NOTES ON

CLASSICAL POTENTIAL THEORY

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Forword

During the fall semester of the academic year 1990-1991 I gave a course on Classical Potential Theory attended by an excellent class of graduate students of the Department of Mathematics of Washington University. That was my first time to teach such a course and, I have to say, besides sporadic knowledge of a few facts directly related to complex analysis, I had no serious knowledge of the subject. The result was: many sleepless nights reading books, trying to choose the material to be presented and preparing hand-written notes for the students.

The books I found very useful and which determined the choice of material were the superb "Éléments de la Théorie Classique du Potentiel" by M. Brelot and the "Selected Problems on Exceptional Sets" by L. Carleson. Other sources were: "Some Topics in the Theory of Functions of One Complex Variable" by W. Fuchs, unpublished notes on "Harmonic Measures" by J. Garnett, "Subharmonic Functions" by W. Hayman and P. Kennedy, "Introduction to Potential Theory" by L. Helms, "Foundations of Modern Potential Theory" by N. Landkof, "Subharmonic Functions" by T. Rado and "Potential Theory in Modern Function Theory" by M. Tsuji.

This is a slightly expanded version of the original notes with very few changes. The principle has remained the same, namely *to present an overview of the classical theory at the level of a graduate course*. The part called "Preliminaries" is new and its contents were silently taken for granted during the original course. The main material is the Divergence Theorem and Green's Formula, a short course on holomorphic functions (, since their real parts are the main examples of harmonic functions in the plane and, also, since one of the central results is the proof of the Riemann Mapping Theorem through potential theory), some basic facts about semi-continuous functions and very few elementary results about distributions and the Fourier transform. Except for the Divergence Theorem, the Arzela-Ascoli Theorem, the Radon-Riesz Representation Theorem and, of course, the basic facts of measure theory and functional analysis, all of which are used but not proved here, all other material contained in these notes is proved with sufficient detail.

Material which was not included in the original notes: the section on harmonic conjugates in the first chapter (it, actually, contains a new proof of the existence of a harmonic conjugate in a simply-connected subset of the plane); the section on the differentiability of potentials in the second chapter; the sections on superharmonic functions at ∞ and on Poisson integrals at ∞ in the fourth chapter; an additional proof of the result about the direct connection between Green's function and harmonic measure in the fifth chapter (indicating the role of the normal derivative of Green's function as an approximation to the identity); the subadditivity of capacity in the eigth chapter; the sections on polar sets and thin sets in the ninth chapter. The definition of the notion of "quasi-almost everywhere" in the eigth chapter has been changed. The proof of the Riemann Mapping Theorem in the ninth chapter is corrected and given in full detail, not relying on "obvious" topological facts any more. In the original course the proof (taken from the notes of J. Garnett) of Wiener's Theorem was presented only in dimension 2. Now, the proof is given in all dimensions.

A short and very classical application of potential theory in dimension 1 on the convergence of trigonometric series is missing from this set of notes, since it is quite specialized. *What is, also, missing is a short chapter on the metrical properties of capacity and an example of a Cantor-like set. But this will be included very soon, after it is expanded as a chapter on "Capacity, capacitability and Hausdorff measures".*

Besides the new material, there is a re-organization which results, I hope, to better exposition.

Here, I would like to thank the Department of Mathematics of Washington University for giving me the opportunity to teach the original course and the graduate class which attended it with great care and enthusiasm.

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Part I

Preliminaries

0.1 Euclidean Spaces

1. We work in the Euclidean space $\mathbf{R^n}$ and denote the Euclidean norm of $x \in \mathbf{R^n}$ by $|x|$ and the Euclidean inner product of $x, y \in \mathbf{R^n}$ by $x \cdot y$.

$B(x;r)$ is the open ball with center x and radius r, $\overline{B(x;r)}$ is the closed ball and the sphere $S(x;r)$ is the boundary of $B(x;r)$.

$d(x,B) = \inf_{y\in B} |x-y|$ denotes the Euclidean distance of the point $x \in \mathbf{R^n}$ from the non-empty subset B of $\mathbf{R^n}$ and $d(A,B) = \inf_{x\in A, y\in B} |x-y|$ denotes the Euclidean distance between the non-empty sets A and B. If B is closed and $x \notin B$, then $d(x,B) > 0$ and, if A is compact, B is closed and the two sets are disjoint, then $d(A,B) > 0$.

2. Suppose Ω is an open subset of $\mathbf{R^n}$ and consider the open sets

$$\Omega_{(m)} \;=\; \left\{x \in \Omega : d(x, \partial\Omega) > \frac{1}{m}\ ,\ |x| < m\right\}\ , \qquad m \in \mathbf{N}.$$

It is easy to check the following four properties:

1. every $\overline{\Omega_{(m)}}$ is a compact subset of Ω,
2. $\overline{\Omega_{(m)}} \subseteq \Omega_{(m+1)}$ for all m,
3. $\cup_{m=1}^{+\infty} \Omega_{(m)} = \Omega$ and
4. every compact subset of Ω is contained in $\Omega_{(m)}$ for a sufficiently large m.

This increasing sequence $\{\Omega_{(m)}\}$ of open sets, or any other with the same four properties, is called an **open exhaustion** of Ω.

The increasing sequence $\{K_{(m)}\}$, with $K_{(m)} = \overline{\Omega_{(m)}}$, where $\{\Omega_{(m)}\}$ is an open exhaustion of Ω, is called a **compact exhaustion** of Ω.

3. $V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of $B_n = B(0;1)$. Hence,

$$V_{2m} = \frac{\pi^m}{m!}\ , \qquad V_{2m+1} = \frac{2(2\pi)^m}{1\cdot 3\cdot 5\cdots(2m+1)}\ .$$

Also, $\omega_{n-1} = nV_n$ is the standard surface area of $S^{n-1} = S(0;1)$.

4. If dm is the Lebesgue measure in $\mathbf{R^n}$ and $d\sigma$ is the standard surface measure in S^{n-1}, then we have the formula

$$\int_{B(x;R)} f(y)\ dm(y) \;=\; \int_0^R \int_{S^{n-1}} f(x+rt)\ d\sigma(t)\ r^{n-1} dr$$

5. We define the **surface-mean-value** of f over $S(x;r)$ by

$$\mathcal{M}_f^r(x) \;=\; \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x+rt)\ d\sigma(t) \;=\; \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x;r)} f(y)\ dS(y)$$

for all f integrable with respect to dS, the surface measure in $S(x;r)$.

We also define the **space-mean-value** of f over $B(x;r)$ by

$$\mathcal{A}_f^r(x) \;=\; \frac{1}{V_n}\int_{B_n} f(x+ry)\; dm(y) \;=\; \frac{1}{V_n r^n}\int_{B(x;r)} f(y)\; dm(y)$$

for all f integrable with respect to dm in $B(x;r)$.

By the formula in paragraph 4,

$$\mathcal{A}_f^R(x) \;=\; \frac{n}{R^n}\int_0^R \mathcal{M}_f^r(x) r^{n-1}\; dr\ .$$

6. Define $\overline{\mathbf{R^n}}$, the **one-point compactification** of $\mathbf{R^n}$, by adjoining the point at ∞ to $\mathbf{R^n}$:

$$\overline{\mathbf{R^n}} \;=\; \mathbf{R^n} \cup \{\infty\}\ .$$

The ϵ-neighborhoods of points $x \in \mathbf{R^n}$ are the usual balls $B(x;\epsilon)$, while the ϵ-neighborhood of ∞ is defined to be the set $\{x \in \mathbf{R^n} : |x| > \frac{1}{\epsilon}\} \cup \{\infty\}$.

We define open sets in $\overline{\mathbf{R^n}}$ through these neighborhoods, in the usual way, and we, also, define closed sets (complements of open sets) and the notion of convergent sequence: a sequence in $\overline{\mathbf{R^n}}$ converges to some point of $\overline{\mathbf{R^n}}$, if the sequence is, eventually, contained in every ϵ-neighborhood of this point.

Hence, if the limit point is in $\mathbf{R^n}$, then the new notion of convergence coincides with the usual one, while $x_m \to \infty$ is equivalent to $|x_m| \to +\infty$.

Whenever we write $\overline{A}$ and ∂A, for any $A \subseteq \overline{\mathbf{R^n}}$, we mean the closure and the boundary, respectively, of A *with respect to* $\overline{\mathbf{R^n}}$.

Hence, if $A \subseteq \mathbf{R^n}$ is bounded, then these two sets coincide with the usual closure and boundary in $\mathbf{R^n}$, while if $A \subseteq \mathbf{R^n}$ is unbounded, then the two sets are the usual closure and boundary in $\mathbf{R^n}$ with the point ∞ adjoined to them.

Any $A \subseteq \mathbf{R^n}$ is open in $\mathbf{R^n}$ in the usual sense if and only if it is open in $\overline{\mathbf{R^n}}$.

It is easy to see that $\overline{\mathbf{R^n}}$ and, hence, every closed subset of it, is compact.

If A is a bounded subset of $\mathbf{R^n}$, then A is closed in $\mathbf{R^n}$ if and only if it is closed in $\overline{\mathbf{R^n}}$. But, if A is an unbounded subset of $\mathbf{R^n}$, then A is closed in $\mathbf{R^n}$ if and only if $A \cup \{\infty\}$ is closed in $\overline{\mathbf{R^n}}$.

The **spherical metric** in $\overline{\mathbf{R^n}}$ is defined by

$$d_S(x,y) \;=\; \begin{cases} \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\ , & \text{if } x,y \in \mathbf{R^n} \\ \frac{2}{\sqrt{1+|x|^2}}\ , & \text{if } x \in \mathbf{R^n} \text{ and } y=\infty \\ 0\ , & \text{if } x=y=\infty\ . \end{cases}$$

The spherical metric induces exactly the open sets, closed sets and convergent sequences in $\overline{\mathbf{R^n}}$ which were described above. In fact, one can, easily, prove that if $x \in \overline{\mathbf{R^n}}$, then, for every ϵ-neighborhood $B_S(x;\epsilon)$ with respect to the spherical metric, there is some ϵ'-neighborhood of x, as this was defined above, contained in $B_S(x;\epsilon)$ and conversely.

7. A subset E of a metric space (X,d) is called **connected**, if, whenever we write $E = A \cup B$ with $A \cap B = \emptyset$ with A, B being both open *relative to* E, it is implied that one of A, B is empty.

A subset F of E is called a **connected component** of E, if it is a *maximal* connected subset of E. This means that F is connected and that there is no connected subset of E strictly containing F.

Every E can be uniquely decomposed in connected components: there exists a unique (perhaps uncountable) family $\mathcal{F}$ such that

1. every $F \in \mathcal{F}$ is a connected component of E,
2. the elements of $\mathcal{F}$ are pairwise disjoint,
3. $\bigcup_{F \in \mathcal{F}} F = E$.

Especially if $\Omega \subseteq \overline{\mathbf{R^n}}$ is open, then Ω is connected if and only if every two of its points can be connected by a polygonal path which is contained in Ω.

The connected components of an open Ω are all open sets and there are at most countably many of them.

Therefore, for every open $\Omega \subseteq \overline{\mathbf{R^n}}$ there exist (at most countably many) sets U_m such that

1. every U_m is connected,
2. the sets U_m are pairwise disjoint and
3. $\bigcup_m U_m = \Omega$.

A subset E of a metric space (X, d) is called a **continuum** if it is connected, compact and contains at least two points.

0.2 Derivatives

1. If Ω is an open subset of $\mathbf{R^n}$, then $C(\Omega) = C^0(\Omega)$ is the space of all complex-valued functions which are continuous in Ω and $C^k(\Omega)$, $1 \leq k \leq +\infty$, is the space of all functions which are k times continuously differentiable in Ω.

Similarly, we denote by $C^k(\overline{\Omega})$ the space of all functions whose derivatives of all orders up to k are continuous in Ω and can be continuously extended in $\overline{\Omega}$. Note that a complex-valued function continuous in Ω can be continuously extended in $\overline{\Omega}$ if and only if it is uniformly continuous in Ω with respect to the spherical metric d_S (or, equivalently, with respect to the usual Euclidean distance in case the set Ω is bounded).

For all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$, we denote the α-derivative of f by

$$D^\alpha f \;=\; \frac{\partial^{\alpha_1 + \cdots + \alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \;=\; \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\ .$$

The order of this derivative is $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

We denote, as usual, the **gradient** of a real- or complex-valued function by

$$\overrightarrow{grad\, f} = \Big(\frac{\partial f}{\partial x_1}\ , \ldots, \frac{\partial f}{\partial x_n}\Big)$$

and Δ is the **Laplace operator** or **Laplacian**

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \ .$$

If $\overrightarrow{V} = (V_1, V_2, \ldots, V_n)$ is a vector-function, we denote its **divergence** by

$$div\overrightarrow{V} \ = \ \frac{\partial V_1}{\partial x_1} + \cdots + \frac{\partial V_n}{\partial x_n} \ .$$

2. We say that the open set Ω **is C^1 at its boundary point** $y \in \mathbf{R^n}$, if there is an open neighborhood V of y and a $\phi : V \to \mathbf{R}$ which is in $C^1(V)$ with $\overrightarrow{grad\,\phi}(y) \neq 0$, so that $V \cap \partial\Omega = \{x \in V : \phi(x) = 0\}$, $V \cap \Omega = \{x \in V : \phi(x) > 0\}$ and $V \setminus \overline{\Omega} = \{x \in V : \phi(x) < 0\}$.

Such a function ϕ is called **a defining function** for Ω in the neighborhood V of the boundary point y. A defining function is not unique; for example any multiple of it by a positive constant is, also, a defining function.

In case there exists a defining function $\phi \in C^k(V)$ in a neighborhood V of the boundary point y, we say that Ω is C^k at its boundary point y.

If Ω is C^1 at its boundary point $y \in \mathbf{R^n}$, we denote by $\overrightarrow{\eta}(y)$ the unit vector at y which is normal to $\partial\Omega$ and has the direction towards the exterior of Ω. In terms of any defining function ϕ for Ω in a neighborhood of y,

$$\overrightarrow{\eta}(y) \ = \ -\frac{\overrightarrow{grad\,\phi}(y)}{|\overrightarrow{grad\,\phi}(y)|} \ .$$

The directional derivative of an f in the direction of the unit normal $\overrightarrow{\eta}(y)$ is

$$\frac{\partial f}{\partial \eta}(y) = \overrightarrow{grad\,f}(y) \cdot \overrightarrow{\eta}(y) \ .$$

If the bounded open set Ω is C^k at all of its boundary points, we say that Ω **has C^k-boundary**.

If the bounded open set Ω has C^1-boundary, then there is the standard **surface measure** defined in $\partial\Omega$, denoted by dS.

Theorem 0.1 *(The Divergence Theorem and Green's Formulas) Let Ω be a bounded open set with C^1-boundary.*

1.

$$\int_\Omega div\overrightarrow{V}(x) \ dm(x) \ = \ \int_{\partial\Omega} \overrightarrow{V}(y) \cdot \overrightarrow{\eta}(y) \ dS(y)$$

for all vector-functions $\overrightarrow{V}$ whose components are in $C(\overline{\Omega}) \cap C^1(\Omega)$.

2. If f is in $C(\overline{\Omega}) \cap C^1(\Omega)$ and g is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$, then

$$\int_\Omega \big(f(x)\Delta g(x) + \overrightarrow{grad\,f}(x) \cdot \overrightarrow{grad\,g}(x)\big) \ dm(x) \ = \ \int_{\partial\Omega} f(y)\frac{\partial g}{\partial \eta}(y) \ dS(y) \ .$$

3. If g is in $C^1(\overline{\Omega}) \cap C^2(\Omega)$, then

$$\int_{\Omega} \Delta g(x)\ dm(x) \ = \ \int_{\partial\Omega} \frac{\partial g}{\partial \eta}(y)\ dS(y)\ .$$

4. If f and g are in $C^1(\overline{\Omega}) \cap C^2(\Omega)$, then

$$\int_{\Omega}\big(f(x)\Delta g(x)-g(x)\Delta f(x)\big)\,dm(x) = \int_{\partial\Omega}\Big(f(y)\frac{\partial g}{\partial \eta}(y)-g(y)\frac{\partial f}{\partial \eta}(y)\Big)\,dS(y).$$

Proof:

The proof of 1 is considered known.

Applying 1 to $V_1 = f\frac{\partial g}{\partial x_1}\ , \dots , V_n = f\frac{\partial g}{\partial x_n}$, we prove 2.

We prove 3 from 2, using the constant function $f = 1$ in $\overline{\Omega}$.

Finally, we prove 4, changing places of f and g in 2 and subtracting.

In these notes whenever we refer to the Green's Formula, we understand any one of the above four formulas.

3. Since open sets with C^k-boundary are widely used, (in particular, to apply Green's Formula), we shall, briefly, describe a standard way to produce such sets arbitrarily "close" to other given sets.

Suppose that $K \subseteq \Omega$, where K is compact and Ω is an open subset of $\mathbf{R^n}$. In practice, we are, usually, given K and choose $\Omega = \{x : d(x, K) < \delta\}$ with arbitrarily small δ or we are given Ω and choose $K = \{x \in \Omega : d(x, \partial\Omega) \geq \delta, |x| \leq \frac{1}{\delta}\}$.

We shall construct a bounded open set O with C^k-boundary so that

$$K \ \subseteq \ O \ \subseteq \ \overline{O} \ \subseteq \ \Omega\ .$$

Consider $\delta_0 < \frac{1}{\sqrt{n}}\, d(K, \partial\Omega)$ and observe that, if a cube Q of sidelenghth δ_0 intersects K, then Q is contained in Ω.

Now, consider the n coordinate-hyperplanes together with all other hyperplanes which are parallel to them and at distances which are integer multiples of δ_0. The space $\mathbf{R^n}$ is, thus, divided into a mesh of cubes

$$Q_k \ = \ \{x = (x_1, \dots , x_n) : k_j\delta_0 \leq x_j \leq (k_j + 1)\delta_0, 1 \leq j \leq n\}$$

of sidelenght δ_0, where $k = (k_1, \dots , k_n)$ is an arbitrary multi-index with integer coordinates. The set of all these cubes we denote by $\mathcal{Q}_{\delta_0}$.

Now, K is intersected by finitely many qubes in $\mathcal{Q}_{\delta_0}$ and we define the set

$$F \ = \ \bigcup\{Q \in \mathcal{Q}_{\delta_0} : Q \cap K \neq \emptyset\}\ .$$

F is a compact subset of Ω with $K \subseteq F$ and it is clear that the boundary of F consists of certain of the $(n-1)$-dimensional faces of the cubes that are used to construct F. If a face of one of the cubes contained in F intersects K, then the adjacent cube is, also, one of the cubes contained in F and, hence, this face is not contained in the boundary of F. Therefore, the boundary of F consists of faces which do not intersect K and, thus, K is contained in the interior U of F.

We have produced an open set U so that $K \subseteq U \subseteq \overline{U} \subseteq \Omega$ whose boundary consists of faces of cubes contained in U. The points at which this boundary is not C^k belong to the $(n-2)$-dimensional edges of these faces.

We, now, modify slightly the boundary to make it "smooth" at its edges and we, thus, produce an open set O slightly different from U, containing K and with C^k-boundary. The "smoothening" process is geometrically clear and it is not worth seeing the actual technical details.

4. A subset Γ of $\mathbf{R^n}$ is said to be C^1 at its point y, if there exists some open neighborhood V of y and a real-valued $\phi : V \to \mathbf{R}$ which is in $C^1(V)$ with $\overrightarrow{grad\,\phi}(y) \neq 0$, so that $\Gamma \cap V = \{x \in V : \phi(x) = 0\}$.

Any such ϕ is called a defining function for Γ in the neighborhood of its point y. Again, ϕ is not unique; for example, any multiple of it by a non-zero real constant is, also, a defining function.

If the defining function ϕ can be chosen to be in $C^k(V)$, then we say that Γ is C^k at y.

Γ is called a C^k**-hypersurface**, if it is C^k at all of its points.

A C^1-hypersurface has a standard **surface measure** dS naturally defined on it.

If Γ is C^k at y and $\phi \in C^k(V)$ is a defining function for Γ in a neighborhood V of y, then, because of continuity, we may choose V to be small enough so that $\overrightarrow{grad\,\phi} \neq 0$ everywhere in V. Now, if $M = \sup_V \phi$ and $m = \inf_V \phi$, then $m < 0 < M$ and the sets

$$\Gamma^t \;=\; \{x \in V : \phi(x) = t\} \;, \qquad m < t < M \;,$$

are pairwise disjoint C^k-hypersurfaces constituting a "continuous" partition of V. In fact, ϕ itself is a defining function for each Γ^t at every point of it.

One can, easily, prove that, for every $f \in C(\overline{V})$,

$$\int_{\Gamma^t} f(x)\, dS(x)$$

(where dS is the surface measure in Γ^t) is a continuous function of $t \in (m, M)$.

At every point y where Γ is C^1 there are exactly two unit vectors normal to Γ. These, in terms of any defining function ϕ in a neighborhood of y, are

$$\overrightarrow{\eta}(y) \;=\; \pm \frac{\overrightarrow{grad\,\phi}(y)}{|\overrightarrow{grad\,\phi}(y)|} \;.$$

The C^1-hypersurface Γ is called **orientable**, if, for every $y \in \Gamma$, we can choose one of the two possible $\overrightarrow{\eta}(y)$ so that the resulting $\overrightarrow{\eta} : \Gamma \to \mathbf{R^n}$ is continuous. Then, we call $\overrightarrow{\eta}$ a **continuous unit vector field normal to** Γ.

In such a case, the function $-\overrightarrow{\eta}$ is, also, a continuous unit vector field normal to Γ and, if Γ is connected, these are the only continuous unit vector fields normal to Γ.

If Ω is a bounded open set with C^1-boundary, then $\Gamma = \partial\Omega$ is an orientable C^1-hypersurface and there are two continuous unit vector fields normal to $\partial\Omega$; one of them has the direction towards the exterior of Ω and the other has the opposite direction.

In this special case, we keep the notation $\overrightarrow{\eta}$ only for the vector field with the direction towards the exterior of Ω, in agreement with the discussion in paragraph 2.

5. A function f which is in $C^k(B(x_0;R))$ has a **Taylor-expansion of order** k at the point x_0. Using the notation $\alpha! = \alpha_1! \cdots \alpha_n!$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$, this means that

$$f(x) \;=\; \sum_{|\alpha| \le k} \frac{1}{\alpha!}\, D^\alpha f(x_0)(x-x_0)^\alpha + R_k(x;x_0)\;, \qquad x \in B(x_0;R)\;,$$

where $\frac{|R_k(x;x_0)|}{|x-x_0|^k} \to 0$ as $x \to x_0$. In fact,

$$R_k(x;x_0) \;=\; \sum_{|\alpha|=k} \frac{1}{\alpha!}\, \big(D^\alpha f(x') - D^\alpha f(x_0)\big)(x-x_0)^\alpha$$

for some $x' = x'(x)$ contained in the linear segment $[x_0, x]$ and, thus,

$$|R_k(x;x_0)| \;\le\; C_{k,n}|x-x_0|^k \sup_{|\alpha|=k, |x'-x_0| \le |x-x_0|} |D^\alpha f(x') - D^\alpha f(x_0)|\;,$$

where $C_{k,n}$ is a constant depending only on k and n.

A function f defined in an open neighborhood of $x_0 \in \mathbf{R^n}$ is called **real-analytic** at x_0, if there is some $R > 0$ and constants a_α for all multi-indices α, so that

$$f(x) \;=\; \sum_\alpha a_\alpha (x-x_0)^\alpha$$

for every $x \in B(x_0;R)$.

This series expansion is unique and is called the **Taylor-series** of f at x_0.

Then, f is real-analytic at every other point of $B(x_0;R)$, it is infinitely many times differentiable in $B(x_0;R)$ and we have the formulas

$$a_\alpha \;=\; \frac{1}{\alpha!}\, D^\alpha f(x_0)\;.$$

6. Suppose that Γ is C^1 at its point y and let $\phi \in C^1(V)$ be a defining function for Γ in the neighborhood $V = B(y;r)$ of y.

We write the Taylor-expansion of order 1 at y

$$\phi(x) \;=\; \overrightarrow{grad\,\phi}(y) \cdot (x-y) + R_1(x;y)\;, \qquad x \in B(y;r)\;,$$

we choose t with $0 < t < |\overrightarrow{grad\,\phi}(y)|$ and we take r small enough so that $|R_1(x;y)| \le t|x-y|$ for all $x \in B(y;r)$.

It is, then, a matter of simple calculations to see that the two sets

$$F_\pm \;=\; \{x : t|x-y| < \pm\, \overrightarrow{grad\,\phi}(y) \cdot (x-y) < tr\}$$

are two open truncated cones with common vertex y and contained in the sets

$$\{x : \pm\phi(x) > 0\} \ \cap \ B(y;r)$$

respectively.

As a special case we have that, if the open set Ω is C^1 at its boundary point y, then there are two open truncated cones with common vertex y so that one of them is contained in Ω and the other is contained in $\mathbf{R^n} \setminus \overline{\Omega}$.

Now, suppose that Γ is C^2 at its point y and let $\phi \in C^2(V)$ be a defining function for Γ in the neighborhood $V = B(y;r)$ of y.

We, now, write the Taylor-expansion of order 2 at y in the simplified form

$$\phi(x) \ = \ \overrightarrow{grad\,\phi}(y) \cdot (x-y) + R_2(x;y) \ , \qquad x \in B(y;r) \ ,$$

where

$$|R_2(x;y)| \ \leq \ M|x-y|^2$$

for all $x \in B(y;r)$. In fact, $M = C_n \sup_{|\alpha|=2, x\in B(y;r)} |D^\alpha \phi(x)|$, where C_n is a constant depending only on n.

If we choose $\alpha = \min\Big(\frac{1}{2M} \ , \frac{r}{2\left|\overrightarrow{grad\,\phi}(y)\right|}\Big)$, then it is, again, a matter of calculations to see that the two open balls

$$B_\pm \ = \ \big\{x : \big|x - \big(y \pm \alpha\overrightarrow{grad\,\phi}(y)\big)\big| < \alpha\big|\overrightarrow{grad\,\phi}(y)\big|\big\}$$

are mutually tangent at the point y and they are contained in the sets

$$\{x : \pm\phi(x) > 0\} \ \cap \ B(y;r)$$

respectively.

Hence, if the open set Ω is C^2 at its boundary point y, then there are two open balls mutually tangent at the point y so that one of them is contained in Ω and the other is contained in $\mathbf{R^n} \setminus \overline{\Omega}$.

0.3 Holomorphic Functions

If Ω is an open subset of $\mathbf{C} = \mathbf{R^2}$, a function $f = \Re f + i\Im f : \Omega \to \mathbf{C}$ is called **holomorphic in** Ω, if $\Re f, \Im f \in C^1(\Omega)$ and they satisfy the system of **Cauchy-Riemann equations**

$$\frac{\partial(\Re f)}{\partial x_1} \ = \ \frac{\partial(\Im f)}{\partial x_2} \ , \qquad \frac{\partial(\Re f)}{\partial x_2} \ = \ -\frac{\partial(\Im f)}{\partial x_1}$$

everywhere in Ω or, equivalently, $\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2} = 0$ everywhere in Ω.

It is trivial to see that the property of holomorphy is preserved under addition, multiplication, division and composition of functions and that the usual formulas $(f+g)' = f' + g'$, $(fg)' = f'g + fg'$, $(f \circ g)' = (f' \circ g)g'$ hold.

If f is holomorphic in Ω, writing the Taylor-expansions of order 1 at every $x \in \Omega$ for $\Re f$ and $\Im f$ and using the Cauchy-Riemann equations, we easily find that

$$f(y) \;=\; f(x) + \frac{\partial f}{\partial x_1}(x)(y-x) + R(y;x) \;=\; f(x) - i\frac{\partial f}{\partial x_2}(x)(y-x) + R(y;x)$$

where $\frac{|R(y;x)|}{|y-x|} \to 0$ as $y \to x$. Therefore, the limit $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x}$ exists at every $x \in \Omega$. This limit is called the **(complex) derivative** of f at x and

$$f'(x) \;=\; \frac{\partial f}{\partial x_1}(x) \;=\; -i\frac{\partial f}{\partial x_2}(x) \;, \qquad x \in \Omega \;.$$

In elementary courses on the theory of holomorphic functions it is, actually, proved that the converse is, also, true: if the complex derivative of f exists at every point of Ω, then $\Re f$ and $\Im f$ are in $C^1(\Omega)$ and satisfy the Cauchy-Riemann equations. We shall not need this result.

In this section, our aim is to develop only a very small part of the theory of holomorphic functions leading, through Cauchy's Theorem, up to the Argument Principle and one of its consequences. These results will not be stated in the generality (requiring homological considerations) in which they are presented in the standard courses on the theory of holomorphic functions. We shall be restricted to the study of curvilinear integrals only over boundaries of open sets with C^1-boundary. Our main tool, therefore, is the Divergence Theorem.

Theorem 0.2 *(Cauchy's Theorem) Let f be holomorphic in Ω and the open set Ω_1 with C^1-boundary be such that $\overline{\Omega_1} \subseteq \Omega$. Then,*

$$\int_{\partial\Omega_1} f(y)\; dy \;=\; 0 \;.$$

Remark This integral is called the **curvilinear integral of f over $\partial\Omega_1$ in the positive direction of $\partial\Omega_1$ with respect to Ω_1**. Here, we define

$$dy \;=\; i\eta(y)\; dS(y) \;, \qquad y \in \partial\Omega_1 \;,$$

where $\eta = \eta_1 + i\eta_2$ is the continuous unit vector field normal to $\partial\Omega_1$ and directed towards the exterior of Ω_1.

Proof:

From the Cauchy-Riemann equations and the Divergence Theorem,

$$\begin{aligned} 0 \;&=\; i\int_{\Omega_1}\Big(\frac{\partial f}{\partial x_1}(x) + i\frac{\partial f}{\partial x_2}(x)\Big)\; dm(x) \\ &=\; i\int_{\partial\Omega_1} f(y)\eta_1(y)\; dS(y) - \int_{\partial\Omega_1} f(y)\eta_2(y)\; dS(y) \\ &=\; \int_{\partial\Omega_1} f(y)i\eta(y)\; dS(y) \;=\; \int_{\partial\Omega_1} f(y)\; dy \;. \end{aligned}$$

Theorem 0.3 *(Cauchy's Formula) Let f be holomorphic in Ω and the open set Ω_1 with C^1-boundary be such that $\overline{\Omega_1} \subseteq \Omega$. Then,*

$$f(x) \;=\; \frac{1}{2\pi i}\int_{\partial\Omega_1} \frac{f(y)}{y-x}\,dy$$

for every $x \in \Omega_1$.

Proof:
Let $r > 0$ be small so that $\overline{B(x;r)} \subseteq \Omega_1$ and let $\Omega_2 = \Omega_1 \setminus \overline{B(x;r)}$. We apply Cauchy's Theorem to the function $g(z) = \frac{f(z)}{z-x}$, $z \in \Omega \setminus \{x\}$, which is holomorphic in $\Omega \setminus \{x\}$ and get

$$0 \;=\; \int_{\partial\Omega_2} \frac{f(y)}{y-x}\,dy \;=\; \int_{\partial\Omega_1} \frac{f(y)}{y-x}\,dy - \int_{\partial B(x;r)} \frac{f(y)}{y-x}\,dy\ .$$

Therefore,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial\Omega_1} \frac{f(y)}{y-x}\,dy \;&=\; \frac{1}{2\pi i}\int_{\partial B(x;r)} \frac{f(y)}{y-x}\,dy\\
&=\; \frac{1}{2\pi}\int_{\partial B(x;r)} \frac{f(y)}{y-x}\,\frac{y-x}{r}\,dS(y)\\
&=\; f(x) + \frac{1}{2\pi r}\int_{\partial B(x;r)} \big(f(y)-f(x)\big)\,dS(y)\ .
\end{aligned}$$

The continuity of f at x implies that the last term tends to 0 as $r \to 0+$ and the proof is complete.

Example The first case below is implied by Cauchy's Formula and the second by Cauchy's Theorem:

$$\frac{1}{2\pi i}\int_{\partial B(x_0;r)} \frac{1}{y-x}\,dy \;=\; \begin{cases} 1\ , & \text{if } x \in B(x_0;r)\\ 0\ , & \text{if } x \notin \overline{B(x_0;r)}\ . \end{cases}$$

Theorem 0.4 *Let f be holomorphic in Ω. Then,*
1. f' is holomorphic in Ω and
2. for every $\overline{B(x;r)} \subseteq \Omega$, f can be expanded in $B(x;r)$ in a unique way as an absolutely convergent power series

$$f(z) \;=\; \sum_{n=0}^{+\infty} a_n (z-x)^n\ , \qquad z \in B(x;r)\ .$$

Proof:
1. From Cauchy's Formula $f(z) = \frac{1}{2\pi i}\int_{\partial B(x;r)} \frac{f(y)}{y-z}\,dy$, $z \in B(x;r)$, it is trivial to prove, using difference quotients for both sides and interchanging limit and integration, that

$$f'(z) = \frac{1}{2\pi i}\int_{\partial B(x;r)} \frac{f(y)}{(y-z)^2}\,dy\ , \qquad z \in B(x;r)\ .$$

Interchanging, again, partial derivatives and integration, we easily see that $\frac{\partial f'}{\partial x_1}$ and $\frac{\partial f'}{\partial x_2}$ are continuous in Ω and $\frac{\partial f'}{\partial x_1} + i\frac{\partial f'}{\partial x_2} = 0$ everywhere in Ω.
2. For every $z \in B(x;r)$, the geometric series

$$\frac{1}{y-z} = \frac{1}{y-x}\,\frac{1}{1-\frac{z-x}{y-x}} = \sum_{n=0}^{+\infty} \frac{(z-x)^n}{(y-x)^{n+1}}$$

converges absolutely and uniformly in $\partial B(x;r)$, since $\left|\frac{z-x}{y-x}\right| = \frac{|z-x|}{r} < 1$. This permits us to interchange integration and summation in Cauchy's Formula to get

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-x)^n$$

with $a_n = \frac{1}{2\pi i}\int_{\partial B(x;r)} \frac{f(y)}{(y-x)^{n+1}}\, dy$.

If

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-x)^n = \sum_{n=0}^{+\infty} a'_n (z-x)^n , \qquad z \in B(x;r) ,$$

then, using $z = x$, we get $a_0 = a'_0$. Cancelling a_0 and simplifying, we find

$$\sum_{n=0}^{+\infty} a_{n+1}(z-x)^n = \sum_{n=0}^{+\infty} a'_{n+1}(z-x)^n , \qquad z \in B(x;r)\setminus\{x\} .$$

By the continuity of both power series at x, we get $a_1 = a'_1$. We continue inductively to conclude that $a_n = a'_n$ for all n.

We may, inductively, see that, if f is holomorphic in Ω, then it has (complex) derivatives of all orders and that, for every $\overline{B(x;r)} \subseteq \Omega$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial B(x;r)} \frac{f(y)}{(y-z)^{n+1}}\, dy , \qquad z \in B(x;r) .$$

Therefore, the coefficients in the power series $f(z) = \sum_{n=0}^{+\infty} a_n(z-x)^n$ are given by $a_n = \frac{f^{(n)}(x)}{n!}$.

Suppose, now, that f is holomorphic in Ω and consider its expansion as a power series in any $B(x;r)$ with $\overline{B(x;r)} \subseteq \Omega$.

If all coefficients a_n are equal to 0, then $f = 0$ everywhere in $B(x;r)$ and we say that x is a **zero** of f **of infinite multiplicity**.

If $a_n \neq 0$ for at least one n, and N is the smallest such n, then

$$f(z) = (z-x)^N \sum_{n=0}^{+\infty} a_{n+N}(z-x)^n , \qquad z \in B(x;r) .$$

We, now, define

$$g(z) = \begin{cases} \frac{f(z)}{(z-x)^N} , & \text{if } z \in \Omega\setminus\{x\} \\ a_N \neq 0 , & \text{if } z = x \end{cases}$$

and it is trivial to prove that g is holomorphic in Ω. Therefore,

$$f(z) \;=\; (z-x)^N g(z)\ , \qquad z \in \Omega\ ,$$

where g is holomorphic in Ω and $g(x) \neq 0$. In this case we say that x is a **zero** of f **of finite multiplicity** and the number N is called the **multiplicity of** x **as a zero of** f and it is denoted by $m(x; f)$.

Of course, $f(x) \neq 0$ is equivalent to $m(x; f) = 0$. Whenever we say that x is a **zero** of f, we shall understand that $f(x) = 0$ or, equivalently, that $m(x; f) \geq 1$.

Using the series expansion, we, easily, see that the set of the zeros of finite multiplicity and the set of zeros of infinite multiplicity are both open sets and, hence, we get

Proposition 0.1 *If f is holomorphic in the open connected Ω, then either all points of Ω are zeros of f of finite multiplicity or $f = 0$ everywhere in Ω.*

We, also, see that if f is holomorphic in the open connected Ω and is not $= 0$ identically in Ω, then every zero of f is isolated. Therefore,

Theorem 0.5 *(Analytic Continuation Principle) If f is holomorphic in the open connected Ω and is not $= 0$ identically in Ω, then every compact subset of Ω contains at most finitely many zeros of f.*

Theorem 0.6 *(Argument Principle) Let f be holomorphic in Ω and the open set Ω_1 with C^1-boundary be such that $\overline{\Omega_1} \subseteq \Omega$. If $\partial\Omega_1$ contains no zeros of f, then*

$$\frac{1}{2\pi i}\int_{\partial\Omega_1} \frac{f'(y)}{f(y)}\, dy \;=\; \sum_{x\in\Omega_1} m(x; f)\ .$$

Proof:

By the Analytic Continuation Principle, $\overline{\Omega_1}$ contains at most finitely many zeros of f (which are all contained in Ω_1), $x_1, \ldots, x_N$, and let $m_j = m(x_j; f)$ be the corresponding multiplicities. We may, then, write

$$f(x) \;=\; (x-x_1)^{m_1}\cdots(x-x_N)^{m_N} g(x)\ , \qquad x \in \Omega\ ,$$

where g is holomorphic in Ω and has no zeros in $\overline{\Omega_1}$. Therefore,

$$\frac{f'(x)}{f(x)} \;=\; \sum_{k=1}^{N} \frac{m_k}{x-x_k} + \frac{g'(x)}{g(x)}\ , \qquad x \in \Omega\setminus\{x_1,\ldots,x_N\}\setminus\{x\in\Omega : g(x) = 0\}\ .$$

We consider small closed discs $\overline{B(x_j; r_j)}$ which are pairwise disjoint and are all contained in Ω_1 and the open set $\Omega_2 = \Omega_1 \setminus \cup_{j=1}^N \overline{B(x_j; r_j)}$.

Then, $\frac{f'}{f}$ is holomorphic in an open set containing $\overline{\Omega_2}$, and, by Cauchy's Theorem,

$$0 \;=\; \frac{1}{2\pi i}\int_{\partial\Omega_2} \frac{f'(y)}{f(y)}\, dy \;=\; \frac{1}{2\pi i}\int_{\partial\Omega_1} \frac{f'(y)}{f(y)}\, dy - \sum_{j=1}^{N} \frac{1}{2\pi i}\int_{\partial B(x_j; r_j)} \frac{f'(y)}{f(y)}\, dy\ .$$

Thus,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\Omega_1}\frac{f'(y)}{f(y)}\,dy &= \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{2\pi i}\int_{\partial B(x_j;r_j)}\frac{m_k}{y-x_k}\,dy\\ &\quad+\sum_{j=1}^{N}\frac{1}{2\pi i}\int_{\partial B(x_j;r_j)}\frac{g'(y)}{g(y)}\,dy\\ &= \sum_{j=1}^{N}m_j = \sum_{j=1}^{N}m(x_j;f) = \sum_{x\in\Omega_1}m(x;f)\ ,\end{aligned}$$

from the example after Cauchy's Formula and from Cauchy's Theorem applied to $\frac{g'}{g}$ which is holomorphic in Ω_1.

If $f(x) = w$, then x is a zero of $f - w$ and its multiplicity is denoted by $m(x; f, w)$.

Theorem 0.7 *Let f be holomorphic in Ω and the open set Ω_1 with C^1-boundary be such that $\overline{\Omega_1} \subseteq \Omega$. If $w', w'' \in \mathbf{C}$ are contained in the same component of the complement of the compact set $f(\partial\Omega_1)$, then*

$$\sum_{x\in\Omega_1} m(x; f, w') = \sum_{x\in\Omega_1} m(x; f, w'')\ .$$

Proof:
Applying the Argument Principle to the function $f - w$, we get

$$\sum_{x\in\Omega_1} m(x; f, w) = \frac{1}{2\pi i}\int_{\partial\Omega_1}\frac{f'(y)}{f(y)-w}\,dy\ , \qquad w \notin f(\partial\Omega_1)\ .$$

The integral in the right side is a continuous function of w in the complement of $f(\partial\Omega_1)$ and it is integer-valued, as the left side shows. Therefore, this function is constant in each component of the complement of $f(\partial\Omega_1)$.

An important example of holomorphic function is the **exponential function** $\exp : \mathbf{C} \to \mathbf{C}\setminus\{0\}$ given by

$$\exp(x) = e^{x_1}(\cos x_2 + i\sin x_2)\ , \qquad x = x_1 + ix_2 \in \mathbf{C}\ .$$

(Of course, there is no contradiction to use the notation e^x, instead of $\exp(x)$, and we shall very often do so.)

This satisfies the identity $\exp(x' + x'') = \exp(x)\exp(x'')$ and it is periodic with period $i2\pi$. In fact, the only periods of exp are the numbers $ik2\pi$, $k \in \mathbf{Z}$.

A, perhaps, even more important function is the **logarithmic "function"**, denoted by log, which is the *many-valued* inverse of exp. For each $x \in \mathbf{C}\setminus\{0\}$, we have that $\log(x) = \log|x| + i\theta + ik2\pi$, where θ is any real number so that $e^{i\theta} = \frac{x}{|x|}$ and k takes *all* integer values.

To be strict, one has to talk about branches of the logarithmic function, as follows. A function f in any $A \subseteq \mathbf{C} \setminus \{0\}$ is called a **branch of the logarithm in** A, if it is continuous in A and

$$\exp(f(x)) \;=\; x \;, \qquad x \in A \;.$$

More generally, if $g : A \to \mathbf{C} \setminus \{0\}$ is continuous in A, then $f : A \to \mathbf{C}$ is called a **branch of the logarithm of** g **in** A if it is continuous in A and

$$\exp(f(x)) \;=\; g(x) \;, \qquad x \in A \;.$$

If Ω is open and g is holomorphic in Ω, then it is trivial to show that every branch f of the logarithm of g in Ω is holomorphic in Ω and that

$$f'(x) \;=\; \frac{g'(x)}{g(x)} \;, \qquad x \in \Omega \;.$$

It is, also, trivial to show, by continuity of the branches, that, if A is connected, then any two branches of the logarithm of g in A differ by a constant of the form $ik2\pi$, $k \in \mathbf{Z}$.

As the simplest example, if $\Omega_0 = \mathbf{C} \setminus \{x = x_1 + ix_2 : x_2 = 0, x_1 \leq 0\}$ then the **principal branch** of the logarithm in Ω_0 is $\log_0 : \Omega_0 \to \mathbf{C}$ given by

$$\log_0(x) \;=\; \log|x| + i\theta \;,$$

where θ is the unique real number so that $\cos\theta + i\sin\theta = \frac{x}{|x|}$ and $-\pi < \theta < \pi$. Therefore, the totality of branches of the logarithm in Ω_0 are all functions of the form $\log_0(x) + ik2\pi$, $x \in \Omega_0$, where k runs in $\mathbf{Z}$.

If $\Omega_\phi = \mathbf{C} \setminus \{x = -re^{i\phi} : r \geq 0\}$, then the totality of branches of the logarithm in Ω_ϕ are the functions of the form $\log_0(e^{-i\phi}x) + i\phi + ik2\pi$, $x \in \Omega_\phi$, where k runs in $\mathbf{Z}$.

Here is a negative result.

Lemma 0.1 *Let A_ϕ and $A_{\phi+\pi}$ be connected with $A_\phi \subseteq \Omega_\phi$ and $A_{\phi+\pi} \subseteq \Omega_{\phi+\pi}$. If these two sets have a common point $x' = |x'|e^{i\theta'}$ with $\phi < \theta' < \phi + \pi$ and another common point $x'' = |x''|e^{i\theta''}$ with $\phi - \pi < \theta'' < \phi$, then there is no branch of the logarithm in $A = A_\phi \cup A_{\phi+\pi}$.*

Proof:

Let f be a branch of the logarithm in $A = A_\phi \cup A_{\phi+\pi}$.

Since A_ϕ is connected, $f(x) = \log_0(e^{-i\phi}x) + i\phi + ik2\pi$ for all $x \in A_\phi$ and for some $k \in \mathbf{Z}$. Hence, $f(x') - f(x'') = \log\left|\frac{x'}{x''}\right| + \theta' - \theta''$.

Since $A_{\phi+\pi}$ is connected, $f(x) = \log_0(e^{-i(\phi+\pi)}x) + i(\phi + \pi) + il2\pi$ for all $x \in A_{\phi+\pi}$ and for some $l \in \mathbf{Z}$. Hence, $f(x') - f(x'') = \log\left|\frac{x'}{x''}\right| + \theta' - \theta'' - 2\pi$.

We, thus, get a contradiction.

For instance, there is no branch of the logarithm in $\mathbf{C} \setminus \{0\}$ or, even, in any circle $S(0; r)$.

Lemma 0.1 is used to prove a purely topological result.

Theorem 0.8 *Let the compact sets* A_ϕ *and* $A_{\phi+\pi}$ *be connected with* $A_\phi \subseteq \Omega_\phi$ *and* $A_{\phi+\pi} \subseteq \Omega_{\phi+\pi}$. *Suppose, also, that these two sets have a common point* $x' = |x'|e^{i\theta'}$ *with* $\phi < \theta' < \phi + \pi$ *and another common point* $x'' = |x''|e^{i\theta''}$ *with* $\phi - \pi < \theta'' < \phi$.

Then, 0 *(which is not in* $A = A_\phi \cup A_{\phi+\pi}$*) does not belong to the unbounded component of* $\mathbf{C} \setminus A = \mathbf{C} \setminus \big(A_\phi \cup A_{\phi+\pi}\big)$.

Proof:

Consider the functions $g_a : \mathbf{C} \setminus \{a\}$ given by

$$g_a(x) \;=\; x - a \;, \qquad x \in \mathbf{C} \setminus \{a\} \;.$$

If $|a' - a| < d(a, A)$, then the existence of a branch of the logarithm of g_a in A implies the existence of a branch of the logarithm of $g_{a'}$ in A. Indeed, if f_a is a branch of the logarithm of g_a in A, then, for all $x \in A$,

$$e^{f_a(x)+\log_0(1+\frac{a-a'}{x-a})} \;=\; g_a(x)\Big(1 + \frac{a-a'}{x-a}\Big) \;=\; g_{a'}(x) \;.$$

It is obvious now that, if $|a' - a| < d(a', A)$, then the non-existence of a branch of the logarithm of g_a in A implies the non-existence of a branch of the logarithm of $g_{a'}$ in A.

Now, take any R so that $A \subseteq \overline{B(0;R)}$ and any point $x_0 \notin \overline{B(0;R)}$. Then, x_0 belongs to the unbounded component O of $\mathbf{C} \setminus A$.

The set of points $a \in O$ such that there exists a branch of the logarithm of g_a in A and the set of points $a \in O$ such that there does not exist a branch of the logarithm of g_a in A are, by the previous discussion, both open sets and x_0 is in the first set. By the connectedness of O, the second set is empty. Lemma 0.1, finally, implies that $0 \notin O$.

