbb{S}_+^{n-1}} t^{-p(\theta)}\, |h|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}t \right)^{\frac{2}{2^*(\theta)}}.$$

Therefore we have

$$(n-2+\alpha)^{-\frac{2^*(\theta)+2}{2^*(\theta)}}\, S_{n,\alpha,\theta} \;=\; \inf_{\substack{h \in C^\infty([1,\infty)\times \mathbb{S}_+^{n-1}) \\ h \not\equiv 0,\; h(1,\cdot)=0}} \frac{\int_1^\infty \int_{\mathbb{S}_+^{n-1}} \cos^\alpha \varphi \,\left(h_t^2 \,+\, (n-2+\alpha)^{-2}\, t^{-2}\, |\nabla_\vartheta h|^2 \right) \,\mathrm{d}S\,\mathrm{d}t}{\left(\int_1^\infty \int_{\mathbb{S}_+^{n-1}} t^{-p(\theta)}\, |h|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}t \right)^{\frac{2}{2^*(\theta)}}}. \tag{20}$$

Similarly we will express the quotient I in terms of polar coordinates (17), and then we will make the change of r variable, setting

$$t = \frac{1}{X(\frac{r}{D})} = 1 - \ln\left(\frac{r}{D}\right) \quad \text{and} \quad \upsilon(r, \vartheta) = w(t, \vartheta). \tag{21}$$

Then, direct calculationsyield

$$I_1[\upsilon] = \int_0^D \int_{\mathbb{S}_+^{n-1}} r\, \cos^\alpha \varphi \left(\upsilon_r^2 \,+\, \frac{1}{r^2}\, |\nabla_\theta \upsilon|^2 \right) \,\mathrm{d}S\,\mathrm{d}r = \int_1^\infty \int_{\mathbb{S}_+^{n-1}} \cos^\alpha \varphi \,\left(w_t^2 \,+\, |\nabla_\theta w|^2 \right) \,\mathrm{d}S\,\mathrm{d}t.$$

Similarly the denominator is transformed into

$$I_2[\upsilon] = \left(\int_0^D \int_{\mathbb{S}_+^{n-1}} \frac{X^{p(\theta)}\left(\frac{|x|}{D}\right)}{r}\, |\upsilon|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}r \right)^{\frac{2}{2^*(\theta)}} \;=\; \left(\int_1^\infty \int_{\mathbb{S}_+^{n-1}} t^{-p(\theta)}\, |w|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}t \right)^{\frac{2}{2^*(\theta)}}.$$

Therefore we have

$$c_{n,\alpha,\theta} \;=\; \inf_{\substack{w \in C^\infty([1,\infty)\times \mathbb{S}_+^{n-1}) \\ w \not\equiv 0,\; w(1,\cdot)=0}} \frac{\int_1^\infty \int_{\mathbb{S}_+^{n-1}} \cos^\alpha \varphi \,\left(w_t^2 \,+\, |\nabla_\vartheta w|^2 \right) \,\mathrm{d}S\,\mathrm{d}t}{\left(\int_1^\infty \int_{\mathbb{S}_+^{n-1}} t^{-p(\theta)}\, |w|^{2^*(\theta)} \,\mathrm{d}S\,\mathrm{d}t \right)^{\frac{2}{2^*(\theta)}}}. \tag{22}$$

Comparing now the quotients (20), (22), we obtain that $c_{n,\alpha,\theta} \geq \tau_{n,\alpha,\theta}\, S_{n,\alpha,\theta}$ for some positive constant $\tau_{n,\alpha,\theta}$. This proves (15), which yields (6).

To complete the proof of the Theorem, it remains to verify that the weight $X^{p(\theta)}$, in the remainder term cannot be replaced by a smaller power of X. For the sake of simplicity, let us abbreviate $p = p(\theta)$, $q = 2^*(\theta)$.