0.4 Equicontinuity

Let $\mathcal{F}$ be a family of complex-valued functions defined in a subset E of a metric space (X, d).

The family is called **bounded** at $x \in E$, if $\sup_{f\in\mathcal{F}} |f(x)| < +\infty$.

The family is called **equicontinuous** at $x \in E$, if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, so that for all $f \in \mathcal{F}$ it holds $|f(y) - f(x)| < \epsilon$ whenever $y \in E$ and $d(y, x) < \delta$.

Observe that δ, in this definition, does not depend on $f \in \mathcal{F}$.

The following is one of several versions of the Ascoli-Arzela Theorem.

Theorem 0.9 *(Arzela and Ascoli) Suppose* $\mathcal{F}$ *is a family of functions defined in some compact subset* K *of a metric space and let* $\mathcal{F}$ *be bounded and equicontinuous at every point of* K.

Then, from every sequence in $\mathcal{F}$ *we can extract a subsequence which converges uniformly in* K *to some function (not necessarily belonging to* $\mathcal{F}$*).*

In these notes we shall apply this result when the metric space is either $\mathbf{R^n}$ with the Euclidean metric or $\overline{\mathbf{R^n}}$ with the spherical metric.

0.5 Semi-continuity

Suppose that E is a subset of a metric space (X, d).

Definition 0.1 *A function f is called* ***lower-semicontinuous*** *in E, if*

1. $-\infty < f(x) \leq +\infty$ *for all $x \in E$ and*
2. $f(x) \leq \liminf_{E \ni y \to x} f(y)$ *for all $x \in E$*

or, equivalently, if

1. $-\infty < f(x) \leq +\infty$ *for all $x \in E$ and*
2. $\{y \in E : \lambda < f(x)\}$ *is open relative to E for every real λ.*

We call f ***upper-semicontinuous*** *in E, if $-f$ is lower-semicontinuous in E: in all relations above we just reverse the inequalities, replace $\pm\infty$ by $\mp\infty$ and replace* $\liminf$ *by* $\limsup$.

Properties of semicontinuous functions
(1) *f is continuous in E if and only if it is simultaneously lower- and upper-semicontinuous in E.*
(2) *Linear combinations of lower-semicontinuous functions with non-negative coefficients are lower-semicontinuous. This is true for upper-semicontinuous functions, also.*

The proofs are easy.

(3) *The supremum of any family of lower-semicontinuous functions is lower-semicontinuous. There is a dual statement for upper-semicontinuous functions.*

Suppose that each $f \in \mathcal{F}$ is lower-semicontinuous in E and let $F(x) = sup_{f\in\mathcal{F}} f(x)$ for every $x \in E$. Then, it is obvious that $-\infty < F(x)$ for all $x \in E$ and, since $f(y) \leq F(y)$ for all $y \in E$ and all $f \in \mathcal{F}$, we find

$$f(x) \ \leq \ \liminf_{E \ni y \to x} f(y) \ \leq \ \liminf_{E \ni y \to x} F(y)$$

for $x \in E$. Taking the supremum over all $f \in \mathcal{F}$, we conclude

$$F(x) \ \leq \ \liminf_{E \ni y \to x} F(y) \ .$$

(4) *The minimum of finitely many lower-semicontinuous functions is lower-semicontinuous. There is a dual statement for upper-semicontinuous functions.*

It is enough to consider two lower-semicontinuous functions f_1, f_2 and let $f = \min(f_1, f_2)$. Then, it is obvious that $f(x) > -\infty$ for all $x \in E$.

Take $\lambda < f(x)$ and, hence, $\lambda < f_1(x)$ and $\lambda < f_2(x)$. There exist $\delta_1 > 0$ and $\delta_2 > 0$ so that $\lambda < f_1(y)$ for all $y \in E$ with $d(y, x) < \delta_1$ and $\lambda < f_2(y)$ for all $y \in E$ with $d(y, x) < \delta_2$.

If $\delta = \min(\delta_1, \delta_2)$, then $\lambda < f(y)$ for all $y \in E$ with $d(y, x) < \delta$. Therefore, $f(x) \leq \liminf_{E \ni y \to x} f(y)$.

The following four propositions state properties of semicontinuous functions which we shall make constant use of in later chapters.

Proposition 0.2 *If f is lower-semicontinuous in a compact set K, then f is bounded from below in K and takes a minimum value in K.*

There is an obvious dual statement for upper-semicontinuous functions.

Proof:

Let $m = \inf_{x\in K} f(x)$ and take $\{x_k\}$ in K so that $f(x_k) \to m$. Replacing $\{x_k\}$ by some subsequence, if necessary, we may assume that $x_k \to x$ for some $x \in K$. But then,

$$m \ \le\ f(x) \ \le\ \liminf_{E\ni y\to x} f(y) \ \le\ \lim_{k\to+\infty} f(x_k) \ =\ m$$

and we get that $-\infty < m = f(x)$.

Another proof of the boundedness from below runs as follows. Since f does not take the value $-\infty$,

$$K \ \subseteq\ \bigcup_{k=1}^{+\infty} \{x \in K : -k < f(x)\}\ .$$

The terms of the union are open and increasing with k and, since K is compact, K is contained in one of them.

The following is a partial converse of property (3) of lower-semicontinuous functions.

Proposition 0.3 *If f is lower-semicontinuous in a compact set K, then there exists an increasing sequence of continuous functions $\{f_k\}$ in K which converges to f pointwise in K.*

There is the usual dual statement for upper-semicontinuous functions.

Proof:

If $f = +\infty$ identically in K, we consider $f_k = k$ identically in K.

Otherwise, we define

$$f_k(x) \ =\ \inf_{y\in K} (f(y) + kd(x,y))$$

for all $x \in K$. Since, by Proposition 0.2, both f and $d(x,\cdot)$ are bounded from below in K, we have that $f_k(x)$ is a real number.

From the inequality

$$\big|(f(y) + kd(x,y)) - (f(y) + kd(x',y))\big| \ \le\ kd(x,x')\ ,$$

we, easily, prove that

$$|f_k(x) - f_k(x')| \ \le\ kd(x,x')\ ,$$

implying that f_k is continuous in K.

It is, also, clear that $\{f_k\}$ is increasing and that $f_k(x) \leq f(x)$ for all k and all x: just take $y = x$ in the definition of $f_k(x)$.

Now, fix $x \in K$ and $\lambda < f(x)$.

By the lower-semicontinuity of f, there exists $\delta > 0$ so that $\lambda \leq f(y)$ for all $y \in K$ with $d(x, y) < \delta$.

If we take k large enough, then $\lambda \leq f(y) + kd(x, y)$ for all $y \in K$ with $d(x, y) \geq \delta$. In fact, by Proposition 0.2, $\min_K f$ is finite and, then it is enough to take $k \geq \frac{1}{\delta}(\lambda - \min_K f)$.

Therefore, if k is large enough,

$$\lambda \;\leq\; \inf_{y \in K}(f(y) + kd(x, y)) \;=\; f_k(x) \;.$$

This, together with $f_k(x) \leq f(x)$, implies

$$\lim_{k \to +\infty} f_k(x) \;=\; f(x)$$

for all $x \in K$.

Proposition 0.4 *(First Minimum Principle) Let f be lower-semicontinuous in a connected subset E of a metric space with the property that, if it has a minimum value at some point, then it is constant in some open (relative to E) neighborhood of the same point.*

Then, if f takes a minimum value in E, it is constant in E.

There is a dual Maximum Principle for upper-semicontinuous functions.

Proof:

The assumptions imply that both sets $\{x \in E : \inf_E f = f(x)\}$ and $\{x \in E : \inf_E f < f(x)\}$ are open relative to E. Therefore, since E is connected, one of them is empty and the other is all of E.

Proposition 0.5 *(Second Minimum Principle) Suppose that O is an open connected subset of a metric space with compact closure $\overline{O}$ and with non-empty boundary. Let f be lower-semicontinuous in O with the property that, if it has a minimum value at some point, then it is constant in some open neighborhood of the same point.*

1. *If f takes a minimum value in O, then it is constant in O.*
2. *Let $m = \inf_{y \in \partial O}\big(\liminf_{O \ni x \to y} f(x)\big)$. Then $m \leq f(x)$ for all x in O.*

   If $m = f(x)$ for some $x \in O$, then $f = m$ identically in O.

There is a dual Maximum Principle for upper-semicontinuous functions.

Proof:

The first part is straightforward from the First Minimum Principle.

As for the second part, extend f in ∂O, defining

$$f(y) \;=\; \liminf_{O \ni x \to y} f(x)$$

for all $y \in \partial O$.

Then, it is easy to see that f becomes lower-semicontinuous in $\overline{O}$ and, from Proposition 0.2, this extended f takes a minimum value in $\overline{O}$, say m^*.

If $m^* < m$, then m^* is taken at a point of O and, by the first part, f must be constant, $f = m^*$, in O. Therefore, taking any boundary point y,

$$m \;\le\; f(y) \;=\; \liminf_{O \ni x \to y} f(x) \;=\; m^*$$

and we get a contradiction.

Hence, $m \le m^*$.

If for some $x \in O$ we have $m = f(x)$, then, by the first part, $f = m$ in O.

0.6 Borel Measures

1. The **Borel σ-algebra** $\mathcal{B}(\mathbf{R^n})$ is the smallest σ-algebra of subsets of $\mathbf{R^n}$ which contains all open sets. Its elements are called **Borel sets**.

If A is any Borel set, then we define the Borel σ-algebra of A by

$$\mathcal{B}(A) \;=\; \{B \in \mathcal{B}(\mathbf{R^n}) : B \subseteq A\} \;.$$

A complex measure $d\mu$ on $\mathcal{B}(A)$ is called a **complex Borel measure in** A.

A non-negative measure $d\mu$ on $\mathcal{B}(A)$ is called a **non-negative Borel measure in** A if, additionally, $d\mu(K) < +\infty$ for every compact $K \subseteq A$.

A **signed Borel measure in** A is any difference of two non-negative Borel measures in A at least one of which is finite.

2. A Borel measure $d\mu$ of any kind (complex, signed, non-negative) in $\mathbf{R^n}$ is said to be **supported** in a Borel set A, if $d\mu(B) = 0$ for all Borel sets B such that $B \cap A = \emptyset$.

For any kind of Borel measure $d\mu$ in $\mathbf{R^n}$ and any Borel set A, we define the **restriction** of $d\mu$ in A by

$$d\mu_A(B) \;=\; d\mu(B \cap A)$$

for all Borel sets B. This is a Borel measure supported in A.

One should observe and keep in mind the difference between a Borel measure *in* a Borel set A and a Borel measure in $\mathbf{R^n}$ *supported in* the Borel set A.

3. Every signed Borel measure $d\mu$ has a **non-negative variation** $d\mu^+$ and a **non-positive variation** $d\mu^-$, which are *both non-negative* Borel measures, they are supported in disjoint Borel sets, at least one of them is finite and satisfy $d\mu = d\mu^+ - d\mu^-$. Then, the non-negative Borel measure $|d\mu| = d\mu^+ + d\mu^-$ is called the **absolute variation** of $d\mu$.

It is true that, for every signed Borel measure $d\mu$ and every Borel set A, $|d\mu|(A) = \sup \sum_{k=1}^{m} |d\mu(B_k)|$ over all $m \in \mathbf{N}$ and all partitions $A = \cup_{k=1}^{m} B_k$ of A into pairwise disjoint Borel subsets of it.

We extend to the case of complex Borel measures and, for every complex Borel measure $d\mu$ and every Borel set A, we define

$$|d\mu|(A) \;=\; \sup \sum_{k=1}^{m} |d\mu(B_k)|$$

over all $m \in \mathbf{N}$ and all partitions $A = \cup_{k=1}^{m} B_k$ of A into pairwise disjoint Borel subsets of it.

Then, $|d\mu|$ is a finite non-negative Borel measure and the finite number

$$\|d\mu\| \;=\; |d\mu|(\mathbf{R^n})$$

is called the **total variation** of $d\mu$.

4. If f is integrable in $\mathbf{R^n}$, then a complex Borel measure $d\mu_f$ is defined by

$$d\mu_f(A) \;=\; \int_A f(x)\; dm(x)$$

for all Borel sets A. This measure is denoted, also, by

$$f\, dm \;=\; d\mu_f \;.$$

The integrable function f is called the **density function** or the **Radon-Nikodym derivative with respect to** dm of the complex Borel measure $f\, dm$.

If $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$, whence $f = f^+ - f^-$ and $|f| = f^+ + f^-$, then

$$(f\, dm)^+ \;=\; f^+\, dm \;, \quad (f\, dm)^- \;=\; f^-\, dm \;, \quad |f\, dm| \;=\; |f|\, dm$$

and

$$\|f\, dm\| \;=\; \int_{\mathbf{R^n}} |f(x)|\; dm(x) \;.$$

We, also, have that $f\, dm$ is supported in the Borel set A if and only if $f = 0$ almost everywhere in $\mathbf{R^n} \setminus A$.

The complex Borel measures of the form $d\mu = f\, dm$ are called **absolutely continuous** and are characterized by the property:

$$d\mu(A) \;=\; 0 \quad \text{for all Borel sets } A \text{ with } dm(A) = 0 \;.$$

For every $x \in \mathbf{R^n}$ we define the **Dirac mass at** x as the measure

$$d\delta_x(A) \;=\; \begin{cases} 1 \;, & \text{if } x \in A \\ 0 \;, & \text{if } x \notin A \end{cases}$$

for all Borel sets A. This is not absolutely continuous and is supported in $\{x\}$.
5. The space $\mathcal{M}(\mathbf{R^n})$ of all complex Borel measures in $\mathbf{R^n}$ is a linear space and $\|\cdot\|$ is a norm on it. Under this norm, $\mathcal{M}(\mathbf{R^n})$ becomes a Banach space.

If the complex Borel measure $d\mu$ is supported in the Borel set A, then, obviously, $|d\mu|$ is also supported in A and $\|d\mu\| = |d\mu|(A)$. We, then, denote by $\mathcal{M}(A)$ the space of all complex Borel measures supported in A. This space is just a closed linear subspace of $\mathcal{M}(\mathbf{R^n})$ and, hence, a Banach space itself.
6. If A is a subset of a metric space, then $C_B(A)$ is the Banach space of all complex-valued functions continuous and bounded in A, with the norm

$$\|f\|_\infty \;=\; \sup_{x\in A} |f(x)| \;, \qquad f \in C_B(A) \;.$$

If A is compact, then $C_B(A) = C(A)$.

We shall need the following simple version of an important theorem.

Theorem 0.10 *(Representation Theorem of J. Radon and F. Riesz) Let L be a bounded linear functional on the Banach space $C(K)$, where K is any compact subset of $\mathbf{R^n}$. I.e.*

$$L(af + bg) \;=\; aL(f) + bL(g)$$

for all $a, b \in \mathbf{C}$ and all $f, g \in C(K)$ and

$$|L(f)| \;\le\; \kappa \,\|f\|_\infty$$

for all $f \in C(K)$ and some κ not depending on f.

Then, there exists a unique complex Borel measure $d\mu$ supported in K so that

$$L(f) \;=\; \int_K f(x)\; d\mu(x)$$

for all $f \in C(K)$. We, also, have that

$$\|d\mu\| \;=\; \|L\| \;=\; \sup\{|L(f)| : f \in C(K),\ \|f\|_\infty \le 1\} \;.$$

If L is, also, non-negative, i.e.

$$L(f) \;\ge\; 0$$

for all $f \in C(K)$ with $f \ge 0$ everywhere in K, then $d\mu$ is non-negative.

If for every $d\mu \in \mathcal{M}(K)$ we define the function

$$L_{d\mu}(f) \;=\; \int_K f(x)\; d\mu(x) \;, \qquad f \in C(K) \;,$$

then $L_{d\mu}$ is a continuous linear functional on the Banach space $C(K)$, i.e. an element of $C(K)^*$.

The content of the Representation Theorem of J. Radon and F. Riesz is that the (obviously, linear) mapping

$$\mathcal{M}(K) \ni d\mu \;\mapsto\; L_{d\mu} \in C(K)^*$$

is a bijective isometry.
7. If $\{d\mu_m\}$ is a sequence of Borel measures, then we say that it **converges weakly on the compact set** K to the Borel measure $d\mu$, if for all $f \in C(K)$,

$$\int_K f(x)\ d\mu_m(x) \ \to \ \int_K f(x)\ d\mu(x)\ .$$

In this case, by the Uniform Boundedness Theorem, it is true that there is a constant M so that $|d\mu|(K) \leq M$ and $|d\mu_m|(K) \leq M$ for all m.

As an application of the Banach-Alaoglou Theorem in the Banach space $C(K)$, we get that for any sequence $\{d\mu_m\}$ of complex Borel measures supported in K with $\|d\mu_m\| \leq M$ for some $M < +\infty$ not depending on m, there is some subsequence converging weakly on K to some complex Borel measure supported in K.

If the sequences $\{d\mu_m^1\}$ and $\{d\mu_m^2\}$ converge weakly on the compact subsets K_1 and K_2 of $\mathbf{R^n}$ to $d\mu^1$ and $d\mu^2$, respectively, then the product measures $d\mu_m^1 \times d\mu_m^2$ converges weakly on $K_1 \times K_2$ to $d\mu^1 \times d\mu^2$. To show this, consider an arbitrary product $f_1(x)f_2(y)$ of continuous functions in K_1 and K_2. Then, by the Theorem of Fubini, it is clear that

$$\int_{K_1\times K_2} f_1(x)f_2(y)\ d\mu_m^1 \times d\mu_m^2(x,y) \ \to \ \int_{K_1\times K_2} f_1(x)f_2(y)\ d\mu^1 \times d\mu^2(x,y)\ .$$

For the arbitrary continuous $f(x,y)$ in $K_1 \times K_2$, we use the Stone-Weierstrass Theorem to approximate f by a polynomial $\sum x^\alpha y^\beta$ uniformly in $K_1 \times K_2$ and that $\|d\mu_m^1 \times d\mu_m^2\| \leq \|d\mu_m^1\|\|d\mu_m^2\| \leq M_1M_2 < +\infty$, to prove

$$\int_{K_1\times K_2} f(x,y)\ d\mu_m^1 \times d\mu_m^2(x,y) \ \to \ \int_{K_1\times K_2} f(x,y)\ d\mu^1 \times d\mu^2(x,y)\ .$$

8. All kinds of Borel measures $d\mu$ are **regular**. This means that, for every Borel set B with finite $d\mu(B)$ and every $\epsilon > 0$, there is an open set U and a compact set K so that $K \subseteq B \subseteq U$ and $|d\mu|(U \setminus K) < \epsilon$.
9. Let $d\mu$ be a non-negative Borel measure and let the extended-real-valued f be defined in the Borel set A. We, then, define the **upper integral** of f in A by

$$\overline{\int}_A f(x)\ d\mu(x) \ = \ \inf \int_A \phi(x)\ d\mu(x)\ ,$$

where the infimum is taken over all bounded from below lower-semicontinuous ϕ in A with $\phi \geq f$ everywhere in A. We, also, define the **lower integral**

$$\underline{\int}_A f(x)\ d\mu(x) \ = \ \sup \int_A \psi(x)\ d\mu(x)\ ,$$

where the supremum is taken over all bounded from above upper-semicontinuous ψ in A with $\psi \leq f$ everywhere in A.

It is, then, true that

$$\underline{\int}_A f(x)\ d\mu(x)\ \leq\ \overline{\int}_A f(x)\ d\mu(x)$$

It is, also, true that f is $d\mu$-integrable in A if and only if

$$-\infty\ <\ \underline{\int}_A f(x)\ d\mu(x)\ =\ \overline{\int}_A f(x)\ d\mu(x)\ <\ +\infty\ ,$$

and, in this case, the common value of the upper and the lower integral is equal to $\int_A f(x)\ d\mu(x)$.

0.7 Distributions

The following is only a short exposition of very few elementery facts about distributions with brief proofs of only those of them which are considered new or not easy enough to be proved by the inexperienced reader.

If f is a measurable function in $\mathbf{R^n}$, then $x \in \mathbf{R^n}$ is said to be a **support-point** for f, if f is almost everywhere 0 in *no* neighborhood of x.

The **support** of a measurable function $f : \mathbf{R^n} \to \mathbf{C}$ is the smallest closed set in $\mathbf{R^n}$ outside of which f is almost everywhere 0 and it is defined by

$$supp(f)\ =\ \{x \in \mathbf{R^n} : x \text{ is a support-point for } f\}\ .$$

Saying that the support of a measurable function is bounded is equivalent to saying that it is a compact subset of $\mathbf{R^n}$ and equivalent to saying that the function vanishes almost everywhere outside a compact subset of $\mathbf{R^n}$. In this case, we say that the function **is compactly supported** or that it **has compact support**.

In case f is continous in $\mathbf{R^n}$, then it is easy to see that its support is the smallest closed set in $\mathbf{R^n}$ outside of which f is everywhere 0.

If $d\mu$ is a Borel measure (of any kind), we say that the point $x \in \mathbf{R^n}$ is a **support-point** of $d\mu$, if $d\mu$ is the zero measure in *no* neighborhood of x. The **support** or, sometimes called, **closed-support** of $d\mu$ is defined by

$$supp(d\mu)\ =\ \{x \in \mathbf{R^n} : x \text{ is a support-point of } d\mu\}\ .$$

It is easy to see, using the regularity of $d\mu$, that $supp(d\mu)$ is the smallest closed set in $\mathbf{R^n}$ outside of which $d\mu$ is the zero measure.

Example

The finite non-negative Borel measure $d\mu = \sum_{k=1}^{+\infty} 2^{-k} d\delta_{\frac{1}{k}}$ is supported in the set $\{\frac{1}{k} : k \in \mathbf{N}\}$, but $supp(d\mu) = \{\frac{1}{k} : k \in \mathbf{N}\} \cup \{0\}$.

Whenever we say that $d\mu$ is **compactly supported** or that it **has compact support**, we mean that $supp(d\mu)$ is a compact subset of $\mathbf{R^n}$.

In case $d\mu = f\,dm$ is an absolutely continuous compex Borel measure with density function $f \in L^1(\mathbf{R^n})$, then, clearly,

$$supp(f\,dm) \;=\; supp(f)\;.$$

Let $f, g \in L^1(\mathbf{R^n})$. Then, the **convolution**

$$f * g(x) \;=\; \int_{\mathbf{R^n}} f(x-y)g(y)\;dm(y)$$

is defined for almost every x and it is true that $f * g \in L^1(\mathbf{R^n})$. Also,

$$\|f * g\|_{L^1(\mathbf{R^n})} \;\le\; \|f\|_{L^1(\mathbf{R^n})}\|g\|_{L^1(\mathbf{R^n})}\;.$$

It is true that

$$f * g \;=\; g * f\;, \qquad f * (g * h) \;=\; (f * g) * h$$

and

$$supp(f * g) \;\subseteq\; \overline{supp(f) + supp(g)}\;,$$

where $A + B = \{x + y : x \in A, y \in B\}$ for $A, B \subseteq \mathbf{R^n}$.

The convolution is defined in other instances also.

Let f be measurable in $\mathbf{R^n}$ and integrable in all compact subsets of $\mathbf{R^n}$. We, then, say that f is **locally integrable** in $\mathbf{R^n}$ and the space of all such f is denoted by $L^1_{loc}(\mathbf{R^n})$.

If f is locally integrable and g is integrable and compactly supported, then $f * g(x)$ is, obviously, well-defined for almost every x.

The convolution is a useful device and one reason is that it preserves the regularity properties of its component functions.

For instance,

Proposition 0.6 *Suppose f is locally integrable and g is continuous and compactly supported. Then $f * g$ is everywhere defined and continuous.*

If, moreover, g is in $C^k(\mathbf{R^n})$, then $f * g$ is, also, in $C^k(\mathbf{R^n})$ and

$$D^\alpha(f * g) \;=\; f * D^\alpha g$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with order $|\alpha| \le k$.

The same conclusions hold, if we assume that f is locally integrable and compactly supported and g is continuous or in $C^k(\mathbf{R^n})$.

Proof:
(i) Let $supp(g)$ be contained in the compact $K \subseteq \mathbf{R^n}$. Then, for every x,

$$\int_{\mathbf{R^n}} |f(x-y)g(y)|\;dm(y) \;\le\; \|g\|_\infty \int_{x-K} |f(y)|\;dm(y) \;<\; +\infty\;,$$

since $x - K = \{x - y : y \in K\}$ is compact.
(ii) For fixed x and for $|x' - x| \le \delta \le 1$, the function $g(x' - \cdot) - g(x - \cdot)$ has its support contained in the compact subset $L = \{y : d(y, x - K) \le 1\}$ of $\mathbf{R^n}$.

Therefore,

$$\begin{aligned} |f*g(x')-f*g(x)| &\le \int_L |f(y)||g(x'-y)-g(x-y)|\;dm(y) \\ &\le \sup_{|a-b|\le\delta} |g(a)-g(b)| \int_L |f(y)|\;dm(y) \end{aligned}$$

and, since g is uniformly continuous, the last quantity tends to 0 as $\delta\to 0$.

Thus, $f*g$ is continuous at x.

(iii) Now, suppose that $g\in C^1(\mathbf{R^n})$. Then, as before, for fixed x and for $h\in\mathbf{R}$ with $|h|\le\delta\le 1$,

$$\begin{aligned} &\Big|\frac{f*g(x+he_j)-f*g(x)}{h} - \int_{\mathbf{R^n}} f(y)\frac{\partial g}{\partial x_j}(x-y)\;dm(y)\Big| \\ &\le \int_L |f(y)|\Big|\frac{g(x+he_j-y)-g(x-y)}{h} - \frac{\partial g}{\partial x_j}(x-y)\Big|\;dm(y) \\ &\le \sup_{|a-b|\le\delta}\Big|\frac{\partial g}{\partial x_j}(a)-\frac{\partial g}{\partial x_j}(b)\Big|\int_L |f(y)|\;dm(y) \end{aligned}$$

and, since $\frac{\partial g}{\partial x_j}$ is uniformly continuous, the last quantity tends to 0 as $\delta\to 0$.

Thus, $\frac{\partial(f*g)}{\partial x_j}(x)=f*\frac{\partial g}{\partial x_j}(x)$ for every x and, from part (ii), $f*g\in C^1(\mathbf{R^n})$.

Using induction, we extend to derivatives of higher order.

The proof of the second part is, after trivial modifications, similar to the proof of the first part.

A useful technical tool is the function

$$\phi_0(t) \;=\; \begin{cases} \exp(-\frac{1}{t})\;, & \text{if } t>0 \\ 0\;, & \text{if } t\le 0\;. \end{cases}$$

It is easy to prove that ϕ_0 is in $C^\infty(\mathbf{R})$ and $supp(\phi_0)=\mathbf{R}_0^+$. Therefore, if $C>0$, the function

$$\Phi(x) \;=\; \begin{cases} C\exp(-\frac{1}{1-|x|^2})\;, & \text{if } x\in B(0;1) \\ 0\;, & \text{if } x\notin B(0;1) \end{cases}$$

has the properties:

1. Φ is in $C^\infty(\mathbf{R^n})$,
2. Φ is non-negative and $supp(\Phi)=\overline{B(0;1)}$,
3. Φ is radial. I.e. $\Phi(x)=\Phi(y)$ whenever $|x|=|y|$,
4. $\int_{\mathbf{R^n}}\Phi(x)\;dm(x)=1$, if we choose the constant C appropriately.

Now, the functions ($\delta > 0$)

$$\Phi_\delta(x) \;=\; \frac{1}{\delta^n}\,\Phi(\frac{x}{\delta})\;, \qquad x \in \mathbf{R^n}\;,$$

have the same properties 1, 3 and 4 and property 2 replaced by

$$supp(\Phi_\delta) = \overline{B(0;\delta)}\;.$$

Definition 0.2 *The family of functions $\{\Phi_\delta : \delta > 0\}$, or any other similar family coming from any Φ with properties 1-4, is called an* ***approximation to the identity****.*

Starting with any radial function F, the following function is well-defined for all $r \in \mathbf{R}_0^+$ for which there is at least one x in the domain of definition of F with $|x| = r$:

$$F_*(r) \;=\; F(x)\;, \qquad \text{for any } x \text{ with } r = |x|\;.$$

This is useful whenever we use polar coordinates to evaluate integrals.

Let f be measurable in Ω, an open subset of $\mathbf{R^n}$, and integrable in all compact subsets of Ω. We call f **locally integrable in** Ω and the space of all such f is denoted by $L^1_{loc}(\Omega)$.

Proposition 0.7 *If f is in $L^1_{loc}(\Omega)$, g is in $C^k(\mathbf{R^n})$ and $supp(g) \subseteq \overline{B(0;\delta)}$, then the convolution*

$$f * g(x) \;=\; \int_{\mathbf{R^n}} f(x-y)g(y)\; dm(y)$$

is defined for all x in the open set

$$\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}$$

and it is in $C^k(\Omega_\delta)$. Moreover,

$$D^\alpha(f * g) \;=\; f * D^\alpha g$$

in Ω_δ, for all α with $|\alpha| \le k$.

Proof:

A minor modification of the proof of Proposition 0.6.

Definition 0.3 *Let Ω be an open subset of $\mathbf{R^n}$. Then the* ***space of test-functions*** *in Ω, denoted by $\mathcal{D}(\Omega)$, is the space of all infinitely differentiable complex-valued functions with compact support contained in Ω.*

$\mathcal{D}(\Omega)$ is a linear space and $f\phi \in \mathcal{D}(\Omega)$ for every $f \in C^\infty(\Omega)$ and every $\phi \in \mathcal{D}(\Omega)$. In fact, since $C^\infty(\Omega)$ is an algebra, we see that $\mathcal{D}(\Omega)$ is an ideal in $C^\infty(\Omega)$.

The next result is technically very helpful.

Lemma 0.2 *Suppose that Ω is an open subset of $\mathbf{R^n}$ and K is a compact subset of Ω. Then, there exists a $\phi \in \mathcal{D}(\Omega)$ so that*

1. $0 \leq \phi \leq 1$ *everywhere in Ω and*

2. $\phi = 1$ *everywhere in K.*

Proof:

Consider $\delta_0 < \frac{1}{4}\, d(K, \partial\Omega)$ and the compact set

$$K_{2\delta_0} \;=\; \{x : d(x,K) \leq 2\delta_0\} \; .$$

We also consider any approximation to the identity $\{\Phi_\delta : \delta > 0\}$ and the convolution

$$\phi \;=\; \chi_{K_{2\delta_0}} * \Phi_{\delta_0} \; ,$$

where $\chi_{K_{2\delta_0}}$ is the characteristic function of $K_{2\delta_0}$.

By Proposition 0.6, $\phi \in C^\infty(\mathbf{R^n})$ and $supp(\phi) \subseteq K_{2\delta_0} + \overline{B(0;\delta_0)}$ which is a compact subset of Ω. Hence, $\phi \in \mathcal{D}(\Omega)$.

Also, for every x,

$$0 \;\leq\; \phi(x) \;=\; \int_{K_{2\delta_0}} \Phi_{\delta_0}(x-y)\; dm(y) \;\leq\; \int_{\mathbf{R^n}} \Phi_{\delta_0}(x-y)\; dm(y) \;=\; 1 \; .$$

Finally, if $x \in K$, then $B(x;\delta_0) \subseteq K_{2\delta_0}$ and, thus,

$$\phi(x) \;=\; \int_{B(x;\delta_0)} \chi_{K_{2\delta_0}}(y)\Phi_{\delta_0}(x-y)\; dm(y) \;=\; \int_{B(x;\delta_0)} \Phi_{\delta_0}(x-y)\; dm(y) \;=\; 1 \; .$$

We define a notion of **convergence** for sequences in $\mathcal{D}(\Omega)$ as follows.

Definition 0.4 *Let $\{\phi_m\}$ and ϕ be in $\mathcal{D}(\Omega)$. We write*

$$\phi_m \to \phi \quad in \quad \mathcal{D}(\Omega) \; ,$$

if

1. *there exists some compact $K \subseteq \Omega$ so that $supp(\phi_m) \subseteq K$ for all m and*

2. *$D^\alpha\phi_m \to D^\alpha\phi$ uniformly in Ω for every multi-index α .*

Of course, 1 and 2 imply that $supp(\phi) \subseteq K$.

Proposition 0.8 1. *If $\phi_m \to \phi$ and $\psi_m \to \psi$ in $\mathcal{D}(\Omega)$, then $\lambda\phi_m + \mu\psi_m \to \lambda\phi + \mu\psi$ and $\phi_m\psi_m \to \phi\psi$ in $\mathcal{D}(\Omega)$ for all $\lambda, \mu \in \mathbf{C}$.*

2. *If $\phi_m \to \phi$ in $\mathcal{D}(\Omega)$ and $f \in C^\infty(\Omega)$, then $f\phi_m \to f\phi$ in $\mathcal{D}(\Omega)$.*

Proof:

If the supports of all ϕ_m are contained in the compact K and the supports of all ψ_m are contained in the compact L, then the supports of all $\lambda\phi_m + \mu\psi_m$ are contained in the compact $K \cup L$, the supports of all $\phi_m\psi_m$ are contained in the compact $K \cap L$ and the supports of all $f\phi_m$ are contained in K.

The rest is an easy application of the rule of Leibniz for the derivatives of products.

Proposition 0.9 *For every ϕ in $\mathcal{D}(\Omega)$ and any approximation to the identity $\{\Phi_\delta : \delta > 0\}$ we have that $\phi * \Phi_\delta$ belongs to $\mathcal{D}(\Omega)$ for small enough δ and*

$$\phi * \Phi_\delta \to \phi \qquad \textit{in} \quad \mathcal{D}(\Omega)$$

as $\delta \to 0+$.

Proof:

Observe that ϕ, defined to be equal to 0 outside Ω, is in $C^\infty(\mathbf{R^n})$. Therefore, from Proposition 0.6,

$$D^\alpha(\phi * \Phi_\delta) \;=\; D^\alpha\phi * \Phi_\delta$$

in $\mathbf{R^n}$.

If we consider $K = supp(\phi)$ and $\delta_0 = \frac{1}{2}\, d(K, \partial\Omega) > 0$, then, for all $\delta \le \delta_0$, the support of $\phi * \Phi_\delta$ is contained in the compact set $L = K + \overline{B(0;\delta_0)} = \{x : d(x,K) \le \delta_0\} \subseteq \Omega$.

Hence, $\phi * \Phi_\delta$ belongs to $\mathcal{D}(\Omega)$ for all $\delta \le \delta_0$.

Now, for every x,

$$\begin{aligned} |D^\alpha\big(\phi * \Phi_\delta\big)(x) - D^\alpha\phi(x)| &= |D^\alpha\phi * \Phi_\delta(x) - D^\alpha\phi(x)| \\ &\le \int_{\mathbf{R^n}} |D^\alpha\phi(x-y) - D^\alpha\phi(x)|\Phi_\delta(y)\; dm(y) \\ &= \int_{\mathbf{R^n}} |D^\alpha\phi(x-\delta y) - D^\alpha\phi(x)|\Phi_1(y)\; dm(y) \\ &\le \sup_{|a-b|\le\delta} |D^\alpha\phi(a) - D^\alpha\phi(b)| \end{aligned}$$

and the last term tends to 0 as $\delta \to 0$.

Therefore, $D^\alpha\big(\phi * \Phi_\delta\big) \to D^\alpha\phi$ uniformly in $\mathbf{R^n}$.

This notion of convergence defines a topology in the space $\mathcal{D}(\Omega)$, which then becomes a locally convex topological vector space, but we shall not touch and not need this more abstract matter.

We call distribution in Ω any continuous linear functional T on $\mathcal{D}(\Omega)$. Thus,

Definition 0.5 *A function*

$$T \;:\; \mathcal{D}(\Omega) \to \mathbf{C}$$

is called ***distribution in*** Ω*, if*

1. *$T(\lambda\phi + \mu\psi) = \lambda T(\phi) + \mu T(\psi)$ for all $\phi, \psi \in \mathcal{D}(\Omega)$ and all $\lambda, \mu \in \mathbf{C}$.*
2. *$T(\phi_m) \to T(\phi)$ whenever $\phi_m \to \phi$ in $\mathcal{D}(\Omega)$.*

We denote by $\mathcal{D}^*(\Omega)$ the set of all distributions in Ω .

The space $\mathcal{D}^*(\Omega)$ becomes a linear space, when we define

$$(\lambda T + \mu S)(\phi) = \lambda T(\phi) + \mu S(\phi)$$

for all $\phi \in \mathcal{D}(\Omega)$. In fact, it is trivial to prove that $\lambda T + \mu S$ is a distribution in Ω whenever T and S are distributions in Ω .

We can, also, define the notion of **(w^*-)convergence** in $\mathcal{D}^*(\Omega)$ as follows.

Definition 0.6 *We write*

$$T_m \to T \quad in \quad \mathcal{D}^*(\Omega) ,$$

if

$$T_m(\phi) \to T(\phi)$$

for all $\phi \in \mathcal{D}(\Omega)$.

The two examples which will be introduced in Definitions 0.7 and 0.10 show how certain concrete objects, like functions and measures, can be viewed as distributions.

Definition 0.7 *For every $f \in L^1_{loc}(\Omega)$, we define*

$$T_f(\phi) = \int_\Omega \phi(x) f(x)\ dm(x)$$

for all $\phi \in \mathcal{D}(\Omega)$.

It is trivial to prove that T_f is a distribution in Ω.

The (obviously) linear map

$$L^1_{loc}(\Omega) \ni f \mapsto T_f \in \mathcal{D}^*(\Omega)$$

is a **continuous injection**.

This means that,

1. if $T_f = T_g$, then $f = g$ almost everywhere in Ω and
2. $T_{f_m} \to T_f$ in $\mathcal{D}^*(\Omega)$, whenever $f_m \to f$ in $L^1_{loc}(\Omega)$ or, equivalently, whenever $\int_K |f_m(x) - f(x)|\ dm(x) \to 0$ for every compact $K \subseteq \Omega$.

The second statement is trivial to prove, and the first is equivalent to

Proposition 0.10 *If f and g are locally integrable in Ω and if*

$$\int_\Omega f(x)\phi(x)\ dm(x) = \int_\Omega g(x)\phi(x)\ dm(x)$$

for all $\phi \in \mathcal{D}(\Omega)$, then $f = g$ almost everywhere in Ω.

Proof:

If K is a compact subset of Ω, consider $U_m = \{x : d(x,K) < \frac{1}{m}\}$.
Then, $K = \cap_{m=1}^{+\infty} U_m$, implying $dm(U_m \setminus K) < \epsilon$ for large m.
By Lemma 0.2, there exists a $\phi \in \mathcal{D}(U_m)$ so that

1. $0 \leq \phi \leq 1$ everywhere and
2. $\phi = 1$ in K.

Then,

$$\Big|\int_\Omega f(x)\phi(x)\ dm(x) - \int_K f(x)\ dm(x)\Big| \ \leq\ \int_{U_m \setminus K} |f(x)|\ dm(x)\ ,$$

with a similar inequality for g.

Letting $\epsilon \to 0$ and using the hypothesis, we get

$$\int_K f(x)\ dm(x)\ =\ \int_K g(x)\ dm(x)$$

for all K.

By the regularity of dm, the proof is complete.

We may, thus, "identify" locally integrable functions in Ω with the corresponding distributions in Ω.

Definition 0.8 *If for some distribution T there exists a (necessarily unique) locally integrable f so that $T = T_f$, then we say that T **is represented by** f or that T **is identified with** f.*

Because of this identification, it is *common practice* to use the symbol f for both the function and the distribution represented by it. We, thus, write

$$f(\phi)\ =\ \int_\Omega \phi(x)f(x)\ dm(x)\ .$$

Therefore, the symbol f has two meanings; the "function" meaning, when f acts on points of Ω, and the "functional" meaning, when f acts on functions $\phi \in \mathcal{D}(\Omega)$.

Let $d\mu$ be a complex-valued function defined on the union of $\mathcal{B}(K)$ for all compact $K \subseteq \Omega$ (where $\mathcal{B}(K)$ is the Borel σ-algebra of K), so that it is a complex Borel measure in every such compact K.

To say the same thing in a different way,

Definition 0.9 *Let $d\mu(A)$ be defined as a complex number for every Borel set A contained in a compact subset of Ω and let*

$$d\mu\Big(\bigcup_{k=1}^{+\infty} A_k\Big)\ =\ \sum_{k=1}^{+\infty} d\mu(A_k)\ ,$$

whenever the Borel sets A_k are pairwise disjoint and are all contained in the same compact subset of Ω.

*Every such $d\mu$ is called a **locally finite complex Borel measure** in Ω and the set of all such $d\mu$ is denoted $M_{loc}(\Omega)$.*

If $d\mu$ is a non-negative Borel measure in Ω, then $d\mu(A)$ is defined for all Borel subsets A of Ω and, by definition, it is a (finite) non-negative number whenever A is contained in a compact subset of Ω. Thus, by just restricting its domain of definition, $d\mu$ becomes a locally finite complex Borel measure in Ω with non-negative values.

The converse is the content of the next result.

Proposition 0.11 *Suppose that $d\mu$ is a locally finite complex Borel measure in Ω with non-negative values.*

Then, the domain of definition of $d\mu$ can be extended to $\mathcal{B}(\Omega)$ so that $d\mu$ becomes a non-negative Borel measure in Ω.

The extension of $d\mu$ on $\mathcal{B}(\Omega)$ is unique.

Proof:

If A is an arbitrary Borel subset of Ω, define

$$d\mu_0(A) \;=\; \sup d\mu(B)$$

over all Borel sets $B \subseteq A$ which are contained in compact subsets of Ω.

(i) It is obvious that, if A itself is contained in a compact subset of Ω, then $d\mu_0(A) = d\mu(A)$ and, thus, $d\mu_0$ is an extension of $d\mu$.

(ii) Now, let the sets A_k, $k \in \mathbf{N}$, be pairwise disjoint Borel subsets of Ω.

If $B \subseteq \cup_{k=1}^{+\infty} A_k$ is contained in a compact subset of Ω, then all $B \cap A_k$ are contained in the same compact subset of Ω and, thus,

$$d\mu(B) \;=\; \sum_{k=1}^{+\infty} d\mu(B \cap A_k) \;\leq\; \sum_{k=1}^{+\infty} d\mu_0(A_k)\ .$$

This implies that

$$d\mu_0\Big(\bigcup_{k=1}^{+\infty} A_k\Big) \;\leq\; \sum_{k=1}^{+\infty} d\mu_0(A_k)\ .$$

Now, for each k with $1 \leq k \leq m$, consider arbitrary Borel sets $B_k \subseteq A_k$ contained in compact subsets of Ω. Then $\cup_{k=1}^{m} B_k$ is, also, contained in a compact subset of Ω and, hence, $\sum_{k=1}^{m} d\mu(B_k) = d\mu(\cup_{k=1}^{m} B_k) \leq d\mu_0(\cup_{k=1}^{+\infty} A_k)$.

Taking the supremum over each B_k independently, we find $\sum_{k=1}^{m} d\mu_0(A_k) \leq d\mu_0(\cup_{k=1}^{+\infty} A_k)$ and, letting $m \to +\infty$,

$$\sum_{k=1}^{+\infty} d\mu_0(A_k) \;\leq\; d\mu_0\Big(\bigcup_{k=1}^{+\infty} A_k\Big)\ .$$

Therefore, $d\mu_0$ is a non-negative Borel measure in Ω.
(iii) Now, let $d\mu_1$ be another extension of $d\mu$ and consider a compact exhaustion $\{K_{(m)}\}$ of Ω. Then, for every Borel subset A of Ω, the sets $B_m = K_{(m)} \cap A$ increase towards A and each is contained in a compact subset of Ω. Therefore,

$$d\mu_1(A) \;=\; \lim_{m\to+\infty} d\mu_1(B_m) \;=\; \lim_{m\to+\infty} d\mu(B_m) \;=\; \lim_{m\to+\infty} d\mu_0(B_m) \;=\; d\mu_0(A).$$

Because of the last proposition *we shall never distinguish between non-negative Borel measures in Ω and locally finite Borel measures in Ω with non-negative values.*

If $d\mu$ is a locally finite complex Borel measure in Ω, then through the extension procedure of Proposition 0.11, we can define the non-negative Borel measures $|d\mu|$, $d\mu^+$ and $d\mu^-$ in Ω satisfying $|d\mu|(A) = d\mu^+(A) + d\mu^-(A)$ for all $A \in \mathcal{B}(\Omega)$ and $d\mu(A) = d\mu^+(A) - d\mu^-(A)$ for all $A \in \mathcal{B}(\Omega)$ contained in a compact subset of Ω.

If f is a locally integrable function in Ω, then a locally finite complex Borel measure $f\,dm$ in Ω is defined, as usual, by

$$f\,dm(A) \;=\; \int_A f(x)\;dm(x)$$

for all Borel sets A which are contained in compact subsets of Ω.

It is obvious that we may integrate any continuous function with compact support contained in Ω against any locally finite complex Borel measure in Ω.

Definition 0.10 *For every $d\mu \in M_{loc}(\Omega)$ we define*

$$T_{d\mu}(\phi) \;=\; \int_\Omega \phi(x)\;d\mu(x)$$

for all $\phi \in \mathcal{D}(\Omega)$.

It is clear that $T_{d\mu}$ is a distribution in Ω and that the linear map

$$M_{loc}(\Omega) \ni d\mu \;\mapsto\; T_{d\mu} \in \mathcal{D}^*(\Omega)$$

is a continuous injection.

This means that,

1. if $T_{d\mu} = T_{d\nu}$, then $d\mu = d\nu$ and
2. $T_{d\mu_m} \to T_{d\mu}$ in $\mathcal{D}^*(\Omega)$, whenever $d\mu_m \to d\mu$ in $M_{loc}(\Omega)$ or, equivalently, whenever $|d\mu_m - d\mu|(K) \to 0$ for every compact $K \subseteq \Omega$.

The first statement is the same as

Proposition 0.12 *If $d\mu$ and $d\nu$ are two locally finite complex Borel measures in Ω and*

$$\int_\Omega \phi(x)\;d\mu(x) \;=\; \int_\Omega \phi(x)\;d\nu(x)$$

for all $\phi \in \mathcal{D}(\Omega)$, then $d\mu = d\nu$.

Proof:
The proof is identical to the proof of Proposition 0.10, replacing $f(x)dm(x)$ by $d\mu(x)$ and $g(x)dm(x)$ by $d\nu(x)$ and using the regularity of Borel measures.

Therefore, we identify locally finite complex Borel measures in Ω with the corresponding distributions in Ω and, as above for functions,

Definition 0.11 *If for a distribution T in Ω there exists some (necessarily unique) locally finite complex Borel measure $d\mu$ in Ω so that $T = T_{d\mu}$, then we say that T* ***is represented by*** *$d\mu$ or that T* ***is identified with*** *$d\mu$.*

As in the case of functions and because of this identification between measures and distributions, it is *common practice* to use the same symbol $d\mu$ for both the measure and the distribution represented by it. We, therefore, very often write

$$d\mu(\phi) \;=\; \int_\Omega \phi(x)\; d\mu(x)\;.$$

Hence, the symbol $d\mu$ has two meanings; one is the meaning of a "measure", when $d\mu$ acts on certain Borel subsets of Ω, and the other is the "functional" meaning, when $d\mu$ acts on functions $\phi \in \mathcal{D}(\Omega)$.

A particular case is when the measure is the $d\delta_a$, the Dirac-mass at a point $a \in \Omega$. Then,

$$d\delta_a(\phi) \;=\; T_{d\delta_a}(\phi) \;=\; \int_\Omega \phi(x)\; d\delta_a(x) \;=\; \phi(a)$$

for all $\phi \in \mathcal{D}(\Omega)$.

Another particular case is when the measure is of the form $f\,dm$ for some locally integrable function f in Ω. Then,

$$f\,dm(\phi) \;=\; T_{f\,dm}(\phi) \;=\; \int_\Omega \phi(x)f(x)\; dm(x) \;=\; T_f(\phi) \;=\; f(\phi)$$

for all $\phi \in \mathcal{D}(\Omega)$, implying $T_{f\,dm} = T_f$ or, informally, $f\,dm = f$.

Definition 0.12 *A distribution T in Ω is called* ***non-negative*** *and we write*

$$T \geq 0\;,$$

if

$$T(\phi) \;\geq\; 0$$

for all $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$ everywhere in Ω.

Theorem 0.11 *If T is a non-negative distribution in Ω, then there exists a non-negative Borel measure $d\mu$ in Ω so that $T = T_{d\mu}$.*

Proof:
Consider any open exhaustion $\{\Omega_{(m)}\}$ of Ω and fix m.

By Lemma 0.2, there exists $\phi_0 \in \mathcal{D}(\Omega)$ such that $0 \leq \phi_0 \leq 1$ everywhere and $\phi_0 = 1$ in the compact $\overline{\Omega_{(m)}}$.

Then, for every $\phi \in \mathcal{D}(\Omega_{(m)})$, we have $|\phi| \leq \|\phi\|_\infty \phi_0$ everywhere in Ω and, since $T \geq 0$,

$$|T(\phi)| \ \leq \ |T(\Re\phi)| + |T(\Im\phi)| \ \leq \ 2T(\phi_0)\|\phi\|_\infty \ .$$

Therefore, T is a bounded linear functional on the linear subspace $\mathcal{D}(\Omega_{(m)})$ of the normed space $C(\overline{\Omega_{(m)}})$. By the Hahn-Banach Theorem, T can be extended as a bounded linear functional on $C(\overline{\Omega_{(m)}})$.

By the Representation Theorem of J. Radon and F. Riesz, there exists a complex Borel measure $d\mu_m$ supported in $\overline{\Omega_{(m)}}$ so that

$$T(\phi) \ = \ \int_{\overline{\Omega_{(m)}}} \phi(x) \ d\mu_m(x)$$

for all $\phi \in \mathcal{D}(\Omega_{(m)})$.

If K is any compact subset of $\Omega_{(m)}$, then, as in the proof of Proposition 0.10, there exists $\phi \in \mathcal{D}(\Omega_{(m)})$ so that $0 \leq \phi \leq 1$ in $\Omega_{(m)}$, $\phi = 1$ in K and $d\mu_m\big(supp(\phi) \setminus K\big) < \epsilon$. This implies

$$|d\mu_m(K) - T(\phi)| \ \leq \ \int_{\overline{\Omega_{(m)}}} |\chi_K(x) - \phi(x)| \ d\mu_m(x) \ \leq \ \epsilon \ ,$$

and, since $T(\phi) \geq 0$ and ϵ is arbitrary,

$$d\mu_m(K) \ \geq \ 0 \ .$$

By the regularity of $d\mu_m$, we conclude that it is a non-negative measure.

Now, since $\int_{\overline{\Omega_{(m)}}} \phi(x) \ d\mu_m(x) \ = \ T(\phi) \ = \ \int_{\overline{\Omega_{(m+1)}}} \phi(x) \ d\mu_{m+1}(x)$ for every $\phi \in \mathcal{D}(\Omega_{(m)})$, Proposition 0.12 implies that

$$d\mu_m \ = \ d\mu_{m+1}$$

in $\Omega_{(m)}$. Therefore, the measures $d\mu_m$ define a locally finite complex Borel measure $d\mu$ in Ω with non-negative values so that the restriction of $d\mu$ in each $\Omega_{(m)}$ coincides with $d\mu_m$; simply, define, for all Borel sets A which are contained in compact subsets of Ω,

$$d\mu(A) \ = \ d\mu_m(A)$$

for any m with $A \subseteq \Omega_{(m)}$.

Finally, if $\phi \in \mathcal{D}(\Omega)$, then, since $supp(\phi)$ is compact, $\phi \in \mathcal{D}(\Omega_{(m)})$ for large m and

$$T_{d\mu}(\phi) \ = \ \int_\Omega \phi(x) \ d\mu(x) \ = \ \int_{\Omega_{(m)}} \phi(x) \ d\mu_m(x) \ = \ T(\phi) \ .$$

In the rest of this section we define two operations on distributions, the differentiation and the convolution with infinitely differentiable functions with compact support in $\mathbf{R^n}$, and study their calculus.

Definition 0.13 *If T is a distribution in Ω, we define*

$$\frac{\partial T}{\partial x_j}(\phi) \;=\; -T\Big(\frac{\partial \phi}{\partial x_j}\Big)$$

*for all $\phi \in \mathcal{D}(\Omega)$ and call $\frac{\partial T}{\partial x_j}$ the j-**derivative** of T.*

Then, $\frac{\partial T}{\partial x_j}$ is a new distribution in Ω, as one can, easily, see.

Inductively, we see that a distribution in Ω has derivatives of all orders and these are, also, distributions in Ω. In fact, for every multi-index α,

$$D^\alpha T(\phi) \;=\; (-1)^{|\alpha|} T(D^\alpha \phi)\;.$$

Definition 0.14 *If T_f is the distribution identified with the locally integrable function f, then the distribution $\frac{\partial T_f}{\partial x_j}$ is called the **distributional** j**-derivative** of f.*

By Definitions 0.7 and 0.13, this means nothing more than that this distributional j-derivative is a functional on $\mathcal{D}(\Omega)$, defined by

$$\frac{\partial T_f}{\partial x_j}(\phi) \;=\; -T_f\Big(\frac{\partial \phi}{\partial x_j}\Big) \;=\; -\int_\Omega f(x)\frac{\partial \phi}{\partial x_j}(x)\; dm(x)\;.$$

Therefore, a locally integrable function has distributional derivatives of all orders which are, in general, distributions.

To avoid a conflict, we must find the relation between the distributional derivative and the classical derivative, whenever this last one exists. Therefore, assume that f is in $C^1(\Omega)$ and let $\frac{\partial f}{\partial x_j}$ be its usual j-derivative in Ω. Then, for every $\phi \in \mathcal{D}(\Omega)$,

$$\frac{\partial T_f}{\partial x_j}(\phi) \;=\; -\int_\Omega f(x)\frac{\partial \phi}{\partial x_j}(x)\; dm(x) \;=\; \int_\Omega \frac{\partial f}{\partial x_j}(x)\phi(x)\; dm(x) \;=\; T_{\frac{\partial f}{\partial x_j}}(\phi)\;.$$

The second equality is the well-known **Integration by Parts formula** and can be proved using Green's Formula in a bounded open set G with C^1-boundary and with *supp*$(\phi) \subseteq G \subseteq \overline{G} \subseteq \Omega$ (so that ϕ and $\frac{\partial \phi}{\partial \eta}$ vanish in ∂G).

Thus,

Proposition 0.13 *The distributional j-derivative of any f in $C^1(\Omega)$ is identified with its classical j-derivative.*

Based on this last result and on the identification between f and T_f, we, often, use the customary symbol $\frac{\partial f}{\partial x_j}$ to denote $\frac{\partial T_f}{\partial x_j}$, even though f may not be differentiable in the classical sense. Thus, the informal notation is

$$\frac{\partial f}{\partial x_j}(\phi) \;=\; -\int_\Omega f(x)\frac{\partial \phi}{\partial x_j}(x)\; dm(x)\;.$$

If $T \in \mathcal{D}^*(\Omega)$, then

$$\frac{\partial^2 T}{\partial x_j^2}(\phi) \;=\; -\frac{\partial T}{\partial x_j}\Big(\frac{\partial \phi}{\partial x_j}\Big) \;=\; T\Big(\frac{\partial^2 \phi}{\partial x_j^2}\Big) \;.$$

Therefore, the **Laplacian** of a distribution T is given by

$$\Delta T(\phi) \;=\; T(\Delta \phi)$$

for all $\phi \in \mathcal{D}(\Omega)$.

Another case which we shall often consider is the distributional Laplacian Δf of a locally integrable function. The correct notation is, of course, ΔT_f and it is the distribution which, for every $\phi \in \mathcal{D}(\Omega)$, satisfies

$$\Delta T_f(\phi) \;=\; T_f(\Delta\phi) \;=\; \int_\Omega f(x)\Delta\phi(x)\; dm(x) \;.$$

The informal notation is, of course,

$$\Delta f(\phi) \;=\; \int_\Omega f(x)\Delta\phi(x)\; dm(x) \;,$$

even if f is not differentiable in the classical sense.

In case $f \in C^k(\Omega)$, using the formula of Integration by Parts,

$$\begin{aligned} D^\alpha T_f(\phi) \;&=\; (-1)^{|\alpha|} T_f(D^\alpha \phi) \;=\; (-1)^{|\alpha|}\int_\Omega f(x) D^\alpha\phi(x)\; dm(x) \\ &=\; \int_\Omega D^\alpha f(x)\phi(x)\; dm(x) \;=\; T_{D^\alpha f}(\phi) \end{aligned}$$

for all $\phi \in \mathcal{D}(\Omega)$ and all multi-indices α of order at most k. Thus,

Proposition 0.14 *The distributional α-derivative of an f in $C^k(\Omega)$ is identified with its classical α-derivative, for all α with $|\alpha| \le k$.*

Again, the informal notation, even if f is not differentiable, is

$$D^\alpha f(\phi) \;=\; (-1)^{|\alpha|}\int_\Omega f(x) D^\alpha \phi(x)\; dm(x) \;.$$

Let $\delta > 0$ and consider any $f \in C^\infty(\mathbf{R^n})$ with $supp(f) \subseteq \overline{B(0;\delta)}$. Denote

$$\widetilde{f}(x) \;=\; f(-x) \;, \qquad x \in \mathbf{R^n} \;.$$

Definition 0.15 *For any $T \in \mathcal{D}^*(\Omega)$ and any $\phi \in \mathcal{D}(\Omega_\delta)$, we define*

$$T * f(\phi) \;=\; T(\widetilde{f} * \phi) \;.$$

$T * f$ is called the ***convolution*** *of the distribution T and the function f.*

This is well-defined. Indeed, $\widetilde{f}*\phi \in C^\infty(\mathbf{R^n})$ and $supp(\widetilde{f}*\phi) \subseteq \overline{B(0;\delta)}+\Omega_\delta \subseteq \Omega$ and, hence, $\widetilde{f}*\phi \in \mathcal{D}(\Omega)$.

Proposition 0.15 *If $T \in \mathcal{D}^*(\Omega)$ and $f \in C^\infty(\mathbf{R^n})$ with $supp(f) \subseteq \overline{B(0;\delta)}$, then $T*f$ is a distribution in Ω_δ.*

Proof:

The linearity is trivial to prove and we consider $\phi_m \to \phi$ in $\mathcal{D}(\Omega_\delta)$.

If all $supp(\phi_m)$ are contained in the compact $K \subseteq \Omega_\delta$, then all $supp(\widetilde{f}*\phi_m)$ are contained in the compact $L = K + \overline{B(0;\delta)} \subseteq \Omega$. Moreover,

$$\begin{aligned}
&|D^\alpha(\widetilde{f}*\phi_m)(x) - D^\alpha(\widetilde{f}*\phi)(x)| \\
&\leq \int_{\mathbf{R^n}} |D^\alpha\phi_m(y) - D^\alpha\phi(y)||\widetilde{f}(x-y)|\ dm(y) \\
&\leq \sup_{y\in\mathbf{R^n}} |D^\alpha\phi_m(y) - D^\alpha\phi(y)| \int_{\mathbf{R^n}} |f(z)|\ dm(z)\ .
\end{aligned}$$

and, hence, $D^\alpha(\widetilde{f}*\phi_m) \to D^\alpha(\widetilde{f}*\phi)$ uniformly in $\mathbf{R^n}$.

This implies that $\widetilde{f}*\phi_m \to \widetilde{f}*\phi$ in $\mathcal{D}(\Omega)$ and, since $T \in \mathcal{D}^*(\Omega)$, we, finally, get

$$T*f(\phi_m) \ = \ T(\widetilde{f}*\phi_m) \ \to \ T(\widetilde{f}*\phi) \ = \ T*f(\phi)\ .$$

Guided by the "principle" that convolution preserves the regularity of its components, we shall prove that $T*f$ can be identified with some function $g \in C^\infty(\Omega_\delta)$.

Remark

In the proof that follows there are four "why"s and "how"s. The interested reader is advised to answer them, providing the easy but technical proofs.

Define $f^x(y) = f(x-y)$ and observe that $f^x \in \mathcal{D}(\Omega)$, if $x \in \Omega_\delta$. Therefore, the function

$$g(x) \ = \ T(f^x)\ , \qquad x \in \Omega_\delta\ ,$$

is well-defined.

If $x_m \to x$, then $f^{x_m} \to f^x$ in $\mathcal{D}(\Omega)$ (why?) and, hence, $g(x_m) \to g(x)$.

Therefore, g is continuous in Ω_δ.

Next, we see easily (how?) that, for all $x \in \Omega_\delta$, if $h \to 0$ in $\mathbf{R}$, then

$$\frac{f^{x+e_jh} - f^x}{h} \to \Big(\frac{\partial f}{\partial x_j}\Big)^x$$

in $\mathcal{D}(\Omega)$, where e_j is the j-coordinate unit vector.

From this,

$$\lim_{h\to 0}\frac{g(x+e_jh)-g(x)}{h} \ = \ \lim_{h\to 0} T\Big(\frac{f^{x+e_jh} - f^x}{h}\Big) \ = \ T\Big(\Big(\frac{\partial f}{\partial x_j}\Big)^x\Big)\ .$$

Hence,
$$\frac{\partial g}{\partial x_j}(x) \;=\; T\Big(\Big(\frac{\partial f}{\partial x_j}\Big)^x\Big) \;, \qquad x \in \Omega_\delta \;,$$
implying that g has continuous partial derivatives of first order and, by induction, of any order:
$$g \in C^\infty(\Omega_\delta) \;.$$

Take, now, any $\phi \in \mathcal{D}(\Omega_\delta)$.

Since the function ϕg is continuous with compact support contained in Ω_δ,
$$\int_{\Omega_\delta} \phi(x)g(x)\; dm(x) \;=\; \lim \sum \phi(x_k)g(x_k)\Delta m(x_k) \;,$$
where the last sum is a Riemann-sum defined, as usual (how?), by a mesh of small cubes and the limit is taken as the size of the cubes tends to 0.

Now, it is easy to prove (how?) that, also,
$$\int_{\Omega_\delta} \phi(x)f^x(\cdot)\; dm(x) \;=\; \lim \sum \phi(x_k)f^{x_k}(\cdot)\Delta m(x_k) \qquad \text{in } \mathcal{D}(\Omega) \;.$$

Finally,
$$\begin{aligned}
\int_{\Omega_\delta} \phi(x)g(x)\; dm(x) \;&=\; \lim \sum \phi(x_k)g(x_k)\Delta m(x_k) \\
&=\; \lim \sum \phi(x_k)T(f^{x_k})\Delta m(x_k) \\
&=\; \lim T\Big(\sum \phi(x_k)f^{x_k}(\cdot)\Delta m(x_k)\Big) \\
&=\; T\Big(\int_{\Omega_\delta} \phi(x)f^x(\cdot)\; dm(x)\Big) \\
&=\; T(\widetilde{f} * \phi) \;=\; T * f(\phi) \;,
\end{aligned}$$
where the third equality holds because the sum is finite.

We conclude that $T * f$ is identified with the function $g \in C^\infty(\Omega_\delta)$. Thus,

Proposition 0.16 *The convolution of a distribution in Ω and an infinitely differentiable function supported in $\overline{B(0;\delta)}$ is a distribution in Ω_δ identified with a function in $C^\infty(\Omega_\delta)$.*

We can easily prove

Proposition 0.17
$$T * (f * g) \;=\; (T * f) * g$$
for all $T \in \mathcal{D}^*(\Omega)$ and $f, g \in C^\infty(\mathbf{R^n})$ with supp$(f) \subseteq \overline{B(0;\delta)}$ and supp$(g) \subseteq \overline{B(0;\epsilon)}$.

In this equality, both sides are distributions in $\Omega_{\delta+\epsilon}$.

Proof:
For any $\phi \in \mathcal{D}(\Omega_{\delta+\epsilon})$, and since $supp(f*g) \subseteq \overline{B(0;\delta+\epsilon)}$,

$$\begin{aligned}\big(T*(f*g)\big)(\phi) &= T\big(\widetilde{(f*g)}*\phi\big) = T\big(\widetilde{f}*(\widetilde{g}*\phi)\big)\\ &= (T*f)(\widetilde{g}*\phi) = \big((T*f)*g\big)(\phi)\ .\end{aligned}$$

Proposition 0.18 *For all $T \in \mathcal{D}^*(\Omega)$ and $f \in C^\infty(\mathbf{R^n})$ with $supp(f) \subseteq \overline{B(0;\delta)}$,*

$$D^\alpha(T*f) = D^\alpha T*f = T*D^\alpha f\ ,$$

as distributions in Ω_δ .

Proof:
For every $\phi \in \mathcal{D}(\Omega_\delta)$,

$$D^\alpha(T*f)(\phi) = (-1)^{|\alpha|}(T*f)(D^\alpha\phi) = (-1)^{|\alpha|}T(\widetilde{f}*D^\alpha\phi)\ .$$

This is, on one hand, equal to

$$(-1)^{|\alpha|}T\big(D^\alpha(\widetilde{f}*\phi)\big) = D^\alpha T(\widetilde{f}*\phi) = (D^\alpha T*f)(\phi)$$

and, on the other hand, equal to

$$T(\widetilde{D^\alpha f}*\phi) = (T*D^\alpha f)(\phi)\ .$$

Proposition 0.19 *If $h \in L^1_{loc}(\Omega)$ and $f \in C^\infty(\mathbf{R^n})$ with $supp(f) \subseteq \overline{B(0;\delta)}$, then*

$$T_h*f = T_{h*f}$$

as distributions in Ω_δ .

Proof:
By Proposition 0.7, the function $h*f$ is in $C^\infty(\Omega_\delta)$ and, hence, both sides are distributions in Ω_δ. If $\phi \in \mathcal{D}(\Omega_\delta)$ and $supp(\phi)$ is contained in the compact $K \subseteq \Omega_\delta$, then $supp(\widetilde{f}*\phi)$ is contained in the compact $L = K+\overline{B(0;\delta)} \subseteq \Omega$ and, using the Theorem of Fubini,

$$\begin{aligned}(T_h*f)(\phi) &= T_h(\widetilde{f}*\phi) = \int_\Omega h(x)\int_K f(y-x)\phi(y)\ dm(y)\ dm(x)\\ &= \int_L h(x)\int_K f(y-x)\phi(y)\ dm(y)\ dm(x)\\ &= \int_K (h*f)(y)\phi(y)\ dm(y) = T_{h*f}(\phi)\ .\end{aligned}$$

Proposition 0.20 *Consider any approximation to the identity* $\{\Phi_\delta : \delta > 0\}$, *let* T *be a distribution in* Ω *and fix an arbitrary* $\epsilon > 0$.

Then, for all $\delta \leq \epsilon$, $T * \Phi_\delta$ *is a distribution in* Ω_ϵ *and*

$$T * \Phi_\delta \to T \qquad \text{in } \mathcal{D}^*(\Omega_\epsilon)$$

as $\delta \to 0$.

Proof:

Take any $\phi \in \mathcal{D}(\Omega_\epsilon)$ and write

$$(T * \Phi_\delta)(\phi) \;=\; T(\widetilde{\Phi_\delta * \phi}) \;=\; T((\widetilde{\Phi})_\delta * \phi) \to T(\phi) \ ,$$

since, by Proposition 0.9, $(\widetilde{\Phi})_\delta * \phi \to \phi$ in $\mathcal{D}(\Omega)$ as $\delta \to 0$.

0.8 Concavity

An extended-real-valued function f defined in an interval $I \subseteq [-\infty, +\infty]$ is called **concave** in I, if it takes only real values in the interior of I and

$$f(ta + (1-t)b) \;\geq\; tf(a) + (1-t)f(b)$$

for all $a, b \in I$ and all $t \in (0, 1)$.

f is called **convex** in I, if $-f$ is concave in I.

If f is concave in the interval I, then f is continuous at every interior point of I and, at each of the endpoints, the one-sided limit of f exists. In case the endpoint is contained in I, this limit is no smaller than the value of f there.

If f is concave in the open interval I, then f is differentiable at all points of a subset D of I, where $I \setminus D$ is at most countable. At every $a \in I \setminus D$ the one-sided derivatives of f exist and $f'_-(a) \geq f'_+(a)$. The functions f'_- and f'_+ are decreasing in I and, hence, f' is decreasing in D. The corresponding Lebesgue-Stieltjes measure df' is a non-positive Borel measure in I, which is the zero measure in I if and only if f is a linear function in I.

Also, at every interior point a_0 of I, f has a line supporting its graph from above. This means that for some $l, k \in \mathbf{R}$,

$$f(a) \;\leq\; la + k$$

for every $a \in I$, and

$$f(a_0) \;=\; la_0 + k \ .$$

As l we may consider any number between the one-sided derivatives of f at a_0.

If f is twice continuously differentiable in the open interval I, then f is concave in I if and only if $f'' \leq 0$ everywhere in I.

Theorem 0.12 *(Inequality of Jensen) Let* $d\nu$ *be a non-negative measure in the measure space* $(X, \mathcal{A})$ *with* $d\nu(X) = 1$ *and* ϕ *be an extended-real-valued function in* X *which is in* $L^1(d\nu)$.

Then, for every f which is concave in an interval containing the values $\phi(x)$ for almost all (with respect to $d\nu$) $x \in X$,

$$f\Big(\int_X \phi(x)\ d\nu(x)\Big)\ \geq\ \int_X f\circ\phi(x)\ d\nu(x)\ .$$

The inequality is reversed, if f is convex instead of concave.

Proof:

Let I be an interval such that $\phi(x) \in I$ for almost all (with respect to $d\nu$) $x \in X$.

If the real number $a_0 = \int_X \phi(x)\ d\nu(x)$ is an endpoint of I, then ϕ is almost everywhere (with respect to $d\nu$) equal to a_0 and Jensen's inequality is trivial to prove.

Now, suppose that $a_0 = \int_X \phi(x)\ d\nu(x)$ is an interior point of I.

Then, there exist l, k so that $f(a_0) = la_0 + k$ and $f(a) \leq la + k$ for all $a \in I$.

Hence, for almost all (with respect to $d\nu$) $x \in X$, $f\big(\phi(x)\big) \leq l\phi(x) + k$ and, integrating,

$$\int_X f\circ\phi(x)\ d\nu(x)\ \leq\ l\int_X \phi(x)\ d\nu(x) + k\ =\ f\Big(\int_X \phi(x)\ d\nu(x)\Big)\ .$$

0.9 The Fourier Transform

In this section we shall state and prove only a few elementary properties of the Fourier transform of functions and measures which we shall need later on in these notes.

Definition 0.16 *If $f \in L^1(\mathbf{R^n})$, we define*

$$\widehat{f}(\xi)\ =\ \int_{\mathbf{R^n}} e^{-2\pi i\xi\cdot x} f(x)\ dm(x)$$

for all $\xi \in \mathbf{R^n}$.

It is clear that the Lebesgue integral of the definition is well-defined as a complex number.

Definition 0.17 *If $f \in L^1(\mathbf{R^n})$, then the function*

$$\widehat{f} : \mathbf{R^n} \to \mathbf{C}$$

*is called **the Fourier transform** of f.*

Example

If

$$f(x)\ =\ e^{-\alpha|x|^2}\ ,\qquad x \in \mathbf{R^n}\ ,$$

for any $\alpha > 0$, then

$$\widehat{f}(\xi)\ =\ \Big(\frac{\pi}{\alpha}\Big)^{\frac{n}{2}} e^{-\frac{\pi^2}{\alpha}|\xi|^2}\ ,\qquad \xi \in \mathbf{R^n}\ .$$

For the proof we use the Theorem of Fubini to reduce the calculation to the case of dimension $n = 1$ and, then,

$$\widehat{f}(\xi) \ = \ \int_{-\infty}^{+\infty} e^{-2\pi i\xi x} e^{-\alpha x^2} \ dm(x) \ .$$

Taking derivative and using integration by parts we may, easily, prove that

$$\frac{\partial \widehat{f}}{\partial \xi}(\xi) \ = \ -\frac{2\pi^2}{\alpha} \ \xi \widehat{f}(\xi) \ , \qquad \xi \in \mathbf{R} \ .$$

Therefore,

$$\frac{\partial}{\partial \xi}\Big(\widehat{f}(\xi) e^{\frac{\pi^2}{\alpha}\xi^2}\Big) \ = \ -\frac{2\pi^2}{\alpha} \ \xi \widehat{f}(\xi) e^{\frac{\pi^2}{\alpha}\xi^2} + \widehat{f}(\xi)\frac{2\pi^2}{\alpha} \ \xi e^{\frac{\pi^2}{\alpha}\xi^2} \ = \ 0 \ , \qquad \xi \in \mathbf{R} \ .$$

Since $\widehat{f}(0) = \sqrt{\frac{\pi}{\alpha}}$, we find

$$\widehat{f}(\xi) \ = \ \sqrt{\frac{\pi}{\alpha}} \ e^{-\frac{\pi^2}{\alpha}\xi^2} \ .$$

The proof of the next result is trivial.

Proposition 0.21 *If $f \in L^1(\mathbf{R^n})$, then*

$$\sup_{\xi \in \mathbf{R^n}} |\widehat{f}(\xi)| \ \leq \ \|f\|_{L^1(\mathbf{R^n})}$$

and, thus,

$$\widehat{f} \ \in \ L^\infty(\mathbf{R^n}) \ .$$

Moreover, for every f_1 and f_2 in $L^1(\mathbf{R^n})$ and all λ_1 and λ_2 in $\mathbf{C}$,

$$(\lambda_1 f_1 + \lambda_2 f_2)\widehat{\ } = \ \lambda_1 \widehat{f_1} + \lambda_2 \widehat{f_2} \ .$$

Definition 0.18 *The linear operator*

$$\mathcal{F} : L^1(\mathbf{R^n}) \ \to \ L^\infty(\mathbf{R^n})$$

defined by

$$\mathcal{F}(f) \ = \ \widehat{f}$$

*is called the **Fourier transform on** $L^1(\mathbf{R^n})$.*

Proposition 0.22 *The norm of the linear operator $\mathcal{F} : L^1(\mathbf{R^n}) \to L^\infty(\mathbf{R^n})$ is equal to* 1.

Proof:
The norm is, by Proposition 0.21, at most 1.

Taking any non-negative $f \in L^1(\mathbf{R^n})$, we have, for all $\xi \in \mathbf{R^n}$,

$$|\widehat{f}(\xi)| \;\leq\; \int_{\mathbf{R^n}} |f(x)|\; dm(x) \;=\; \int_{\mathbf{R^n}} f(x)\; dm(x) \;=\; \widehat{f}(0)\;.$$

Therefore,

$$\sup_{\xi \in \mathbf{R^n}} |\widehat{f}(\xi)| \;=\; \|f\|_{L^1(\mathbf{R^n})}\;,$$

implying that the norm of the Fourier transform on $L^1(\mathbf{R^n})$ is equal to 1.

Theorem 0.13 *(The Lemma of Riemann and Lebesgue) If $f \in L^1(\mathbf{R^n})$, then the function $\widehat{f}$ is continuous in $\mathbf{R^n}$ and*

$$\lim_{\xi \to \infty} \widehat{f}(\xi) \;=\; 0\;.$$

Proof:
If $\xi_k \to \xi$, then

$$\widehat{f}(\xi_k) - \widehat{f}(\xi) \;=\; \int_{\mathbf{R^n}} \big(e^{-2\pi i \xi_k \cdot x} - e^{-2\pi i \xi \cdot x}\big) f(x)\; dm(x) \;\to\; 0$$

by the Dominated Convergence Theorem and this proves the continuity of $\widehat{f}$.
(i) If

$$f \;=\; \chi_I$$

is the characteristic function of an n-dimensional interval

$$I \;=\; [a_1, b_1] \times \cdots \times [a_n, b_n]\;,$$

then, for every $\xi = (\xi_1, \ldots, \xi_n)$, we calculate

$$\widehat{\chi_I}(\xi) \;=\; \prod_{j=1}^{n} \frac{e^{-2\pi i \xi_j b_j} - e^{-2\pi i \xi_j a_j}}{-2\pi i \xi_j}\;.$$

We, now, easily, show that

$$\lim_{\xi \to \infty} \widehat{\chi_I}(\xi) \;=\; 0\;.$$

This property extends, by the linearity in Proposition 0.21, to all linear combinations

$$f \;=\; \sum_{k=1}^{m} \lambda_k \chi_{I_k}\;.$$

(ii) If $f \in L^1(\mathbf{R^n})$ and $\epsilon > 0$ is arbitrary, we take a linear combination of characteristic functions of n-dimensional intervals $g = \sum_{k=1}^{m} \lambda_k \chi_{I_k}$ so that

$$\|f - g\|_{L^1(\mathbf{R^n})} \;\leq\; \epsilon\;.$$

Then, from Proposition 0.21 and the result of (i),

$$\begin{aligned}
\limsup_{\xi\to\infty} |\widehat{f}(\xi)| &\leq \limsup_{\xi\to\infty} |\widehat{f}(\xi) - \widehat{g}(\xi)| + \limsup_{\xi\to\infty} |\widehat{g}(\xi)| \\
&\leq \|f - g\|_{L^1(\mathbf{R^n})} \\
&\leq \epsilon \ .
\end{aligned}$$

Since ϵ is arbitrary, the proof is complete.

Proposition 0.23 *If f and g are in $L^1(\mathbf{R^n})$, then*

$$\widehat{f * g} = \widehat{f}\widehat{g} \ .$$

Proof:
In fact, by the Theorem of Fubini,

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbf{R^n}} e^{-2\pi i\xi\cdot x} \int_{\mathbf{R^n}} f(x-y)g(y) \ dm(y) \ dm(x) \\
&= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} e^{-2\pi i\xi\cdot(x-y)} f(x-y) \ dm(x) \ e^{-2\pi i\xi\cdot y} g(y) \ dm(y) \\
&= \int_{\mathbf{R^n}} \widehat{f}(\xi) e^{-2\pi i\xi\cdot y} g(y) \ dm(y) \\
&= \widehat{f}(\xi)\widehat{g}(\xi) \ .
\end{aligned}$$

Definition 0.19 *If $d\mu$ is a complex Borel measure in $\mathbf{R^n}$, then we define*

$$\widehat{d\mu}(\xi) = \int_{\mathbf{R^n}} e^{-2\pi i\xi\cdot x} \ d\mu(x)$$

for all $\xi \in \mathbf{R^n}$.

The function

$$\widehat{d\mu} : \mathbf{R^n} \to \mathbf{C}$$

is called the ***Fourier transform*** *of $d\mu \in \mathcal{M}(\mathbf{R^n})$.*

Proposition 0.24 *If $d\mu \in \mathcal{M}(\mathbf{R^n})$, then*

$$\sup_{\xi\in\mathbf{R^n}} |\widehat{d\mu}(\xi)| \leq \|d\mu\|$$

and, thus,

$$\widehat{d\mu} \in L^\infty(\mathbf{R^n}) \ .$$

Moreover, for all $d\mu_1$ and $d\mu_2$ in $\mathcal{M}(\mathbf{R^n})$ and all λ_1 and λ_2 in $\mathbf{C}$,

$$(\lambda_1 d\mu_1 + \lambda_2 d\mu_2)\widehat{\ } = \lambda_1\widehat{d\mu_1} + \lambda_2\widehat{d\mu_2} \ .$$

The proof is clear.

Definition 0.20 *The linear operator*

$$\mathcal{F} : \mathcal{M}(\mathbf{R^n}) \ \rightarrow \ L^\infty(\mathbf{R^n}) \ ,$$

defined by

$$\mathcal{F}(d\mu) \ = \ \widehat{d\mu} \ ,$$

*is called the **Fourier transform on** $\mathcal{M}(\mathbf{R^n})$.*

Proposition 0.25 *If $d\mu \in \mathcal{M}(\mathbf{R^n})$, then $\widehat{d\mu}$ is continuous in $\mathbf{R^n}$.*

Proof:
The proof, based on the Dominated Convergence Theorem, is identical to the proof of the first part of Theorem 0.13.

The second part of Theorem 0.13 is, in general, not true for measures.

Example

$$\widehat{d\delta_a}(\xi) \ = \ e^{-2\pi i \xi \cdot a}$$

for all $\xi \in \mathbf{R^n}$ and, hence,

$$|\widehat{d\delta_a}(\xi)| \ = \ 1$$

for all $\xi \in \mathbf{R^n}$.

Example
For every absolutely continuous measure $f\,dm$ with density function $f \in L^1(\mathbf{R^n})$, it is clear that

$$\widehat{f\,dm} \ = \ \widehat{f} \ .$$

Proposition 0.26 *The Fourier transform on $\mathcal{M}(\mathbf{R^n})$ has norm equal to 1.*

Proof:
The proof is identical to the proof of Proposition 0.22 and uses any non-negative Borel measure in $\mathcal{M}(\mathbf{R^n})$.

Definition 0.21 *For every $d\mu$ and $d\nu$ in $\mathcal{M}(\mathbf{R^n})$ and every Borel set A, we define*

$$d\mu * d\nu(A) \ = \ \int_{\mathbf{R^n}\times\mathbf{R^n}} \chi_A(x+y)\ d\mu(x) \times d\nu(y) \ ,$$

where $d\mu \times d\nu$ is the product measure and the double integral is well-defined as a complex number by the Theorem of Fubini.
*$d\mu * d\nu$ is called the **convolution** of the measures $d\mu$ and $d\nu$.*

The proof of the next result is easy and is based on the definition of total variation.

Proposition 0.27 *If $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$, then $d\mu * d\nu$ is σ-additive and, thus,*

$$d\mu * d\nu \ \in \ \mathcal{M}(\mathbf{R^n}) \ .$$

Moreover,

$$\|d\mu * d\nu\| \ \leq \ \|d\mu\|\|d\nu\| \ .$$

Proposition 0.28 *If $d\mu$, $d\nu$ and $d\rho$ are in $\mathcal{M}(\mathbf{R^n})$, then*

$$d\mu * d\nu \ = \ d\nu * d\mu$$

and

$$(d\mu * d\nu) * d\rho \ = \ d\mu * (d\nu * d\rho) \ .$$

Proof:
By the Theorem of Fubini, for every Borel set A,

$$\begin{aligned} d\mu * d\nu(A) &= \int_{\mathbf{R^n}\times\mathbf{R^n}} \chi_A(x+y)\ d\mu(x) \times d\nu(y) \\ &= \int_{\mathbf{R^n}}\int_{\mathbf{R^n}} \chi_{A-y}(x)\ d\mu(x)\ d\nu(y) \\ &= \int_{\mathbf{R^n}} d\mu(A-y)\ d\nu(y) \\ &= \int_{\mathbf{R^n}} d\mu(A-x)\ d\nu(x) \\ &= \int_{\mathbf{R^n}}\int_{\mathbf{R^n}} \chi_{A-x}(y)\ d\mu(y)\ d\nu(x) \\ &= \int_{\mathbf{R^n}}\int_{\mathbf{R^n}} \chi_{A-x}(y)\ d\nu(x)\ d\mu(y) \\ &= \int_{\mathbf{R^n}\times\mathbf{R^n}} \chi_A(x+y)\ d\nu(x) \times d\mu(y) \\ &= d\nu * d\mu(A) \ . \end{aligned}$$

The proof of the second identity is similar and, even, easier.

Definition 0.22 *If $f \in L^1(\mathbf{R^n})$ and $d\nu \in \mathcal{M}(\mathbf{R^n})$, we define*

$$f * d\nu(x) \ = \ \int_{\mathbf{R^n}} f(x-y)\ d\nu(y) \ , \qquad x \in \mathbf{R^n} \ ,$$

and call the function $f * d\nu$ the ***convolution*** *of the function f and the measure $d\nu$.*

It is trivial to prove that $f * d\nu$ is in $L^1(\mathbf{R^n})$ and that

$$\|f * d\nu\|_{L^1(\mathbf{R^n})} \ \leq \ \|f\|_{L^1(\mathbf{R^n})}\|d\nu\| \ .$$

Proposition 0.29 *Suppose that $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$ and that $d\mu = f\,dm$ is absolutely continuous with density function f. Then, $d\mu * d\nu$ is, also, absolutely continuous and its density function is*

$$f * d\nu(\cdot) \ = \ \int_{\mathbf{R^n}} f(\cdot \ - y)\ d\nu(y) \ .$$

Proof:

For every Borel set A,

$$\begin{aligned} d\mu * d\nu(A) \ &= \ \int_{\mathbf{R^n}} d\mu(A - y)\ d\nu(y) \\ &= \ \int_{\mathbf{R^n}} \int_{A-y} f(x)\ dm(x)\ d\nu(y) \\ &= \ \int_{\mathbf{R^n}} \int_{A} f(x-y)\ dm(x)\ d\nu(y) \\ &= \ \int_{A} \int_{\mathbf{R^n}} f(x-y)\ d\nu(y)\ dm(x) \ . \end{aligned}$$

Proposition 0.30 *If $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$, $d\mu$ is supported in the Borel set M and $d\nu$ is supported in the Borel set N, then $d\mu * d\nu$ is supported in $M + N$.*

Hence,

$$supp(d\mu * d\nu) \ \subseteq \ \overline{supp(d\mu) + supp(d\nu)} \ .$$

Proof:

If the Borel set A is disjoint from $M + N$ and if $y \in N$, then

$$M \cap (A - y) \ = \ \emptyset$$

and, hence,

$$d\mu * d\nu(A) \ = \ \int_N \int_M \chi_{A-y}(x)\ d\mu(x)\ d\nu(y) \ = \ \int_N d\mu\big(M \cap (A-y)\big)\ d\nu(y) \ = \ 0 \ .$$

Lemma 0.3 *If $d\mu \in \mathcal{M}(\mathbf{R^n})$, then*

$$\lim_{R \to +\infty} \|d\mu - d\mu_{B(0;R)}\| \ = \ 0 \ .$$

Proof:

$$\|d\mu - d\mu_{B(0;R)}\| \ = \ \|d\mu_{\mathbf{R^n} \setminus B(0;R)}\| \ = \ |d\mu|\big(\mathbf{R^n} \setminus B(0;R)\big) \ \to \ 0 \ ,$$

since $|d\mu|$ is a finite non-negative Borel measure and $B(0;R) \uparrow \mathbf{R^n}$ as $R \uparrow +\infty$.

Proposition 0.31 *If $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$ and f is any bounded continuous function in $\mathbf{R^n}$, then*

$$\int_{\mathbf{R^n}} f(z)\ d\mu * d\nu(z) \ = \ \int_{\mathbf{R^n} \times \mathbf{R^n}} f(x+y)\ d\mu(x) \times d\nu(y) \ .$$

Proof:

By Lemma 0.3, we may take large enough R so that

$$\|d\mu - d\mu_{B(0;R)}\| \ \leq\ \epsilon\ , \qquad \|d\nu - d\nu_{B(0;R)}\| \ \leq\ \epsilon\ .$$

Because of the uniform continuity of f in $B(0;2R)$, we may, also, take a linear combination

$$g \ =\ \sum_{k=1}^{m} \lambda_k \chi_{A_k}\ ,$$

where $B(0;2R) = \cup_{k=1}^{m} A_k$ is a partition of $B(0;2R)$ in pairwise disjoint Borel subsets of small diameter, so that

$$\sup_{z\in B(0;2R)} |f(z) - g(z)| \ \leq\ \epsilon$$

and, hence,

$$\sup_{z\in\mathbf{R^n}} |g(z)| \ \leq\ \sup_{z\in\mathbf{R^n}} |f(z)| + \epsilon\ .$$

The measure $d\mu_{B(0;R)} * d\nu_{B(0;R)}$ is supported in $B(0;R)+B(0;R) = B(0;2R)$. Therefore,

$$\Big|\int_{\mathbf{R^n}} f(z)\ d\mu_{B(0;R)} * d\nu_{B(0;R)}(z) - \int_{\mathbf{R^n}} g(z)\ d\mu_{B(0;R)} * d\nu_{B(0;R)}(z)\Big|$$
$$\leq\ \epsilon\|d\mu_{B(0;R)} * d\nu_{B(0;R)}\| \ \leq\ \epsilon\|d\mu\|\|d\nu\|\ .$$

We, clearly, have that

$$\begin{aligned}&\Big|\int_{\mathbf{R^n}} f(z)\ d\mu * d\nu(z) - \int_{\mathbf{R^n}} f(z)\ d\mu_{B(0;R)} * d\nu_{B(0;R)}(z)\Big| \\ &\leq\ \sup_{z\in\mathbf{R^n}} |f(z)|\|d\mu - d\mu_{B(0;R)}\|\|d\nu\| \\ &+\ \sup_{z\in\mathbf{R^n}} |f(z)|\|d\mu_{B(0;R)}\|\|d\nu - d\nu_{B(0;R)}\| \\ &\leq\ \epsilon \sup_{z\in\mathbf{R^n}} |f(z)|\big(\|d\mu\| + \|d\nu\|\big)\ .\end{aligned}$$

In the same manner,

$$\begin{aligned}&\Big|\int_{\mathbf{R^n}} g(z)\ d\mu * d\nu(z) - \int_{\mathbf{R^n}} g(z)\ d\mu_{B(0;R)} * d\nu_{B(0;R)}(z)\Big| \\ &\leq\ \epsilon \sup_{z\in\mathbf{R^n}} |g(z)|\big(\|d\mu\| + \|d\nu\|\big) \\ &\leq\ \epsilon\big(\sup_{z\in\mathbf{R^n}} |f(z)| + \epsilon\big)\big(\|d\mu\| + \|d\nu\|\big)\ .\end{aligned}$$

With similar calculations, we find that

$$\begin{aligned}&\Big|\int_{B(0;R)\times B(0;R)} f(x+y)\ d\mu(x)\times d\nu(y) \\ &- \int_{B(0;R)\times B(0;R)} g(x+y)\ d\mu(x)\times d\nu(y)\Big| \\ &\leq\ \epsilon\|d\mu\|\|d\nu\|\ ,\end{aligned}$$

that

$$\begin{aligned}
&\Big| \int_{\mathbf{R^n} \times \mathbf{R^n}} f(x+y)\ d\mu(x) \times d\nu(y) \\
&- \int_{B(0;R)\times B(0;R)} f(x+y)\ d\mu(x) \times d\nu(y) \Big| \\
&\le\ \epsilon \sup_{z \in \mathbf{R^n}} |f(z)| \big(\|d\mu\| + \|d\nu\| \big)
\end{aligned}$$

and that

$$\begin{aligned}
&\Big| \int_{\mathbf{R^n} \times \mathbf{R^n}} g(x+y)\ d\mu(x) \times d\nu(y) \\
&- \int_{B(0;R)\times B(0;R)} g(x+y)\ d\mu(x) \times d\nu(y) \Big| \\
&\le\ \epsilon \big(\sup_{z \in \mathbf{R^n}} |f(z)| + \epsilon \big) \big(\|d\mu\| + \|d\nu\| \big)\ .
\end{aligned}$$

Finally, from the definition of $d\mu * d\nu$ and from linearity, we have

$$\int_{\mathbf{R^n}} g(z)\ d\mu * d\nu(z)\ =\ \int_{\mathbf{R^n} \times \mathbf{R^n}} g(x+y)\ d\mu(x) \times d\nu(y)\ .$$

Combining all the above estimates with the last equality, we conclude that

$$\Big| \int_{\mathbf{R^n}} f(z)\ d\mu * d\nu(z) - \int_{\mathbf{R^n} \times \mathbf{R^n}} f(x+y)\ d\mu(x) \times d\nu(y) \Big|\ \le\ C\epsilon\ ,$$

where C is a constant independent of ϵ, and this finishes the proof.

Proposition 0.32 *If $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$, then*

$$\widehat{d\mu * d\nu}\ =\ \widehat{d\mu}\widehat{d\nu}\ .$$

Also, if $f \in L^1(\mathbf{R^n})$ and $d\nu \in \mathcal{M}(\mathbf{R^n})$, then

$$\widehat{f * d\nu}\ =\ \widehat{f}\widehat{d\nu}\ .$$

Proof:

For the first part, we apply Proposition 0.31 to the function

$$f(z)\ =\ e^{-2\pi i \xi \cdot z}\ , \qquad z \in \mathbf{R^n}\ .$$

For the last part, we just imitate the proof of Proposition 0.23.

Lemma 0.4 *Suppose that f and g are in $L^1(\mathbf{R^n})$ and that $\int_{\mathbf{R^n}} g(x)\ dm(x) = 1$. Then,*

$$f * g_\delta\ \to\ f$$

in $L^1(\mathbf{R^n})$ as $\delta \to 0+$.

Proof:

$$\begin{aligned}
&\int_{\mathbf{R^n}} |f * g_\delta(x) - f(x)| \; dm(x) \\
&= \int_{\mathbf{R^n}} \Big| \int_{\mathbf{R^n}} \big(f(x-\delta y) - f(x)\big) g(y) \; dm(y) \Big| \; dm(x) \\
&\le \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \big| f(x-\delta y) - f(x) \big| \; dm(x) \; |g(y)| \; dm(y) \; .
\end{aligned}$$

The function

$$\phi_\delta(y) \;=\; \int_{\mathbf{R^n}} \big| f(x-\delta y) - f(x) \big| \; dm(x) \; , \qquad y \in \mathbf{R^n} \; ,$$

has the properties:

1. $\sup_{y \in \mathbf{R^n}} |\phi_\delta(y)| \le 2\|f\|_{L^1(\mathbf{R^n})}$ and
2. $\lim_{\delta \to 0} \phi_\delta(y) = 0$ for all $y \in \mathbf{R^n}$.

By the Dominated Convergence Theorem,

$$\lim_{\delta \to 0} \int_{\mathbf{R^n}} |f * g_\delta(x) - f(x)| \; dm(x) \;=\; 0 \; .$$

Theorem 0.14 *(The Inversion Formula) If $f \in L^1(\mathbf{R^n})$ and $\widehat{f} \in L^1(\mathbf{R^n})$, then*

$$f(x) \;=\; \int_{\mathbf{R^n}} e^{2\pi i \xi \cdot x} \widehat{f}(\xi) \; dm(\xi)$$

for almost every $x \in \mathbf{R^n}$.

Proof:
For every $\alpha > 0$ and every $x \in \mathbf{R^n}$,

$$\begin{aligned}
&\int_{\mathbf{R^n}} e^{2\pi i \xi \cdot x} e^{-\alpha |\xi|^2} \widehat{f}(\xi) \; dm(\xi) \\
&= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} e^{-2\pi i \xi \cdot (y-x)} e^{-\alpha |\xi|^2} \; dm(\xi) \; f(y) \; dm(y) \\
&= \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \int_{\mathbf{R^n}} e^{-\frac{\pi^2}{\alpha}|x-y|^2} f(y) \; dm(y) \; .
\end{aligned}$$

Since $\widehat{f} \in L^1(\mathbf{R^n})$, by the Dominated Convergence Theorem, the left side converges to $\int_{\mathbf{R^n}} e^{2\pi i \xi \cdot x} \widehat{f}(\xi) \; dm(\xi)$ as $\alpha \to 0$.

From Lemma 0.4, the right side converges to f in $L^1(\mathbf{R^n})$ as $\alpha \to 0$ and the proof is complete.