In order to verify the optimality of the power of X, we have to show that there are no constants $0 < \epsilon < p$, $c > 0$, such that the following inequality is valid

$$H(n,\alpha) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x' + c \left(\int\limits_{U^+} \frac{X^{p-\epsilon}}{|x|^\theta} |u|^q \,\mathrm{d}x \right)^{2/q} \leq \int\limits_{U^+} x_n^\alpha\, |\nabla u|^2 \,\mathrm{d}x, \quad \forall u \in C_0^\infty(U).$$

Note also that it suffices to prove the claim, only for the case $0 < \epsilon < p-1$, since $X^{p-\epsilon_0} > X^{p-\epsilon}$, $\forall \epsilon_0 > \epsilon$.

The result will follow after showing that there exists a sequence $\{u_k\} \subset C_0^\infty(U)$, such that

$$\frac{\int\limits_{U^+} x_n^\alpha\, |\nabla u_k|^2 \,\mathrm{d}x \;-\; H(n,\alpha) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u_k^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x'}{\left(\int\limits_{U^+} \frac{X^{p-\epsilon}}{|x|^\theta} |u_k|^q \,\mathrm{d}x \right)^{2/q}} \xrightarrow{k\to\infty} 0.$$

Let now $\upsilon \in C_0^\infty(U)$. In view of (13), the substitution $u = \upsilon\,\psi$, transforms the numerator of the above quotient into $\int_{U^+} x_n^\alpha\, |\nabla \upsilon_k|^2\, \psi^2 \,\mathrm{d}x$. Moreover (10), that is $\psi \sim |x|^{\frac{2-n-\alpha}{2}}$ in $\mathbb{R}^n_+$, it suffices to fix a sequence $\{\upsilon_k\} \subset C_0^\infty(U)$ such that

$$Q[\upsilon_k] = \frac{N[\upsilon_k]}{D[\upsilon_k]} := \frac{\int_{U^+} \frac{x_n^\alpha\, |\nabla \upsilon_k|^2}{|x|^{n+\alpha-2}}\,\mathrm{d}x}{\left(\int_{U^+} \frac{X^{p-\epsilon}\; |\upsilon_k|^q}{|x|^n}\,\mathrm{d}x \right)^{2/q}} \longrightarrow 0, \;\text{ as }\; k \to \infty. \tag{23}$$

The domain U contains a ball B_r centered at the origin and without loss of generality we may assume $r = 1$. Let us also abbreviate $V(x) = \frac{X^{p-\epsilon}}{|x|^n}$, $w(x) = \frac{x_n^\alpha}{|x|^{n+\alpha-2}}$ and define the space $D_0^{1,2}(B_1, w(x)\mathrm{d}x)$ as the completion of $C_0^\infty(B_1)$ with respect to the norm $\|\upsilon\| = (\int_{B_1^+} |\nabla \upsilon|^2\, w(x)\,\mathrm{d}x)^{1/2}$. Then, by a standard approximation, it suffices to fix a sequence $\{\upsilon_k\} \subset D_0^{1,2}(B_1, w(x)\mathrm{d}x)$ with $\int_{B_1^+} V(x)\,|\upsilon_k|^q \,\mathrm{d}x < \infty$, such that $Q[\upsilon_k] \to 0$, as $k \to \infty$.

To this end, we choose δ such that $0 < \epsilon < \delta < p-1$, which eventually will be sent to ϵ, we set $R_m = e^{1-m}$ so that

$$\frac{1}{m} \leq X(|x|) \leq 1 \Leftrightarrow R_m \leq |x| \leq 1, \quad m = 1,2,3,\ldots$$

and define the radial functions f_m as follows

$$f_m(x) = \begin{cases} X^{\frac{\delta}{q}-\frac{1}{2}}(|x|), & R_m \leq |x| \leq 1, \\ m^{\frac{3}{2}-\frac{\delta}{q}}\, X(|x|), & |x| \leq R_m, \end{cases} \qquad \text{whence} \qquad \nabla f_m(x) = \begin{cases} \left(\frac{\delta}{q} - \frac{1}{2}\right) X^{\frac{\delta}{q}+\frac{1}{2}}(|x|)\, \frac{x}{|x|^2}, & R_m < |x| \leq 1, \\ m^{\frac{3}{2}-\frac{\delta}{q}}\, X^2\, \frac{x}{|x|^2}, & |x| < R_m. \end{cases}$$