Theorem 0.15 *(The Fourier transform is injective) If f and g are in $L^1(\mathbf{R^n})$ and $\widehat{f} = \widehat{g}$, then $f = g$.*

If $d\mu$ and $d\nu$ are in $\mathcal{M}(\mathbf{R^n})$ and $\widehat{d\mu} = \widehat{d\nu}$, then $d\mu = d\nu$.

Proof:

It is, clearly, enough to consider the case when $g = 0$ and $d\nu$ is the zero measure.

If $\widehat{f} = 0$, then the Inversion Formula gives that $f = 0$.

Now, assume that $\widehat{d\mu} = 0$.

By Proposition 0.32, for all $f \in L^1(\mathbf{R^n})$,

$$\widehat{f * d\mu} \;=\; \widehat{f}\widehat{d\mu} \;=\; 0$$

and, thus, by the first part,

$$f * d\mu \;=\; 0\ .$$

Considering any approximation to the identity $\{\Phi_\delta : \delta > 0\}$ and any $\phi \in \mathcal{D}(\mathbf{R^n})$,

$$\begin{aligned} 0 &= \int_{\mathbf{R^n}} \Phi_\delta * d\mu(x)\ \phi(x)\ dm(x) \\ &= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} \Phi_\delta(x)\phi(y-x)\ dm(x)\ d\mu(y) \\ &\to \int_{\mathbf{R^n}} \phi(y)\ d\mu(y)\ , \end{aligned}$$

as $\delta \to 0$, by the Dominated Convergence Theorem and Proposition 0.9.

Therefore,

$$\int_{\mathbf{R^n}} \phi(y)\ d\mu(y) \;=\; 0$$

for all $\phi \in \mathcal{D}(\mathbf{R^n})$ and, by Proposition 0.12, $d\mu$ is the zero measure.

Theorem 0.16 *Suppose that $(1 + |\cdot|)^k f(\cdot) \in L^1(\mathbf{R^n})$. Then $\widehat{f} \in C^k(\mathbf{R^n})$ and*

$$D^\alpha \widehat{f}(\xi) \;=\; \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}(-2\pi i x)^\alpha f(x)\ dm(x) \;=\; (-2\pi i)^\alpha \big[(\cdot)^\alpha f(\cdot)\big]\widehat{\ }(\xi)$$

for all $\xi \in \mathbf{R^n}$ and all α with $|\alpha| \le k$.

Proof:

By induction, it is enough to consider the case $|\alpha| = 1$. We, thus, suppose that $(1 + |\cdot|)f(\cdot) \in L^1(\mathbf{R^n})$ and we get, for all j with $1 \le j \le n$,

$$\begin{aligned} \lim_{h\to 0} \frac{\widehat{f}(\xi + he_j) - \widehat{f}(\xi)}{h} &= \lim_{h\to 0} \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\ \frac{e^{-2\pi i x_j h} - 1}{h}\ f(x)\ dm(x) \\ &= \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}(-2\pi i x_j) f(x)\ dm(x)\ . \end{aligned}$$

The use of the Dominated Convergence Theorem in the above limit is permitted by the inequality $\left|\frac{e^{-2\pi i x_j h}-1}{h}\right| \leq 2\pi |x_j|$.

Theorem 0.17 *Suppose that* $f \in C^k(\mathbf{R^n})$ *and* $D^\alpha f \in L^1(\mathbf{R^n})$ *for all* α *with* $|\alpha| \leq k$. *Then*

$$\widehat{D^\alpha f}(\xi) \;=\; (2\pi i \xi)^\alpha \widehat{f}(\xi)$$

for all $\xi \in \mathbf{R^n}$ *and all* α *with* $|\alpha| \leq k$.

Proof:

By induction, it is enough to consider the case $|\alpha| = 1$.

We estimate

$$\begin{aligned}
&\Big|\int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{f(x+he_j)-f(x)}{h}\; dm(x) - \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{\partial f}{\partial x_j}(x)\; dm(x)\Big| \\
&= \;\Big|\int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi} \int_0^1 \Big(\frac{\partial f}{\partial x_j}(x+the_j) - \frac{\partial f}{\partial x_j}(x)\Big)\; dt\; dm(x)\Big| \\
&\leq \;\int_0^1 \int_{\mathbf{R^n}} \Big|\frac{\partial f}{\partial x_j}(x+the_j) - \frac{\partial f}{\partial x_j}(x)\Big|\; dm(x)\; dt \\
&\leq \;\sup_{|y|\leq|h|} \Big\|\frac{\partial f}{\partial x_j}(\cdot+y) - \frac{\partial f}{\partial x_j}(\cdot)\Big\|_{L^1(\mathbf{R^n})} .
\end{aligned}$$

Therefore,

$$\lim_{h\to 0}\int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{f(x+he_j)-f(x)}{h}\; dm(x) \;=\; \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{\partial f}{\partial x_j}(x)\; dm(x) .$$

On the other hand,

$$\begin{aligned}
&\lim_{h\to 0}\int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{f(x+he_j)-f(x)}{h}\; dm(x) \\
&= \;\lim_{h\to 0}\int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi}\; \frac{e^{2\pi i \xi_j h}-1}{h}\; f(x)\; dm(x) \\
&= \;2\pi i \xi_j \widehat{f}(\xi)
\end{aligned}$$

and we conclude that

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) \;=\; 2\pi i \xi_j \widehat{f}(\xi)$$

for all $\xi \in \mathbf{R^n}$.

Corollary 0.1 *1. If* $|\cdot|^k f(\cdot) \in L^1(\mathbf{R^n})$ *for all* $k \geq 0$, *then* $\widehat{f} \in C^\infty(\mathbf{R^n})$ *and* $D^\alpha \widehat{f} \in L^\infty(\mathbf{R^n})$ *for all* α. *In particular, if* f *is compactly supported in* $\mathbf{R^n}$, *then* $\widehat{f} \in C^\infty(\mathbf{R^n})$.
2. If $f \in C^\infty(\mathbf{R^n})$ *and* $D^\alpha f \in L^1(\mathbf{R^n})$ *for all* α, *then* $|\cdot|^k \widehat{f}(\cdot) \in L^\infty(\mathbf{R^n})$ *for all* $k \geq 0$.

The most symmetric result comes after the following definition.

Definition 0.23 *f is called a* ***Schwartz-function*** *if it belongs to $C^\infty(\mathbf{R^n})$ and $|\cdot|^k|D^\alpha f(\cdot)| \in L^\infty(\mathbf{R^n})$ for all $k \geq 0$ and all α.*

The space of all Schwartz-functions is denoted by $\mathcal{S}(\mathbf{R^n})$ and it is called the ***Schwartz-class****.*

For example, every $\phi \in \mathcal{D}(\mathbf{R^n})$ belongs to $\mathcal{S}(\mathbf{R^n})$.

Theorem 0.18 *The Fourier transform is a linear, one-to-one mapping from $\mathcal{S}(\mathbf{R^n})$ onto itself:*

$$\mathcal{F} \ : \ \mathcal{S}(\mathbf{R^n}) \to \mathcal{S}(\mathbf{R^n}) \ .$$

Proof:

If $f \in \mathcal{S}(\mathbf{R^n})$, then $|\cdot|^k|f(\cdot)| \in L^\infty(\mathbf{R^n})$ for all $k \geq 0$ and, hence, $|\cdot|^k|f(\cdot)| \in L^1(\mathbf{R^n})$ for all $k \geq 0$. By Corollary 0.1, $\widehat{f}$ belongs to $C^\infty(\mathbf{R^n})$.

A combination of Theorems 0.16 and 0.17 gives

$$(2\pi i)^{|\beta|}\xi^\beta D^\alpha \widehat{f}(\xi) \ = \ (-2\pi i)^{|\alpha|}\big[D^\beta\big\{(\cdot)^\alpha f(\cdot)\big\}\big]\widehat{\ }(\xi)$$

for all ξ and all α and β. From $f \in \mathcal{S}(\mathbf{R^n})$ we get that $D^\beta\big\{(\cdot)^\alpha f(\cdot)\big\} \in L^1(\mathbf{R^n})$ and, thus, $|\cdot|^{|\beta|}D^\alpha\widehat{f}(\cdot) \in L^\infty(\mathbf{R^n})$ for all α and β.

Hence, $\widehat{f} \in \mathcal{S}(\mathbf{R^n})$ and $\mathcal{F}$ is a linear and, by Theorem 0.15, one-to-one mapping of $\mathcal{S}(\mathbf{R^n})$ into itself.

If $g \in \mathcal{S}(\mathbf{R^n})$, we consider the function $f = \widetilde{\widehat{g}}$. This belongs to $\mathcal{S}(\mathbf{R^n})$ and, by the Inversion Formula, $g = \widehat{f}$.

There exists a natural metric on $\mathcal{S}(\mathbf{R^n})$ and $\mathcal{F}$ is a homeomorphism of $\mathcal{S}(\mathbf{R^n})$. We shall not work in this direction.

Part II

Main Theory

Chapter 1

Harmonic Functions

1.1 Definition

Suppose Ω is an open subset of $\mathbf{R^n}$.

Definition 1.1 *A complex-valued function u defined in Ω is called **harmonic** in Ω, if*

1. *u is continuous in Ω and*
2. *$u(x) = \mathcal{M}_u^r(x)$ for all $x \in \Omega$ and all $r < d(x, \partial\Omega)$*

or, if

1. *u is continuous in Ω and*
2. *$u(x) = \mathcal{A}_u^r(x)$ for all $x \in \Omega$ and all $r < d(x, \partial\Omega)$.*

Both conditions 2 are called the **mean-value property** of harmonic functions. The first is the surface-mean-value property and the second the space-mean-value property.

The two definitions are equivalent because, if $u(x) = \mathcal{M}_u^r(x)$ for all $x \in \Omega$ and all $r < d(x, \partial\Omega)$, then

$$\mathcal{A}_u^r(x) \;=\; \frac{n}{r^n}\int_0^r \mathcal{M}_u^s(x)s^{n-1}ds \;=\; \frac{n}{r^n}\int_0^r u(x)s^{n-1}ds \;=\; u(x)$$

and, if $u(x) = \mathcal{A}_u^r(x)$ for all $x \in \Omega$ and all $r < d(x, \partial\Omega)$, then

$$r^n u(x) = n\int_0^r \mathcal{M}_u^s(x)s^{n-1}ds$$

and, taking derivatives with respect to r, we get

$$nr^{n-1}u(x) \;=\; nr^{n-1}\mathcal{M}_u^r(x)$$

and, hence,

$$u(x) = \mathcal{M}_u^r(x) \ .$$

Properties of harmonic functions

(1) *Linear combinations of harmonic functions are harmonic.*

(2) *Harmonic functions are preserved by rigid motions of the space.*

Suppose u is harmonic in the open Ω and $x' = O(x) + b$ is a rigid motion, where O is an orthogonal linear transformation and $b \in \mathbf{R^n}$.

Then $\Omega' = O(\Omega) + b = \{O(x) + b : x \in \Omega\}$ is open and $u'(x') = u(x)$ is defined and continuous on Ω'. By the invariance of distances and of Lebesgue measure under rigid motions,

$$u'(x') \ = \ u(x) \ = \ \frac{1}{V_n r^n} \int_{B(x;r)} u(y)\ dm(y) \ = \ \frac{1}{V_n r^n} \int_{B(x';r)} u'(y')\ dm(y')$$

for every $x' \in \Omega'$ and every $r < d(x', \partial\Omega')$.

(3) *Locally uniform limits of harmonic functions are harmonic.*

Because, suppose $u_k \to u$ locally uniformly in Ω, i.e. uniformly on every compact subset of Ω. Then, u is, obviously, continuous on Ω. Also, $S(x;r)$ is compact in Ω if $x \in \Omega$ and $r < d(x, \partial\Omega)$. Therefore,

$$u(x) \leftarrow u_k(x) \ = \ \mathcal{M}_{u_k}^r(x) \to \mathcal{M}_u^r(x) \ ,$$

because of uniform convergence on $S(x;r)$.

(4) *(Picard) If the real-valued u is harmonic in $\mathbf{R^n}$ and bounded from above or from below, then it is constant.*

It is enough to assume $u \geq 0$ in $\mathbf{R^n}$.

Take $x \neq y$ and $B(x;r_1) \subseteq B(y;r_2)$. Let, in fact,

$$r_2 \ = \ r_1 + |x - y| \ .$$

Then,

$$V_n r_1^n \mathcal{A}_u^{r_1}(x) \ = \ \int_{B(x;r_1)} u(z)\ dm(z) \ \leq \ \int_{B(y;r_2)} u(z)\ dm(z) \ = \ V_n r_2^n \mathcal{A}_u^{r_2}(y)$$

and, hence,

$$r_1^n u(x) \ \leq \ r_2^n u(y) \ .$$

Letting $r_1, r_2 \to +\infty$, we get, since $\frac{r_1}{r_2} \to 1$,

$$u(x) \ \leq \ u(y)$$

and, symmetrically, $u(y) \leq u(x)$. Therefore,

$$u(x) \ = \ u(y)$$

and u is constant.

(5) *If the real-valued harmonic u has a local extremum at x, then it is constant in an open neighborhood of x.*

Suppose $u(y) \le u(x)$ for all $y \in B(x;r)$ and u is harmonic in an open set containing $\overline{B(x;r)}$. Then,

$$u(x) \ = \ \frac{1}{V_n r^n} \int_{B(x;r)} u(y)\ dm(y) \ \le \ \frac{1}{V_n r^n} \int_{B(x;r)} u(x)\ dm(y) \ = \ u(x) \ .$$

The inequality becomes equality only if $u(y) = u(x)$ for almost every y in $B(x;r)$ and, since u is continuous,

$$u(y) \ = \ u(x)$$

for every $y \in B(x;r)$.

1.2 Maximum-minimum principle

Theorem 1.1 *(Maximum-Minimum Principle for Harmonic Functions) Let u be real-valued and harmonic in the open $\Omega \subseteq \mathbf{R^n}$.*

1. *If u takes one of its extremum values at some x, then u is constant in the connected component of Ω which contains x.*
2. *If*
$$M = \sup_{y\in\partial\Omega} \big(\limsup_{\Omega\ni x\to y} u(x)\big) \ , \qquad m = \inf_{y\in\partial\Omega} \big(\liminf_{\Omega\ni x\to y} u(x)\big) \ ,$$
then $m \le u(x) \le M$ for all x in Ω.

   If $u(x) = m$ or if $u(x) = M$ at some $x \in \Omega$, then u is constant in the connected component of Ω which contains x.

Proof:
By the fifth property of harmonic functions and, since $-u$ is harmonic when u is, the result is an immediate application of Proposition 0.5.

If O is any connected component of Ω, then its closure $\overline{O}$ is compact and its boundary ∂O is a non-empty subset of $\partial\Omega$. By Proposition 0.5,

$$m \ \le \ \inf_{y\in\partial O} \big(\liminf_{O\ni x\to y} u(x)\big) \ \le \ u(x) \ \le \ \sup_{y\in\partial O} \big(\limsup_{O\ni x\to y} u(x)\big) \ \le \ M$$

for all $x \in O$.

Corollary 1.1 *Let u be harmonic in the open $\Omega \subseteq \mathbf{R^n}$. If $\lim_{\Omega\ni x\to y} u(x) = 0$ for all $y \in \partial\Omega$, then u is identically 0 in Ω.*

Later on we shall study the

Problem of Dirichlet *Given the open $\Omega \subseteq \mathbf{R^n}$ and the complex-valued f defined in $\partial\Omega$, find u harmonic in Ω so that*

$$\lim_{\Omega\ni x\to y} u(x) \ = \ f(y)$$

for all $y \in \partial\Omega$. This is the ***Dirichlet Problem for Ω with boundary function f.***

The only remark we shall make at this point is that, if a solution exists, then, by the Corollary 1.1, it is unique.

1.3 Differentiability of harmonic functions

If u is harmonic in the open subset Ω of $\mathbf{R^n}$ and $\{\Phi_\delta : \delta > 0\}$ is any approximation to the identity, then, for every $x \in \Omega_\delta$,

$$\begin{aligned}
\Phi_\delta * u(x) &= \int_{B(0;\delta)} u(x-y)\Phi_\delta(y)\ dm(y) \\
&= \int_0^\delta \Phi_{\delta *}(r) \int_{S^{n-1}} u(x+rt)\ d\sigma(t)\ r^{n-1}dr \\
&= \omega_{n-1} \int_0^\delta \mathcal{M}_u^r(x)\Phi_{\delta *}(r)\ r^{n-1}dr \\
&= \omega_{n-1} u(x) \int_0^\delta \Phi_{\delta *}(r)\ r^{n-1}dr \\
&= u(x) \int_{B(0;\delta)} \Phi_\delta(y)\ dm(y) \\
&= u(x)\ .
\end{aligned}$$

From this and from Proposition 0.7, we get that u is in $C^\infty(\Omega_\delta)$ and, since δ is arbitrary, that u is in $C^\infty(\Omega)$. Therefore, we proved the

Theorem 1.2 *If u is harmonic in the open set $\Omega \subseteq \mathbf{R^n}$, then u is infinitely differentiable in Ω.*

Theorem 1.3 *u is harmonic in the open set $\Omega \subseteq \mathbf{R^n}$ if and only if u is in $C^2(\Omega)$ and satisfies the Laplace equation $\Delta u = 0$ everywhere in Ω.*

Proof:
Suppose u is harmonic in Ω. By Theorem 1.2, u is in $C^2(\Omega)$ and, taking any $\overline{B(x;R)} \subseteq \Omega$ and applying Green's Formula, we get

$$\begin{aligned}
\int_{B(x;R)} \Delta u(z)\ dm(z) &= \int_{S(x;R)} \frac{\partial u}{\partial \eta}(y)\ dS(y) \\
&= R^{n-1} \int_{S^{n-1}} \frac{d}{dr}\big(u(x+rt)\big)_{r=R}\ d\sigma(t) \\
&= R^{n-1} \frac{d}{dr}\Big(\int_{S^{n-1}} u(x+rt)\ d\sigma(t)\Big)_{r=R} \\
&= \omega_{n-1} R^{n-1} \frac{d}{dr}\big(\mathcal{M}_u^r(x)\big)_{r=R}
\end{aligned}$$

$$\begin{aligned} &= \omega_{n-1} R^{n-1} \frac{d}{dr}\big(u(x)\big)_{r=R} \\ &= 0 \ . \end{aligned}$$

Since Δu is continuous in Ω and the ball is arbitrary, we conclude that

$$\Delta u(x) \ = \ \lim_{R\to 0+} \frac{1}{V_n R^n} \int_{B(x;R)} \Delta u(z)\ dm(z) \ = \ 0$$

for all $x \in \Omega$.

Now, suppose $\Delta u = 0$ everywhere in Ω, fix an arbitrary $x \in \Omega$ and take any $R < d(x, \partial\Omega)$.

By reversing the calculations above, we find

$$\frac{d}{dr}\big(\mathcal{M}_u^r(x)\big)_{r=R} \ = \ 0$$

and, therefore, $\mathcal{M}_u^R(x)$ is constant in the interval $0 < R < d(x, \partial\Omega)$.

By the continuity of u at x,

$$u(x) \ = \ \lim_{R\to 0+} \mathcal{M}_u^R(x)$$

and we conclude that

$$u(x) \ = \ \mathcal{M}_u^R(x)$$

for all $R < d(x, \partial\Omega)$.

Lemma 1.1 *If f is twice continuously differentiable in some neighborhood of the point $x \in \mathbf{R^n}$, then*

$$\Delta f(x) \ = \ \lim_{r\to 0+} \frac{2n}{r^2}\ \big(\mathcal{M}_f^r(x) - f(x)\big) \ = \ \lim_{r\to 0+} \frac{2(n+2)}{r^2}\ \big(\mathcal{A}_f^r(x) - f(x)\big)\ .$$

Proof:

We write the Taylor-expansion of order 2 of f at x and observe the resulting cancellation in $\mathcal{M}_f^r(x)$ and $\mathcal{A}_f^r(x)$.

In case f is not differentiable but satisfies the necessary integrability conditions on spheres or balls, then either of the limits in Lemma 1.1 is used to *define* the so-called **generalized Laplacian** of f.

1.4 Holomorphy and harmonic conjugates

The standard examples of harmonic functions in dimension 2 are the real and imaginary parts of holomorphic functions in open sets.

Let f be holomorphic in the open set $\Omega \subseteq \mathbf{R^2}$. Then, if $u = \Re f$ and $v = \Im f$, the functions u and v are infinitely differentiable in Ω and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x_1} \ = \ \frac{\partial v}{\partial x_2}\ , \qquad \frac{\partial u}{\partial x_2} \ = \ -\frac{\partial v}{\partial x_1}\ .$$

Therefore,

$$\Delta u = \Delta v = 0$$

in Ω.

In another way, using the Cauchy's Formula,

$$f(x) = \frac{1}{2\pi i}\int_{S(x;r)} \frac{f(y)}{y-x}\, dy = \frac{1}{\omega_1 r}\int_{S(x;r)} f(y)\, dS(y)$$

and taking real and imaginary parts, we prove the mean-value property of u and v.

At least locally, these are the only examples of harmonic functions in dimension 2. In fact, if u is real-valued and harmonic in an open rectangle $\mathcal{R} = (a_1, b_1) \times (a_2, b_2)$, then there exists a holomorphic f in this rectangle so that $u = \Re f$.

To see this, we fix a point $(c_1, c_2) \in \mathcal{R}$ and define

$$v(X_1, X_2) = -\int_{c_1}^{X_1} \frac{\partial u}{\partial x_2}(x_1, c_2)\, dx_1 + \int_{c_2}^{X_2} \frac{\partial u}{\partial x_1}(X_1, x_2)\, dx_2$$

for all $X = (X_1, X_2) \in \mathcal{R}$.

We, clearly, have

$$\frac{\partial v}{\partial x_2}(X_1, X_2) = \frac{\partial u}{\partial x_1}(X_1, X_2)$$

and

$$\begin{aligned}
\frac{\partial v}{\partial x_1}(X_1, X_2) &= -\frac{\partial u}{\partial x_2}(X_1, c_2) + \int_{c_2}^{X_2} \frac{\partial^2 u}{\partial x_1^2}(X_1, x_2)\, dx_2 \\
&= -\frac{\partial u}{\partial x_2}(X_1, c_2) - \int_{c_2}^{X_2} \frac{\partial^2 u}{\partial x_2^2}(X_1, x_2)\, dx_2 \\
&= -\frac{\partial u}{\partial x_2}(X_1, c_2) - \frac{\partial u}{\partial x_2}(X_1, X_2) + \frac{\partial u}{\partial x_2}(X_1, c_2) \\
&= -\frac{\partial u}{\partial x_2}(X_1, X_2)\ .
\end{aligned}$$

Therefore, the functions u and v satisfy the Cauchy-Riemann equations in $\mathcal{R}$ and, thus, the function $f = u + iv$ is holomorphic in $\mathcal{R}$ with $u = \Re f$ there.

If $g = u + iw$ is another function holomorphic in $\mathcal{R}$ with $u = \Re g$ there, then, by the Cauchy-Riemann equations, we get that v and w differ by some (real) constant in $\mathcal{R}$ and, hence, f and g differ by an imaginary constant in $\mathcal{R}$.

This result is, as we shall see in a moment, true, even when we replace the rectangle by an arbitrary simply-connected open $\Omega \subseteq \mathbf{R}^2$. On the other hand, the function $u : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}$ given by

$$u(x) = \log|x|\ , \qquad x \neq 0$$

is harmonic in $\mathbf{R^2} \setminus \{0\}$, but there is no f holomorphic in $\mathbf{R^2} \setminus \{0\}$ so that $\Re f = u$ everywhere in $\mathbf{R^2} \setminus \{0\}$.

In fact, if there was such an f, then we would have that

$$f(x) \ = \ \log_0(x) + ia \ , \qquad x \in \Omega_0$$

for some constant $a \in \mathbf{R}$, where $\Omega_0 = \mathbf{R^2} \setminus \{x = (x_1, 0) : x_1 \leq 0\}$ and $\log_0$ is the principal branch of the logarithm in Ω_0. This, clearly, contradicts the continuity of f at the points of the negative-x_1-axis.

Definition 1.2 *Let u be real-valued and harmonic in the open $\Omega \subseteq \mathbf{R^2}$. If v is real-valued and harmonic in Ω and*

$$\frac{\partial u}{\partial x_1} \ = \ \frac{\partial v}{\partial x_2} \ , \qquad \frac{\partial u}{\partial x_2} \ = \ -\frac{\partial v}{\partial x_1}$$

*everywhere in Ω, then v is called a **harmonic conjugate of** u **in** Ω.*

It is obvious that the existence of a harmonic conjugate of u in Ω is equivalent to the existence of a holomorphic function in Ω whose real part is u. Furthermore, if Ω is, also, connected, then any two harmonic conjugates of u differ by a real constant in Ω.

If v is a harmonic conjugate of u, then $-u$ is a harmonic conjugate of v.

The many-valued "function" called **argument** is defined in $\mathbf{R^2} \setminus \{0\}$ by $\arg(x) = \theta + k2\pi$, where θ is any number with $e^{i\theta} = \frac{x}{|x|}$ and k takes all values in $\mathbf{Z}$.

If $g : A \to \mathbf{R^2} \setminus \{0\}$ is continuous in A, then any function $v : A \to \mathbf{R}$ which is continuous in A and is such that

$$\exp\bigl(iv(x)\bigr) \ = \ \frac{g(x)}{|g(x)|}$$

for all $x \in A$, is called **a branch of the argument of** g **in** A**.** It is easy to see, by continuity, that if A is a connected set and a branch v of the argument of g exists in A, then the totality of branches of the argument of g in A are the functions $v + k2\pi$, where $k \in \mathbf{Z}$.

It is obvious that, through $f = \log|g| + iv$ every branch of the logarithm of g defines a branch of the argument of g and *vice versa*.

Proposition 1.1 *If $g : \Omega \to \mathbf{R^2} \setminus \{0\}$ is holomorphic in the connected open set $\Omega \subseteq \mathbf{R^2}$, then every branch of the argument of g in Ω is a harmonic conjugate of $\log|g|$ in Ω and, conversely, every harmonic conjugate of $\log|g|$ in Ω is, except for an additive constant, a branch of the argument of g in Ω.*

If v is any branch of the argument of g in Ω, then, clearly, the function $f = \log|g| + iv$ is a branch of the logarithm of g in Ω and, hence, v is a harmonic conjugate of $\log|g|$ in Ω. Conversely, if v is a harmonic conjugate of $\log|g|$ in Ω, then the function $f = \log|g| + iv$ is holomorphic in Ω and the functions $\exp(f)$ and g have the same absolute value everywhere in Ω. It is, now, easy to show, through the Cauchy-Riemann equations, that, in each component of Ω, $\exp(f) = \lambda g$, where $\lambda = e^{i\theta}$ is a constant with $|\lambda| = 1$. Hence, $v - \theta$ is a branch of the argument of g in Ω.

Definition 1.3 *An open set $\Omega \subseteq \mathbf{R^2}$ is called **simply-connected** if Ω is connected and $\overline{\mathbf{R^2}} \setminus \Omega$ is connected.*

Lemma 1.2 *Suppose that $\mathcal{R}_1$ and $\mathcal{R}_2$ are two open rectangles with sides parallel to the x_1- and x_2-axes with one common side I and let $\mathcal{R}$ be the open rectangle $\mathcal{R}_1 \cup \mathcal{R}_2 \cup int(I)$. If u is harmonic in $\mathcal{R}$ and v_1 is a harmonic conjugate of u in $\mathcal{R}_1$, then there exists a harmonic conjugate v of u in $\mathcal{R}$ so that $v = v_1$ in $\mathcal{R}_1$.*

Proof:

Let v_0 be any harmonic conjugate of u in $\mathcal{R}$. This is, also, a harmonic conjugate of u in $\mathcal{R}_1$ and, hence, $v_0 = v_1 + a$ everywhere in $\mathcal{R}_1$ for some real constant a. Then $v = v_0 - a$ is the harmonic conjugate of u in $\mathcal{R}$ with $v = v_1$ in $\mathcal{R}_1$.

Theorem 1.4 *Suppose that u is real-valued and harmonic in the simply-connected $\Omega \subseteq \mathbf{R^2}$. Then, there exists a harmonic conjugate of u in Ω. Two such harmonic conjugates of u differ by a real constant in Ω.*

Proof:

Fix a small $\delta > 0$ and consider the collection of all closed squares

$$Q_k = \{x = (x_1, x_2) : k_1\delta \leq x_1 \leq (k_1 + 1)\delta,\ k_2\delta \leq x_2 \leq (k_2 + 1)\delta\} ,$$

where $k = (k_1, k_2) \in \mathbf{Z} \times \mathbf{Z}$.

Fix, also, a large $K \in \mathbf{Z}$ and consider the finite subcollection $\mathcal{Q}_\delta^K$ of all Q_k with $|k_1| \leq K$ and $|k_2| \leq K$ which are contained in Ω. Form the set

$$\Omega_\delta^K = int \bigcup \{Q_k : Q_k \in \mathcal{Q}_\delta^K\} ,$$

which consists of the interiors of all $Q_k \in \mathcal{Q}_\delta^K$ together with the interiors of all common sides of all these squares.

We separate, in the obvious way, Ω_δ^K into horizontal layers of height δ where each layer consists of finitely many open rectangles arranged horizontally and having no common sides. These layers are “glued” at the common parts of their horizontal sides to form the set Ω_δ^K.

Starting with the first (bottom) layer, we construct a harmonic conjugate of u in each of its rectangles and we, thus, have a harmonic conjugate v_1 of u in this first layer.

Suppose that we have constructed a harmonic conjugate v_k of u in the part $\Omega_\delta^{K,k}$ of Ω_δ^K which lies below its $k+1$ layer and consider the first (to the left) rectangle $\mathcal{R}$ of the $k+1$ layer of Ω_δ^K. If $\mathcal{R}$ has no common side with any of the rectangles of the k layer, we construct, arbitrarily, a harmonic conjugate of u in $\mathcal{R}$. If the lower side of $\mathcal{R}$ has a common part with the upper side of *only one* rectangle $\mathcal{R}'$ of the k layer, we use Lemma 1.2 (at most three times) to extend v_1 as a harmonic conjugate of u in the union $\Omega_\delta^{K,k} \cup \mathcal{R}$.

If the lower side of $\mathcal{R}$ has a common part with the upper side of *at least two* rectangles $\mathcal{R}', \mathcal{R}'', \ldots$ of the k layer, then we consider the sets $O_k', O_k'', \ldots$ defined as the components of $\Omega_\delta^{K,k}$ which contain $\mathcal{R}', \mathcal{R}'', \ldots$, respectivelly.

Observe, now, that these components are *pairwise disjoint* as a trivial application of Theorem 0.8 shows!

We may now extend v_k in the union of, say, O_k' with $\mathcal{R}$ and, next, modify v_k in the other components $O_k'', \ldots$, adding an appropriate (for each component) constant to it, so that the, already constructed extension of v_k from O_k into $\mathcal{R}$ coincides (in $\mathcal{R}$) with its similarly constructed extensions from $O_k'', \ldots$ into $\mathcal{R}$. After all this, we end up with a harmonic conjugate of u in the union $\Omega_\delta^{K,k} \cup \mathcal{R}$.

We, now, continue with the second from the left rectangle in the $k+1$ layer and, with the same procedure, we modify and extend the already constructed harmonic conjugate into this new rectangle. After finitely many steps, we shall end up with a harmonic conjugate v_{k+1} of u in $\Omega_\delta^{K,k+1}$.

Finally, after finitely many steps we find a harmonic conjugate of u in Ω_δ^K.

Now, we consider any open exhaustion $\{\Omega_{(m)}\}$ of Ω so that every $\Omega_{(m)}$ is connected. Each $\Omega_{(m)}$ is contained in some Ω_δ^K, provided δ is small and K is large. Therefore, there is some harmonic conjugate $v_{(m)}$ of u in $\Omega_{(m)}$.

Since $\Omega_{(m)}$ is connected, modifying $v_{(m+1)}$ by some appropriate constant, we may arrange so that $v_{(m+1)} = v_{(m)}$ in $\Omega_{(m)}$. Therefore, a function v is defined in Ω which, clearly, is a harmonic conjugate of u in Ω.

Proposition 1.2 *Consider the open subsets Ω_1 and Ω_2 of $\mathbf{R^2}$, a function f holomorphic in Ω_1 and assume that $f(\Omega_1) \subseteq \Omega_2$.*

If u is harmonic in Ω_2, then $u \circ f$ is harmonic in Ω_1 .

Proof:

Direct calculation, using the Cauchy-Riemann equations, shows that

$$\Delta(u \circ f)(x) \ = \ |f'(x)|^2 \, \Delta u(f(x)) \ = \ 0$$

for every $x \in \Omega_1$.

Example

If the open set $\Omega \subseteq \mathbf{R^2}$ does not contain 0 and u is harmonic in Ω, we consider the open set $\Omega^* = \{x : \frac{1}{x} \in \Omega\}$ and define

$$u^*(x) \ = \ u\Big(\frac{1}{x}\Big)$$

for all $x \in \Omega^*$. Then u^* is harmonic in Ω^*.

1.5 Fundamental solution

Now, suppose that u is defined in a **ring** $B(0; R_1, R_2) = \{x \in \mathbf{R^n} : R_1 < |x| < R_2\}$ and that it is a radial function:

$$u(x) \ = \ u(y)$$

for all $x, y \in B(0; R_1, R_2)$ with $|x| = |y|$.

Then the function

$$u_*(r) \;=\; u(x) \;, \qquad \text{for any } x \in B(0; R_1, R_2) \text{ with } |x| = r \;,$$

is well-defined in the interval $R_1 < r < R_2$.

By trivial calculations, we can prove that u is twice continuously differentiable in $B(0; R_1, R_2)$ if and only if u_* is twice continuously differentiable in the interval $R_1 < r < R_2$ and, if this is true,

$$\Delta u(x) \;=\; \frac{d^2 u_*}{dr^2}(|x|) + \frac{n-1}{r}\,\frac{du_*}{dr}(|x|)$$

identically in $B(0; R_1, R_2)$.

Proposition 1.3 *Let u be a radial function defined in $B(0; R_1, R_2)$. Then u is harmonic there if and only if*

$$u(x) \;=\; \begin{cases} A \log \frac{1}{|x|} + B \;, & n = 2 \\ A \frac{1}{|x|^{n-2}} + B \;, & n > 2 \end{cases}$$

for some constants A and B.

If u is radial and harmonic in $B(0; R)$, then u is constant there.

Proof:

We, easily, solve the second order ordinary differential equation

$$\frac{d^2 u_*}{dr^2}(r) + \frac{n-1}{r}\,\frac{du_*}{dr}(r) \;=\; 0$$

in the interval $R_1 < r < R_2$.

The last statement is proved from the first, using the continuity of u at 0.

Definition 1.4 *We call the function*

$$h(x) \;=\; \begin{cases} \log \frac{1}{|x|} \;, & n = 2 \\ \frac{1}{|x|^{n-2}} \;, & n > 2 \end{cases}$$

the fundamental solution *of the Laplace equation in $\mathbf{R^n} \setminus \{0\}$.*

We, also, define

$$h_z(x) \;=\; h(x - z) \;, \qquad x \in \mathbf{R^n} \;,$$

for every $z \in \mathbf{R^n}$.

In any case the fundamental solution becomes $+\infty$ at 0, but observe a crucial difference between $n = 2$ and $n > 2$. The fundamental solution is bounded from below (by 0) when $n > 2$, while it tends to $-\infty$ when x tends to ∞ in $\mathbf{R^2}$.

Lemma 1.3

$$\frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_x(y)\ dS(y) \;=\; \begin{cases} h_*(R)\ , & \textit{if } |x| = r \le R \\ h(x) = h_*(r)\ , & \textit{if } |x| = r \ge R\ . \end{cases}$$

Proof:

Consider the function

$$u(x) \;=\; \frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_x(y)\ dS(y)\ , \qquad x \in \mathbf{R^n}\ .$$

It is clear that u is well-defined for all $x \notin S(0;R)$, since h_x is bounded in $S(0;R)$. For the same reason, using the ideas in the proof of Proposition 0.6, we may interchange integration and differentiation and prove that $\Delta u = 0$ in $\mathbf{R^n} \setminus S(0;R)$.

u is also well-defined for all $x \in S(0;R)$, since, as it is easy to see, h_x is integrable in $S(0;R)$.

If $|x_1| = |x_2|$, we consider any orthogonal transformation O so that $x_2 = O(x_1)$ and, since dS is rotation invariant and O preserves distances,

$$\begin{aligned} u(x_2) \;&=\; \frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_{O(x_1)}(y)\ dS(y) \\ &=\; \frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_{O(x_1)}\big(O(z)\big)\ dS(z) \\ &=\; \frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_{x_1}(z)\ dS(z) \\ &=\; u(x_1)\ . \end{aligned}$$

Hence, u is radial and this, by Proposition 1.4, implies that u is constant in $B(0;R)$. This constant is $u(0) = h_*(R)$.

h_x is harmonic in $\mathbf{R^n} \setminus \{x\}$ and, if $|x| > R$, we may apply the mean-value property to get

$$\frac{1}{\omega_{n-1}R^{n-1}}\int_{S(0;R)} h_x(y)\ dS(y) \;=\; h_x(0) \;=\; h(x)\ .$$

Now, consider any x with $|x| = R$ and the points $\frac{r}{R}\,x$ for all $r > R$. It is trivial to see that

$$h\Big(\frac{r}{R}\,x - y\Big) \;\le\; h(x-y)$$

for all $y \in S(0;R)$ and all $r > R$.

Therefore, by the Dominated Convergence Theorem,

$$u(x) \;=\; \lim_{r\to R+} u\Big(\frac{r}{R}\,x\Big) \;=\; \lim_{r\to R+} h\Big(\frac{r}{R}\,x\Big) \;=\; h(x) \;=\; h_*(R)$$

and this finishes the proof.

1.6 Potentials

Definition 1.5 *Let $d\mu$ be a compactly supported complex Borel measure. The function*

$$U_h^{d\mu}(x) \;=\; \int_{\mathbf{R^n}} h(x-y)\; d\mu(y) \;=\; \int_{\mathbf{R^n}} h_x(y)\; dm(y)\;, \qquad x \in \mathbf{R^n} \setminus supp(d\mu)\;,$$

*is called the h-**potential** of $d\mu$. More precisely, the **logarithmic potential** in case $n = 2$ and the **Newtonian potential** in case $n > 2$.*

*In case the support of $d\mu$ is contained in a C^1-hypersurface Γ, then the h-potential is, also, called **single-layer potential** of $d\mu$.*

Definition 1.6 *Let $d\mu$ be a complex Borel measure with compact support contained in an orientable C^1-hypersurface Γ. Then the function*

$$U(x) \;=\; \int_\Gamma \frac{\partial h_x}{\partial \eta}(y)\; d\mu(y)\;, \qquad x \in \mathbf{R^n} \setminus supp(d\mu)\;,$$

*where $\overrightarrow{\eta}$ is a continuous unit vector field normal to Γ, is called **double-layer potential** of $d\mu$.*

Comments A Newtonian potential can be defined for locally finite complex Borel measures $d\mu$ which are not necessarily supported in compact subsets of $\mathbf{R^n}$, provided they are of a definite sign, non-negative or non-positive. This is because the fundamental solution is positive and we end up integrating a quantity of a definite sign. The integral is, then, well-defined *for all* $x \in \mathbf{R^n}$, although it may take the value $\pm\infty$.

The situation is different for the logarithmic potential, because the fundamental solution is, on the one hand, bounded from below in every bounded set but, on the other hand, is not bounded from below in any unbounded set. Therefore, a Borel measure not supported in a compact subset of $\mathbf{R^n}$, even if it is of a definite sign, may not have a well-defined logarithmic potential.

Hence, we never consider logarithmic potentials of Borel measures which are not compactly supported.

Example: The h-potential of the Dirac mass $d\delta_a$ is equal to the translate of the fundamental solution at the point a :

$$U_h^{d\delta_a} \;=\; h_a\;.$$

Proposition 1.4 *Any of the above defined potentials of a compactly supported complex Borel measure $d\mu$ is harmonic in the open set $\mathbf{R^n} \setminus supp(d\mu)$.*

Proof:

We, easily, prove that $\Delta U(x) = 0$ for every $x \notin supp(d\mu)$, by passing the derivatives inside the integrals. To do this, we use a simple variant of the proof of Proposition 0.6.

1.7 Flux

The next three theorems in this and the next section are applications of the Green's Formulas. The first is for a special kind of domains, but for general harmonic functions. The second and third theorems together with their corollary are quite general and fundamental.

Theorem 1.5 *If u is harmonic in the ring $B(x; r_1, r_2)$, then the quantity*

$$\tau \;=\; \int_{S(x;r)} \frac{\partial u}{\partial \eta}(y)\; dS(y)$$

is constant as a function of r in the interval $r_1 < r < r_2$. Here, $\overrightarrow{\eta}$ is the continuous unit vector field normal to $S(x;r)$ in the direction towards the exterior of $B(x;r)$.

Also, for some constant λ,

$$\mathcal{M}_u^r(x) \;=\; \begin{cases} -\frac{\tau}{\omega_1}\log\frac{1}{r} + \lambda \;, & \text{if } \; n = 2 \\ -\frac{\tau}{(n-2)\omega_{n-1}}\frac{1}{r^{n-2}} + \lambda \;, & \text{if } \; n > 2 \end{cases}$$

in the interval $r_1 < r < r_2$.

If the ring becomes a ball $B(x; r_2)$, then $\tau = 0$ in all the above.

Proof:

Take $r_1 < r < r' < r_2$ and $\Omega = B(x; r, r')$. Then

$$\int_{\partial\Omega} \frac{\partial u}{\partial \eta}(y)\; dS(y) \;=\; \int_{\Omega} \Delta u(x)\; dm(x) \;=\; 0 \;,$$

where $\overrightarrow{\eta}$ is the continuous unit vector field normal to $\partial\Omega$ in the direction towards the exterior of Ω. This implies

$$-\int_{S(x;r)} \frac{\partial u}{\partial \eta}(y)\; dS(y) \;+\; \int_{S(x;r')} \frac{\partial u}{\partial \eta}(y)\; dS(y) \;=\; 0 \;,$$

where, in each integral, $\overrightarrow{\eta}$ is the continuous unit vector field normal to the sphere in the direction towards the exterior of the corresponding ball.

Therefore,

$$\tau \;=\; \int_{S(x;r)} \frac{\partial u}{\partial \eta}(y)\; dS(y)$$

is constant.

Now, the last equality, easily, becomes

$$\frac{d}{dr}\mathcal{M}_u^r(x) \;=\; \frac{\tau}{\omega_{n-1} r^{n-1}}$$

and this implies the formula for $\mathcal{M}_u^r(x)$.

If u is harmonic in $B(x; r_2)$, then $\mathcal{M}_u^r(x) = u(x)$ is constant as a function of r and, hence, $\tau = 0$.

Theorem 1.6 *Suppose Ω is an open subset of $\mathbf{R^n}$, K is a compact subset of Ω and u is harmonic in the open set $\Omega \setminus K$. Consider a variable bounded open Ω_* with C^1-boundary such that $K \subseteq \Omega_* \subseteq \overline{\Omega_*} \subseteq \Omega$. Then, the quantity*

$$\tau \;=\; \int_{\partial\Omega_*} \frac{\partial u}{\partial \eta}(y)\; dS(y)\;,$$

where $\overrightarrow{\eta}$ is the continuous unit vector field normal to $\partial\Omega_*$ in the direction towards the exterior of Ω_*, does not depend on Ω_*.

In case K is empty, which means that u is harmonic in Ω, then $\tau = 0$.

Proof:

We consider a third bounded open Ω_1 with C^1-boundary such that

$$K \subseteq \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_*$$

and apply Green's theorem in $\Omega_* \setminus \overline{\Omega_1}$.

Then

$$\int_{\partial\Omega_*} \frac{\partial u}{\partial \eta}(y)\; dS(y) \;=\; \int_{\partial\Omega_1} \frac{\partial u}{\partial \eta}(y)\; dS(y)\;.$$

Now, given two sets, Ω_* and Ω_{**}, as in the statement of the theorem, we can use an Ω_1 such that $\overline{\Omega_1} \subseteq \Omega_* \cap \Omega_{**}$ and finish the proof, by comparing the τ's for these two sets with the τ for the third set.

The last result is proved, by applying Green's Formula in Ω_*.

Definition 1.7 *Suppose Γ is an orientable C^1-hypersurface, $\overrightarrow{\eta}$ is a continuous unit vector field normal to Γ and u is harmonic in an open set containing Γ.*

Then the quantity

$$\tau \;=\; \int_{\Gamma} \frac{\partial u}{\partial \eta}(y)\; dS(y)$$

is called ***flux*** *of u* ***through*** *Γ* ***in the direction determined by the vector field*** *$\overrightarrow{\eta}$.*

1.8 The representation formula

Notation

$$\kappa_n \;=\; \begin{cases} -\omega_1\;, & \text{if } \; n = 2 \\ -(n-2)\omega_{n-1}\;, & \text{if } \; n > 2\;. \end{cases}$$

Theorem 1.7 *If Ω is a bounded open set with C^1-boundary, u is harmonic in Ω and belongs to $C^1(\overline{\Omega})$, then*

$$u(x) \;=\; \frac{1}{\kappa_n} \int_{\partial\Omega} \Big(\frac{\partial h_x}{\partial \eta}(y) u(y) - h_x(y) \frac{\partial u}{\partial \eta}(y) \Big)\; dS(y)$$

for every $x \in \Omega$, where $\overrightarrow{\eta}$ is the continuous unit vector field normal to $\partial\Omega$ in the direction towards the exterior of Ω.

Therefore, u is represented as a difference between a double- and a single-layer potential.

Proof:
Take $x \in \Omega$ and $\overline{B(x;r)} \subseteq \Omega$ and apply Green's Formula in the open set $\Omega^* = \Omega \setminus \overline{B(x;r)}$ with $f = u$ and $g = h_x$:

$$\begin{aligned} 0 &= \int_{\partial\Omega} \Big(u(y) \frac{\partial h_x}{\partial \eta}(y) - h_x(y) \frac{\partial u}{\partial \eta}(y) \Big) \, dS(y) \\ &+ \int_{S(x;r)} \Big(u(y) \frac{\partial h_x}{\partial \eta}(y) - h_x(y) \frac{\partial u}{\partial \eta}(y) \Big) \, dS(y) \, , \end{aligned}$$

where $\overrightarrow{\eta}$ is the continuous unit vector field normal to $\partial\Omega$ in the direction towards the exterior of Ω and normal to $S(x;r)$ in the direction towards the interior of $B(x;r)$.

In case $n = 2$, using Theorem 1.5, the second integral becomes

$$\frac{1}{r} \int_{S(x;r)} u(y) \, dS(y) \; - \; \log \frac{1}{r} \int_{S(x;r)} \frac{\partial u}{\partial \eta}(y) \, dS(y) \; = \; \omega_1 \mathcal{M}_u^r(x) \; = \; \omega_1 u(x) \, .$$

Similarly, in case $n > 2$, the second integral becomes

$$\begin{aligned} &(n-2) \frac{1}{r^{n-1}} \int_{S(x;r)} u(y) \, dS(y) \; - \; \frac{1}{r^{n-2}} \int_{S(x;r)} \frac{\partial u}{\partial \eta} \, dS(y) \\ &= \; (n-2) \omega_{n-1} \mathcal{M}_u^r(x) \; = \; (n-2) \omega_{n-1} u(x) \, . \end{aligned}$$

The result of the theorem is, now, obvious.

Corollary 1.2 *Let $d\mu$ be a compactly supported complex Borel measure and Ω_* a bounded open set with C^1-boundary such that $supp(d\mu) \subseteq \Omega_*$. Then*

$$\tau \; = \; \int_{\partial\Omega_*} \frac{\partial U_h^{d\mu}}{\partial \eta}(y) \, dS(y) \; = \; \kappa_n d\mu(\mathbf{R^n}) \, ,$$

where $U_h^{d\mu}$ is the h-potential of $d\mu$.

Proof:

$$\begin{aligned} \int_{\partial\Omega_*} \frac{\partial U_h^{d\mu}}{\partial \eta}(y) \, dS(y) \; &= \; \int_{\partial\Omega_*} \frac{\partial}{\partial \eta} \Big(\int_{supp(d\mu)} h_x(\cdot) \, d\mu(x) \Big)(y) \, dS(y) \\ &= \; \int_{\partial\Omega_*} \int_{supp(d\mu)} \frac{\partial h_x}{\partial \eta}(y) \, d\mu(x) \, dS(y) \\ &= \; \int_{supp(d\mu)} \int_{\partial\Omega_*} \frac{\partial h_x}{\partial \eta}(y) \, dS(y) \, d\mu(x) \, . \end{aligned}$$

Observe that, if $x \in supp(d\mu)$, then $x \in \Omega_*$ and, applying Theorem 1.7 with $u = 1$,

$$\int_{\partial\Omega_*} \frac{\partial h_x}{\partial \eta}(y) \, dS(y) \; = \; \kappa_n \, .$$

Therefore,

$$\int_{\partial\Omega_*} \frac{\partial U_h^{d\mu}}{\partial\eta}(y)\, dS(y) \;=\; \int_{supp(d\mu)} \kappa_n d\mu(x) \;=\; \kappa_n d\mu(\mathbf{R^n}) \ .$$

Combining Corrolary 1.2 with the last part of Theorem 1.6, we may state a general

Principle: *The flux of the h-potential of a compactly supported complex Borel measure $d\mu$ through the boundary of a bounded, open set Ω with C^1-boundary disjoint from supp$(d\mu)$ and in the direction towards the exterior of Ω is equal to a (negative) constant times the total mass of the part of the measure which lies inside the open set.*

1.9 Poisson integrals

Definition 1.8 *If $x \neq x_0$, then the point*

$$x^* \;=\; x_0 + \frac{R^2}{|x-x_0|^2}(x-x_0)$$

*is called the **symmetric** of x with respect to the sphere $S(x_0;R)$.*

We define ∞ as the symmetric of x_0 and x_0 as the symmetric of ∞.

It is easy to see that x and x^* are on the same half-line having x_0 as vertex and the product of their distances from x_0 is equal to R^2. This, in an extended sense, happens, also, for the pair of x_0 and ∞.

Observe that x is the symmetric of x^*, $x = (x^*)^*$, and that $x = x^*$ if and only if $x \in S(x_0;R)$.

Suppose u is harmonic in $B(x_0;R)$ and belongs to $C^1(\overline{B(x_0;R)})$. Then, applying Theorem 1.7 to $\Omega = B(x_0;R)$, we find, for every $x \in B(x_0;R)$,

$$u(x) \;=\; \frac{1}{\kappa_n}\int_{S(x_0;R)} \Big(\frac{\partial h_x}{\partial\eta}(y)u(y) - h_x(y)\frac{\partial u}{\partial\eta}(y)\Big)\, dS(y) \ ,$$

where $\overrightarrow{\eta}$ is the continuous unit vector field normal to $S(x_0;R)$ directed towards the exterior of $B(x_0;R)$.

If $x \neq x_0$, consider, also, the symmetric x^* of x with respect to the sphere $S(x_0;R)$. Then h_{x^*} is harmonic in an open set containing $\overline{B(x_0;R)}$ and Green's Formula implies

$$0 \;=\; \frac{1}{\kappa_n}\int_{S(x_0;R)} \Big(\frac{\partial h_{x^*}}{\partial\eta}(y)u(y) - h_{x^*}(y)\frac{\partial u}{\partial\eta}(y)\Big)\, dS(y) \ .$$

Let $n = 2$.

Then $h_{x^*}(y) - h_x(y) = \log \frac{|x-y|}{|x^*-y|} = \log \frac{|x-x_0|}{R}$ for all $y \in S(x_0;R)$ and, subtracting the two equations above, we get

$$\begin{aligned} u(x) &= \frac{1}{\kappa_2} \int_{S(x_0;R)} \frac{\partial(h_x - h_{x^*})}{\partial \eta}(y) u(y)\, dS(y) \\ &+ \frac{1}{\kappa_2} \log \frac{|x-x_0|}{R} \int_{S(x_0;R)} \frac{\partial u}{\partial \eta}(y)\, dS(y) . \end{aligned}$$

The last integral is 0 and a trivial calculation of the directional derivative of $h_x(\cdot) - h_{x^*}(\cdot) = \log \frac{|x^* - \cdot|}{|x - \cdot|}$ in the direction of $\overrightarrow{\eta}$ gives

$$u(x) = \frac{1}{\omega_1 R} \int_{S(x_0;R)} \frac{R^2 - |x-x_0|^2}{|x-y|^2}\, u(y)\, dS(y) .$$

Now, let $n > 2$.

Then $h_x(y) - \frac{R^{n-2}}{|x-x_0|^{n-2}}\, h_{x^*}(y) = 0$ for all $y \in S(x_0;R)$ and, if we subtract the two equations, after multiplying the second by the factor $\frac{R^{n-2}}{|x-x_0|^{n-2}}$, we find

$$u(x) = \frac{1}{\kappa_n} \int_{S(x_0;R)} \frac{\partial}{\partial \eta}\Big(h_x - \frac{R^{n-2}}{|x-x_0|^{n-2}}\, h_{x^*}\Big)(y) u(y)\, dS(y) .$$

Again, a calculation gives

$$u(x) = \frac{1}{\omega_{n-1} R} \int_{S(x_0;R)} \frac{R^2 - |x-x_0|^2}{|x-y|^n}\, u(y)\, dS(y) .$$

Of course, this formula covers the case $n = 2$ above and it is, trivially, true for $x = x_0$.

Definition 1.9 *If $x \in B(x_0;R)$, then the function*

$$P(y;x,x_0,R) = \frac{1}{\omega_{n-1} R} \frac{R^2 - |x-x_0|^2}{|x-y|^n} , \qquad y \in S(x_0;R) ,$$

*is called the **Poisson kernel** of the ball $B(x_0;R)$ with respect to x.*

We have proved the

Theorem 1.8 *If u is harmonic in $B(x_0;R)$ and it belongs to $C^1(\overline{B(x_0;R)})$, then*

$$u(x) = \int_{S(x_0;R)} P(y;x,x_0,R) u(y)\, dS(y) , \qquad x \in B(x_0;R) .$$

*This is called the **Poisson integral formula**.*

Properties of the Poisson kernel

(1) *P is positive.*

(2) *P is a harmonic function of x in $B(x_0;R)$.*
This is a matter of calculation of the Laplacian.

(3) $\int_{S(x_0;R)} P(y;x,x_0,R)\ dS(y) \ = \ 1$.
We just apply the Poisson integral formula to $u=1$.

(4) *If V is an open neighborhood of $y_0 \in S(x_0;R)$, then*

$$\lim_{B(x_0;R)\ni x\to y_0} P(y;x,x_0,R) \ = \ 0$$

uniformly in $y \in S(x_0;R)\setminus V$.

If $|y-y_0| \geq \delta_0$ for all $y \in S(x_0;R)\setminus V$, then, when $|x-y_0| \leq \frac{1}{2}\delta_0$, we have

$$0 \ \leq \ P(y;x,x_0,R) \ \leq \ \frac{1}{\omega_{n-1}R}\frac{R^2-(R-|x-y_0|)^2}{(\frac{1}{2}\delta_0)^n} \ \to \ 0$$

as $|x-y_0| \to 0$.

Definition 1.10 *Let f be integrable in $S(x_0;R)$ with respect to the surface measure dS and define*

$$P_f(x) \ = \ P_f(x;x_0,R) \ = = P_f(x;B) \ = \ \int_{S(x_0;R)} P(y;x,x_0,R)f(y)\ dS(y)$$

for all $x \in B(x_0;R)$.

*This is called the **Poisson integral** of f in $B = B(x_0;R)$.*

Theorem 1.9 *$P_f(\cdot\ ;x_0,R)$ is harmonic in $B(x_0;R)$.*

If f is continuous at some $y_0 \in S(x_0;R)$, then

$$\lim_{B(x_0;R)\ni x\to y_0} P_f(x;x_0,R) \ = \ f(y_0) \ .$$

Therefore, if f is continuous in $S(x_0;R)$, then $P_f(\cdot\ ;x_0,R)$ is the solution of the Dirichlet problem in $B(x_0;R)$ with boundary function f.

Proof:

The harmonicity of $P_f(\cdot\ ;x_0,R)$ results from property (2) of the Poisson kernel.

For the limit, it is enough, by linearity, to assume that f is real-valued and, then, to prove

$$\limsup_{B(x_0;R)\ni x\to y_0} P_f(x;x_0,R) \ \leq \ f(y_0) \ .$$

Indeed, we then apply this to $-f$, find

$$f(y_0) \leq \liminf_{B(x_0;R)\ni x\to y_0} P_f(x;x_0,R)$$

and combine the two inequalities to get the equality we want to prove.

Let $M > f(y_0)$ and fix an open neighborhood V of y_0 so that $f < M$ in $S(x_0; R) \cap V$. Then,

$$\begin{aligned} P_f(x) &= \int_{S(x_0;R)\cap V} P(y; x, x_0, R) f(y)\ dS(y) \\ &\quad + \int_{S(x_0;R)\setminus V} P(y; x, x_0, R) f(y)\ dS(y) \ . \end{aligned}$$

Now, by properties (1) and (3) of the Poisson kernel, the first integral is

$$\leq M \int_{S(x_0;R)\cap V} P(y; x, x_0, R)\ dS(y) \ = \ M - M \int_{S(x_0;R)\setminus V} P(y; x, x_0, R)\ dS(y)$$

and, by property (4) of the Poisson kernel, the integrals over $S(x_0; R) \setminus V$ tend to 0 as $x \to y_0$.

Hence,

$$\limsup_{B(x_0;R)\ni x \to y_0} P_f(x) \ \leq \ M$$

and, since M can be taken arbitrarily close to $f(y_0)$, the proof is complete.

Remark *There is a second way to prove the Poisson Integral Formula, without going through Green's Formula* (in case we wish to avoid its heavy technical machinery).

Suppose u is harmonic in $B(x_0; R)$ and it belongs to $C^1(\overline{B(x_0; R)})$ and consider the function $P_u(\cdot\ ; x_0, R)$, the solution of the Dirichlet problem in $B(x_0; R)$ with the restriction of u in $S(x_0; R)$ as boundary function. By the uniqueness of the solution, we conclude that

$$u(x) \ = \ P_u(x; x_0, R) \ , \qquad x \in B(x_0; R) \ .$$

1.10 Consequences of the Poisson formula

Proposition 1.5 *If u is harmonic in the open $\Omega \subseteq \mathbf{R^n}$, then it is real-analytic in Ω.*

Proof:

Let $x_0 \in \Omega$ and take $\overline{B(x_0; R)} \subseteq \Omega$. Write $u(x) = P_u(x; x_0, R)$ in $B(x_0; R)$ and use the real-analyticity of $P(y; x, x_0, R)$:

$$\begin{aligned} P(y; x, x_0, R) &= \frac{1}{\omega_{n-1} R} \frac{R^2 - |x - x_0|^2}{|x - y|^n} \\ &= \frac{1}{\omega_{n-1} R} \frac{R^2 - |x - x_0|^2}{|(x - x_0) - (y - x_0)|^n} \\ &= \sum_\alpha a_\alpha(y)(x - x_0)^\alpha \ , \end{aligned}$$

where the sum is over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, a_α is a polynomial in y and the series converges uniformly in $y \in S(x_0; R)$ and in $x \in B\big(x_0; \frac{1}{2}R\big)$.

Therefore,

$$u(x) \;=\; \sum_\alpha \beta_\alpha (x - x_0)^\alpha \;, \qquad x \in B\Big(x_0; \frac{1}{2}R\Big) \;,$$

where $\beta_\alpha = \int_{S(x_0;R)} a_\alpha(y) u(y)\, dS(y)$ and, hence, u is real-analytic at the arbitrary point x_0 of Ω.

Theorem 1.10 *Suppose that u is harmonic in the connected open set $\Omega \subseteq \mathbf{R^n}$ and that $u = 0$ in an open subset O of Ω. Then $u = 0$ identically in Ω.*

Proof:

It is obvious that, for all $x \in O$, $D^\alpha u(x) = 0$ for every multi-index α.

We consider the sets

$$A \;=\; \{x \in \Omega : D^\alpha u(x) = 0 \text{ for all } \alpha\} \;, \qquad B \;=\; \Omega \setminus A \;.$$

If $x_0 \in B$, then $D^{\alpha_0} u(x_0) \neq 0$ for some multi-index α_0 and, by continuity, this is true in some neighborhood of x_0. Hence, the set B is open.

If $x_0 \in A$, then the Taylor-series expansion of u in a neighborhood of x_0 has all its coefficients equal to 0, and, thus, $u = 0$ in this neighborhood. Therefore, this neighborhood is contained in A and A is an open set.

By the connectivity of Ω, we get that $\Omega = A$ and, thus, $u = 0$ in Ω.

Theorem 1.11 *(Weak mean-value property) If u is continuous in the open $\Omega \subseteq \mathbf{R^n}$ and if for every $x \in \Omega$ there exists a sequence $\{r_k(x)\}$ such that $r_k(x) \to 0$ and $u(x) = \mathcal{M}_u^{r_k(x)}(x)$ for all k, then u is harmonic in Ω.*

The same is true, if we replace the surface-means with the space-means in the above statement.

Proof:

Take $\overline{B(x_0; R)} \subseteq \Omega$ and consider

$$\phi(x) \;=\; u(x) - P_u(x; x_0, R) \;, \qquad x \in B(x_0; R) \;.$$

By the continuity of u and Theorem 1.9, we have that ϕ is continuous in $B(x_0; R)$ and that $lim_{B(x_0;R) \ni x \to y} \phi(x) = 0$ for every $y \in S(x_0; R)$.

Since $P_u(\cdot\, ; x_0, R)$ is harmonic in $B(x_0; R)$, for every $x \in B(x_0; R)$ we have that

$$\phi(x) = \mathcal{M}_\phi^{r_k(x)}(x)$$

for every k which is large enough so that $\overline{B(x; r_k(x))} \subseteq B(x_0; R)$.

From this, in the same way in which we proved the fifth property of harmonic functions, we can prove that, if ϕ takes one of its extremal values at some point in $B(x_0; R)$, then ϕ is constant in a neighborhood of this point.

Now, applying Proposition 0.5 to ϕ and $-\phi$, we conclude that

$$\phi(x) \;=\; 0$$

for every $x \in B(x_0; R)$.

Hence, u is equal to the harmonic P_u in $B(x_0; R)$ and, since the ball is arbitrary in Ω, we get that u is harmonic in Ω.

Exactly the same argument applies when we replace $\mathcal{M}_u^r$ by $\mathcal{A}_u^r$.

The following theorem gives a necessary and sufficient condition for a point to be a removable isolated singularity of a harmonic function.

Theorem 1.12 *(Riemann) Suppose u is harmonic in $B(x_0; R) \setminus \{x_0\}$.*

Then, u can be defined at x_0 so that it becomes harmonic in $B(x_0; R)$ if and only if

$$\lim_{x \to x_0} \frac{u(x)}{h_{x_0}(x)} \;=\; 0 \;.$$

Proof:

The necessity of the condition is trivial.

Fix $R_1 < R$ and consider

$$\phi(x) \;=\; u(x) - P_u(x; x_0, R_1) - \epsilon\big(h_{x_0}(x) - h_*(R_1)\big) \;, \qquad x \in B(x_0; R_1) \setminus \{x_0\} \;,$$

for an arbitrary $\epsilon > 0$.

By Theorem 1.9, we have that ϕ is harmonic in $\Omega = B(x_0; R_1) \setminus \{x_0\}$ and, also, by the hypothesis, that $\limsup_{\Omega \ni x \to y} \phi(x) \leq 0$ for every $y \in \partial\Omega$.

Theorem 1.1 implies that $\phi(x) \leq 0$ for all $x \in B(x_0; R_1) \setminus \{x_0\}$. Hence,

$$u(x) \;\leq\; P_u(x; x_0, R_1) + \epsilon\big(h_{x_0}(x) - h_*(R_1)\big)$$

for all $x \in B(x_0; R_1) \setminus \{x_0\}$ and every $\epsilon > 0$ and, finally,

$$u(x) \;\leq\; P_u(x; x_0, R_1) \;, \qquad x \in B(x_0; R_1) \setminus \{x_0\} \;.$$

Working in the same manner with $-u$, we find the opposite inequality. Therefore,

$$u(x) \;=\; P_u(x; x_0, R_1) \;, \qquad x \in B(x_0; R_1) \setminus \{x_0\} \;.$$

But, $P_u(\cdot\; ; x_0, R_1)$ is harmonic in $B(x_0; R_1)$ and, defining

$$u(x_0) \;=\; P_u(x_0; x_0, R_1) \;=\; \mathcal{M}_u^{R_1}(x_0) \;,$$

u becomes harmonic in $B(x_0; R_1)$ and, hence, in $B(x_0; R)$.

Theorem 1.13 *(Reflection Principle of H. Schwartz) Let L be a hyperplane in $\mathbf{R^n}$ and $\Omega \subseteq \mathbf{R^n}$ be open and symmetric with respect to L. Let L^+ and L^- be the two open half-spaces on the two sides of L and let $\Omega^+ = \Omega \cap L^+$, $\Omega^- = \Omega \cap L^-$ and $\Gamma = \Omega \cap L$.*

Suppose u is harmonic in Ω^+ and

$$\lim_{\Omega^+ \ni x \to y} u(x) = 0$$

for every $y \in \Gamma$.

Then, u can be defined in $\Gamma \cup \Omega^-$ so that it becomes harmonic in Ω.

Proof:

Define

$$u(x) \;=\; \begin{cases} 0\,, & \text{if } x \in \Gamma \\ -u(x^*)\,, & \text{if } x \in \Omega^- \end{cases}\quad ,$$

where x^* is the symmetric of x with respect to L.

Since u is harmonic in Ω^+, it satisfies the mean-value property for all balls which are contained in Ω^+.

Since symmetry with respect to a hyperplane is a rigid motion, we have, by the second property of harmonic functions, that the extended u is harmonic in Ω^- and it satisfies the mean-value property for all balls contained in Ω^-.

We also have that the extended u is continuous in Ω and, if we take any point $x \in \Gamma$ and any $\overline{B(x;R)} \subseteq \Omega$, then

$$u(x) \;=\; 0 \;=\; \mathcal{A}_u^r(x)$$

for all $r < R$. This holds because the two integrals over the two half-balls, $B(x;r) \cap L^+$ and $B(x;r) \cap L^-$, cancel.

Therefore, by Theorem 1.11, u is harmonic in Ω.

Theorem 1.14 *(Harnack's Inequalities) Let u be positive and harmonic in the open set $\Omega \subseteq \mathbf{R^n}$ and let $\overline{B(x_0;R)} \subseteq \Omega$. Then,*

$$\frac{R - |x - x_0|}{(R + |x - x_0|)^{n-1}}\, R^{n-2} \;\leq\; \frac{u(x)}{u(x_0)} \;\leq\; \frac{R + |x - x_0|}{(R - |x - x_0|)^{n-1}}\, R^{n-2}$$

for all $x \in B(x_0;R)$.

Also,

$$3^{-n} \;\leq\; \frac{u(x)}{u(x')} \;\leq\; 3^n$$

for all x, x' in $B(x_0; \frac{1}{2}R)$.

Moreover, if Ω is connected and K is any compact subset of Ω, then there is a constant $C = C_{n,K,\Omega} > 0$ depending only on n, K and Ω so that

$$\frac{1}{C} \;\leq\; \frac{u(x)}{u(x')} \;\leq\; C$$

for all x, x' in K.

Proof:

The first set of inequalities is an immediate application of the Poisson integral formula, when we make use of the

$$R-|x-x_0| \ \leq\ |x-y| \ \leq\ R+|x-x_0| \ , \qquad y\in S(x_0;R) \ ,$$

of the positivity of u and of the mean-value property of u at x_0.

If $|x-x_0|\leq \frac{1}{2}R$, then from these inequalities we get

$$\frac{2^{n-2}}{3^{n-1}} \ \leq\ \frac{u(x)}{u(x_0)} \ \leq\ 3\cdot 2^{n-2} \ .$$

If, also, $|x'-x_0|\leq \frac{1}{2}R$, then we immediately get

$$3^{-n} \ \leq\ \frac{u(x)}{u(x')} \ \leq\ 3^n \ .$$

For the last set of inequalities, we observe that it is enough to prove the inequality in the right.

For each $x\in K$ take $\overline{B(x;R(x))}\subseteq\Omega$ and find $x_1,\ldots,x_N$ so that $K\subseteq \cup_{k=1}^N B(x_k;\frac{1}{2}R(x_k))$.

Now, set

$$M \ =\ \max(u(x_1),\ldots,u(x_N)) \ , \qquad m \ =\ \min(u(x_1),\ldots,u(x_N)) \ .$$

From the first part, we get that

$$\frac{u(x)}{u(x')} \ \leq\ 3^{2n}\frac{M}{m}$$

for every x,x' in K.

It is obvious that there is some $p\in\mathbf{N}$, depending only on the points $x_1,\ldots,x_N$ and Ω, so that for every two x_k and x_l there are at most p successive points in Ω, the first being x_k and the last being x_l and every two consecutive ones of which are contained in the same closed ball whose double is contained in Ω.

Applying the first part, we see that, for all k and l, $\frac{u(x_k)}{u(x_l)}\leq 3^{pn}$ and, finally,

$$\frac{u(x)}{u(x')} \ \leq\ 3^{(p+2)n}$$

for all x,x' in K.

Theorem 1.15 *(Gradient Estimates) Let u be harmonic in the open set $\Omega\subseteq$ $\mathbf{R^n}$ and $\overline{B(x_0;R)}\subseteq\Omega$.*

1. $|\overrightarrow{grad}\,u(x_0)|\leq \frac{n}{R}\frac{1}{\omega_{n-1}R^{n-1}}\int_{S(x_0;R)}|u(y)|\,dS(y)$.

2. If, also, u is positive in $S(x_0;R)$, then $|\overrightarrow{grad\, u}(x_0)| \leq \frac{n}{R}\, u(x_0)$.

3. If $m \leq u \leq M$ in $S(x_0;R)$, then $|\overrightarrow{grad\, u}(x_0)| \leq \frac{n}{R}\, \frac{M-m}{2}$.

Proof:
If $x_0 = (x_{0,1}, \ldots, x_{0,n})$, then, by an easy calculation in the Poisson integral,

$$\frac{\partial u}{\partial x_j}(x_0) \;=\; -\frac{n}{\omega_{n-1}R^{n+1}} \int_{S(x_0;R)} (x_{0,j} - y_j)u(y)\; dS(y)\ .$$

Multiplying both sides by $\frac{\partial u}{\partial x_j}(x_0)$, then summing over $j = 1, \ldots, n$ and using Cauchy's inequality inside the integral, we prove 1.

If $u > 0$ in $S(x_0;R)$, then

$$\frac{1}{\omega_{n-1}R^{n-1}} \int_{S(x_0;R)} |u(y)|\; dS(y) \;=\; u(x_0)$$

and 2 is implied by 1.

Finally, if $m \leq u(y) \leq M$ for all $y \in S(x_0;R)$, then we apply 1 to the function $u - \frac{M+m}{2}$ and prove 3.

1.11 Monotone sequences

Theorem 1.16 *(Monotone sequences of harmonic functions) Suppose $\{u_m\}$ is a sequence of real-valued functions harmonic in the connected open set $\Omega \subseteq \mathbf{R^n}$ and $u_1 \leq u_2 \leq \ldots$ in Ω.*

Then, either $u_m \uparrow +\infty$ uniformly on compact subsets of Ω or there exists some u harmonic in Ω so that $u_m \uparrow u$ uniformly on compact subsets of Ω.

There is a dual result for decreasing sequences of harmonic functions.

Proof:
Subtracting u_1 from all u_m, we may assume that

$$0 \;\leq\; u_1 \;\leq\; u_2 \;\leq\; u_3 \;\leq \ldots$$

in Ω. By the Maximum-Minimum Principle, we may even assume that all inequalities are strict everywhere in Ω.

(1) Let $x_0 \in \Omega$ with $u_m(x_0) \uparrow +\infty$ and consider any compact $K \subseteq \Omega$. By Theorem 1.14 applied to $K \cup \{x_0\}$, we have that, for some $C > 0$ independent of m,

$$\frac{1}{C}\, u_m(x_0) \;\leq\; u_m(x)$$

for all $x \in K$. This implies that $u_m \uparrow +\infty$ uniformly in K.

(2) Let $x_0 \in \Omega$ with $u_m(x_0) \uparrow M$ for some real M and consider any compact $K \subseteq \Omega$. Again by Theorem 1.14, we have that

$$u_k(x) - u_m(x) \;\leq\; C(u_k(x_0) - u_m(x_0))$$

for all $x \in K$ and all m, k with $m < k$. This means that $\{u_m\}$ is uniformly Cauchy in K and, hence, it converges uniformly in K to some real-valued function.

If we, now, define $u(x) = \lim_{m\to+\infty} u_m(x)$ for all $x \in \Omega$, then $u_m \uparrow u$ uniformly on compact subsets of Ω. By the third property of harmonic functions, u is harmonic in Ω.

1.12 Normal families of harmonic functions

If we drop the assumption of monotonicity, the results are not that clear, but we still get some "normal families"-type of results.

Theorem 1.17 *If $\{u_m\}$ is a sequence of harmonic functions in the open $\Omega \subseteq \mathbf{R^n}$ and $u_m \to u$ uniformly on compact subsets of Ω, then the derivatives of the u_m converge to the corresponding derivatives of u uniformly on compact subsets of Ω.*

Proof:

Take $\overline{B(x_0; R)} \subseteq \Omega$ and observe that, if $x \in B(x_0; \frac{1}{2}R)$, then $\overline{B(x; \frac{1}{2}R)} \subseteq B(x_0; R)$. By Theorem 1.15(1),

$$|\overrightarrow{grad(u_m - u)}(x)| \;\leq\; \frac{2n}{R} \sup_{B(x_0;R)} |u_m - u| \, .$$

Therefore, $\frac{\partial u_m}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ uniformly on $B(x_0; \frac{1}{2}R)$ and, since x_0 is arbitrary, the convergence is uniform on all compact subsets of Ω.

By Theorem 1.3, all derivatives of harmonic functions are harmonic and, by induction, we can prove uniform convergence on compact sets for all derivatives.

Theorem 1.18 *Let $\mathcal{U}$ be a family of harmonic functions in the open $\Omega \subseteq \mathbf{R^n}$ which are uniformly bounded on compact subsets of Ω.*

Then, their derivatives are also uniformly bounded on compact subsets of Ω and from every sequence in $\mathcal{U}$ we can extract a subsequence converging uniformly on compact subsets of Ω to some harmonic function.

Proof:

Take $\overline{B(x_0; R)} \subseteq \Omega$. Then, there is some $M = M(x_0, R)$ so that

$$|u(x)| \;\leq\; M$$

for every $x \in B(x_0; R)$ and every $u \in \mathcal{U}$.

By the same argument as in the proof of the previous theorem, we get that

$$|\overrightarrow{grad}\, u(x)| \;\leq\; \frac{2n}{R} \sup_{B(x_0;R)} |u| \;\leq\; \frac{2n}{R} M$$

for every $x \in B(x_0; \frac{1}{2}R)$ and every $u \in \mathcal{U}$.

Since x_0 is arbitrary, we conclude that the derivatives of first order of the functions in $\mathcal{U}$ are uniformly bounded on compact subsets of Ω.

From the mean value theorem of the differential calculus,

$$|u(x) - u(x_0)| \leq \frac{2n}{R} M |x - x_0|$$

for every $x \in B(x_0; \frac{1}{2}R)$ and every $u \in \mathcal{U}$. Therefore, $\mathcal{U}$ is equicontinuous (and bounded) at every point in Ω.

Now, take any compact exhaustion $\{K_{(m)}\}$ of Ω.

Given $\{u_k\}$ in $\mathcal{U}$, we use the Arzela-Ascoli theorem for each $K_{(m)}$ to extract a subsequence converging uniformly on $K_{(m)}$. Then, by the usual diagonal argument, we find a subsequence converging uniformly on every $K_{(m)}$ and, hence, on every compact subset of Ω. By the third property of harmonic functions, the limit function is harmonic in Ω.

Definition 1.11 *Suppose $\mathcal{F}$ is a family of extended-real-valued functions defined in the set E. Then, the function*

$$F(x) = \sup_{f \in \mathcal{F}} f(x) , \qquad x \in E ,$$

*is called the **upper envelope** of the family $\mathcal{F}$.*

*The **lower envelope** is similarly defined.*

Theorem 1.19 *Suppose $\mathcal{U}$ is a family of positive harmonic functions in the connected open $\Omega \subseteq \mathbf{R^n}$. Then, the upper envelope of the family is either identically $+\infty$ in Ω or everywhere finite and continuous in Ω.*

In the first case, there exists a sequence in $\mathcal{U}$ diverging to $+\infty$ uniformly on compact subsets of Ω.

In the second case, from every sequence in $\mathcal{U}$ we can extract a subsequence converging uniformly on compact subsets of Ω to some harmonic function.

There is a dual statement for lower envelopes of families of negative harmonic functions.

Proof:

Let U be the upper envelope of $\mathcal{U}$.

(1) If there is some $x_0 \in \Omega$ with $U(x_0) = +\infty$, then there exists $\{u_m\}$ in $\mathcal{U}$ so that $u_m(x_0) \to +\infty$. By Theorem 1.14 and in the same manner as in the proof of Theorem 1.16, we prove that $u_m(x) \to +\infty$ uniformly on compact subsets of Ω and, thus, $U = +\infty$ everywhere in Ω.

(2) Now, suppose that $U(x_0) < +\infty$ for some $x_0 \in \Omega$.

Again by Theorem 1.14, for every compact $K \subseteq \Omega$, there is a $C = C_{n,K,\Omega} > 0$ so that

$$u \leq Cu(x_0) \leq CU(x_0)$$

everywhere in K for all $u \in \mathcal{U}$ and, thus,

$$U \leq CU(x_0)$$

in K.

This implies that U is bounded on every compact subset of Ω and, in particular, everywhere finite in Ω. By Theorem 1.18, we immediately get that from every sequence in $\mathcal{U}$ we can extract a subsequence converging uniformly on compact subsets of Ω to some harmonic function.

It, only, remains to prove the continuity of U.

Consider $x_0 \in \Omega$ and take $\{u_m\}$ in $\mathcal{U}$ so that $u_m(x_0) \to U(x_0)$. Then, for every m, by the continuity of u_m,

$$u_m(x_0) \;=\; \liminf_{x \to x_0} u_m(x) \;\leq\; \liminf_{x \to x_0} U(x)$$

and, letting $m \to +\infty$,

$$U(x_0) \;\leq\; \liminf_{x \to x_0} U(x) \ .$$

Consider $M = \limsup_{x \to x_0} U(x)$.

Then, there exist $x_m \to x_0$ so that $U(x_m) \to M$ and, by the definition of U, there exists $\{u_m\}$ in $\mathcal{U}$ so that $u_m(x_m) \to M$.

Extracting, if necessary, a subsequence, we may assume that u_m converges uniformly on compact subsets of Ω to some u harmonic in Ω. Then,

$$M \;=\; \lim_{m \to +\infty} u_m(x_m) \;=\; u(x_0) \;=\; \lim_{m \to +\infty} u_m(x_0) \;\leq\; U(x_0)$$

and, thus,

$$\limsup_{x \to x_0} U(x) \;\leq\; U(x_0) \ ,$$

implying the continuity of U at x_0.

The hypotheses of Theorem 1.19 can be slightly weakened. Instead of the positivity of all the functions in $\mathcal{U}$, it is enough to assume that on every compact subset of Ω the family $\mathcal{U}$ is uniformly bounded from below.

We then work with an open exhaustion $\{\Omega_{(m)}\}$ of Ω. In each $\Omega_{(m)}$ the family is uniformly bounded from below by some constant and we may apply Theorem 1.19 there. The passage from the $\Omega_{(m)}$'s to Ω presents absolutely no difficulty, except that we must apply a diagonal argument when we extract subsequences.

The interested reader may, easily, complete the details.

1.13 Harmonic distributions

And, now, we prove the famous

Theorem 1.20 *(Lemma of H. Weyl and L. Schwartz) If T is a distribution in the open $\Omega \subseteq \mathbf{R^n}$ and $\Delta T = 0$, then T is identified with a harmonic function u in Ω. This means $T = T_u$ or, more specifically,*

$$T(\phi) \;=\; \int_\Omega \phi(x) u(x) \; dm(x)$$

for every $\phi \in \mathcal{D}(\Omega)$.

Proof:

Consider an approximation to the identity $\{\Phi_\delta : \delta > 0\}$.

Fix $\delta > 0$ and consider variable $\delta_1, \delta_2 < \frac{1}{2}\,\delta$.

Then $T * \Phi_{\delta_1}$ and $T * \Phi_{\delta_2}$ are both distributions in $\Omega_{\frac{1}{2}\delta}$ and $(T * \Phi_{\delta_1}) * \Phi_{\delta_2} = (T * \Phi_{\delta_2}) * \Phi_{\delta_1}$ is a distribution in Ω_δ .

Observe that, by Proposition 0.16, all these distributions are identified with infinitely differentiable functions.

By Proposition 0.18,

$$\Delta(T * \Phi_{\delta_1}) \;=\; \Delta T * \Phi_{\delta_1} \;=\; 0$$

as a distribution in $\Omega_{\frac{1}{2}\delta}$.

Let v be the infinitely differentiable function which represents $T * \Phi_{\delta_1}$ in $\Omega_{\frac{1}{2}\delta}$. Then $T_{\Delta v} = \Delta T_v = 0$, implying $\Delta v = 0$ and, hence, v is harmonic in $\Omega_{\frac{1}{2}\delta}$.

From the proof of Theorem 1.2, we get that

$$v * \Phi_{\delta_2} \;=\; v \qquad \text{in } \Omega_\delta \; .$$

This implies, of course,

$$(T * \Phi_{\delta_1}) * \Phi_{\delta_2} \;=\; T_v * \Phi_{\delta_2} \;=\; T_{v*\Phi_{\delta_2}} \;=\; T_v \;=\; T * \Phi_{\delta_1} \qquad \text{in } \Omega_\delta$$

and, hence,

$$(T * \Phi_{\delta_2}) * \Phi_{\delta_1} \;=\; T * \Phi_{\delta_1} \qquad \text{in } \Omega_\delta \; .$$

Now, let $\delta_1 \to 0$ and get

$$T * \Phi_{\delta_2} \;=\; T \qquad \text{in } \Omega_\delta \; .$$

The same argument applied to $T * \Phi_{\delta_2}$ concludes that $T * \Phi_{\delta_2}$ and, hence, T is identified, as a distribution in Ω_δ, with some function u_δ harmonic in Ω_δ .

If $\delta' < \delta$, then in Ω_δ, which is smaller than $\Omega_{\delta'}$, we have that u_δ and $u_{\delta'}$ represent the same distribution. Therefore, $u_{\delta'}$ is an extension of u_δ and, since $\cup_{\delta>0}\Omega_\delta = \Omega$, we conclude that all the u_δ's define a single u harmonic in Ω which T is identified to.

Suppose f is a locally integrable function in Ω whose distributional Laplacian vanishes in Ω. I.e.

$$\int_\Omega f(x)\Delta\phi(x)\; dm(x) \;=\; 0$$

for all $\phi \in \mathcal{D}(\Omega)$.

The Lemma of Weyl and Schwartz implies that there exists some u harmonic in Ω so that $T_f = T_u$ and, hence,

$$f(x) \;=\; u(x)$$

for almost every $x \in \Omega$.

In other words, we can change f in at most a set of measure zero and make it harmonic in Ω.

If f is continuous to begin with, then it is identical to u and, hence, it is harmonic.

Proposition 1.6 *If the distributional Laplacian of a locally integrable function f in the open $\Omega \subseteq \mathbf{R^n}$ is zero, then f is almost everywhere equal to a harmonic function in Ω.*

If, in addition, f is continuous in Ω, then it is harmonic in Ω.

Chapter 2

Superharmonic Functions

2.1 Definition

Let Ω be open in $\mathbf{R^n}$.

Definition 2.1 *A function u is called **superharmonic** in Ω, if*

1. *u is lower-semicontinuous in Ω,*

2. *u is not identically $+\infty$ in any connected component of Ω and*

3. *$\mathcal{M}_u^r(x) \leq u(x)$ for all $x \in \Omega$ and all $r < d(x, \partial\Omega)$.*

*The function u is called **subharmonic**, if $-u$ is superharmonic.*

Condition 3 is called the **super-mean-value property** while the corresponding condition for subharmonic functions is called the **sub-mean-value property**.

Comments

1. A superharmonic function is extended-real-valued and may take the value $+\infty$, but not the value $-\infty$.
2. If u is superharmonic in Ω, then $u(x) = \liminf_{y \to x} u(y)$ for all $x \in \Omega$.

In fact, let $\liminf_{y \to x} u(y) > u(x)$ and consider a number λ between these two quantities. Take δ so that $u(y) > \lambda$ for all $y \in B(x; \delta)$. Then $\mathcal{M}_u^r(x) \geq \lambda > u(x)$ for all $r < \delta$, a contradiction to the definition.
3. From Proposition 0.2, we get that $\mathcal{M}_u^r(x)$ is well-defined, either as a real number or as $+\infty$, for all $x \in \Omega$ and all r for which $S(x; r) \subseteq \Omega$.
4. We, easily, see that $\lim_{r \to 0+} \mathcal{M}_u^r(x) = u(x)$.

In fact, take $\lambda < u(x)$ and, by the lower-semicontinuity, find $\delta > 0$ so that $u(y) \geq \lambda$ for all $y \in B(x; \delta)$. Then $u(x) \geq \mathcal{M}_u^r(x) \geq \lambda$ for all $r < \delta$.
5. If a function u satisfies conditions 1 and 3 of the definition, but it is identically $+\infty$ in some connected components of Ω, then we may drop these components and form the set Ω^* as the union of the remaining connected components of Ω. Since all components are open sets, Ω^* is open and u *is* superharmonic in Ω^*.

Properties of superharmonic functions
(1) *Linear combinations with non-negative coefficients of superharmonic functions are superharmonic and the same is true for subharmonic functions.*
(2) *The minimum of finitely many superharmonic functions is superharmonic. There is a dual statement for subharmonic functions.*

The lower-semicontinuity is taken care of by property (4) of lower-semicontinuous functions. As for the super-mean-value property, if $u_1, \ldots, u_k$ have it and $u = \min(u_1, \ldots, u_k)$, then for every j we have $\mathcal{M}_u^r(x) \leq \mathcal{M}_{u_j}^r(x) \leq u_j(x)$ and, hence, $\mathcal{M}_u^r(x) \leq u(x)$.
(3) *Increasing limits of superharmonic functions are superharmonic, dropping, if necessary, the connected components where the limits are identically $+\infty$.*

There is a dual statement for subharmonic functions.

In fact, let $\{u_m\}$ be an increasing sequence of superharmonic functions in Ω and let $u_m(x) \uparrow u(x)$ for all $x \in \Omega$. The lower-semicontinuity of u comes from the third property of lower semi-continuous functions.

Also, for every m, $\mathcal{M}_{u_m}^r(x) \leq u_m(x) \leq u(x)$ and we prove $\mathcal{M}_u^r(x) \leq u(x)$, using Proposition 0.2 and the Monotone Convergence Theorem.
(4) *If u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and if u has a local minimum at $x \in \Omega$, then u is constant in some open neighborhood of x.*

A dual result is true for subharmonic functions.

Suppose $u(x) \leq u(y)$ for all $y \in B(x; R)$, where $\overline{B(x; R)} \subseteq \Omega$.

Then, for all $r < R$, $u(x) \leq \mathcal{M}_u^r(x) \leq u(x)$ and for equality to hold we must have $u(y) = u(x)$ for all $y \in S(x; r)$, except for at most a set $E \subseteq S(x; r)$ of zero surface measure.

Now, take $y \in E$. Then, there is some $\{y_m\}$ in $S(x; r) \setminus E$ so that $y_m \to y$. Hence, $u(x) \leq u(y) \leq \liminf_{z \to y} u(z) \leq \lim_{m \to +\infty} u(y_m) = u(x)$ and we get that $u(y) = u(x)$ for all $y \in S(x; r)$.

Therefore, since r is arbitrary with $r < R$, we conclude that u is constant in $B(x; R)$.
(5) *Superharmonic and subharmonic functions are preserved by rigid motions of $\mathbf{R^n}$.*

The proof is exactly the same as the proof of property (2) of harmonic functions.

2.2 Minimum principle

Theorem 2.1 *(The Minimum Principle for superharmonic functions) Suppose u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$.*

1. *If u takes its minimum value at some $x \in \Omega$, then u is constant in the connected component of Ω which contains x.*

2. *If*
$$m = \inf_{y \in \partial\Omega} \left(\liminf_{\Omega \ni x \to y} u(x) \right) ,$$
then $m \leq u(x)$ for all $x \in \Omega$.

If $m = u(x)$ for some x, then u is constant, $u = m$, in the connected component of Ω which contains x.

Proof:

The proof is an application of the fourth property of superharmonic functions and of Proposition 0.5 and is identical to the proof of Theorem 1.1.

The next result provides a characterization of superharmonicity and it is fundamental.

Theorem 2.2 *Suppose u is lower-semicontinuous in the open $\Omega \subseteq \mathbf{R^n}$ and it is not identically $+\infty$ in any connected component of Ω. Then the following are equivalent.*

1. *u is superharmonic in Ω.*
2. *For every $\overline{B(x_0; R)} \subseteq \Omega$ and every v harmonic in $B(x_0; R)$, the validity of $\liminf_{B(x_0;R)\ni x\to y}(u(x) - v(x)) \geq 0$ for all $y \in S(x_0; R)$ implies that $u \geq v$ in $B(x_0; R)$.*

Proof:

Since v being harmonic implies that $-v$ is superharmonic, one direction is a trivial application of Theorem 2.1.

Now, take arbitrary $\overline{B(x_0; R)} \subseteq \Omega$.

Proposition 0.3 implies that there exist f_m continuous in $S(x_0; R)$ so that

$$f_m(y) \;\uparrow\; u(y)$$

for all $y \in S(x_0; R)$.

Consider $P_{f_m}(\cdot\,; x_0, R)$, the Poisson integral of f_m in $B(x_0; R)$. Then, by Theorem 1.9,

$$\liminf_{B(x_0;R)\ni x\to y}(u(x) - P_{f_m}(x; x_0, R)) \;\geq\; u(y) - f_m(y) \;\geq\; 0$$

for all $y \in S(x_0; R)$.

By the hypothesis,

$$u(x_0) \;\geq\; P_{f_m}(x_0; x_0, R) \;=\; \mathcal{M}^R_{f_m}(x_0)\ .$$

Finally, by the Monotone Convergence Theorem and Proposition 0.2,

$$u(x_0) \;\geq\; \mathcal{M}^R_u(x_0)$$

and u is superharmonic in Ω.

If u is superharmonic in an open set containing a ball $\overline{B(x_0; R)}$, then it is bounded from below on the ball and, hence, the Poisson integral of u at every point of $B(x_0; R)$ is well-defined either as a real number or as $+\infty$.

Proposition 2.1 *Let u be superharmonic in an open set containing $\overline{B(x_0;R)}$. Consider $P_u(\cdot\,;x_0,R)$, the Poisson integral in $B(x_0;R)$ of the restriction of u on $S(x_0;R)$. Then,*

1. *$P_u(x;x_0,R) \le u(x)$ for every $x \in B(x_0;R)$.*
2. *Either $P_u(x;x_0,R) = +\infty$ for every $x \in B(x_0;R)$ or $P_u(\cdot\,;x_0,R)$ is harmonic in $B(x_0;R)$.*

In particular, if $\mathcal{M}_u^R(x_0) = +\infty$, then $u = +\infty$ identically in $B(x_0;R)$.

Proof:
Consider f_m continuous in $S(x_0;R)$ so that $f_m(y) \uparrow u(y)$ for every $y \in S(x_0;R)$ and their Poisson integrals $P_{f_m}(\cdot\,;x_0,R)$. Then,

$$\liminf_{B(x_0;R)\ni x\to y} (u(x) - P_{f_m}(x;x_0,R)) \ \ge\ u(y) - f_m(y) \ \ge\ 0$$

for all $y \in S(x_0;R)$ and, from Theorem 2.1 or Theorem 2.2, we get

$$P_{f_m}(x;x_0,R) \le u(x)$$

for every $x \in B(x_0;R)$. By the Monotone Convergence Theorem and the positivity of the Poisson kernel, we prove statement 1.

Let $-\infty < m \le u(y)$ for every $y \in S(x_0;R)$ and write $v = u - m$.

Then, either (i) $\mathcal{M}_u^R(x_0) = +\infty$ or (ii) $m \le \mathcal{M}_u^R(x_0) < +\infty$ and, hence, either (i)$\mathcal{M}_v^R(x_0) = +\infty$ or (ii) $0 \le \mathcal{M}_v^R(x_0) < +\infty$.

In case (i): let $x \in B(x_0;R)$ and consider $k = min_{y\in S(x_0;R)} P(y;x,x_0,R) > 0$. Then, since $0 \le v(y)$ for all $y \in S(x_0;R)$,

$$u(x) \ \ge\ P_u(x;x_0,R) \ =\ P_v(x;x_0,R) + m \ \ge k \int_{S(x_0;R)} v(y)\, dS(y) + m \ =\ +\infty\ .$$

In case (ii): $P_u(x;x_0,R) = P_v(x;x_0,R) + m$ is harmonic in $B(x_0;R)$, since v is integrable in $S(x_0;R)$.

2.3 Blaschke-Privaloff parameters

Definition 2.2 *If f is extended-real-valued and $f(x)$ is a real number, we define:*

$$\overline{M}_f(x) = \limsup_{r\to 0+} \frac{2n}{r^2}\big(\mathcal{M}_f^r(x) - f(x)\big)\ ,\quad \overline{A}_f(x) = \limsup_{r\to 0+} \frac{2(n+2)}{r^2}\big(\mathcal{A}_f^r(x) - f(x)\big)$$

and

$$\underline{M}_f(x) = \liminf_{r\to 0+} \frac{2n}{r^2}\big(\mathcal{M}_f^r(x) - f(x)\big)\ ,\quad \underline{A}_f(x) = \liminf_{r\to 0+} \frac{2(n+2)}{r^2}\big(\mathcal{A}_f^r(x) - f(x)\big)\ ,$$

whenever the mean values that appear are defined for all small enough r.

*These four numbers are called **Blaschke-Privaloff parameters** of f at x.*

Lemma 1.1 says that, if f is twice continuously differentiable in some neighborhood of x, then all four Blaschke-Privaloff parameters of f at x are equal to $\Delta f(x)$.

Lemma 2.1 *If f is extended-real-valued and $f(x)$ is real, the four Blaschke-Privaloff parameters of f at x satisfy:*

$$\underline{M}_f(x) \;\leq\; \underline{A}_f(x) \;\leq\; \overline{A}_f(x) \;\leq\; \overline{M}_f(x)\ .$$

Proof:

The middle inequality is obvious and it is enough to prove the third one, since the first is implied by this, using $-f$ in place of f.