We have

$$N[f_m] = \left(\frac{\delta}{q} - \frac{1}{2}\right)^2 \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{x_n^\alpha X^{\frac{2\delta}{q}+1}(|x|)}{|x|^{n+\alpha}}\,\mathrm{d}x \;+\; m^{3-\frac{2\delta}{q}} \int\limits_{B_{R_m}^+} \frac{x_n^\alpha X^4(|x|)}{|x|^{n+\alpha}}\,\mathrm{d}x \;=:\; N_1[f_m] + N_2[f_m]$$

and

$$D^{q/2}[f_m] = \int\limits_{B_1^+ \setminus B_{R_m}^+} \frac{X^{\delta-\epsilon+1}}{|x|^n}\,\mathrm{d}x \;+\; m^{3q/2-\delta} \int\limits_{B_{R_m}^+} \frac{X^{p-\epsilon+q}}{|x|^n}\,\mathrm{d}x \;=:\; D_1[f_m] + D_2[f_m].$$

We next estimate the terms D_1, D_2, N_1, N_2, using polar coordinates (17), then making the change of variable

$$t \;=\; X(r), \quad \text{thus} \;\; \mathrm{d}t \;=\; \frac{X^2(r)}{r}\,\mathrm{d}r$$

to get

$$\begin{aligned}
N_1[f_m] &= \left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \int\limits_{\mathbb{S}^{n-1}_+} x_n^\alpha \,\mathrm{d}S \int\limits_{R_m}^{1} \frac{X^{\frac{2\delta}{q}+1}(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha}\left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \int\limits_{1/m}^{1} t^{\frac{2\delta}{q}-1}\,\mathrm{d}t \\
&= \mathcal{C}_{n,\alpha}\left(\frac{\delta}{q}-\frac{1}{2}\right)^2 \frac{q(1-m^{-2\delta/q})}{2\delta} = \mathcal{C}_{n,\alpha}\,(\delta+1-p)^2\,\frac{1-m^{-2\delta/q}}{2q\delta}, \\
N_2[f_m] &= m^{3-\frac{2\delta}{q}} \int\limits_{\mathbb{S}^{n-1}_+} x_n^\alpha \,\mathrm{d}S \int\limits_{0}^{R_m} \frac{X^4(r)}{r}\,\mathrm{d}r = \mathcal{C}_{n,\alpha}\, m^{3-\frac{2\delta}{q}} \int\limits_0^{1/m} t^2\,\mathrm{d}t = \frac{\mathcal{C}_{n,\alpha}\, m^{-2\delta/q}}{3},
\end{aligned}$$

and

$$\begin{aligned}
D_1[f_m] &= \int\limits_{\mathbb{S}^{n-1}_+} 1\,\mathrm{d}S \int\limits_{R_m}^{1} \frac{X^{\delta-\epsilon+1}(r)}{r}\,\mathrm{d}r = \gamma_n \int\limits_{1/m}^{1} t^{\delta-\epsilon-1}\,\mathrm{d}t = \frac{\gamma_n\,(1-m^{\epsilon-\delta})}{\delta-\epsilon}, \\
D_2[f_m] &= m^{3q/2-\delta} \int\limits_{\mathbb{S}^{n-1}_+} 1\,\mathrm{d}S \int\limits_{0}^{R_m} \frac{X^{p-\epsilon+q}(r)}{r}\,\mathrm{d}r = \gamma_n\, m^{3q/2-\delta} \int\limits_0^{1/m} t^{p-\epsilon+q-2}\,\mathrm{d}t = \frac{\gamma_n\, m^{\epsilon-\delta}}{3(p-1)-\epsilon}.
\end{aligned}$$