In case $\overline{M}_f(x) = +\infty$, the result is obvious. Therefore, assume $\overline{M}_f(x) < +\infty$.

Let $\overline{M}_f(x) < \lambda$, implying that, for some R,

$$\frac{2n}{r^2}\big(\mathcal{M}_f^r(x) - f(x)\big) \;\leq\; \lambda$$

for all $r < R$.

Then, for $r < R$,

$$\begin{aligned}
\frac{2(n+2)}{r^2}\big(\mathcal{A}_f^r(x) - f(x)\big) \quad &= \quad \frac{2(n+2)}{r^2}\Big(\frac{n}{r^n}\int_0^r \mathcal{M}_f^s(x)\ s^{n-1}ds - f(x)\Big) \\
&\leq \quad \frac{2(n+2)}{r^2}\,\frac{n}{r^n}\int_0^r \lambda\ \frac{s^2}{2n}\ s^{n-1}ds \\
&= \quad \lambda\ .
\end{aligned}$$

Hence,

$$\overline{A}_f(x) \;\leq\; \lambda\ .$$

Letting $\lambda \downarrow \overline{M}_f(x)$, we conclude that

$$\overline{A}_f(x) \;\leq\; \overline{M}_f(x)\ .$$

The following is a general characterization of superharmonicity. Observe, to begin with, that the Blaschke-Privaloff parameters are well-defined for a lower-semicontinuous u, whenever $u(x) < +\infty$.

Theorem 2.3 *(Blaschke and Privaloff) If u is lower-semicontinuous in the open $\Omega \subseteq \mathbf{R^n}$ and not identically $+\infty$ in any connected component of Ω, then u is superharmonic in Ω if and only if $\underline{M}_u(x) \leq 0$ for all x with $u(x) < +\infty$.*

Proof:

The necessity is trivial and, for the sufficiency, take arbitrary $\overline{B(x_0;R)} \subseteq \Omega$ and consider f_m continuous in $S(x_0;R)$, so that $f_m(y) \uparrow u(y)$ for all $y \in S(x_0;R)$.

Consider, also, the auxiliary function

$$w(x) \;=\; |x-x_0|^2 - R^2 \ .$$

Using w, define, for every $\epsilon > 0$,

$$v(x) \;=\; \begin{cases} u(x) - P_{f_m}(x;x_0,R) - \epsilon\, w(x) \ , & x \in B(x_0;R) \\ u(x) - f_m(x) \ , & x \in S(x_0;R) \ . \end{cases}$$

Then, v is lower-semicontinuous in $\overline{B(x_0;R)}$ and, thus, takes a minimum value in there. Its values on $S(x_0;R)$ are non-negative and, hence, if we assume that its minimum value is < 0, then it is taken at some $x \in B(x_0;R)$.

But, then $\underline{M}_v(x) \geq 0$, while $\underline{M}_v(x) = \underline{M}_u(x) - 0 - \epsilon\, 2n \leq -\epsilon\, 2n$.

We, thus, get a contradiction and conclude that $v(x) \geq 0$ for all $x \in B(x_0;R)$.

Now, letting first $\epsilon \to 0$ and then $m \to +\infty$, we find for $x = x_0$,

$$u(x_0) \;\geq\; \mathcal{M}_u^R(x_0) \ .$$

Corollary 2.1 *Suppose* u *is lower-semicontinuous in the open* $\Omega \subseteq \mathbf{R^n}$ *and not identically* $+\infty$ *in any connected component of* Ω.
1. If, for every $x \in \Omega$*, there is some sequence* $\{r_m(x)\}$ *so that* $r_m(x) \to 0$ *and* $\mathcal{M}_u^{r_m(x)}(x) \leq u(x)$ *or* $\mathcal{A}_u^{r_m(x)}(x) \leq u(x)$ *for all* m*,* u *is superharmonic in* Ω.
2. If u *is in* $C^2(\Omega)$*,* u *is superharmonic in* Ω *if and only if* $\Delta u(x) \leq 0$ *for all* $x \in \Omega$.

In view of the extra regularity, Corollary 2.1(2) has an additional proof which uses Green's Formula, in exactly the same way as in the proof of the similar Theorem 1.3.

In fact, we use the formula

$$\int_{B(x;r)} \Delta u(x)\; dm(x) \;=\; \omega_{n-1} r^{n-1} \frac{d}{dr} \mathcal{M}_u^r(x)$$

which was derived in that proof.

If $\Delta u \leq 0$ identically in Ω, then, by the above formula, $\mathcal{M}_u^r(x)$ is decreasing in the interval $0 < r < d(x,\partial\Omega)$ and, taking the limit as $r \to 0+$, we find $\mathcal{M}_u^r(x) \leq u(x)$ for every $x \in \Omega$ and every $r < d(x,\partial\Omega)$.

If, conversely, $\mathcal{M}_u^r(x) \leq u(x)$ for every $x \in \Omega$ and every $r < d(x,\partial\Omega)$, then $\Delta u(x) = \lim_{r\to 0+} \frac{2n}{r^2}\big(\mathcal{M}_u^r(x) - u(x)\big) \leq 0$.

We, thus, get a weakened version of the original definition of superharmonicity, while the next result is the version of the original definition, having the surface-mean-values replaced by the space-mean-values.

Theorem 2.4 *Suppose u is lower-semicontinuous in the open $\Omega \subseteq \mathbf{R^n}$ and not identically $+\infty$ in any connected component of Ω.*

Then u is superharmonic in Ω if and only if, for every $x \in \Omega$ and all $r < d(x, \partial\Omega)$,

$$\mathcal{A}_u^r(x) \;\leq\; u(x) \;.$$

Proof:

The necessity follows from $\mathcal{A}_u^r(x) = \frac{n}{r^n}\int_0^r \mathcal{M}_u^s(x)\; s^{n-1}ds$.

For the sufficiency, we may observe that the above assumption, together with Lemma 2.1, implies $\underline{M}_u(x) \leq \underline{A}_u(x) \leq 0$ and the proof is concluded by Theorem 2.3.

Or, else, we may use Theorem 2.2 as follows. Take $\overline{B(x_0;R)} \subseteq \Omega$ and v harmonic in $B(x_0;R)$ so that $\liminf_{B(x_0;R)\ni x\to y}(u(x) - v(x)) \geq 0$ for all $y \in S(x_0;R)$.

Now, $\mathcal{A}_u^r(x) \leq u(x)$ implies

$$\mathcal{A}_{(u-v)}^r(x) \;\leq\; (u-v)(x) \;,$$

for all $x \in B(x_0;R)$ and all $r < R - |x - x_0|$.

But, from this, in the usual manner (we repeat the proof of the fourth property of superharmonic functions; this is even simpler), we get that, if $u - v$ takes a minimum value at some point in $B(x_0;R)$, then it is constant in a neighborhood of this point.

Proposition 0.5 implies that $u \geq v$ everywhere in $B(x_0;R)$.

By Theorem 2.2, u is superharmonic in Ω.

Example

If

$$u(x) \;=\; |x|^\alpha \;, \qquad x \in \mathbf{R^n} \setminus \{0\} \;,$$

then, using Corollary 2.1(2) and the formula

$$\Delta u(x) \;=\; \frac{d^2 u_*}{dr^2}(|x|) + \frac{n-1}{r}\,\frac{du_*}{dr}(|x|)$$

which holds for all twice continuously differentiable radial functions, we find that u is superharmonic if and only if $2 - n \leq \alpha \leq 0$ and subharmonic if and only if $\alpha \leq 2 - n$ or $0 \leq \alpha$.

Theorem 2.5 *Suppose u is superharmonic in $B(x_0;R) \setminus \{x_0\}$.*

Then, u can be defined at x_0 so that it becomes superharmonic in $B(x_0;R)$ if and only if

$$\liminf_{x\to x_0} \frac{u(x)}{h_{x_0}(x)} \;\geq\; 0 \;.$$

Proof:

The nessecity comes from the fact that if u is superharmonic in $B(x_0;R)$, then it is bounded from below in some neighborhood of x_0.

For the sufficiency, fix $r < R$ and consider functions f_m continuous in $S(x_0; r)$ so that $f_m(y) \uparrow u(y)$ for every $y \in S(x_0; r)$.

We, also, consider the function

$$\phi(x) \;=\; u(x) - P_{f_m}(x; x_0, r) + \epsilon\big(h_{x_0}(x) - h_*(r)\big) \;, \qquad x \in B(x_0; r) \setminus \{x_0\} \;,$$

for an arbitrary $\epsilon > 0$.

By Theorem 1.9, we have that ϕ is superharmonic in $\Omega = B(x_0; r) \setminus \{x_0\}$ and, also by the hypothesis, that $\liminf_{\Omega \ni x \to y} \phi(x) \geq 0$ for every $y \in \partial\Omega$.

Theorem 2.1 implies that $\phi(x) \geq 0$ for all $x \in B(x_0; r) \setminus \{x_0\}$. Hence,

$$u(x) \;\geq\; P_{f_m}(x; x_0, r) - \epsilon\big(h_{x_0}(x) - h_*(r)\big)$$

for all $x \in B(x_0; r) \setminus \{x_0\}$ and every $\epsilon > 0$ and, finally,

$$u(x) \;\geq\; P_{f_m}(x; x_0, r) \;, \qquad x \in B(x_0; r) \setminus \{x_0\} \;.$$

By the Dominated Convergence Theorem,

$$u(x) \;\geq\; P_u(x; x_0, r)$$

for all $x \in B(x_0; r) \setminus \{x_0\}$ and, thus,

$$\liminf_{x \to x_0} u(x) \;\geq\; P_u(x_0; x_0, r) \;=\; \mathcal{M}_u^r(x_0) \;.$$

Now, since r is arbitrary, if we define $u(x_0) = \liminf_{x \to x_0} u(x)$, we immediately conclude that u is lower-semicontinuous on $B(x_0; R)$ and an application of Corollary 2.1(1) concludes the proof.

In fact, the extended u satisfies the super-mean-value property at x_0, and, since it coincides with the original superharmonic u in $B(x_0; R) \setminus \{x_0\}$, it satisfies the super-mean-value property at all other points of $B(x_0; R)$ with respect to small enough balls centered at these points.

Theorem 2.6 *If u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$, then,*

1. *$u(x) < +\infty$ for almost every $x \in \Omega$ and*
2. *for every $\overline{B(x; r)} \subseteq \Omega$,*

$$\mathcal{A}_u^r(x) \;< +\infty \;, \qquad \mathcal{M}_u^r(x) \;\; < +\infty \;.$$

Therefore, u is locally integrable in Ω.

Proof:

It is enough to work separately in the various connected components of Ω and, hence, suppose that Ω is connected.

There exists some $x \in \Omega$ so that $u(x) < +\infty$. Therefore, $\mathcal{A}_u^r(x) < +\infty$ for $r < d(x, \partial\Omega)$ and, hence, u is integrable in $B(x; r)$ for all these r.

Define

$$A = \{x \in \Omega : u \text{ is integrable in some neighborhood of } x\} ,$$

$$B = \{x \in \Omega : u \text{ is not integrable in any neighborhood of } x\} .$$

A is open and non-empty and we shall prove that B is, also, open.

Take $x_0 \in B$. Then, $\mathcal{A}_u^r(x_0) = +\infty$ for $r < d(x_0, \partial\Omega)$.

Now, for all $x \in B(x_0; \frac{r}{2})$ it is true that $x_0 \in B(x; \frac{r}{2})$ and, hence,

$$u(x) \ \geq \ \mathcal{A}_u^{\frac{r}{2}}(x) = +\infty .$$

Therefore, $B(x_0; \frac{r}{2}) \subseteq B$.

Since Ω is connected, u is integrable in a neighborhood of any point in Ω. This implies that $u(x) < +\infty$ for almost every $x \in \Omega$. It, also, implies that $\mathcal{A}_u^r(x) < +\infty$ for every $x \in \Omega$ and all $r < d(x, \partial\Omega)$. In fact, as we proved above, if $\mathcal{A}_u^r(x) = +\infty$ for some $x \in \Omega$ and some r, then $u = +\infty$ in $B(x; \frac{r}{2})$.

If, for some $\overline{B(x;r)} \subseteq \Omega$, we have $\mathcal{M}_u^r(x) = +\infty$, then, from Proposition 2.1, $u = +\infty$ identically in $B(x;r)$ and this is false.

2.4 Poisson modification

Theorem 2.7 *Suppose u is superharmonic in the open subset Ω of $\mathbf{R^n}$ and $\overline{B(x_0;R)} \subseteq \Omega$. Define*

$$u_{B(x_0;R)}(x) \ = \ \begin{cases} P_u(x; x_0, R) \ , & \text{if } x \in B(x_0;R) \\ u(x) \ , & \text{if } x \in \Omega \setminus B(x_0;R) \ . \end{cases}$$

Then

1. *$u_{B(x_0;R)} \ \leq \ u$ in Ω and*

2. *$u_{B(x_0;R)}$ is superharmonic in Ω and harmonic in $B(x_0;R)$.*

Proof:

The first part and the harmonicity of $u_{B(x_0;R)}$ in $B(x_0;R)$ are consequences of Proposition 2.1 and Theorem 2.6.

That $-\infty < u_{B(x_0;R)}(x)$ for all $x \in \Omega$, is obvious.

Now, take f_m continuous in $S(x_0;R)$ so that $f_m(y) \uparrow u(y)$ for all $y \in S(x_0;R)$. Then, for any $y \in S(x_0;R)$,

$$f_m(y) \ = \ \liminf_{B(x_0;R) \ni x \to y} P_{f_m}(x; x_0, R) \ \leq \ \liminf_{B(x_0;R) \ni x \to y} P_u(x; x_0, R)$$

and, letting $m \to +\infty$,

$$u(y) \ \leq \ \liminf_{B(x_0;R) \ni x \to y} P_u(x; x_0, R) \ = \ \liminf_{B(x_0;R) \ni x \to y} u_{B(x_0;R)}(x) \ .$$

We, also, have

$$u(y) \ \leq \ \liminf_{\Omega \ni x \to y} u(x) \ \leq \ \liminf_{\Omega \setminus B(x_0;R) \ni x \to y} u_{B(x_0;R)}(x) \ .$$

Therefore,

$$u(y) \ \leq \ \liminf_{\Omega \ni x \to y} u_{B(x_0;R)}(x)$$

for all $y \in S(x_0;R)$ and, thus, $u_{B(x_0;R)}$ is lower-semicontinuous in Ω.

As for the super-mean-value property, this is obvious, by the harmonicity of the function in $B(x_0;R)$, at points $x \in B(x_0;R)$ with respect to small enough balls centered at x.

If $x \in \Omega \setminus B(x_0;R)$, then

$$u_{B(x_0;R)}(x) \ = \ u(x) \ \geq \ \mathcal{M}_u^r(x) \ \geq \ \mathcal{M}_{u_{B(x_0;R)}}^r(x) \ .$$

Definition 2.3 *Suppose that Ω is an open subset of $\mathbf{R^n}$ and $\overline{B(x_0;R)} \subseteq \Omega$. If u is superharmonic or subharmonic in Ω, then the function*

$$u_{B(x_0;R)}(x) \ = \ \begin{cases} P_u(x;x_0,R) \ , & \text{if } x \in B(x_0;R) \\ u(x) \ , & \text{if } x \in \Omega \setminus B(x_0;R) \end{cases}$$

is called the ***Poisson modification*** *of u with respect to $B(x_0;R)$.*

Proposition 2.2 *Superharmonicity is a local property: if u is superharmonic in a neighborhood of every point of the open $\Omega \subseteq \mathbf{R^n}$, then it is superharmonic in Ω.*

Proof:

It is immediate from Corrolary 2.1(1).

Example

Let f be holomorphic in the open $\Omega \subseteq \mathbf{R^2}$ and not identically 0 in any connected component of Ω. Then the function $-\log|f|$ is superharmonic in Ω.

$-\log|f|$ is defined as $+\infty$ at the points where $f = 0$. In fact, these points are isolated and $-\log|f|$ is not identically $+\infty$ in any connected component of Ω.

$-\log|f|$ is, trivially, lower-semicontinuous in Ω, it is, by Proposition 1.2, harmonic in a neighborhood of every point at which $f \neq 0$ and it satisfies the super-mean-value property at every x where $f(x) = 0$, simply because $-\log|f(x)| = +\infty$.

Example

Suppose u is real-valued and harmonic in the open $\Omega \subseteq \mathbf{R^n}$ and ϕ is concave in the real interval (m,M), where $m = \inf_\Omega u$ and $M = \sup_\Omega u$. Then $\phi \circ u$ is superharmonic in Ω.

Without loss of generality, we suppose that Ω is connected. If $u = m$ or $u = M$ at some point, then u is constant and, no matter how ϕ is defined at the endpoints m, M, $\phi \circ u$ is constant and, hence, superharmonic. Therefore, we may assume that $m < u < M$ in Ω.

ϕ is continuous in (m, M), implying that the composition is also continuous.

As for the super-mean-value property, by the inequality of Jensen,

$$\phi \circ u(x) \;=\; \phi\big(\mathcal{M}_u^r(x)\big) \;\geq\; \mathcal{M}_{\phi\circ u}^r(x)$$

for every $x \in \Omega$ and all $r < d(x, \partial\Omega)$.

Example

If $\alpha > 0$, $p \geq 1$ and u is real-valued and harmonic in the open $\Omega \subseteq \mathbf{R^n}$, then $u^+, u^-, e^{\alpha u}$ and $|u|^p$ are subharmonic in Ω.

Example

Suppose u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and ϕ is increasing and concave in the real interal $(m, M]$, where $m = \inf_\Omega u$ and $M = \sup_\Omega u$. Observe that, necessarily, $\phi(M) = \lim_{t\to M-} \phi(t)$. Then $\phi \circ u$ is superharmonic in Ω.

Except for minor modifications, the proof is the same as the proof in the second example.

Example

If u is subharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and $\alpha > 0$, then $u^+, e^{\alpha u}$ are subharmonic in Ω.

Also, if $p > 0$ and f is holomorphic in the open $\Omega \subseteq \mathbf{R^2}$ and not identically 0 in any connected component of Ω, then $|f|^p$ is subharmonic in Ω. This is a special case of the last example, when we use the increasing convex function $\phi(t) = e^{pt}$ and the subharmonic $\log |f|$.

Example

The function h_{x_0} is superharmonic in $\mathbf{R^n}$.

The function is continuous in $\mathbf{R^n} \setminus \{x_0\}$ and $\lim_{x\to x_0} h_{x_0}(x) = +\infty$. Hence, h_{x_0} is lower-semicontinuous in $\mathbf{R^n}$.

It is harmonic in $\mathbf{R^n}\setminus\{x_0\}$ and, thus, satisfies the super-mean-value property at every $x \neq x_0$ with respect to all sufficiently small balls centered at x.

It satisfies the super-mean-value property, also, at x_0 simply because its value there is $+\infty$.

2.5 Potentials

The next example is, in a sense, the most general and deserves to be stated as a theorem.

Theorem 2.8 *Suppose $d\mu$ is any compactly supported non-negative Borel measure. Then, its h-potential*

$$U_h^{d\mu}(x) \;=\; \int_{\mathbf{R^n}} h_x(y)\; d\mu(y)\;, \qquad x \in \mathbf{R^n}\;,$$

is superharmonic in $\mathbf{R^n}$.

In case $n > 2$, we get the same result for any non-negative Borel measure $d\mu$, assuming only that $U_h^{d\mu}(x) < +\infty$ for at least one point x.

In any case, $U_h^{d\mu}$ is harmonic in $\mathbf{R^n} \setminus supp(d\mu)$.

Proof:

Suppose that $d\mu$ has compact support in $\mathbf{R^n}$.

The statement about harmonicity is the content of Proposition 1.4 and, because of this harmonicity, $U_h^{d\mu}$ is not identically $+\infty$ in $\mathbf{R^n}$.

If d is such that $supp(d\mu) \subseteq \overline{B(0;d)}$, then, for any $\overline{B(x_0;r)}$,

$$\begin{aligned}
&\int_{\overline{B(x_0;r)}} \int_{supp(d\mu)} |h(x-y)|\; d\mu(y)\; dm(x) \\
&= \int_{supp(d\mu)} \int_{\overline{B(x_0;r)}} |h(x-y)|\; dm(x)\; d\mu(y) \\
&= \int_{supp(d\mu)} \int_{\overline{B(x_0-y;r)}} |h(x)|\; dm(x)\; d\mu(y) \\
&\leq \int_{supp(d\mu)} \int_{\overline{B(0;|x_0|+d+r)}} |h(x)|\; dm(x)\; d\mu(y) \\
&= \kappa_{|x_0|+d+r,n} d\mu(\mathbf{R^n}) \;<\; +\infty\;,
\end{aligned}$$

where $\kappa_{t,n}$ is a finite number depending only on t and n.

Therefore, we may use Fubini's Theorem and, from the last example of section 2.4,

$$\begin{aligned}
\mathcal{A}^r_{U_h^{d\mu}}(x_0) &= \int_K \mathcal{A}^r_{h_y}(x_0)\; d\mu(y) \\
&\leq \int_K h_y(x_0)\; d\mu(y) \\
&= U_h^{d\mu}(x_0)\;.
\end{aligned}$$

By Lemma 1.3, there is another way to prove the super-mean-value property.

$$\begin{aligned}
\mathcal{M}^r_{U_h^{d\mu}}(x_0) &= \int_{supp(d\mu)} \mathcal{M}^r_{h_y}(x_0)\; d\mu(y) \\
&= \int_{supp(d\mu)\cap B(x_0;r)} h_*(r)\; d\mu(y) \\
&\quad + \int_{supp(d\mu)\setminus B(x_0;r)} h_{x_0}(y)\; d\mu(y) \\
&\leq U_h^{d\mu}(x_0)\;,
\end{aligned}$$

where the use of the Theorem of Fubini is justified by the proof of Lemma 1.3.

It remains to prove that $U_h^{d\mu}$ is lower-semicontinuous and let $x_m \to x$.

If $|x_m - x| \leq 1$, then $|x_m - y| \leq |x| + d + 1$ for all m and all $y \in supp(d\mu)$. Therefore, the functions h_{x_m} are all bounded from below in $supp(d\mu)$ by the same constant and an application of the Lemma of Fatou gives

$$\liminf_{m\to+\infty} \int_{supp(d\mu)} h_{x_m}(y)\, d\mu(y) \;\geq\; \int_{supp(d\mu)} h_x(y)\, d\mu(y) \;.$$

Thus, $U_h^{d\mu}$ is lower-semicontinuous.

When $n > 2$, the function h is positive everywhere and, in this case, we may freely interchange integrations and apply Fatou's Lemma without having to assume that $d\mu$ is supported in a compact set.

Or, in a different way, consider the restrictions $d\mu_{B(0;m)}$ of $d\mu$ in the balls $B(0;m)$.

By the first part, $U_h^{d\mu_{B(0;m)}}$ is superharmonic in $\mathbf{R^n}$ and harmonic in $\mathbf{R^n} \setminus \big(\overline{B(0;m)} \cap supp(d\mu)\big)$ and, hence, in $\mathbf{R^n} \setminus supp(d\mu)$.

Since h is positive, $U_h^{d\mu_{B(0;m)}} \uparrow U_h^{d\mu}$ everywhere in $\mathbf{R^n}$. By the assumption, $U_h^{d\mu}$ is not identically $+\infty$ and, finally, by the third property of superharmonic functions and Theorem 1.16, $U_h^{d\mu}$ is superharmonic in $\mathbf{R^n}$ and harmonic in $\mathbf{R^n} \setminus supp(d\mu)$.

Later on we shall prove the fundamental theorem of F. Riesz stating that the most general superharmonic function is, more or less, the sum of a harmonic function and the h-potential of a non-negative Borel measure.

2.6 Differentiability of potentials

Proposition 2.3 *Under the hypotheses of Theorem 2.8, $U_h^{d\mu}$ is absolutely continuous on almost every line parallel to the principal x_j-axes, $1 \leq j \leq n$, it has partial derivatives at almost every point of $\mathbf{R^n}$ and these partial derivatives are locally integrable.*

Proof:

We fix an arbitrary $m \in \mathbf{N}$ and we shall work in the cube

$$Q_m \;=\; \{x = (x_1, \ldots, x_n) : |x_j| \leq m \text{ for all } j\} \;.$$

In case $n = 2$, the measure is supported in a compact set. In case $n \geq 3$ the measure need not be supported in a compact set, but we may split $d\mu = d\mu_{Q_{2m}} + d\mu_{\mathbf{R^n} \setminus Q_{2m}}$ and observe that the h-potential of the second term is harmonic in Q_m and, hence, infinitely differentiable there.

We may, therefore, assume that $d\mu$ is supported in some compact set and, in particular, that it is finite.

By a trivial calculation, there is a constant C_n, depending only on the dimension, so that

$$\left|\frac{\partial h_y}{\partial x_j}(x)\right| \;\le\; \frac{C_n}{|x-y|^{n-1}}$$

for all $x, y \in \mathbf{R^n}$.

Therefore,

$$\begin{aligned}
\int_{Q_m}\int_{\mathbf{R^n}}\left|\frac{\partial h_y}{\partial x_j}(x)\right| d\mu(y)\,dm(x) &\le C_n\int_{\mathbf{R^n}}\int_{Q_m}\frac{1}{|x-y|^{n-1}}\,dm(x)\,d\mu(y)\\
&< C_{n,m}d\mu(\mathbf{R^n}) \;<\; +\infty\;.
\end{aligned}$$

Thus, the function

$$u_j(x) \;=\; \int_{\mathbf{R^n}}\frac{\partial h_y}{\partial x_j}(x)\;d\mu(y)\;, \qquad x\in Q_m\;,$$

is integrable in Q_m and, by Fubini's Theorem, it is integrable on almost every line segment parallel to the x_j-axis and extending between the two sides of Q_m which are perpendicular to this axis.

If $[a,b]$ is any part of such a segment, then

$$\begin{aligned}
\int_a^b u_j(x)\;dx_j &= \int_{\mathbf{R^n}}\int_a^b\frac{\partial h_y}{\partial x_j}(x)\;dx_j\;d\mu(y)\\
&= \int_{\mathbf{R^n}}\big(h_y(b)-h_y(a)\big)\;d\mu(y)\\
&= U_h^{d\mu}(b)-U_h^{d\mu}(a)\;.
\end{aligned}$$

This says that $U_h^{d\mu}$ is absolutely continuous on almost every line segment parallel to the x_j-axis and extending between the two sides of Q_m and that $\frac{\partial U_h^{d\mu}}{\partial x_j} = u_j$ almost everywhere on such a line segment.

By Fubini's Theorem, again, $\frac{\partial U_h^{d\mu}}{\partial x_j}$ is defined and equal to u_j almost everywhere in Q_m.

Since m is arbitrary, the proof is finished.

Definition 2.4 *Suppose that the non-negative function f is locally integrable in $\mathbf{R^n}$. The h-**potential of** f is defined to be the h-potential of the non-negative Borel measure $f\,dm$. We denote it by*

$$U_h^f \;=\; U_h^{f\,dm}\;.$$

According to Theorem 2.8, if $n=2$, we assume that f is compactly supported and, if $n\ge 3$, we assume only that $U_h^f(x)<+\infty$ for at least one x.

Proposition 2.4 *Under the hypotheses of the previous definition, if the non-negative function f is in $C^k(\mathbf{R^n})$, $0\le k\le+\infty$, then the h-potential of f is in $C^{k+1}(\mathbf{R^n})$.*

Proof:

If $n = 2$, then f has compact support. If $n \geq 3$, then f need not have compact support, but, since we want to study the h-potential in a neighborhood of an arbitrary point x, we may take a large R so that $x \in B(0;R)$ and split $f = f\phi + f(1-\phi)$, where ϕ is in $\mathcal{D}(\mathbf{R^n})$ with $\phi = 1$ identically in $B(0;2R)$ and $0 \leq \phi \leq 1$ everywhere. The existence of ϕ is due to Lemma 0.2.

Then, the h-potential of $f(1-\phi)$ is harmonic in $B(0;R)$ and, hence, infinitely differentiable in a neighborhood of x. Thus, we need to study the h-potential of the function $f\phi$ which is in $C^k(\mathbf{R^n})$ and has compact support.

We, therefore, assume that f has compact support.

Since h is locally integrable, by Proposition 0.6, the convolution $U_h^f = h * f$ is in $C^k(\mathbf{R^n})$ and

$$D^\alpha\big(U_h^f\big) \;=\; h * D^\alpha f$$

for all multiidices α with $|\alpha| \leq k$.

Hence, the proof reduces to showing that, if f is in $C(\mathbf{R^n})$ with compact support, then $h * f$ is in $C^1(\mathbf{R^n})$.

We observe that

$$\frac{\partial h}{\partial x_j}(x) \;=\; -(n-2)\frac{x_j}{|x|^n}$$

for all $x \neq 0$ and, thus, $\frac{\partial h}{\partial x_j}$ is locally integrable.

We write, now,

$$\frac{h*f(x+te_j) - h*f(x)}{t} \;=\; \int_{\mathbf{R^n}} \frac{h(x-y+te_j) - h(x-y)}{t}\; f(y)\; dm(y)\ .$$

If z is not on the x_j-axis, then

$$\frac{h(z+te_j) - h(z)}{t} \;=\; \int_0^1 \frac{\partial h}{\partial x_j}(z + ste_j)\; ds$$

and, hence,

$$\begin{aligned}
\frac{h*f(x+te_j) - h*f(x)}{t} \;&=\; \int_{\mathbf{R^n}} \int_0^1 \frac{\partial h}{\partial x_j}(x-y+ste_j)\; ds\; f(y)\; dm(y) \\
&=\; \int_0^1 \int_{\mathbf{R^n}} \frac{\partial h}{\partial x_j}(x-y+ste_j) f(y)\; dm(y)\; ds \\
&=\; \int_0^1 \int_{\mathbf{R^n}} \frac{\partial h}{\partial x_j}(y) f(x-y+ste_j)\; dm(y)\; ds\ .
\end{aligned}$$

Therefore, if $|t| \leq 1$,

$$\begin{aligned}
&\Big|\frac{h*f(x+te_j) - h*f(x)}{t} - \frac{\partial h}{\partial x_j} * f(x)\Big| \\
&\leq \int_0^1 \int_{\mathbf{R^n}} \Big|\frac{\partial h}{\partial x_j}(y)\Big|\; |f(x-y+ste_j) - f(x-y)|\; dm(y)\; ds \\
&\leq \int_{B(x;1)+supp(f)} \Big|\frac{\partial h}{\partial x_j}(z)\Big|\; dm(z) \sup_{|a-b|\leq|t|} |f(a) - f(b)|
\end{aligned}$$

and, thus,

$$\frac{\partial(h*f)}{\partial x_j} \;=\; \frac{\partial h}{\partial x_j} * f \; .$$

Since $\frac{\partial h}{\partial x_j}$ is locally integrable and f is continuous with compact support, by Proposition 0.6, the last convolution is continuous.

2.7 Approximation, properties of means

Theorem 2.9 *(First property of the means) Let u be superharmonic in the open $\Omega \subseteq \mathbf{R^n}$. Then, for every $x \in \Omega$, $\mathcal{M}_u^r(x)$ is a decreasing function of r in the interval $0 < r < d(x, \partial\Omega)$. We, also, have that $\lim_{r\to 0+} \mathcal{M}_u^r(x) = u(x)$.*

Exactly the same results hold for the space-means $\mathcal{A}_u^r(x)$.

Proof:

Consider $r_1 < r_2 < d(x, \partial\Omega)$ and the Poisson integral $P_u(\cdot\, ; x, r_2)$ in the ball $B(x; r_2)$. From Proposition 2.1, $u(\cdot) \geq P_u(\cdot\, ; x, r_2)$ in this ball and, hence,

$$\mathcal{M}_u^{r_1}(x) \;\geq\; \mathcal{M}_{P_u(\cdot\, ; x, r_2)}^{r_1}(x) \;=\; P_u(x; x, r_2) \;=\; \mathcal{M}_u^{r_2}(x) \; .$$

The equality $\lim_{r\to 0+} \mathcal{M}_u^r(x) = u(x)$ is just Comment 4 after the definition of superharmonic functions.

For the space-means,

$$\begin{aligned} \mathcal{A}_u^{r_1}(x) &= \frac{n}{r_1^n} \int_0^{r_1} \mathcal{M}_u^r(x) r^{n-1}\, dr \\ &= \frac{n}{r_2^n} \int_0^{r_2} \mathcal{M}_u^{\frac{r_1}{r_2} r}(x) r^{n-1}\, dr \\ &\geq \frac{n}{r_2^n} \int_0^{r_2} \mathcal{M}_u^r(x) r^{n-1}\, dr \\ &= \mathcal{A}_u^{r_2}(x) \; , \end{aligned}$$

where in the last inequality we used the result about the surface-means.

The limit $\lim_{r\to 0+} \mathcal{A}_u^r(x) = u(x)$ is trivial and can be proved in the same way as the limit of the surface-means above.

Corollary 2.2 *If u and v are superharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and $u(x) = v(x)$ for almost every $x \in \Omega$, then u and v are identically equal in Ω.*

Theorem 2.10 *(Approximation) Suppose u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and $\{\Phi_\delta : \delta > 0\}$ is an approximation to the identity. Then, for every δ, the function $u * \Phi_\delta \in C^\infty(\Omega_\delta)$ is superharmonic in Ω_δ and,*

$$u * \Phi_\delta(x) \;\uparrow\; u(x)$$

as $\delta \downarrow 0$, for every $x \in \Omega$.

Proof:

From Proposition 0.7 and Theorem 2.6 we find that $u * \Phi_\delta$ is in $C^\infty(\Omega_\delta)$.

For every $x \in \Omega_\delta$ and $\overline{B(x;R)} \subseteq \Omega_\delta$,

$$\begin{aligned} \mathcal{M}^R_{u*\Phi_\delta}(x) &= \frac{1}{\omega_{n-1}R^{n-1}} \int_{S(x;R)} \int_{B(0;\delta)} u(y-z)\Phi_\delta(z)\ dm(z)\ dS(y) \\ &= \int_{B(0;\delta)} \frac{1}{\omega_{n-1}R^{n-1}} \int_{S(x;R)} u(y-z)\ dS(y)\ \Phi_\delta(z)\ dm(z) \\ &\le \int_{B(0;\delta)} u(x-z)\Phi_\delta(z)\ dm(z) \\ &= u * \Phi_\delta(x)\ , \end{aligned}$$

where the last inequality is true because $\frac{1}{\omega_{n-1}R^{n-1}} \int_{S(x;R)} u(y-z)\ dS(y)$ is the surface-mean of the function $u(\cdot - z)$ which is superharmonic in the open set $\Omega - z$ containing $\overline{B(x;R)}$ whenever $z \in B(0;\delta)$.

Therefore, $u * \Phi_\delta$ is superharmonic in Ω_δ .

Now, since Φ_δ is radial,

$$u * \Phi_\delta(x) \ = \omega_{n-1} \int_0^\delta \Phi_{\delta *}(r) \mathcal{M}^r_u(x)\ r^{n-1} dr\ .$$

By Theorem 2.9, for any $\lambda < u(x)$ we have $\lambda < \mathcal{M}^r_u(x) \le u(x)$ for small enough r. Thus, if δ is small,

$$\lambda\omega_{n-1} \int_0^\delta \Phi_{\delta *}(r)\ r^{n-1} dr \ \le\ u * \Phi_\delta(x) \ \le\ u(x)\omega_{n-1} \int_0^\delta \Phi_{\delta *}(r)\ r^{n-1} dr\ ,$$

implying

$$\lambda \ \le\ u * \Phi_\delta(x) \ \le\ u(x)\ .$$

We conclude that $u * \Phi_\delta(x) \to u(x)$ as $\delta \to 0$.

Also, taking $\delta < \delta'$,

$$\begin{aligned} u * \Phi_\delta(x) &= \omega_{n-1} \int_0^\delta \Phi_{\delta *}(r) \mathcal{M}^r_u(x)\ r^{n-1} dr \\ &= \omega_{n-1} \int_0^\delta \frac{1}{\delta^n}\ \Phi_{1*}(\frac{r}{\delta}) \mathcal{M}^r_u(x)\ r^{n-1} dr \\ &= \omega_{n-1} \int_0^{\delta'} \frac{1}{\delta'^n}\ \Phi_{1*}(\frac{r}{\delta'}) \mathcal{M}^{\frac{\delta}{\delta'} r}_u(x)\ r^{n-1} dr \\ &\ge \omega_{n-1} \int_0^{\delta'} \frac{1}{\delta'^n}\ \Phi_{1*}(\frac{r}{\delta'}) \mathcal{M}^r_u(x)\ r^{n-1} dr \\ &= u * \Phi_{\delta'}(x)\ . \end{aligned}$$

The last inequality is true by Theorem 2.9.

Proposition 2.5 *Consider the open subsets Ω_1 and Ω_2 of $\mathbf{R^2}$, the holomorphic function f in Ω_1 and let $f(\Omega_1) \subseteq \Omega_2$. If u is superharmonic in Ω_2, then $u \circ f$ is superharmonic in Ω_1.*

The only exception is when f is constant in some connected component of Ω_1 and u takes the value $+\infty$ at this value of f.

Proof:

Assuming that u is twice continuously differentiable in Ω_2, we use the formula $\Delta(u \circ f)(z) = |f'(z)|^2 \Delta u(f(z))$ and Corollary 2.1(2) to prove the result.

Otherwise, consider an arbitrary closed disc $\overline{B(z_0; R)} \subseteq \Omega_1$ and the compact set $f\big(\overline{B(z_0; R)}\big) \subseteq \Omega_2$.

We consider a bounded open set V so that $\overline{B(z_0; R)} \subseteq V \subseteq \overline{V} \subseteq \Omega$ and using Theorem 2.10, we approximate u in V by an increasing sequence $\{u_m\}$ of functions which are twice continuously differentiable and superharmonic in V.

Then, the functions $u_m \circ f$ are, by the first part, superharmonic in $B(z_0; R)$ and increase towards $u \circ f$ there.

Thus, by the third property of superharmonic functions, $u \circ f$ is superharmonic in $B(z_0; R)$ and, since the ball is arbitrary, $u \circ f$ is superharmonic in Ω_1.

Example

If u is superharmonic in the open $\Omega \subseteq \mathbf{R^2}$ which does not contain 0 and if in the set $\Omega^* = \{x : \frac{1}{x} \in \Omega\}$ we define the function

$$u^*(x) \;=\; u\Big(\frac{1}{x}\Big) \;,$$

then u^* is superharmonic in Ω^*.

Theorem 2.11 *(Second property of the means) Let u be superharmonic in the open $\Omega \subseteq \mathbf{R^n}$ and $B(x; R_1, R_2) \subseteq \Omega$. Then,*

$$\mathcal{M}_u^r(x) \;<\; +\infty$$

for all r with $R_1 < r < R_2$ and $\mathcal{M}_u^r(x)$ is a concave function of $h(r)$ in the interval $R_1 < r < R_2$.

In particular, $\mathcal{M}_u^r(x)$ is a continuous function of r in the same interval.

Proof:

Assume, first, that u is in $C^2(\Omega)$.

Then, from Corrolary 2.1(2), $\Delta u \le 0$ everywhere in Ω.

Consider $R_1 < r_1 < r_2 < R_2$ and apply Green's Formula in $B(x; r_1, r_2)$ with $\overrightarrow{\eta}$ being the continuous unit vector field normal to $\partial B(x; r_1, r_2)$ in the direction towards the exterior of this ring.

$$\begin{aligned}
\int_{B(x;r_1,r_2)} \Delta u(z)\; dm(z) \;&=\; \int_{S(x;r_2)} \frac{\partial u}{\partial \eta}(y)\; dS(y) + \int_{S(x;r_1)} \frac{\partial u}{\partial \eta}(y)\; dS(y) \\
&=\; r_2^{n-1} \int_{S^{n-1}} \frac{d}{dr}\big(u(x+ry)\big)_{r=r_2}\; d\sigma(y)
\end{aligned}$$

$$
\begin{aligned}
& -r_1^{n-1}\int_{S^{n-1}} \frac{d}{dr}\big(u(x+ry)\big)_{r=r_1}\, d\sigma(y) \\
= \;& r_2^{n-1}\frac{d}{dr}\big(\omega_{n-1}\mathcal{M}_u^r(x)\big)_{r=r_2} \\
& -r_1^{n-1}\frac{d}{dr}\big(\omega_{n-1}\mathcal{M}_u^r(x)\big)_{r=r_1} \,.
\end{aligned}
$$

Now, $\int_{B(x;r_1,r_2)} \Delta u(z)\, dm(z)$ is a decreasing function of r_2 and, hence, $r^{n-1}\frac{d}{dr}\mathcal{M}_u^r(x)$ is a decreasing function of r. Therefore,

$$
\frac{d}{dr}\Big(r^{n-1}\frac{d}{dr}\mathcal{M}_u^r(x)\Big) \;\le\; 0
$$

in $R_1 < r < R_2$.

If we write $h = h(r)$, the last relation becomes

$$
\frac{d^2}{dh^2}\mathcal{M}_u^h(x) \;\le\; 0 \,,
$$

implying

$$
\mathcal{M}_u^{th_1+(1-t)h_2}(x) \;\ge\; t\mathcal{M}_u^{h_1}(x) + (1-t)\mathcal{M}_u^{h_2}(x)
$$

for all $t \in (0,1)$ and $h_1 = h(r_1),\ h_2 = h(r_2)$ with $R_1 < r_1 < r_2 < R_2$.

In the general case, we use the approximation Theorem 2.10 to get a sequence of functions u_m superharmonic and twice continuously differentiable in $B(x;r_1',r_2')$, with $R_1 < r_1' < r_1 < r_2 < r_2' < R_2$ and such that $u_m \uparrow u$ in $B(x;r_1',r_2')$.

We, then, apply the last inequality to each u_m and prove it for u by the Monotone Convergence Theorem.

If we assume that, for some $r_0 \in (R_1,R_2)$, $\mathcal{M}_u^{r_0}(x) = +\infty$, then, using the above inequality, it is easy to show that $\mathcal{M}_u^r(x) = +\infty$ for all r in (R_1,R_2). Taking $R_1 < r_1 < r_2 < R_2$, we get

$$
\int_{\overline{B(x;r_1,r_2)}} u(y)\, dm(y) \;=\; +\infty
$$

contradicting the local integrability of u which was proved in Theorem 2.6.

Thus,

$$
\mathcal{M}_u^r(x) \;<\; +\infty
$$

for all r in (R_1,R_2), finishing the proof of the concavity.

Theorem 2.12 *Suppose u is superharmonic in the open subset Ω of $\mathbf{R^n}$ and let $\overline{B(x_0;R)} \subseteq \Omega$.*

Then, the only function which is superharmonic in Ω, harmonic in $B(x_0;R)$ and coincides with u in $\Omega \setminus B(x_0;R)$ is the Poisson modification $u_{B(x_0;R)}$.

Also, $u_{B(x_0;R)}$ is the upper envelope of the family of functions v which are superharmonic in Ω, harmonic in $B(x_0;R)$ and satisfy $v \le u$ everywhere in Ω.

Proof:

Consider any v with all properties in the first part of the statement.

From the definition of the Poisson modification, we have

$$v_{B(x_0;R)} \;=\; u_{B(x_0;R)}$$

in Ω and, hence, it is enough to prove that $v = v_{B(x_0;R)}$ in $B(x_0;R)$.

From Theorem 2.7, we have that $v \geq v_{B(x_0;R)}$ in $B(x_0;R)$ and, if we prove that $v(x_0) = v_{B(x_0;R)}(x_0)$, then, since both functions are harmonic in $B(x_0;R)$, the Maximum-Minimum Principle will finish the proof.

From the harmonicity of v and of $v_{B(x_0;R)}$ in $B(x_0;R)$ and from Theorem 2.11,

$$v(x_0) \;=\; \lim_{r\to R-} \mathcal{M}_v^r(x_0) \;=\; \mathcal{M}_v^R(x_0) \;=\; v_{B(x_0;R)}(x_0)$$

and the proof of the first part is complete.

If v satisfies the assumptions of the second part, then, by the first part,

$$v \;=\; v_{B(x_0;R)} \;\leq\; u_{B(x_0;R)}$$

in Ω.

2.8 The Perron process

In the proof of the next result we introduce the important **Perron process**.

Theorem 2.13 *Let $\mathcal{V}$ be a non-empty family of functions v subharmonic in the open connected $\Omega \subseteq \mathbf{R^n}$ with the following two properties*

1. *If $v_1, v_2 \in \mathcal{V}$, then $\max(v_1, v_2) \in \mathcal{V}$.*
2. *If $v \in \mathcal{V}$ and $\overline{B(x_0;R)} \subseteq \Omega$, then $v_{B(x_0;R)} \in \mathcal{V}$.*

Then, the upper envelope V of the family $\mathcal{V}$ is either identically $+\infty$ in Ω or it is harmonic in Ω.

There is a dual result about lower envelopes of families of superharmonic functions satisfying the duals of properties 1 and 2.

Proof:

Fix an arbitrary $\overline{B(x_0;R)} \subseteq \Omega$ and consider a countable set $\{x_i : i \in \mathbf{N}\}$ dense in $B(x_0;R)$.

For each x_i, take a sequence $\{v_i^{(m)}\}$ in the family $\mathcal{V}$ so that $v_i^{(m)}(x_i) \uparrow V(x_i)$ as $m \to +\infty$.

Modify, defining

$$u_i^{(m)} \;=\; \max(v_i^{(1)}, \ldots, v_i^{(m)}) \;, \qquad m \in \mathbf{N} \;.$$

$\{u_i^{(m)}\}$ is an increasing sequence in $\mathcal{V}$ with $u_i^{(m)}(x_i) \uparrow V(x_i)$ as $m \to +\infty$.

Modify the new sequence, defining

$$w^{(1)} = u_1^{(1)}, \qquad w^{(m)} = \max(u_1^{(m)}, u_2^{(m)}, \ldots, u_m^{(m)}) .$$

Now, $\{w^{(m)}\}$ is an increasing sequence in $\mathcal{V}$ with $w^{(m)}(x_i) \uparrow V(x_i)$ for all x_i.

Modify once more, taking the sequence

$$v_m = (w^{(m)})_{B(x_0;R)} .$$

This is a new increasing sequence in $\mathcal{V}$ such that $v_m(x_i) \uparrow V(x_i)$ for all x_i, with the additional property that all v_m are harmonic in $B(x_0;R)$.

Set

$$v(x) = \lim_{m\to+\infty} v_m(x)$$

for all x in $B(x_0;R)$.

By Theorem 1.16, either v is harmonic in $B(x_0;R)$ or $v = +\infty$ identically in $B(x_0;R)$. It is obvious that, if $v = +\infty$ in $B(x_0;R)$, then the same is true with V.

Suppose, for the moment, that v is harmonic in $B(x_0;R)$ and, by its construction, $v(x_i) = V(x_i)$ for all x_i.

Consider, now, a point $x \in B(x_0;R)$ different from all x_i and repeat the previous construction with the set $\{x\} \cup \{x_i : i \in \mathbf{N}\}$. A new function v' will be produced, harmonic in $B(x_0;R)$ with $v'(x_i) = V(x_i)$ for all x_i and, also, $v'(x) = V(x)$.

The functions v, v' which are continuous in $B(x_0;R)$ agree on the dense set $\{x_i : i \in \mathbf{N}\}$ of $B(x_0;R)$ and, hence, are identically equal on this ball. Therefore, $v(x) = v'(x) = V(x)$ at the additional point x. Since x is arbitrary, this proves that $V = v$ identically in $B(x_0;R)$ and, finally, that V is harmonic in $B(x_0;R)$.

Now, we define the sets

$$\begin{aligned} A &= \{x \in \Omega : V \text{ is harmonic in some neighborhood of } x\} , \\ B &= \{x \in \Omega : V = +\infty \text{ in some neighborhood of } x\} . \end{aligned}$$

By what we proved before, $\Omega = A \cup B$, $A \cap B = \emptyset$ and both A and B are open sets. Since Ω is connected, either $A = \Omega$ or $B = \Omega$.

2.9 The largest harmonic minorant

Definition 2.5 *If f, g are extended-real-valued functions defined on the same set E and $f(x) \le g(x)$ for all $x \in E$, we say that f is a* ***minorant*** *of g and that g is a* ***majorant*** *of f in E.*

If the same inequality is true for all f in a family $\mathcal{F}$, we say that g is a majorant of $\mathcal{F}$ in E and, if the inequality is true for all g in a family $\mathcal{G}$, we say that f is a minorant of $\mathcal{G}$ in E.

Theorem 2.14 *Let $\mathcal{U}$ be a non-empty family of functions superharmonic in the open $\Omega \subseteq \mathbf{R}^\mathbf{n}$. Suppose that there exists at least one subharmonic minorant of $\mathcal{U}$ in Ω.*

Then the upper envelope of all subharmonic minorants of $\mathcal{U}$ is a function harmonic in Ω.

Proof:

Let $\mathcal{V}$ be the non-empty family of all subharmonic minorants of the family $\mathcal{U}$ and let V be the upper envelope of $\mathcal{V}$.

If $u \in \mathcal{U}$, then $V \leq u$ and, hence, $V(x) < +\infty$ for almost every $x \in \Omega$.

Since it is very easy to see that the family $\mathcal{V}$ satisfies the assumptions of Theorem 2.13 in all connected components of Ω, we conclude that V is harmonic in Ω.

Definition 2.6 *Let $\mathcal{U}$ be a non-empty family of superharmonic functions in the open $\Omega \subseteq \mathbf{R}^\mathbf{n}$ and suppose that $\mathcal{U}$ has at least one subharmonic minorant in Ω. Then the upper envelope of all the subharmonic minorants of $\mathcal{U}$ is called the **largest harmonic minorant** of $\mathcal{U}$.*

*We, similarly, define the **smallest harmonic majorant** of a non-empty family of subharmonic functions.*

As an example, we prove the

Proposition 2.6 *Let u be superharmonic in $B(x_0; R)$. Then u has at least one subharmonic minorant if and only if $\lim_{r\to R-} \mathcal{M}_u^r(x_0) > -\infty$.*

If this condition is satisfied, then the largest harmonic minorant of u is the function $\lim_{r\to R-} u_{B(x_0;r)}$.

Proof:

If u has some subharmonic minorant v in $B(x_0; R)$, then, by Theorem 2.6, for every $r < R$, $\mathcal{M}_u^r(x_0) \geq \mathcal{M}_v^r(x_0) > -\infty$. Since, by Theorem 2.9, the left side decreases while the right side increases when $r \uparrow R$, we see that $\lim_{r\to R-} \mathcal{M}_u^r(x_0) > -\infty$.

Now, let $\lim_{r\to R-} \mathcal{M}_u^r(x_0) > -\infty$ and fix some $r_0 < R$.

It is clear from Theorem 2.7, that, when $r_0 < r < R$, the functions $u_{B(x_0;r)}$ are superharmonic in $B(x_0; R)$, harmonic in $B(x_0; r_0)$ and decrease as r increases.

Since $u_{B(x_0;r)}(x_0) = \mathcal{M}_u^r(x_0)$, Theorem 1.16 implies that the function

$$V = \lim_{r\to R-} u_{B(x_0;r)}$$

is harmonic in $B(x_0; r_0)$.

Since r_0 is arbitrary, V is harmonic in $B(x_0; R)$.

Clearly, $V \leq u_{B(x_0;r)} \leq u$ for all $r < R$ and, hence, V is a harmonic minorant of u in $B(x_0; R)$.

Suppose, now, that v' is any subharmonic minorant of u in $B(x_0; R)$. We, easily, see that Theorem 2.7 implies $v' \leq v'_{B(x_0;r)} \leq u_{B(x_0;r)}$ for all $r < R$.

Therefore $v' \leq V$ in $B(x_0; R)$ and, finally, V is, exactly, the largest harmonic minorant of u in $B(x_0; R)$.

2.10 Superharmonic distributions

The rest of this chapter is devoted to the study of the distributional Laplacian of superharmonic functions and the relevant characterization of them and to the proof of the related Decomposition Theorem of F. Riesz.

In the statement of the next theorem we denote the distributional Laplacian of the function u by Δu, instead of the more correct ΔT_u, but this, as we have already noted, is allowed by the standard convention to identify a function with the corresponding distribution. In the proof, though, we shall be more formal.

Proposition 2.7 *If u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$, then its distributional Laplacian Δu is a non-positive distribution in Ω and, hence, it is identified with a non-positive Borel measure in Ω.*

Proof:

Let u be superharmonic in Ω and consider any approximation to the identity $\{\Phi_\delta : \delta > 0\}$. From Theorem 2.10, we know that $u * \Phi_\delta$ is in $C^\infty(\Omega_\delta)$ and that it is superharmonic in Ω_δ.

Corrolary 2.1(2) implies that $\Delta(u * \Phi_\delta)(x) \leq 0$ for all $x \in \Omega_\delta$.

Take, now, any $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$ everywhere in Ω. Since $supp(\phi) \subseteq \Omega_\delta$ if δ is small enough, using Proposition 0.9 and the calculus of distributions, we find

$$\begin{aligned} \Delta T_u(\phi) &= \lim_{\delta\to 0+} \Delta T_u(\widetilde{\Phi_\delta} * \phi) \;=\; \lim_{\delta\to 0+} T_u\big(\Delta(\widetilde{\Phi_\delta} * \phi)\big) \\ &= \lim_{\delta\to 0+} T_u(\widetilde{\Phi_\delta} * \Delta\phi) \;=\; \lim_{\delta\to 0+} (T_u * \Phi_\delta)(\Delta\phi) \\ &= \lim_{\delta\to 0+} T_{u*\Phi_\delta}(\Delta\phi) \;=\; \lim_{\delta\to 0+} \Delta T_{u*\Phi_\delta}(\phi) \\ &= \lim_{\delta\to 0+} T_{\Delta(u*\Phi_\delta)}(\phi) \;=\; \lim_{\delta\to 0+} \int_{\Omega_\delta} \Delta(u * \Phi_\delta)(x)\phi(x)\; dm(x) \\ &\leq 0\;. \end{aligned}$$

Therefore, $\Delta T_u \leq 0$ and the last statement is a consequence of Theorem 0.11.

The next two results are the main examples.

Proposition 2.8 $\Delta h_{x_0} = \kappa_n d\delta_{x_0}$.

Proof:

Consider any $\phi \in \mathcal{D}(\mathbf{R^n})$. In the following calculations the third equality is true because the integrand is in $L^1(\mathbf{R^n})$, the fourth equality is an application of Green's Formula in the set $B(x_0; r, R)$ for some R which is large enough so that ϕ and $\frac{\partial\phi}{\partial\eta}$ vanish on $S(x_0;R)$ and the sixth equality is true because $\lim_{r\to 0+} r^{n-1}h_*(r) = 0$, $\overrightarrow{grad\,\phi}$ is bounded in a neighborhood of x_0 and ϕ is continuous at x_0.

$$\begin{aligned}
\Delta T_{h_{x_0}}(\phi) &= T_{h_{x_0}}(\Delta\phi) \;=\; \int_{\mathbf{R^n}} h_{x_0}(x)\Delta\phi(x)\; dm(x) \\
&= \lim_{r\to 0+} \int_{\mathbf{R^n}\setminus\overline{B(x_0;r)}} h_{x_0}(x)\Delta\phi(x)\; dm(x) \\
&= \lim_{r\to 0+} \Big(\int_{S(x_0;r)} \frac{\partial\phi}{\partial\eta}(y) h_{x_0}(y)\; dS(y) \\
&\quad - \int_{S(x_0;r)} \phi(y)\frac{\partial h_{x_0}}{\partial\eta}(y)\; dS(y)\Big) \\
&= \lim_{r\to 0+} \Big(-r^{n-1}h_*(r) \int_{S^{n-1}} \frac{d}{dr}\,\phi(x_0+rz)\; d\sigma(z) + \kappa_n \mathcal{M}_\phi^r(x_0)\Big) \\
&= \kappa_n\phi(x_0) \\
&= \kappa_n T_{d\delta_{x_0}}(\phi)\ .
\end{aligned}$$

Therefore, $\Delta T_{h_{x_0}} = \kappa_n T_{d\delta_{x_0}}$ or, less formally,

$$\Delta h_{x_0} \;=\; \kappa_n d\delta_{x_0}\ .$$

Theorem 2.15 *Let $d\mu$ be a compactly supported non-negative Borel measure and consider the superharmonic function $U_h^{d\mu}$, the h-potential of $d\mu$. Then,*

$$\Delta U_h^{d\mu} \;=\; \kappa_n d\mu\ .$$

In case $n > 2$, the non-negative Borel measure $d\mu$ need not be compactly supported and we, only, assume that $U_h^{d\mu}(x) < +\infty$ for at least one x.

Proof:

Using the formal notation for distributions, for every $\phi \in \mathcal{D}(\mathbf{R^n})$,

$$\begin{aligned}
\Delta T_{U_h^{d\mu}}(\phi) &= \int_{\mathbf{R^n}} U_h^{d\mu}(x)\Delta\phi(x)\; dm(x) \\
&= \int_{supp(d\mu)} \int_{\mathbf{R^n}} h(x-y)\Delta\phi(x)\; dm(x)\; d\mu(y) \\
&= \int_{supp(d\mu)} T_{h_y}(\Delta\phi)\; d\mu(y) \\
&= \int_{supp(d\mu)} \Delta T_{h_y}(\phi)\; d\mu(y) \\
&= \kappa_n \int_{supp(d\mu)} T_{d\delta_y}(\phi)\; d\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&= \kappa_n \int_{supp(d\mu)} \phi(y)\ d\mu(y) \\
&= \kappa_n T_{d\mu}(\phi)\ .
\end{aligned}
$$

The use of the Theorem of Fubini, justifying the second equality, is permitted by the calculation at the beginning of the proof of Theorem 2.8 and the fifth equality is just Proposition 2.8.

In case $n > 2$, even if $supp(d\mu)$ is not compact in $\mathbf{R^n}$, the use of Fubini's Theorem is still justified, since, by the positivity of h, the compactness of $supp(\phi)$ and the local integrability of the superharmonic $U_h^{d\mu}$,

$$
\begin{aligned}
&\int_{supp(\phi)} \int_{\mathbf{R^n}} |h(x-y)|\ d\mu(y)\ |\Delta\phi(x)|\ dm(x) \\
&\le\ M \int_{supp(\phi)} U_h^{d\mu}(x)\ dm(x)\ <\ +\infty\ ,
\end{aligned}
$$

where M is a bound of $|\Delta\phi|$.

Observe that this result agrees with the fact that $U_h^{d\mu}$ is harmonic in the set $\mathbf{R^n} \setminus supp(d\mu)$.

Theorem 2.16 *Let T be a distribution in the open $\Omega \subseteq \mathbf{R^n}$. Then, T is identified with a superharmonic function in Ω if and only if $\Delta T \le 0$.*

Proof:

One direction is just Proposition 2.7.

Hence, suppose $\Delta T \le 0$ and apply Theorem 0.11 to get a non-negative Borel measure $d\mu$ in Ω so that

$$
T_{d\mu}\ =\ \frac{1}{\kappa_n}\ \Delta T\ .
$$

Consider, now, an arbitrary open G so that $\overline{G}$ is a compact subset of Ω, the restriction $d\mu_G$ of $d\mu$ in G and its h-potential $U_h^{d\mu_G}$.

If we define the distribution

$$
S_G\ =\ T - T_{U_h^{d\mu_G}}\ ,
$$

then, taking Laplacians, by Theorem 2.15, we have

$$
\Delta S_G\ =\ \Delta T - \kappa_n T_{d\mu_G}\ =\ 0
$$

as a distribution in G.

By Theorem 1.20, S_G is identified with some harmonic function in G and, hence, $T = S_G + T_{U_h^{d\mu_G}}$ is identified with a superharmonic function in G.

Now, consider an open exhaustion $\{\Omega_{(m)}\}$ of Ω and apply the previous result to $G = \Omega_{(m)}$. For each m, there is a superharmonic u_m in $\Omega_{(m)}$ so that $T = T_{u_m}$ in $\Omega_{(m)}$. By Proposition 0.10, $u_m = u_{m+1}$ in $\Omega_{(m)}$ and, hence, the u_m's define a single function u superharmonic in Ω which, for every m, coincides with u_m on $\Omega_{(m)}$.

Now, for every $\phi \in \mathcal{D}(\Omega)$ we have $\phi \in \mathcal{D}(\Omega_{(m)})$ for large enough m and, thus,

$$T(\phi) \;=\; T_{u_m}(\phi) \;=\; \int_{\Omega_{(m)}} u_m(x)\phi(x)\; dm(x) \;=\; \int_{\Omega} u(x)\phi(x)\; dm(x) \;=\; T_u(\phi)\ .$$

Hence, T is identified with u in Ω.

2.11 The theorem of F. Riesz

Theorem 2.17 *(Decomposition Theorem of F. Riesz) Suppose u is superharmonic in the open $\Omega \subseteq \mathbf{R^n}$. Then, there exists a unique non-negative Borel measure $d\mu$ in Ω so that, for every open G with $\overline{G}$ being a compact subset of Ω,*

$$u(x) \;=\; U_h^{d\mu_G}(x) + v_G(x)$$

for every $x \in G$, where $d\mu_G$ is the restriction of $d\mu$ in G and v_G is some harmonic function in G.

In the case $n > 2$, we, also, have

$$u(x) = U_h^{d\mu}(x) + v(x)$$

for all $x \in \Omega$ for some v harmonic in Ω, provided $U_h^{d\mu}(x) < +\infty$ for at least one $x \in \mathbf{R^n}$.

Proof:

Most of the work was done in the proof of the previous theorem. In fact,

$$T_u \;=\; S_G + T_{U_h^{d\mu_G}}\ ,$$

where S_G is identified with a harmonic function, say v_G, in G. I.e.

$$T_u \;=\; T_{v_G} + T_{U_h^{d\mu_G}}\ ,$$

as distributions in G.

Proposition 0.10 implies

$$u(x) \;=\; v_G(x) + U_h^{d\mu_G}(x)$$

for almost every $x \in G$ and, finally, Corollary 2.2 implies that the equality is true everywhere in Ω.

The uniqueness is proved by taking distributional Laplacians in G. By Theorem 2.15,

$$\Delta T_u \;=\; \kappa_n T_{d\mu_G}$$

in G. From this, the distribution $T_{d\mu_G}$ is uniquely determined in G, and, hence, the restriction $d\mu_G$ is, also, uniquely determined. Taking as G's the terms of any open exhaustion of Ω, we prove the uniqueness of $d\mu$ in Ω.

In case $n > 2$, we may adjust the proof of the previous Theorem.

Consider the non-negative Borel measure $d\mu$ in Ω so that

$$T_{d\mu} \ = \ \frac{1}{\kappa_n} \, \Delta T_u \ .$$

Assuming $U_h^{d\mu}(x) < +\infty$ for at least one x, Theorem 2.15 implies that the distribution $S = T_u - T_{U_h^{d\mu}}$ satisfies $\Delta S = 0$ in Ω and, by Theorem 1.20, S is identified with a harmonic v in Ω.

From this, as before,

$$u(x) \ = \ v(x) + U_h^{d\mu}(x)$$

for almost every x and, hence, for every $x \in \Omega$.

Or, in another way, we may start from the restricted result of the first part applied to the terms of an open exhaustion $\{\Omega_{(m)}\}$ of Ω,

$$u(x) \ = \ U_h^{d\mu_{\Omega_{(m)}}}(x) + v_{\Omega_{(m)}}(x) \ , \qquad x \in \Omega_{(m)} \ .$$

For every $x \in \Omega$, because of the positivity of h,

$$U_h^{d\mu_{\Omega_{(m)}}}(x) \uparrow U_h^{d\mu}(x)$$

and the limit is finite for almost every $x \in \mathbf{R^n}$. Therefore, $\{v_{\Omega_{(m)}}\}$ is (eventually) a decreasing sequence of harmonic functions in every fixed $\Omega_{(k)}$ with a limit v which is finite almost everywhere in $\Omega_{(k)}$. Therefore, v is harmonic in $\Omega_{(k)}$ and

$$u(x) \ = \ U_h^{d\mu}(x) + v(x)$$

everywhere in $\Omega_{(k)}$ and, since k is arbitrary, everywhere in Ω.

2.12 Derivatives of superharmonic functions

Theorem 2.18 *Let u be superharmonic in the open $\Omega \subseteq \mathbf{R^n}$. Then, u is absolutely continuous on almost every line parallel to the principal x_j-axes, $1 \leq j \leq n$, it has partial derivatives at almost every point of Ω and these partial derivatives are locally integrable in Ω.*

If the distributional Laplacian of u is a (non-negative) function $f \in C^k(\Omega)$, then u is in $C^{k+1}(\Omega)$.

Proof:

This is a trivial application of the Representation Theorem of F. Riesz and of Propositions 2.3 and 2.4.

Chapter 3

The Problem of Dirichlet

3.1 The generalized solution

Let's remember that the boundary and the closure of subsets of $\mathbf{R^n}$ is taken relative to $\overline{\mathbf{R^n}}$.

Definition 3.1 *Let Ω be an open subset of $\mathbf{R^n}$ and f any extended-real-valued function defined in $\partial\Omega$.*

We denote by Φ_f^Ω the family of all functions u defined in Ω with the properties:

1. *in each connected component of Ω either u is superharmonic or u is identically $+\infty$,*
2. *u is bounded from below in Ω and*
3. $\liminf_{\Omega\ni x\to y} u(x) \geq f(y)$ *for all $y \in \partial\Omega$.*

The lower envelope of the family Φ_f^Ω is denoted by $\overline{H}_f^\Omega$.

In a dual manner, Ψ_f^Ω denotes the family of all v defined in Ω such that:

1. *in each conected component of Ω either v is subharmonic or v is identically $-\infty$,*
2. *v is bounded from above in Ω and*
3. $\limsup_{\Omega\ni x\to y} v(x) \leq f(y)$ *for all $y \in \partial\Omega$.*

The upper envelope of the family Ψ_f^Ω is denoted by $\underline{H}_f^\Omega$.

Both families are non-empty, since $+\infty \in \Phi_f^\Omega$ and $-\infty \in \Psi_f^\Omega$.

Comment It is easy to see that if G is any connected component of the open set $\Omega \subseteq \mathbf{R^n}$, then the restriction of $\overline{H}_f^\Omega$ in G coincides with $\overline{H}_f^G$ and the restriction of $\underline{H}_f^\Omega$ in G coincides with $\underline{H}_f^G$.

This allows us, in many problems, to reduce the study of these functions to the case of the set Ω being connected.

Proposition 3.1 *In each connected component of* Ω, $\overline{H}_f^\Omega$ *is either harmonic or identically* $+\infty$ *or identically* $-\infty$.

The same is true for $\underline{H}_f^\Omega$.

Proof:

Consider the family $\mathcal{V}$ of the *subharmonic restrictions* of the functions in Ψ_f^Ω in any particular connected component G of Ω.

In case all functions in Ψ_f^Ω are identically $-\infty$ in G, then $\mathcal{V}$ is empty and, obviously, $\underline{H}_f^\Omega = -\infty$ in G.

In case there is at least one function in Ψ_f^Ω with subharmonic restriction to G, then $\mathcal{V}$ is non-empty and it is almost obvious that $\mathcal{V}$ satisfies the assumptions in Theorem 2.13 for the connected open G. Therefore, the upper envelope of $\mathcal{V}$ in G, which coincides with the restriction of $\underline{H}_f^\Omega$ in G, is either harmonic in G or identically $+\infty$ in G.

Proposition 3.2 $\overline{H}_f^\Omega \geq \underline{H}_f^\Omega$ *everywhere in* Ω.

Proof:

Fix an arbitrary connected component G of Ω.

In case either $\overline{H}_f^\Omega = +\infty$ everywhere in G or $\underline{H}_f^\Omega = -\infty$ everywhere in G, then the result is obvious. Hence, assume that there is some $u \in \Phi_f^\Omega$ which is not $+\infty$ everywhere in G and some $v \in \Psi_f^\Omega$ which is not $-\infty$ everywhere in G.

Then, $u - v$ is superharmonic in G and

$$\liminf_{G \ni x \to y} \big(u(x) - v(x)\big) \geq 0$$

for all $y \in \partial G$.

This last inequality is obvious for every y for which $f(y)$ is real. In case $f(y) = +\infty$, then the boundedness from above of v is used and, if $f(y) = -\infty$, then the boundedness from below of u has to be used.

From Theorem 2.1, we find that $u \geq v$ in G. Since u is arbitrary in Φ_f^Ω and v is arbitrary in Ψ_f^Ω, we conclude that $\overline{H}_f^\Omega \geq \underline{H}_f^\Omega$ in G.

Definition 3.2 *Let the extended-real-valued* f *be defined in* $\partial\Omega$, *where* $\Omega \subseteq \mathbf{R^n}$ *is open.*

1. *If* $\overline{H}_f^\Omega = \underline{H}_f^\Omega$ *everywhere in* Ω, *then this common function is denoted by*

$$H_f^\Omega \ .$$

2. f *is called* ***resolutive*** *with respect to* Ω *if*

$$\overline{H}_f^\Omega \;=\; \underline{H}_f^\Omega$$

everywhere in Ω and if this common function is not identically $+\infty$ or $-\infty$ in any connected component of Ω.

This common function H_f^Ω is harmonic in Ω and is called the ***generalized solution to the Problem of Dirichlet in Ω with boundary function f.***

To motivate this definition, suppose that f is real-valued in $\partial\Omega$ and that the Problem of Dirichlet in Ω with boundary function f *is* solvable. Therefore, there exists some u harmonic in Ω so that $\lim_{\Omega\ni x\to y} u(x) = f(y)$ for all $y \in \partial\Omega$.

The first thing to observe is that f is, then, continuous on $\partial\Omega$. In fact, take any sequence $\{y_m\}$ in $\partial\Omega$ with $y_m \to y$. Then, there is another $\{x_m\}$ in Ω so that $d_S(x_m, y_m) < \frac{1}{m}$ and $|u(x_m) - f(y_m)| < \frac{1}{m}$. The first inequality gives $d_S(x_m, y) \to 0$ and, hence, $u(x_m) \to f(y)$. Thus, the second inequality gives $f(y_m) \to f(y)$.

Since f is continuous in $\partial\Omega$, it is bounded in $\partial\Omega$ and Theorem 1.1 implies that u is bounded in Ω.

Therefore, $u \in \Phi_f^\Omega \cap \Psi_f^\Omega$ and, hence,

$$u \;\leq\; \underline{H}_f^\Omega \;\leq\; \overline{H}_f^\Omega \;\leq\; u\ .$$

This implies

$$u \;=\; H_f^\Omega$$

in Ω.

Summarizing: *if the Problem of Dirichlet in Ω with the real-valued boundary function f is solvable, then f is continuous in $\partial\Omega$ and resolutive with respect to Ω and the solution to the Problem of Dirichlet coincides with the generalized solution.*

Therefore, to solve the Problem of Dirichlet, we must suppose that the given boundary function f is continuous in $\partial\Omega$ and, then, prove

1. that f is resolutive and

2. that $\lim_{\Omega\ni x\to y} H_f^\Omega(x) = f(y)$ for all $y \in \partial\Omega$.

In the case of any bounded open Ω, item 1 will be answered completely by a theorem of Wiener, which we shall prove in a while, and says that every continuous f is resolutive.

On the other hand, even if f is continuous, the Problem of Dirichlet may not be solvable. Here is an instructive example.

Example

Consider $\Omega = \{x \in \mathbf{R^n} : 0 < |x| < 1\}$ and

$$f(y) \;=\; \begin{cases} 0\ , & \text{if } |y| = 1 \\ 1\ , & \text{if } y = 0\ . \end{cases}$$

Take $u_m(x) = \frac{1}{m}\big(h(x) - h_*(1)\big)$ for all $x \in \Omega$. Then $u_m \in \Phi_f^\Omega$ and, hence,

$$\overline{H}_f^\Omega(x) \;\le\; \lim_{m\to+\infty} u_m(x) \;=\; 0 \;.$$

On the other hand, $0 \in \Psi_f^\Omega$, implying

$$0 \le \underline{H}_f^\Omega(x) \;.$$

Thus, f is resolutive and continuous in $\partial\Omega$, with $H_f^\Omega = 0$ everywhere in Ω, but

$$\lim_{\Omega\ni x\to 0} H_f^\Omega(x) \;=\; 0 \;\ne\; f(0) \;=\; 1 \;.$$

3.2 Properties of the generalized solution

In all statements that follow, the boundary functions are all extended-real-valued.

1. $\overline{H}_{f+c}^\Omega = \overline{H}_f^\Omega + c$ *and* $\underline{H}_{f+c}^\Omega = \underline{H}_f^\Omega + c$ *for all* $c \in \mathbf{R}$.
2. $\overline{H}_{\lambda f}^\Omega = \lambda \overline{H}_f^\Omega$ *and* $\underline{H}_{\lambda f}^\Omega = \lambda \underline{H}_f^\Omega$ *for* $\lambda > 0$ *and* $\overline{H}_{\lambda f}^\Omega = \lambda \underline{H}_f^\Omega$ *for* $\lambda < 0$. *Under the convention* $0(\pm\infty) = 0$, *these formulas hold in case* $\lambda = 0$ *also.*
3. *If* $f_1 \le f_2$, *then* $\overline{H}_{f_1}^\Omega \le \overline{H}_{f_2}^\Omega$ *and* $\underline{H}_{f_1}^\Omega \le \underline{H}_{f_2}^\Omega$.
4. $\inf_{\partial\Omega} f \le \underline{H}_f^\Omega \le \overline{H}_f^\Omega \le \sup_{\partial\Omega} f$.

   All these properties are trivial to prove.

   Convention *When we add boundary functions f and g, we assign any value, whatsoever, to the indeterminate forms* $(\pm\infty) + (\mp\infty)$.

   But, when we add $\overline{H}_f^\Omega$ *and* $\overline{H}_g^\Omega$, *we assign the value* $+\infty$ *to these forms and, when we add* $\underline{H}_f^\Omega$ *and* $\underline{H}_g^\Omega$, *we assign the value* $-\infty$ *to them.*

   In any case, we assign the value 0 *to* $0(\pm\infty)$.

5. $\overline{H}_{f+g}^\Omega \le \overline{H}_f^\Omega + \overline{H}_g^\Omega$ *and* $\underline{H}_f^\Omega + \underline{H}_g^\Omega \le \underline{H}_{f+g}^\Omega$.

   Working in each connected component of Ω separately, it is enough to assume that Ω is connected. Observe that, by our convention, the first inequality is trivial in case at least one of $\overline{H}_f^\Omega$ and $\overline{H}_g^\Omega$ is $+\infty$ identically in Ω.

   Hence, take any $u \in \Phi_f^\Omega$ and any $v \in \Phi_g^\Omega$ which are not identically $+\infty$. Then, $u + v \in \Phi_{f+g}^\Omega$ and, thus, $\overline{H}_{f+g}^\Omega \le u + v$. This is enough to conclude the proof of the first inequality and the second is proved similarly.

6. *If both f and g are resolutive with respect to Ω, then $cf + dg$ is resolutive and $H^\Omega_{cf+dg} = cH^\Omega_f + dH^\Omega_g$ everywhere in Ω, for all $c, d \in \mathbf{R}$.*

   This is a simple combination of properties 2 and 5.

7. *Let $f_m \uparrow f$ and $\overline{H}^\Omega_{f_1}$ be not identically $-\infty$ in a connected component G of Ω. Then $\overline{H}^\Omega_{f_m} \uparrow \overline{H}^\Omega_f$ in G.*

   The dual result is, also, true.

   It is enough to assume that $\Omega = G$ is connected and that, for all m, $\overline{H}^\Omega_{f_m}$ is not identically $+\infty$ in Ω. Therefore, we assume that $\overline{H}^\Omega_{f_m}$ is harmonic in Ω, for all m.

   Fix $x_0 \in \Omega$ and consider $u_m \in \Phi^\Omega_{f_m}$, superharmonic in Ω with

$$u_m(x_0) \ \le \ \overline{H}^\Omega_{f_m}(x_0) + \frac{\epsilon}{2^m} \ .$$

   Now, $u_m - \overline{H}^\Omega_{f_m}$ is superharmonic and non-negative in Ω and, by the third property of superharmonic functions, the series

$$\sum_{m=1}^{+\infty} \big(u_m - \overline{H}^\Omega_{f_m}\big)$$

   either converges to a superharmonic function in Ω or it diverges to $+\infty$ everywhere in Ω. But its value at x_0 is $\le \epsilon$ and, thus, it is superharmonic in Ω.

   By Theorem 1.16, the $\lim_{m\to+\infty} \overline{H}^\Omega_{f_m}$ is either $+\infty$ identically in Ω or it is harmonic in Ω. In the first case, what we want to prove is clear and we assume that this limit is harmonic in Ω.

   Now, the function

$$w \ = \ \lim_{m\to+\infty} \overline{H}^\Omega_{f_m} + \sum_{m=1}^{+\infty} \big(u_m - \overline{H}^\Omega_{f_m}\big)$$

   is superharmonic in Ω. Furthermore, for every m,

$$w \ \ge \ \overline{H}^\Omega_{f_m} + \big(u_m - \overline{H}^\Omega_{f_m}\big) \ = \ u_m \ .$$

   Therefore, w is bounded from below in Ω and, also,

$$\liminf_{\Omega\ni x\to y} w(x) \ \ge \ \liminf_{\Omega\ni x\to y} u_m(x) \ \ge \ f_m(y)$$

   for all $y \in \partial\Omega$ and, since m is arbitrary,

$$\liminf_{\Omega\ni x\to y} w(x) \ \ge \ f(y)$$

for all $y \in \partial\Omega$.

Therefore, $w \in \Phi_f^\Omega$, implying that $w \geq \overline{H}_f^\Omega$. Then

$$\overline{H}_f^\Omega(x_0) \ \leq \ w(x_0) \ \leq \ \lim_{m\to+\infty} \overline{H}_{f_m}^\Omega(x_0) + \epsilon$$

and, since ϵ is arbitrary,

$$\overline{H}_f^\Omega(x_0) \ \leq \ \lim_{m\to+\infty} \overline{H}_{f_m}^\Omega(x_0) \ .$$

The inequality $\lim_{m\to+\infty} \overline{H}_{f_m}^\Omega \leq \overline{H}_f^\Omega$ is obvious and , by Theorem 1.1,

$$\lim_{m\to+\infty} \overline{H}_{f_m}^\Omega = \overline{H}_f^\Omega$$

in Ω.

8. *Let G be an open subset of Ω and f a boundary function in $\partial\Omega$. If we consider*

$$F \ = \ \begin{cases} f, & \text{on } \partial G \cap \partial\Omega, \\ \overline{H}_f^\Omega, & \text{on } \partial G \cap \Omega, \end{cases}$$

then $\overline{H}_F^G = \overline{H}_f^\Omega$ everywhere in G.

There is a dual result for $\underline{H}$.

Since every connected component of G is contained in one of the connected components of Ω, it is enough to assume that both G and Ω are connected. If $u \in \Phi_f^\Omega$, then, in case $y \in \partial G \cap \partial\Omega$,

$$\liminf_{G\ni x\to y} u(x) \ \geq \ \liminf_{\Omega\ni x\to y} u(x) \ \geq \ f(y) \ = \ F(y) \ ,$$

while, in case $y \in \partial G \cap \Omega$,

$$\liminf_{G\ni x\to y} u(x) \ \geq \ u(y) \ \geq \ \overline{H}_f^\Omega(y) \ = \ F(y) \ .$$

Hence, $u \in \Phi_F^G$, implying $u \geq \overline{H}_F^G$ in G and, finally,

$$\overline{H}_f^\Omega \ \geq \ \overline{H}_F^G$$

in G.

The opposite inequality is clear in case $\overline{H}_f^\Omega = -\infty$ identically in Ω.

In case $\overline{H}_f^\Omega = +\infty$ identically in Ω, then $F(y) = +\infty$ for all $y \in \partial G \cap \Omega$. Now, taking any $u \in \Phi_F^G$, we have that $\liminf_{G\ni x\to y} u(x) = +\infty$ for all $y \in \partial G \cap \Omega$.

Extending u as identically $+\infty$ on $\Omega \setminus G$, we either get a superharmonic u in Ω (if u is superharmonic in G) or the function $+\infty$ in Ω (if $u = +\infty$ identically in G). But, the first alternative is impossible, since this would imply the existence of a superharmonic function in Φ_f^Ω.

Therefore, $u = +\infty$ identically in G and we get that

$$\overline{H}_F^G \;=\; +\infty \;\geq\; \overline{H}_f^\Omega$$

in G.

Finally, suppose that $\overline{H}_f^\Omega$ is harmonic in Ω.

Take any $u \in \Phi_F^G$ and define

$$V \;=\; \begin{cases} \min\bigl(u, \overline{H}_f^\Omega\bigr), & \text{in } G, \\ \overline{H}_f^\Omega, & \text{in } \Omega \setminus G \end{cases}$$

which is superharmonic in Ω.

Take arbitrary $v \in \Phi_f^\Omega$ and consider the function $V + v - \overline{H}_f^\Omega$ which is superharmonic and bounded from below in Ω.

In case $y \in \partial\Omega \setminus \partial G$,

$$\liminf_{\Omega \ni x \to y} \bigl(V(x) + v(x) - \overline{H}_f^\Omega(x)\bigr) \;=\; \liminf_{\Omega \ni x \to y} v(x) \;\geq\; f(y)\ .$$

In case $y \in \partial\Omega \cap \partial G$,

$$\liminf_{\Omega \setminus G \ni x \to y} \bigl(V(x) + v(x) - \overline{H}_f^\Omega(x)\bigr) \;=\; \liminf_{\Omega \setminus G \ni x \to y} v(x) \;\geq\; f(y)$$

and

$$\liminf_{G \ni x \to y} \bigl(V(x) + v(x) - \overline{H}_f^\Omega(x)\bigr) \;\geq\; f(y)\ .$$

Hence, $V + v - \overline{H}_f^\Omega \geq \overline{H}_f^\Omega$ in Ω.

This implies that $V \geq \overline{H}_f^\Omega$ in Ω, which gives $u \geq \overline{H}_f^\Omega$ in G and, finally,

$$\overline{H}_F^G \;\geq\; \overline{H}_f^\Omega$$

in G.

9. *Let G be an open subset of Ω and f a boundary function on $\partial\Omega$. If f is resolutive with respect to Ω, then the function*

$$F \;=\; \begin{cases} f, & \text{on } \partial G \cap \partial\Omega, \\ H_f^\Omega, & \text{on } \partial G \cap \Omega, \end{cases}$$

is resolutive with respect to G and $H_F^G = H_f^\Omega$ identically in G.

10. *Let each f_m be resolutive and $f_m \to f$ uniformly in $\partial\Omega$.*
Then f is resolutive and $H^\Omega_{f_m} \to H^\Omega_f$ uniformly in Ω.
Fix $\epsilon > 0$. For large m, $f_m - \epsilon \le f \le f_m + \epsilon$ everywhere in $\partial\Omega$ and, hence, $H^\Omega_{f_m} - \epsilon \le \underline{H}^\Omega_f \le \overline{H}^\Omega_f \le H^\Omega_{f_m} + \epsilon$ in Ω. Thus, $0 \le \overline{H}^\Omega_f - \underline{H}^\Omega_f \le 2\epsilon$ and, since ϵ is arbitrary,
$$\underline{H}^\Omega_f \;=\; \overline{H}^\Omega_f$$
in Ω and f is resolutive.
Also, $f - \epsilon \le f_m \le f + \epsilon$ in $\partial\Omega$ implies $H^\Omega_f - \epsilon \le H^\Omega_{f_m} \le H^\Omega_f + \epsilon$ in Ω, from which we get the uniform convergence $H^\Omega_{f_m} \to H^\Omega_f$ in Ω.

3.3 Wiener's Theorem

Lemma 3.1 *Let Ω be a bounded open subset of $\mathbf{R^n}$. Then, every real-valued f continuous in $\partial\Omega$ can be uniformly approximated in $\partial\Omega$ by the difference of the restrictions in $\partial\Omega$ of two functions continuous in $\overline{\Omega}$ and superharmonic in Ω.*

Proof:
Since $\partial\Omega$ is a compact subset of $\mathbf{R^n}$, by the Stone-Weierstrass Theorem, we can approximate f uniformly in $\partial\Omega$ by a real-valued polynomial $P(x_1, \ldots, x_n)$. Take a constant $M > 0$ so that $\Delta P(x) \le M$ for all $x \in \overline{\Omega}$.

Now, the difference of the two functions, $P(x_1, \ldots, x_n) - \frac{M}{2}x_1^2$ and $-\frac{M}{2}x_1^2$, approximates f uniformly on $\partial\Omega$, while both of them are superharmonic in Ω.

Theorem 3.1 *(N. Wiener) Let Ω be a bounded open subset of $\mathbf{R^n}$. Then, every real-valued f continuous in $\partial\Omega$ is resolutive with respect to Ω.*

Proof:
Let the real-valued F be continuous in $\overline{\Omega}$ and superharmonic in Ω.

Then, obviously, $F \in \Phi^\Omega_F$ and, thus, $F \ge \overline{H}^\Omega_F$ in Ω. By property 4 in section 3.2, $\overline{H}^\Omega_F$ is bounded and harmonic in Ω and, for every $y \in \partial\Omega$,
$$\limsup_{\Omega \ni x \to y} \overline{H}^\Omega_F \;\le\; \limsup_{\Omega \ni x \to y} F(x) \;=\; F(y) \;,$$
implying that $\overline{H}^\Omega_F \in \Psi^\Omega_F$. Hence,
$$\overline{H}^\Omega_F \;=\; \underline{H}^\Omega_F$$
and the restriction of F in $\partial\Omega$ is resolutive with respect to Ω.

By Lemma 3.1, there is a sequence $\{F_m - G_m\}$ so that all F_m and G_m are continuous in $\overline{\Omega}$, superharmonic in Ω and the restrictions in $\partial\Omega$ of $F_m - G_m$ converge to f uniformly in $\partial\Omega$.

From the first part and property 6, the restrictions in $\partial\Omega$ of all $F_m - G_m$ are resolutive with respect to Ω. Therefore, from property 10, f is resolutive with respect to Ω.

Example

Let $\Omega = \mathbf{R^2} \setminus \{0\}$ and $f(0) = 0$, $f(\infty) = 1$.

If $u \in \Phi_f^\Omega$, then, for any $\epsilon > 0$ we have that, for all large enough M and N,

$$u(x) \ \geq \ \frac{\log|x| + \log N}{\log M + \log N} - \epsilon$$

for every x with $\frac{1}{N} \leq |x| \leq M$. Let $N \to +\infty$ and then $M \to +\infty$ and get that $u(x) \geq 1 - \epsilon$ for all $x \neq 0, \infty$. Since ϵ is arbitrary, we find $u \geq 1$ and, thus, $\overline{H}_f^\Omega = 1$ in Ω.

If $v \in \Psi_f^\Omega$, then, for any $\epsilon > 0$ we have that, for all large enough M and N,

$$v(x) \ \leq \ \frac{\log|x| + \log N}{\log M + \log N} + \epsilon$$

for every x with $\frac{1}{N} \leq |x| \leq M$. Let $M \to +\infty$ and then $N \to +\infty$ and get that $v(x) \leq \epsilon$ for all $x \neq 0, \infty$. Since ϵ is arbitrary, we find $u \leq 0$ and, thus, $\underline{H}_f^\Omega = 0$ in Ω.

Therefore, f is not resolutive with respect to Ω and Theorem 3.1 cannot be extended to hold for unbounded open sets in $\mathbf{R^2}$.

3.4 Harmonic measure

Let Ω be a bounded open set and $x_0 \in \Omega$. By Theorem 3.1,

$$C(\partial\Omega) \ \ni \ f \ \mapsto \ H_f^\Omega(x_0) \ \in \ \mathbf{C}$$

defines a (complex-)linear functional on $C(\partial\Omega)$.

This is, at first, defined for real-valued f, but it is, trivially, extended to complex-valued f by $H_f^\Omega(x_0) = H_{\Re f}^\Omega(x_0) + iH_{\Im f}^\Omega(x_0)$.

It is easy to prove that

$$H_1^\Omega(x_0) \ = \ 1 \ ,$$
$$H_f^\Omega(x_0) \ \geq \ 0$$

for all $f \in C(\partial\Omega)$ with $f \geq 0$ in $\partial\Omega$ and

$$|H_f^\Omega(x_0)| \ \leq \ \|f\|_\infty$$

for all $f \in C(\partial\Omega)$. In fact, the last inequality is straightforward for real-valued f, while, for complex-valued f, we take θ so that $H_f^\Omega(x_0) = e^{i\theta}|H_f^\Omega(x_0)|$ and write

$$\begin{aligned} |H_f^\Omega(x_0)| \ &= \ e^{-i\theta}H_f^\Omega(x_0) \ = \ H_{e^{-i\theta}f}^\Omega(x_0) \\ &= \ H_{\Re(e^{-i\theta}f)}^\Omega(x_0) \ \leq \ \|\Re(e^{-i\theta}f)\|_\infty \ \leq \ \|f\|_\infty \ . \end{aligned}$$

Therefore, $H^{\Omega}_{(\cdot)}(x_0)$ is a non-negative bounded linear functional on $C(\partial\Omega)$ with norm 1 and, from Theorem 0.10, there exists a unique non-negative Borel measure $d\mu^{\Omega}_{x_0}$ supported in $\partial\Omega$ so that

$$d\mu^{\Omega}_{x_0}(\partial\Omega) \;=\; 1$$

and

$$H^{\Omega}_f(x_0) \;=\; \int_{\partial\Omega} f(y)\; d\mu^{\Omega}_{x_0}(y)$$

for all $f \in C(\partial\Omega)$.

The fact that $d\mu^{\Omega}_{x_0}$ is a non-negative Borel measure with total mass equal to 1 is described by calling it a Borel **probability measure**.

By the process of Caratheodory, the measure $d\mu^{\Omega}_{x_0}$ can be considered as uniquely extended on the σ-algebra of its measurable sets. This σ-algebra is larger than $\mathcal{B}(\partial\Omega)$ and a set $A \subseteq \partial\Omega$ belongs to this σ-algebra if and only if $A = B \cup N$ for some Borel set $B \subseteq \partial\Omega$ and some N with $d\mu^{\Omega}_{x_0}(N) = 0$.

Also, $d\mu^{\Omega}_{x_0}$ is complete on the σ-algebra of its measurable sets.

Definition 3.3 *Let $\Omega \subseteq \mathbf{R^n}$ be a bounded open set and $x_0 \in \Omega$.*

The complete probability measure $d\mu^{\Omega}_{x_0}$ in $\partial\Omega$, constructed above, whose σ-algebra of measurable sets includes all Borel sets in $\partial\Omega$ and satisfies

$$H^{\Omega}_f(x_0) \;=\; \int_{\partial\Omega} f(y)\; d\mu^{\Omega}_{x_0}(y)$$

*for all $f \in C(\partial\Omega)$ is called the **harmonic measure in $\partial\Omega$ with respect to Ω and** x_0.*

Lemma 3.2 *Suppose Ω is a bounded open subset of $\mathbf{R^n}$ and consider f lower-semicontinuous in $\partial\Omega$. Then,*

1. *For every $x_0 \in \Omega$,*

$$H^{\Omega}_f(x_0) \;=\; \int_{\partial\Omega} f(y)\; d\mu^{\Omega}_{x_0}(y)\ .$$

2. *f is resolutive with respect to Ω if and only if it is $d\mu^{\Omega}_{x_0}$-integrable for at least one x_0 in every connected component of Ω.*

   The same is true, if f is upper-semicontinuous.

Proof:

Consider a sequence $\{f_m\}$ of functions continuous in $\partial\Omega$ with $f_m(y) \uparrow f(y)$ for every $y \in \partial\Omega$.

By Theorem 3.1,

$$\overline{H}^{\Omega}_{f_m}(x_0) \;=\; \underline{H}^{\Omega}_{f_m}(x_0) \le \underline{H}^{\Omega}_f(x_0)\ .$$

By property 7 of H, the continuity of f_m and the Monotone Convergence Theorem,

$$\overline{H}_f^\Omega(x_0) = \lim_{m\to+\infty} \overline{H}_{f_m}^\Omega(x_0) = \lim_{m\to+\infty} \int_{\partial\Omega} f_m(y)\, d\mu_{x_0}^\Omega(y) = \int_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \;.$$

From these two relations we get the first result.

Now, since f is bounded from below in $\partial\Omega$, we have that, for every x_0, $\int_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) < +\infty$ if and only if f is $d\mu_{x_0}^\Omega$-integrable.

If f is resolutive with respect to Ω, then all integrals are finite and f is $d\mu_{x_0}^\Omega$-integrable for all $x_0 \in \Omega$. On the other hand, if f is $d\mu_{x_0}^\Omega$-integrable for at least one x_0 in some connected component, then $H_f^\Omega(x_0)$ is finite and, hence, H_f^Ω is harmonic in the same component.

Theorem 3.2 *Suppose Ω is a bounded open subset of $\mathbf{R^n}$. For every extended-real-valued function f defined in $\partial\Omega$,*

1. $$\overline{H}_f^\Omega(x_0) = \overline{\int}_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \;, \qquad \underline{H}_f^\Omega(x_0) = \underline{\int}_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \;,$$

   for all $x_0 \in \Omega$,

2. *f is resolutive with respect to Ω if and only if it is $d\mu_{x_0}^\Omega$-integrable for at least one x_0 in every connected component of Ω and*

3. *f is resolutive with respect to Ω if and only if it is $d\mu_{x_0}^\Omega$-integrable for every x_0 in Ω and, in this case,*

   $$H_f^\Omega(x_0) = \int_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \;,$$

   for all $x_0 \in \Omega$.

In particular, an $E \subseteq \partial\Omega$ is $d\mu_{x_0}^\Omega$-measurable for every x_0 in Ω if and only if χ_E is resolutive with respect to Ω and, in this case,

$$H_{\chi_E}^\Omega(x_0) = d\mu_{x_0}^\Omega(E)$$

for every x_0 in Ω.

Proof:
1. It is enough to prove the first equality and we, first, see that, by Lemma 3.2,

$$\overline{\int}_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) = \inf \int_{\partial\Omega} \phi(y)\, d\mu_{x_0}^\Omega(y) = \inf\; H_\phi^\Omega(x_0) \geq \overline{H}_f^\Omega(x_0) \;,$$

where the infima are taken over all lower-semicontinuous ϕ in $\partial\Omega$ with $\phi \geq f$ everywhere in $\partial\Omega$.