We conclude that

$$J[f_m] = \frac{\mathcal{C}_{n,\alpha}}{\gamma_n^{2/q}}\,\frac{(\delta+1-p)^2\,\frac{1-m^{-2\delta/q}}{2q\delta} + \frac{m^{-2\delta/q}}{3}}{\left(\frac{1-m^{\epsilon-\delta}}{\delta-\epsilon} + \frac{m^{\epsilon-\delta}}{3(p-1)-\epsilon}\right)^{2/q}}.$$

We then take a sequence $\delta_k \searrow \epsilon$ and choose m_k sufficiently large so that $m_k^{\epsilon-\delta_k} < 1/2$. It follows that $Q[f_{m_k}] \to 0$, as $k \to \infty$. The boundary conditions can be fixed by considering a function $\eta \in C_0^\infty(U)$, which is constant, not zero, in a neighbourhood of the origin. It is then straightforward to verify that the sequence $\upsilon_k = f_{m_k}\,\eta$, satisfies $Q[\upsilon_k] \to 0$, as $k \to \infty$, that is the condition (23). $\square$

Notice that once we have proven the optimality of the exponent of the weight X^2, for the special case $\theta = 2-\alpha$, in Theorem 1, then the optimality of $X^{p(\theta)}$ in the remaining cases can be deduced via Hölder inequality, as follows (cf. [14]): Let us suppose, towards contradiction, that there exists a constant $c > 0$, such that the following inequality holds

$$c\left(\int\limits_{U^+} \frac{X^{p-\epsilon}}{|x|^\theta}\,|u|^{2^*(\theta)}\,\mathrm{d}x\right)^{\frac{2}{2^*(\theta)}} \le \int\limits_{U^+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x', \quad \forall u \in C_0^\infty(U). \tag{24}$$

In the left hand side, we will employ the Hölder's inequality with conjugate exponents $q = \frac{2^*(\theta)}{2}$, $q' = \frac{q}{q-1}$. We choose $0 < \delta < 1$ and noting that $p = q+1$ we obtain

$$\int\limits_{U^+} \frac{X^{2-\epsilon(1-\delta)/q}}{|x|^{2-\alpha}}\,u^2\,\mathrm{d}x \le \left(\int\limits_{U^+} \frac{X^{1+\epsilon\delta q'/q}}{|x|^{(2-\alpha-\frac{\theta}{q})q'}}\,\mathrm{d}x\right)^{1/q'} \left(\int\limits_{U^+} \frac{X^{p-\epsilon}}{|x|^\theta}\,|u|^{2^*(\theta)}\,\mathrm{d}x\right)^{1/q}. \tag{25}$$

By (24) and (25), we conclude that there exists a positive constant C such that

$$C\int\limits_{U^+} \frac{X^{2-\epsilon(1-\delta)/q}}{|x|^{2-\alpha}}\,u^2\,\mathrm{d}x \le \int\limits_{U^+} x_n^\alpha\,|\nabla u|^2\,\mathrm{d}x - H(n,\alpha)\int\limits_{\partial\mathbb{R}^n_+\cap U} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x', \quad \forall u \in C_0^\infty(U).$$

This inequality contradicts to the assertion for the case $\theta = 2-\alpha$, that the weight X^2 in the inequality

$$C \int\limits_{U^+} \frac{X^2}{|x|^{2-\alpha}}\, u^2 \,\mathrm{d}x \;\le\; \int\limits_{U^+} x_n^\alpha \,|\nabla u|^2 \,\mathrm{d}x - H(n,\alpha) \int\limits_{\partial\mathbb{R}^n_+ \cap U} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,\mathrm{d}x',$$

cannot be replaced by a smaller power of X.

Remarks and further developments Although the optimal constants in Theorem 1 are not determined, explicit bounds, in terms of the sharp Hardy-Sobolev constants $S_{n,\alpha,\theta}$ in (16), can be deduced from our proof. In the unweighted case $a = 0$, the infimum (16) remains the same even if is just taken over functions with radially symmetry ([2], [10], [12], [17]), which allow us to obtain the value of the best constant in (8); we refer to the works [1], [9] where the optimal constants for the endpoint cases $\theta = 0$, $\theta = 2$, respectively have been determined.

Modifications of (5) with trace remainder terms (cf. [7], [8], [19]) can be also considered as we shall show in a forthcoming work, yielding as an application, inequalities which are translated, via the so called harmonic extension [5] (see also [13], [15], [16]), into refined versions of fractional Hardy inequalities on bounded domains, improving and extending some earlier results.

Let us finally note that the method to prove our results, based on symmetry and homogeneity arguments, can be well suited to handle more general weighted trace Hardy inequalities involving distances to linear subspaces of several codimensions.

Appendix

Let us show here that (3) cannot be improved in the sense that there are no positive constant C, exponent $p > 0$, and nontrivial potential $V \ge 0$ such that the following inequality is valid

$$C\left(\int_{\mathbb{R}^n_+} V(x)\,|u(x)|^p\,dx\right)^{\frac{2}{p}} \le \int_{\mathbb{R}^{n}+} x_n^\alpha\,|\nabla u|^2\,dx - H(n,\alpha)\int_{\partial\mathbb{R}^n_+} \frac{u^2(x',0)}{|x'|^{1-\alpha}}\,dx',$$

for all $u \in C_0^\infty(\mathbb{R}^n)$. It suffices to show that there exist functions u_ε

$$\frac{\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u_\varepsilon|^2\,dx - H(n,\alpha)\int_{\partial\mathbb{R}^n_+} \frac{u_\varepsilon^2(x',0)}{|x'|^{1-\alpha}}\,dx'}{\left(\int_{\mathbb{R}^n_+} V(x)\,|u_\varepsilon|^p\,dx\right)^{\frac{2}{p}}} \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0. \tag{26}$$

To this end we consider for $\varepsilon > 0$ the function

$$u_\varepsilon(x) = \begin{cases} \psi(x)\,|x|^{\varepsilon/2}, & |x| \le 1,\\ \psi(x)\,|x|^{-\varepsilon/2}, & |x| \ge 1, \end{cases}$$

where ψ is introduced in Section 2. Now we make integration by parts in the domains $\mathbb{R}^n_+ \cap B_1$, $\mathbb{R}^n_+ \cap (B_R \setminus B_1)$, where $R > 1$, then send $R \to \infty$ taking into account that $\nabla u_\varepsilon(x)\cdot x = \frac{2-n-\alpha-\varepsilon}{2}\,u_\varepsilon(x)$, the relations (9), (10) jointly with the estimate

$$\int_{\mathbb{R}^n_+\cap\partial B_R} x_n^\alpha\,u_\varepsilon\left(\nabla u_\varepsilon(x)\cdot\frac{x}{|x|}\right)dS = \frac{2-n-\alpha-\varepsilon}{2R}\int_{\mathbb{R}^n_+\cap\partial B_R} x_n^\alpha\,u_\varepsilon^2\,dS \le c(n,\alpha)\,R^{-\varepsilon} \overset{R\to\infty}{\longrightarrow} 0,$$

to obtain

$$\int_{\mathbb{R}^n_+} x_n^\alpha\,|\nabla u_\varepsilon|^2\,dx = H(n,\alpha)\int_{\partial\mathbb{R}^n_+} \frac{u_\varepsilon^2(x',0)}{|x'|^{1-\alpha}}\,dx' - \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n_+} \frac{x_n^\alpha u_\varepsilon^2}{|x|^2}\,dx + \varepsilon\int_{\partial B_1\cap\mathbb{R}^n_+} x_n^\alpha\,\psi^2\,dS.$$

Here dS stands for the $(n-1)-$dimensional Lebesgue measure over the corresponding spheres $\partial B_R = \{x \in \mathbb{R}^n : |x| = R\}$ or $\partial B_1 = \{x \in \mathbb{R}^n :\ |x| = 1\}$. Then, letting $\varepsilon \to 0$, we obtain (26).

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