The opposite inequality is obvious if $\overline{H}_f^\Omega(x_0) = +\infty$ and, thus, assume that $\overline{H}_f^\Omega(x_0) < +\infty$. Now, take arbitrary $\lambda > \overline{H}_f^\Omega(x_0)$ and $u \in \Phi_f^\Omega$ so that

$$u(x_0) \ \le \ \lambda \ .$$

The function defined by $\phi(y) = \liminf_{\Omega \ni x \to y} u(x)$ for all $y \in \partial\Omega$ is lower-semicontinuous in $\partial\Omega$ and satisfies $\phi \ge f$ there. Hence,

$$\overline{\int}_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \ \le \ \int_{\partial\Omega} \phi(y)\, d\mu_{x_0}^\Omega(y) \ = \ H_\phi^\Omega(x_0) \ \le \ u(x_0) \ \le \ \lambda$$

and, since λ is arbitrary, we get

$$\overline{\int}_{\partial\Omega} f(y)\, d\mu_{x_0}^\Omega(y) \ \le \ \overline{H}_f^\Omega(x_0) \ .$$

2 and 3. It is obvious, from 1, that if f is resolutive with respect to Ω, then it is $d\mu_{x_0}^\Omega$-integrable for every $x_0 \in \Omega$.

If f is $d\mu_{x_0}^\Omega$-integrable for some x_0 in Ω, then, from 1, the two functions, $\underline{H}_f^\Omega$ and $\overline{H}_f^\Omega$, are harmonic in the connected component of Ω which contains x_0 and, by the Maximum-Minimum Principle, they are identically equal in the same component.

Theorem 3.3 *Suppose Ω is a bounded open subset of $\mathbf{R^n}$ and x_1, x_2 are in the same connected component of Ω. Then*

1. *$d\mu_{x_1}^\Omega$ and $d\mu_{x_2}^\Omega$ have the same zero-sets.*
2. *$d\mu_{x_1}^\Omega$-measurable sets and functions are the same as the $d\mu_{x_2}^\Omega$-measurable sets and functions.*
3. *$L^1(\partial\Omega, d\mu_{x_1}^\Omega) = L^1(\partial\Omega, d\mu_{x_2}^\Omega)$ and the norms in these two spaces are equivalent.*

Proof:
1. Let $d\mu_{x_1}^\Omega(N) = 0$. From Theorem 3.2,

$$0 \ = \ d\mu_{x_1}^\Omega(N) \ = \ \int_{\partial\Omega} \chi_N(y)\, d\mu_{x_1}^\Omega(y) \ = \ H_{\chi_N}^\Omega(x_1)$$

and, from the Maximum-Minimum Principle, $H_{\chi_N}^\Omega = 0$ identically in the connected component of Ω containing x_1. Now, the same set of equalities for x_2 instead of x_1 give that $d\mu_{x_2}^\Omega(N) = 0$.
2. If E is $d\mu_{x_1}^\Omega$-measurable, then $E = B \cup N$, for some Borel set B and some N with $d\mu_{x_1}^\Omega(N) = 0$. From part 1, we get that $d\mu_{x_2}^\Omega(N) = 0$ and, hence, E is $d\mu_{x_2}^\Omega$-measurable.
3. If f is $d\mu_{x_1}^\Omega$-integrable, then

$$H_{|f|}^\Omega(x_1) \ = \ \int_{\partial\Omega} |f(y)|\, d\mu_{x_1}^\Omega(y) \ < \ +\infty \ .$$

Therefore, $H^{\Omega}_{|f|}$ is harmonic in the connected component of Ω containing x_1 and, then

$$\int_{\partial\Omega} |f(y)|\, d\mu^{\Omega}_{x_2}(y) \;=\; H^{\Omega}_{|f|}(x_2) \;<\; +\infty$$

implies that f is $d\mu^{\Omega}_{x_2}$-integrable.

If f is $d\mu^{\Omega}_{x}$-integrable for some $x \in \Omega$ and, hence, for all x in the same connected component G of Ω, then the function

$$\int_{\partial\Omega} |f(y)|\, d\mu^{\Omega}_{x}(y) \;=\; H^{\Omega}_{|f|}(x) \;, \qquad x \in G \;,$$

is harmonic and non-negative in G. From Harnack's Inequalities we have that for every $x_1, x_2 \in G$ there is a constant C, depending only on these two points and on G, so that

$$\frac{1}{C}\int_{\partial\Omega} |f(y)|\, d\mu^{\Omega}_{x_2}(y) \;\le\; \int_{\partial\Omega} |f(y)|\, d\mu^{\Omega}_{x_1}(y) \;\le\; C\int_{\partial\Omega} |f(y)|\, d\mu^{\Omega}_{x_2}(y)$$

for all f.

Proposition 3.3 *Let Ω be a bounded open subset of $\mathbf{R^n}$.*

If G is one of the connected components of Ω and $x_0 \in G$, then,

$$d\mu^{\Omega}_{x_0}(\partial\Omega \setminus \partial G) \;=\; 0 \;.$$

Hence, the harmonic measure with respect to Ω and x_0 is supported in the boundary of the connected component of Ω which contains x_0.

Proof:

In fact, consider the function

$$u(x) \;=\; \begin{cases} 0 \;, & \text{if } x \in G \\ 1 \;, & \text{if } x \in \Omega \setminus G \;. \end{cases}$$

u is harmonic in Ω and $\liminf_{\Omega \ni x \to y} u(x) \ge \chi_{\partial\Omega\setminus\partial G}(y)$ for all $y \in \partial\Omega$.

Therefore,

$$0 \;\le\; d\mu^{\Omega}_{x_0}(\partial\Omega \setminus \partial G) \;\le\; \overline{H}^{\Omega}_{\chi_{\partial\Omega\setminus\partial G}}(x_0) \;\le\; u(x_0) \;=\; 0 \;.$$

Example

If f is defined and continuous on the sphere $S(x_0; R)$, then $P_f(\cdot\,, x_0; R)$ is the solution to the Problem of Dirichlet in $B(x_0; R)$ with boundary function f. Hence,

$$H^{B(x_0;R)}_f(x) \;=\; \int_{S(x_0;R)} f(y) P(y; x, x_0, R)\, dS(y)$$

for all $x \in B(x_0; R)$.

From the definition of harmonic measure,

$$d\mu_x^{B(x_0;R)} \;=\; P(\cdot\;; x, x_0, R)\; dS\;.$$

This means that $d\mu_x^{B(x_0;R)}$ *and susface measure on* $S(x_0;R)$ *are mutually absolutely continuous and the density-function (Radon-Nikodym derivative) of* $d\mu_x^{B(x_0;R)}$ *with respect to* dS *is exactly the Poisson kernel* $P(\cdot\;, x, x_0; R)$*.*

In particular,

$$d\mu_{x_0}^{B(x_0;R)} \;=\; \frac{1}{\omega_{n-1}R^{n-1}}\; dS\;.$$

The harmonic measure with respect to the ball and its center is the normalized surface measure on its surface.

3.5 Sets of zero harmonic measure

Definition 3.4 *Suppose that* Ω *is a bounded open subset of* $\mathbf{R^n}$*.*

A set $E \subseteq \partial\Omega$ *is said to be* ***of zero harmonic measure with respect to*** Ω*, if* $d\mu_x^\Omega(E) = 0$ *for every* $x \in \Omega$ *or, equivalently, for at least one* x *in every connected component of* Ω*.*

Theorem 3.4 *Suppose that* Ω *is a bounded open subset of* $\mathbf{R^n}$ *and let* $E \subseteq \partial\Omega$*.*

Then, the following are equivalent.

1. *There exists a function* u *superharmonic in* Ω *with* $u \geq 0$ *in* Ω *and* $\lim_{\Omega\ni x\to y} u(x) = +\infty$ *for every* $y \in E$*.*

2. E *is of zero harmonic measure with respect to* Ω*.*

Proof:

1. Assume the existence of a u with the properties in the statement of the theorem. Then $\frac{1}{m}u \in \Phi_{\chi_E}^\Omega$ for every $m \in \mathbf{N}$ and, hence, for every $x \in \Omega$,

$$0 \;\leq\; \underline{H}_{\chi_E}^\Omega(x) \;\leq\; \overline{H}_{\chi_E}^\Omega(x) \;\leq\; \frac{1}{m}\; u(x)\;.$$

Now, let $m \to +\infty$ and get

$$d\mu_x^\Omega(E) \;=\; H_{\chi_E}^\Omega(x) \;=\; 0$$

for every $x \in \Omega$.

2. Assume $d\mu_{x_k}^\Omega(E) = 0$ for at least one x_k in each of the at most countably many connected components Ω_k of Ω. Therefore

$$H_{\chi_E}^\Omega(x_k) \;=\; 0$$

for the same points. This implies that, for every k and every m, there exists a $u_{m,k} \in \Phi_{\chi_E}^\Omega$ so that

$$u_{m,k}(x_k) \;\leq\; \frac{1}{2^{m+1}}\;.$$

Now, modify these functions and define

$$v_m = \min(u_{m,1}, \dots, u_{m,m}) .$$

The functions v_m are superharmonic and non-negative in Ω with

$$\liminf_{\Omega \ni x \to y} v_m(x) \geq 1$$

for all $y \in E$ and

$$v_m(x_k) \leq \frac{1}{2^{m+1}} ,$$

for $k = 1, \dots, m$.

Finally, define

$$v = \sum_{m=1}^{+\infty} v_m .$$

Since $v(x_k) < +\infty$ for every k, by the third property of superharmonic functions, v is superharmonic and non-negative in Ω and

$$\liminf_{\Omega \ni x \to y} v(x) \geq \sum_{m=1}^{+\infty} \liminf_{\Omega \ni x \to y} v_m(x) = +\infty$$

for every $y \in E$.

Theorem 3.5 *Suppose that Ω is a bounded open subset of $\mathbf{R^n}$ and let E be a subset of $\partial\Omega$ which is $d\mu_x^\Omega$-measurable for all $x \in \Omega$. In particular, E can be any Borel subset of $\partial\Omega$. Then, the following are equivalent.*

1. *E is of zero harmonic measure with respect to Ω.*
2. *For every u superharmonic and bounded from below in Ω with*

$$\liminf_{\Omega \ni x \to y} u(x) \geq 0$$

for every $y \in \partial\Omega \setminus E$, it is true that

$$u \geq 0$$

everywhere in Ω.

There is a dual statement for subharmonic functions.

Proof:

Let E be of zero harmonic measure with respect to Ω.

Assume that u satisfies the hypotheses in 2 and take $M > 0$ so that $u \geq -M$ identically in Ω. Then $-\frac{1}{M}u \in \Psi_{\chi_E}^\Omega$ implying

$$-\frac{1}{M} u \leq H_{\chi_E}^\Omega$$

and, since the last function is identically 0 in Ω, the proof of one direction is complete.

Now, let $d\mu_{x_0}^{\Omega}(E) > 0$ for at least one $x_0 \in \Omega$.

Then

$$H_{-\chi_E}^{\Omega}(x_0) \;=\; -d\mu_{x_0}^{\Omega}(E) \;<\; 0$$

and, hence, there exists $u \in \Phi_{-\chi_E}^{\Omega}$ with $u(x_0) < 0$.

This u is superharmonic and bounded from below in Ω and

$$\liminf_{\Omega \ni x \to y} u(x) \;\geq\; 0$$

for every $y \in \partial\Omega \setminus E$.

Theorem 3.5 expresses an extension of the Minimum Principle for superharmonic functions. It appears that the sets of zero harmonic measure with respect to Ω are *the* "negligible" sets when testing the hypotheses of the Minimum Principle. There is an extra "mild" hypothesis: the superharmonic function must be bounded from below.

3.6 Barriers and regularity

Definition 3.5 *Let Ω be open in $\mathbf{R^n}$ and $y_0 \in \partial\Omega$.*

*We say that Ω **has a barrier at** y_0, if there is an open neighborhood V of y_0 and a positive superharmonic function u in $V \cap \Omega$ so that*

$$\lim_{\Omega \ni x \to y_0} u(x) \;=\; 0\;.$$

*This u is called **barrier for** Ω **at** y_0.*

*y_0 is called **regular boundary point of** Ω, if there exists some barrier for Ω at y_0.*

*Ω is called **regular open set**, if all its boundary points are regular boundary points of Ω.*

Observe that the neighborhood of y_0 in the definition may become as small as we like and, hence, the part of Ω outside an arbitrarily small neighborhood of y_0 does not play any role in whether y_0 is regular or not. In other words the definition of regularity of a boundary point is *local* in character.

The next result is a concrete characterization of regularity of boundary points.

Lemma 3.3 *Let Ω be open in $\mathbf{R^n}$, $y_0 \in \partial\Omega$ and $y_0 \neq \infty$.*

Then the following are equivalent.

1. *y_0 is a regular boundary point of Ω.*
2. *There is an $R > 0$ so that*

$$\lim_{\Omega \ni x \to y_0} H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;R)}(x) \;=\; 0\;.$$

3. *The previous condition holds for all* $R > 0$.

If $n \geq 3$ *and* Ω *is not bounded, then* ∞ *is always a regular boundary point of* Ω.

If $n = 2$ *and* Ω *is not bounded, then the following are equivalent*

1. ∞ *is a regular boundary point of* Ω.

2. *There is an* $R > 0$ *so that*

$$\lim_{\Omega \ni x \to \infty} H_{1/|\cdot|}^{\{x \in \Omega : |x| > R\}}(x) \;=\; 0 \;.$$

3. *The previous condition holds for all* $R > 0$.

Proof:
(1) Let $y_0 \neq \infty$ and suppose that the condition in 2 is true for some R.

Since $|\cdot \, - y_0|$ is subharmonic, we have that $H_{|\cdot\,-y_0|}^{\Omega \cap B(y_0;R)}(x) \geq |x - y_0| > 0$ for all $x \in \Omega \cap B(y_0; R)$. Therefore, $H_{|\cdot\,-y_0|}^{\Omega \cap B(y_0;R)}$ is a barrier for Ω at y_0.

Conversely, suppose that there is a neighborhood V of y_0 and a positive superharmonic u in $V \cap \Omega$ with $\lim_{\Omega \ni x \to y_0} u(x) = 0$.

We may assume that $V = B(y_0; r)$ with $r < R$ and we consider an extra $\rho < r$.

Then $\Omega \cap S(y_0; \rho)$ is an open subset of $S(y_0; \rho)$ and we may decompose it as $\Omega \cap S(y_0; \rho) = F \cup A$, where F is compact, A is open in $S(y_0; \rho)$, $F \cap A = \emptyset$ and $dS(A) < \omega_{n-1} \rho^{n-1} \frac{\rho}{r}$.

For this purpose, consider a compact exhaustion $\{K_{(m)}\}$ of Ω and take $F = K_{(m)} \cap S(y_0; \rho)$ for a large enough m.

Now, define the function

$$w(x) \;=\; \frac{r}{\min_F u}\, u(x) + r P_{\chi_A}(x; y_0, \rho) + \rho \;, \qquad x \in \Omega \cap B(y_0; \rho) \;.$$

which is positive and superharmonic in $\Omega \cap B(y_0; \rho)$ and consider an arbitrary $v \in \Psi_{|\cdot\,-y_0|}^{\Omega \cap B(y_0;r)}$. Then, for every $y \in \partial\big(\Omega \cap B(y_0; \rho)\big)$,

$$\limsup_{\Omega \cap B(y_0;\rho) \ni x \to y} v(x) \;\leq\; \liminf_{\Omega \cap B(y_0;\rho) \ni x \to y} w(x) \;.$$

This is easy to prove, by considering the three cases: $y \in \partial\Omega \cap B(y_0; \rho)$, $y \in F$ and $y \in A$.

Therefore, $v \leq w$ everywhere in $\Omega \cap B(y_0; \rho)$ and, thus,

$$H_{|\cdot\,-y_0|}^{\Omega \cap B(y_0;r)} \;\leq\; w$$

everywhere in $\Omega \cap B(y_0; \rho)$.

This implies

$$\limsup_{\Omega \ni x \to y_0} H_{|\cdot\,-y_0|}^{\Omega \cap B(y_0;r)}(x) \;\leq\; \frac{r}{\min_F u} \lim_{\Omega \ni x \to y_0} u(x) + r P_{\chi_A}(y_0; y_0, \rho) + \rho \;\leq\; 2\rho$$

and, since ρ is arbitrarily small,

$$\lim_{\Omega \ni x \to y_0} H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;r)}(x) \; = \; 0 \; .$$

We need to show the same thing, but with r replaced by R.
Since

$$|x - y_0| \; \leq \; H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;r)}(x) \; \leq \; r$$

for all $x \in \Omega \cap B(y_0; r)$, we get

$$\lim_{\Omega \cap B(y_0;r) \ni z \to x} H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;r)}(z) \; = \; r$$

for all $x \in \Omega \cap S(y_0; r)$. This implies that the function

$$\begin{cases} \frac{R}{r} \, H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;r)}(x) \; , & \text{if } x \in \Omega \cap B(y_0; r) \\ R \; , & \text{if } x \in \Omega \cap \big(B(y_0; R) \setminus B(y_0; r)\big) \end{cases}$$

belongs to $\Phi_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;R)}$ and, hence,

$$H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;R)} \; \leq \; \frac{R}{r} \, H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;r)}$$

everywhere in $\Omega \cap B(y_0; r)$. Therefore,

$$\lim_{\Omega \ni x \to y_0} H_{|\cdot\, - y_0|}^{\Omega \cap B(y_0;R)}(x) \; = \; 0 \; .$$

(2) Now, if $n \geq 3$ and Ω is not bounded, the function

$$h(x) \; = \; \frac{1}{|x|^{n-2}} \; , \qquad x \in \Omega \; ,$$

is a barrier for Ω at ∞.
(3) In the case $n = 2$ we may either modify the proof of part (1) or, better, consider the inversion $x^* = \frac{1}{x}$ which transforms $\Omega \setminus \{0\}$ with ∞ as boundary point to the set $\Omega^* = \{x : \frac{1}{x} \in \Omega \setminus \{0\}\}$ having 0 as boundary point.

This inversion, as we have already seen, preserves the properties of harmonicity and superharmonicity. Hence, it is clear that ∞ is regular for Ω if and only if 0 is regular for Ω^* and, also, that

$$\lim_{\Omega \ni x \to \infty} H_{\frac{1}{|\cdot|}}^{\{x \in \Omega : |x| > R\}}(x) \; = \; 0$$

if and only if

$$\lim_{\Omega^* \ni x \to 0} H_{|\cdot|}^{\Omega^* \cap B(0;\frac{1}{R})}(x) \; = \; 0 \; .$$

Now, we may use the result of part (1) and complete the proof.

Theorem 3.6 *(Bouligand) Let Ω be an open subset of $\mathbf{R^n}$ and $y_0 \in \partial\Omega$. Then, the following are equivalent.*

1. *y_0 is a regular boundary point of Ω.*
2. *For every open neighborhood V of y_0, there is a positive u superharmonic in Ω so that $u = 1$ identically in $\Omega \setminus V$ and $\lim_{\Omega\ni x\to y_0} u(x) = 0$.*

Proof:
One direction is trivial. Therefore, let y_0 be regular and V be any open neighborhood of y_0.

Assume, first, that $y_0 \in \mathbf{R^n}$ and consider $R > 0$ small enough to have $B(y_0; R) \subseteq V$. By Lemma 3.3,

$$\lim_{\Omega\ni x\to y_0} H^{\Omega\cap B(y_0;R)}_{|\cdot\, - y_0|}(x) \;=\; 0 \;.$$

We, clearly, have

$$0 \;<\; |x - y_0| \;\leq\; H^{\Omega\cap B(y_0;R)}_{|\cdot\, - y_0|}(x) \;\leq\; R$$

for all $x \in \Omega \cap B(y_0; R)$ and, thus,

$$\lim_{\Omega\cap B(y_0;R)\ni z\to x} H^{\Omega\cap B(y_0;R)}_{|\cdot\, - y_0|}(z) \;=\; R$$

for all $x \in \Omega \cap S(y_0; R)$.

This implies that the function

$$u(x) \;=\; \begin{cases} \frac{1}{R} H^{\Omega\cap B(y_0;R)}_{|\cdot\, - y_0|}(x) \;, & \text{if } x \in \Omega \cap B(y_0; R) \\ 1 \;, & \text{if } x \in \Omega \setminus B(y_0; R) \end{cases}$$

has the desired properties.

If $n \geq 3$ and $y_0 = \infty$, we consider R large enough so that $\{x : |x| > R\} \subseteq V$ and, then, take

$$u(x) \;=\; \min\Big(\frac{R^{n-2}}{|x|^{n-2}}\;, 1\Big) \;, \qquad x \in \Omega \;.$$

If, finally, $n = 2$ and $y_0 = \infty$, then, taking R large enough and

$$u(x) \;=\; \begin{cases} R H^{\Omega\cap\{x:|x|>R\}}_{\frac{1}{|\cdot|}}(x) \;, & \text{if } x \in \Omega \text{ and } |x| > R \\ 1 \;, & \text{if } x \in \Omega \text{ and } |x| \leq R \;, \end{cases}$$

we conclude the proof.

3.7 Regularity and the problem of Dirichlet

Theorem 3.7 *Suppose Ω is a bounded open subset of $\mathbf{R^n}$ and $y_0 \in \partial\Omega$. Then the following are equivalent.*

1. *y_0 is a regular boundary point of Ω.*

2. *For every real-valued f defined and bounded in $\partial\Omega$ and continuous at y_0 it is true that*

$$\lim_{\Omega\ni x\to y_0} \overline{H}_f^{\Omega}(x) = \lim_{\Omega\ni x\to y_0} \underline{H}_f^{\Omega}(x) = f(y_0) .$$

That 1 implies 2 holds without the assumption of boundedness of Ω.

Proof:

By the subharmonicity of the function $f(\cdot) = |\cdot - y_0|$, we find

$$0 < |x - y_0| \le \underline{H}_f^{\Omega}(x) \le diam(\Omega)$$

for all $x \in \Omega$. Therefore, if 2 holds, the function $\underline{H}_f^{\Omega}$ is a barrier for Ω at y_0 and y_0 is a regular boundary point of Ω.

Now, suppose that y_0 is a regular boundary point of Ω and consider any real-valued f defined and bounded in $\partial\Omega$ and continuous at y_0.

Take any $\epsilon > 0$ and let V be a neighborhood of y_0 so that $|f(y) - f(y_0)| < \epsilon$ for all $y \in \partial\Omega \cap \overline{V}$.

Theorem 3.6 implies that there is a positive superharmonic u in Ω so that $\lim_{\Omega\ni x\to y_0} u(x) = 0$ and $u = 1$ identically in $\Omega \setminus V$.

If $|f(y)| \le M$ for all $y \in \partial\Omega$, then the function

$$w = \big(M - f(y_0)\big)u + f(y_0) + \epsilon$$

belongs to Φ_f^{Ω} and, thus, $\limsup_{\Omega\ni x\to y_0} \overline{H}_f^{\Omega}(x) \le f(y_0) + \epsilon$. Since ϵ is arbitrary,

$$\limsup_{\Omega\ni x\to y_0} \overline{H}_f^{\Omega}(x) \le f(y_0) .$$

Applying this to $-f$ we find

$$\liminf_{\Omega\ni x\to y_0} \underline{H}_f^{\Omega}(x) \ge f(y_0)$$

and, combining the two inequalities, we finish the proof.

Theorem 3.8 *If Ω is an open subset of $\mathbf{R}^{\mathbf{n}}$, then the regularity of Ω implies that the Problem of Dirichlet is solvable for every $f \in C(\partial\Omega)$.*

If Ω is bounded, then the converse is, also, true.

Proof:

A direct consequence of Theorem 3.7.

3.8 Criteria for regularity

The next three results give three useful criteria for regularity of boundary points. Much later we shall prove Wiener's characterization of regularity of boundary points.

Proposition 3.4 *(The ball-criterion) Let $\Omega \subseteq \mathbf{R^n}$ be open and $y_0 \in \partial\Omega$, $y_0 \in \mathbf{R^n}$. If there is a ball $B \subseteq \mathbf{R^n} \setminus \overline{\Omega}$ so that $y_0 \in \partial B$, then y_0 is a regular boundary point of Ω.*

If, in particular, Ω is C^2 at y_0, then y_0 is a regular boundary point of Ω. Therefore, if the bounded open Ω has C^2-boundary, then it is a regular set.

Proof:

If y_1 is the center of B and r is its radius, then $-h_{y_1} + h_*(r)$ is a barrier for Ω at y_0.

Proposition 3.5 *(The continuum-criterion) Let $\Omega \subseteq \mathbf{R^2}$ be open and $y_0 \in \partial\Omega$. If there is a continuum containing y_0 and contained in $\overline{\mathbf{R^2}} \setminus \Omega$, then y_0 is a regular boundary point of Ω.*

In particular, if the complement of Ω with respect to $\overline{\mathbf{R^2}}$ has no component reducing to only one point, then Ω is a regular set.

Proof:

Assume, first, that $y_0 \neq \infty$.

Let C be the continuum of the statement and consider $y_1 \in C \cap \mathbf{R^2}$ with $y_1 \neq y_0$. If $R = |y_1 - y_0|$, then all connected components of the open set $B(y_0; R) \setminus C$ are simply-connected. The function $u(x) = \log\left|\frac{x-y_1}{x-y_0}\right|$, $x \in B(y_0; R) \setminus C$, is harmonic in $B(y_0; R) \setminus C$ and, by Theorem 1.4, there is a harmonic conjugate v of it there.

Since $\Omega \cap B(y_0; R) \subseteq B(y_0; R) \setminus C$, it is clear that the function

$$\Re\Big(\frac{1}{u(x) + iv(x)}\Big) , \qquad x \in \Omega \text{ and } \Big|\frac{x-y_1}{x-y_0}\Big| > 1 ,$$

is a barrier for Ω at y_0.

If $y_0 = \infty$, we choose a $y_1 \in C$ with $y_1 \neq \infty$ and, then, all connected components of $\overline{\mathbf{R^2}} \setminus C$ are simply-connected. We define a harmonic conjugate v of the harmonic function $u(x) = \log|x - y_1|$ in $\overline{\mathbf{R^2}} \setminus C$.

Since $\Omega \subseteq \overline{\mathbf{R^2}} \setminus C$, the function

$$\Re\Big(\frac{1}{u(x) + iv(x)}\Big) , \qquad x \in \Omega \text{ and } |x - y_1| > 1 ,$$

is a barrier for Ω at ∞.

Proposition 3.6 *(The cone-criterion) Let $\Omega \subseteq \mathbf{R^n}$ be open and $y_0 \in \partial\Omega$, $y_0 \in \mathbf{R^n}$. If there is an open truncated cone $F \subseteq \mathbf{R^n} \setminus \overline{\Omega}$ with vertex y_0, then y_0 is a regular boundary point of Ω.*

If, in particular, Ω is C^1 at y_0, then y_0 is a regular boundary point of Ω. Hence, if the bounded open Ω has C^1-boundary, then it is a regular set.

Proof:

Let R be the height of the cone F. It is enough to find a barrier for the open set $B(y_0;R)\setminus\overline{F}$ at y_0 and, by Lemma 3.3, it is enough to prove

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y_0} H^{B(y_0;R)\setminus\overline{F}}_{|\cdot\,-y_0|}(x) \;=\; 0\ .$$

Now, set

$$u(x) \;=\; H^{B(y_0;R)\setminus\overline{F}}_{|\cdot\,-y_0|}(x)$$

for all $x\in B(y_0;R)\setminus\overline{F}$ and, then, dilate F by a factor of two, producing the cone

$$F' \;=\; y_0+2(F-y_0)\ ,$$

and consider

$$v(x) \;=\; u\big(y_0+\frac{1}{2}(x-y_0)\big)$$

for all $x\in B(y_0;2R)\setminus\overline{F'}$.

The function v is harmonic in $B(y_0;2R)\setminus\overline{F'}$.

By the ball-criterion, every boundary point of $B(y_0;R)\setminus\overline{F}$ is regular except, perhaps, y_0 and, therefore,

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y} u(x) \;=\; |y-y_0|$$

for all $y\in\partial\big(B(y_0;R)\setminus\overline{F}\big)$ except, perhaps, y_0.

By the Maximum-Minimum Principle, we have that $u<R$ in $B(y_0;R)\setminus\overline{F}$ and, thus,

$$\sup_{S(y_0;R)\setminus\overline{F}} v \;<\; R\ .$$

Hence, we can choose α so that $\frac{1}{2}<\alpha<1$ and

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y} \big(\alpha u(x)-v(x)\big) \;\ge\; 0$$

for all $y\in S(y_0;R)\setminus\overline{F}$. We, also, have

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y} \big(\frac{1}{2}\,u(x)-v(x)\big) \;=\; 0$$

for all $y\in\partial F\setminus\{y_0\}$.

Therefore,

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y} \big(\alpha u(x)-v(x)\big) \;\ge\; 0$$

for all $y\in\partial\big(B(y_0;R)\setminus\overline{F}\big)$ except, perhaps, y_0.

Since the function $\alpha u - v$ is bounded from below, we have that, for all $\epsilon > 0$,

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y}\Big(\alpha u(x) - v(x) + \epsilon(h_{y_0}(x) - h_*(R))\Big) \ \geq \ 0$$

for all $y \in \partial\big(B(y_0;R)\setminus\overline{F}\big)$.

By the Maximum-Minimum Principle

$$\alpha u(x) - v(x) + \epsilon(h_{y_0}(x) - h_*(R)) \ \geq \ 0$$

for all $x \in B(y_0;R)\setminus\overline{F}$,and, since ϵ is arbitrary, we find

$$\alpha u \ \geq \ v$$

in $B(y_0;R)\setminus\overline{F}$.

From this,

$$\begin{aligned}\limsup_{B(y_0;R)\setminus\overline{F}\ni x\to y_0} u(x) \ &\geq \ \frac{1}{\alpha}\limsup_{B(y_0;R)\setminus\overline{F}\ni x\to y_0} u\big(y_0 + \frac{1}{2}(x-y_0)\big) \\ &= \ \frac{1}{\alpha}\limsup_{B(y_0;R)\setminus\overline{F}\ni x\to y_0} u(x)\ ,\end{aligned}$$

implying

$$\lim_{B(y_0;R)\setminus\overline{F}\ni x\to y_0} u(x) \ = \ 0\ .$$

If a boundary point satisfies the ball-criterion, then it satisfies the cone-criterion and, in case $n = 2$, if it satisfies the cone-criterion then it satisfies the continuum-criterion. Therefore, the cone-criterion is the most useful in case $n \geq 3$ and the continuum-criterion is the most useful in case $n = 2$.

Proposition 3.7 *If $\Omega \subseteq \mathbf{R^n}$ is open and y_0 is an isolated point of $\partial\Omega$, then y_0 is not a regular boundary point of Ω. The only exception is when $n \geq 3$ and $y_0 = \infty$.*

Proof:

Let $y_0 \neq \infty$ be a regular boundary point of Ω and consider a small enough R so that $B(y_0;2R)\setminus\{y_0\} \subseteq \Omega$ and a positive superharmonic u in $B(y_0;2R)\setminus\{y_0\}$ with $\lim_{x\to y_0} u(x) = 0$.

Let $m = \min_{S(y_0;R)} u$, being clear that $m > 0$.

By the Minimum Principle, for every $\delta < R$,

$$u(x) \ \geq \ m\frac{h_*(\delta) - h(x)}{h_*(\delta) - h_*(R)}$$

for all x in $B(y_0;\delta,R)$.

Now, let $\delta \to 0+$ and find

$$u(x) \ \geq \ m$$

for all $x \in B(y_0; R) \setminus \{y_0\}$, getting a contradiction.

In case $n = 2$ and $y_0 = \infty$ we modify the previous proof, taking u positive and superharmonic in $\{x : |x| > \frac{1}{2}R\} \subseteq \Omega$ with $\lim_{x \to \infty} u(x) = 0$, defining $m = \min_{|x|=R} u(x) > 0$ and observing that

$$u(x) \;\geq\; m \frac{h(x) - h_*(r)}{h_*(R) - h_*(r)}$$

for all x with $R < |x| < r$.

We get a contradiction, letting $r \to +\infty$ and finding

$$u(x) \;\geq\; m$$

for all x with $R < |x|$.

Chapter 4

The Kelvin Transform

4.1 Definition

Consider any ball $B(x_0; R)$ and the symmetric x^* of any x with respect to $S(x_0; R)$,

$$x^* \ = \ x_0 + \frac{R^2}{|x - x_0|^2}(x - x_0) \ .$$

As usual, we consider each of x_0 and ∞ to be symmetric to the other.

Now, for every set $A \subseteq \overline{\mathbf{R^n}}$, we define its symmetric with respect to $S(x_0; R)$ by

$$A^* \ = \ \{x^* : x \in A\} \ .$$

The new set A^* contains x_0 or ∞ if and only if A contains ∞ or x_0, respectively.

A nice geometric property is that spheres are transformed, by symmetry, onto spheres. In fact, doing some easy calculations, we can prove that if the sphere $S(x_1; R_1)$ does not contain x_0, then its symmetric, $S(x_1; R_1)^*$, is the sphere having the point $\frac{2R^2}{|x_1 - x_0|^2 - R_1^2}(x_1 - x_0)$ as center and the number $\frac{R^2 R_1}{||x_1 - x_0|^2 - R_1^2|}$ as radius.

If the interior $B(x_1; R_1)$ contains x_0, then it is transformed onto the exterior of the image sphere, while the exterior is transformed onto the interior.

If the interior does not contain x_0, then it is transformed onto the interior of the image sphere, while the exterior is transformed onto the exterior.

If the sphere $S(x_1; R_1)$ contains x_0, then it is transformed onto the hyperplane described by the equation $(x^* - x_0) \cdot (x_1 - x_0) = \frac{1}{2}R^2$ and the interior and exterior of the sphere are transformed onto the two half-spaces determined by this hyperplane.

Definition 4.1 *Let Ω be an open subset of $\mathbf{R^n}$ not containing x_0.*

For every function f defined in Ω, we call the function

$$f^*(x^*) \ = \ \frac{|x - x_0|^{n-2}}{R^{2n-4}} \ f(x) \ = \ \frac{1}{|x^* - x_0|^{n-2}} \ f(x) \ ,$$

*defined in Ω^*, the **Kelvin Transform** of f with respect to $S(x_0;R)$.*

Whenever we write about the Kelvin Transform without specifying the sphere, we shall understand that the sphere is $S(0;1)$.

Proposition 4.1 *Let Ω be an open subset of $\mathbf{R^n}$ not containing x_0.*

Then, u is harmonic or superharmonic or suharmonic in Ω if and only if its Kelvin Transform u^* with respect to $S(x_0;R)$ is harmonic or superharmonic or subharmonic, respectively, in Ω^*.

Proof:

If u is in $C^2(\Omega)$, then, by trivial calculations, we can prove

$$\Delta u^*(x^*) \;=\; \frac{|x-x_0|^{n+2}}{R^{2n}}\,\Delta u(x)$$

for all $x^* \in \Omega^*$.

We conclude that, if u is harmonic or twice continuously differentiable and superharmonic in Ω, then u^* is harmonic or superharmonic, respectively, in Ω^*.

For a general superharmonic u in Ω and an arbitrary open ball B with $\overline{B} \subseteq \Omega^*$, we consider the symmetric closed ball $\overline{B}^* \subseteq \Omega$ and, through Theorem 2.10, a sequence $\{u_m\}$ of twice continuously differentiable superharmonic functions in an open set Ω_0 with $\overline{B}^* \subseteq \Omega_0 \subseteq \Omega$ which increase towards u in Ω_0.

Then, the functions u_m^* are superharmonic in Ω_0^* and, hence, in B and increase towards u^* there.

By the third property of superharmonic functions, u^* is superharmonic in B and, since the ball is arbitrary, superharmonic in Ω^*.

4.2 Harmonic functions at ∞

If V is an open neighborhood of $x \in \overline{\mathbf{R^n}}$, then the set $V \setminus \{x\}$ is called a **punctured neighborhood** of x.

Proposition 4.2 *Let u be harmonic in a punctured neighborhood $V \setminus \{\infty\}$ of ∞ and u^* its Kelvin Transform with respect to $S(x_0;R)$. We know that u^* is harmonic in the punctured neighborhood $V^* \setminus \{x_0\}$ of x_0.*

Then, the following are equivalent.

1. *u^* can be defined at x_0 so that it becomes harmonic in V^*.*

2. $$\lim_{x\to\infty} u(x) \;=\; 0 \qquad \textit{in case } n \geq 3$$

   and
   $$\lim_{x\to\infty} \frac{u(x)}{\log|x|} \;=\; 0 \qquad \textit{in case } n = 2\ .$$

Proof:
In case $n \geq 3$,

$$\lim_{x^* \to x_0} \frac{u^*(x^*)}{h_{x_0}(x^*)} = \lim_{x \to \infty} u(x) \ ,$$

while, in case $n = 2$,

$$\lim_{x^* \to x_0} \frac{u^*(x^*)}{h_{x_0}(x^*)} = \lim_{x \to \infty} \frac{u(x)}{\log \frac{|x-x_0|}{R^2}} = \lim_{x \to \infty} \frac{u(x)}{\log |x|} \ .$$

Theorem 1.12 concludes the proof.

Proposition 4.3 *Suppose that $n = 2$ and f is holomorphic in $B(x_0; R) \setminus \{x_0\}$. Then, f can be extended as a holomorphic function in $B(x_0; R)$ if and only if* $\lim_{x \to x_0}(x - x_0)f(x) = 0$.

Proof:
The necessity of the condition is obvious.
Therefore, assume that $\lim_{x \to x_0}(x - x_0)f(x) = 0$, take r so that $0 < r < R$ and consider the function

$$g(x) = \frac{1}{2\pi i} \int_{\partial B(x_0; r)} \frac{f(y)}{y - x} \, dy \ , \qquad x \in B(x_0; r) \ ,$$

which is holomorphic in $B(x_0; r)$.
Now, fix an $x \in B(x_0; r) \setminus \{x_0\}$, take ϵ so that $0 < \epsilon < |x - x_0|$ and apply Cauchy's Formula to f in $B(x_0; r) \setminus \overline{B(x_0; \epsilon)}$ to get

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int_{\partial B(x_0; r)} \frac{f(y)}{y - x} \, dy - \frac{1}{2\pi i} \int_{\partial B(x_0; \epsilon)} \frac{f(y)}{y - x} \, dy \\ &= g(x) - \frac{1}{2\pi i} \int_{\partial B(x_0; \epsilon)} \frac{f(y)}{y - x} \, dy \ . \end{aligned}$$

The last integral tends to 0 as $\epsilon \to 0+$ and, thus, $f(x) = g(x)$ for all $x \in B(x_0; r) \setminus \{x_0\}$.

Proposition 4.4 *Suppose that $n = 2$, f is holomorphic in a punctured neighborhood $V \setminus \{\infty\}$ of ∞ and f^* is its Kelvin Transform with respect to $S(x_0; R)$. Then, f^* is holomorphic in the punctured neighborhood $V^* \setminus \{x_0\}$ of x_0 and it can be extended as a holomorphic function in V^* if and only if* $\lim_{x \to \infty} \frac{f(x)}{x} = 0$.

Proof:
The proof is a trivial application of Proposition 4.3.

Observe that condition 2 of Proposition 4.2 and the analogous condition of Proposition 4.4 are independent of the sphere with respect to which we take the Kelvin Transform. Therefore, in the following definition the use of $S(0; 1)$ is only for reasons of simplicity and the use of any other sphere is equivalent.

Definition 4.2 *Suppose that the open set $\Omega \subseteq \overline{\mathbf{R^n}}$ contains ∞ and u is defined in $\Omega \setminus \{\infty\}$.*

We say that u is harmonic in Ω if it is harmonic in $\Omega \setminus \{\infty\}$ and there is a punctured neighborhood $V \setminus \{\infty\}$ of ∞ so that the Kelvin Transform u^*, harmonic in $V^* \setminus \{0\}$, can be defined at 0 so that it is harmonic in V^*.

We call u harmonic at ∞, if it is harmonic in some neighborhood of ∞.

If $n = 2$, we, similarly, define holomorphic functions at ∞.

Proposition 4.2 gives a necessary and sufficient condition on u, harmonic in a punctured neighborhood of ∞, so that it is harmonic at ∞ and Proposition 4.4 gives a necessary and sufficient condition on f, holomorphic in a punctured neighborhood of ∞, so that it is holomorphic at ∞.

Theorem 4.1 *Let u be harmonic in a punctured neighborhood $V \setminus \{\infty\}$ of ∞.*

If $n \geq 3$, u can be extended as harmonic in V if and only if $\lim_{x\to\infty} u(x) = 0$.

If $n = 2$, u can be extended as harmonic in V if and only if $\lim_{x\to\infty} \frac{u(x)}{\log|x|} = 0$ if and only if $\lim_{x\to\infty} u(x)$ exists in $\mathbf{C}$.

Suppose that $n = 2$ and f is holomorphic in a punctured neighborhood $V \setminus \{\infty\}$ of ∞. Then f can be extended as holomorphic in V if and only if $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ if and only if $\lim_{x\to\infty} f(x)$ exists in $\mathbf{C}$.

In case $n \geq 3$, if u is harmonic at ∞, then, by defining $u(\infty) = 0$, we guarrantee that u is continuous at ∞. Therefore, we may say that, if u is defined and continuous in an open set Ω containing ∞, then it is harmonic in Ω if and only if it is harmonic in $\Omega \setminus \{\infty\}$ and $u(\infty) = 0$.

Writing $u^*(x^*) = u^*(0) + \mathcal{O}(|x^*|)$ when x^* is near 0, we find that

$$u(x) \;=\; \frac{u^*(0)}{|x|^{n-2}} + \mathcal{O}\Big(\frac{1}{|x|^{n-1}}\Big)$$

when x is near ∞.

In case $n = 2$, we get in the same way that

$$u(x) \;=\; u^*(0) + \mathcal{O}\Big(\frac{1}{|x|}\Big)$$

when x is near ∞, implying that, by defining $u(\infty) = u^*(0)$, u becomes continuous at ∞. Therefore, we may say that, if u is defined and continuous in an open set Ω containing ∞, then it is harmonic in Ω if and only if it is harmonic in $\Omega \setminus \{\infty\}$.

In this case there is no *universal* value at ∞ for harmonic functions there, as is the value 0 in case $n \geq 3$.

We summarize.

Proposition 4.5 *Suppose u is defined and continuous in an open set Ω containing ∞.*

If $n \geq 3$, then u is harmonic in Ω if and only if u is harmonic in $\Omega \setminus \{\infty\}$ and $u(\infty) = 0$ if and only if u is harmonic in $\Omega \setminus \{\infty\}$ and there is some complex number a so that $u(x) = \frac{a}{|x|^{n-2}} + \mathcal{O}\Big(\frac{1}{|x|^{n-1}}\Big)$ when x is near ∞.

If $n = 2$, then u is harmonic in Ω if and only if u is harmonic in $\Omega \setminus \{\infty\}$ if and only if u is harmonic in $\Omega \setminus \{\infty\}$ and $u(x) = u(\infty) + \mathcal{O}\left(\frac{1}{|x|}\right)$ when x is near ∞.

If f is holomorphic in an open set Ω containing ∞ and all $x \in \mathbf{R^2}$ with $|x| \geq R$, then a trivial use of the Kelvin Transform with respect to $S(0;1)$ shows that f has a power series expansion

$$f(x) \;=\; \sum_{n=0}^{+\infty} a_n \frac{1}{x^n} \;, \qquad |x| > R \;.$$

Of course, $a_0 = f(\infty)$ and

$$f'(\infty) \;=\; a_1 \;=\; \lim_{x \to +\infty} x(f(x) - f(\infty))$$

is called the **(complex) derivative of f at ∞**. Observe that this coincides with the derivative of f^* at 0.

Either using the Kelvin Transform and the analogous formulas for f^* or integrating the power series of f, we may, easily, prove that

$$a_n \;=\; \frac{1}{2\pi i} \int_{\partial B(0;r)} f(y) y^{n-1} \, dy \;.$$

The following is a direct application of the definitions and Theorem 1.5.

Theorem 4.2 *Let u be harmonic in the neighborhood $\{x : R < |x|\} \cup \{\infty\}$ of ∞.*

In case $n \geq 3$ we have

1. $\tau = \int_{S(0;r)} \frac{\partial u}{\partial \eta}(y) \, dS(y)$ *is constant in the interval $R < r < +\infty$.*
2. $\mathcal{M}_u^r(0) = -\frac{\tau}{(n-2)\omega_{n-1}} \frac{1}{r^{n-2}}$ *in the same interval.*

In case $n = 2$,

1. $0 = \int_{S(0;r)} \frac{\partial u}{\partial \eta}(y) \, dS(y)$ *identically in the interval $R < r < +\infty$.*
2. $\mathcal{M}_u^r(0) = u(\infty)$ *identically in the same interval.*

Here, $\overrightarrow{\eta}$ is the continuous unit vector field which is normal to $S(0;r)$ and in the direction towards the exterior of $B(0;r)$.

The next result regards the representation of a harmonic function as the difference between a single- and a double-layer potential.

The function h_∞ is defined by

$$h_\infty(x) \;=\; \begin{cases} \log|x| \;, & \text{if } n = 2 \\ \frac{1}{|x|^{n-2}} \;, & \text{if } n \geq 3 \;, \end{cases}$$

and it, also, is a fundamental solution of the Laplace equation in $\mathbf{R^n} \setminus \{0\}$.

Theorem 4.3 *Let Ω be an open set containing ∞ and having C^1-boundary and let $\overrightarrow{\eta}$ be the continuous unit vector field normal to $\partial\Omega$ in the direction towards the exterior of Ω. Then,*

1. *For every $x \in \Omega$ with $x \neq \infty$,*

$$u(x) - u(\infty) \;=\; \frac{1}{\kappa_n} \int_{\partial\Omega} \Big(u(y) \frac{\partial h_\infty(x - \cdot)}{\partial \eta}(y) - h_\infty(x - y) \frac{\partial u}{\partial \eta}(y) \Big) \, dS(y) \; .$$

2. *If x_1 is any point outside $\overline{\Omega}$, then*

$$u(\infty) \;=\; \frac{1}{\kappa_n} \int_{\partial\Omega} \Big(u(y) \frac{\partial h_\infty(x_1 - \cdot)}{\partial \eta}(y) - h_\infty(x_1 - y) \frac{\partial u}{\partial \eta}(y) \Big) \, dS(y) \; .$$

Proof:
The proof is a routine application of Green's formula in the open set $\Omega \cap B(0; R)$ or $\Omega \cap B(x_1; R)$, where R is large and eventually tends to $+\infty$. It uses the formulas in Theorem 4.1 and it is left to the interested reader.

Theorem 4.4 *If u is harmonic in $\overline{\mathbf{R^n}}$, then, in case $n \geq 3$, it is identically* 0 *and, in case $n = 2$, it is a constant function.*
Similarly, if f is holomorphic in $\overline{\mathbf{R^2}}$, then f is a constant function.

Proof:
u is bounded from below in $\mathbf{R^n}$ and, from the Theorem of Picard, it is constant.

4.3 Superharmonic functions at ∞

The following is parallel to the definition of harmonicity at ∞.

Definition 4.3 *Suppose that the open set $\Omega \subseteq \overline{\mathbf{R^n}}$ contains ∞ and u is defined in $\Omega \setminus \{\infty\}$.*
We say that u is superharmonic in Ω if it is superharmonic in $\Omega \setminus \{\infty\}$ and there is a punctured neighborhood $V \setminus \{\infty\}$ of ∞ so that the Kelvin Transform u^*, superharmonic in $V^* \setminus \{0\}$, can be defined at 0 *so that it is superharmonic in V^*.*
We say that u is superharmonic at ∞, if it is superharmonic in some neighborhood of ∞.
The definition of subharmonicity is analogous.

If u is superharmonic at ∞, then it is natural to admit

$$u(\infty) \;=\; \liminf_{x \to \infty} u(x)$$

as its value at ∞. This choise makes u lower-semicontinuous in the set where it is superharmonic.

Theorem 4.5 *Let u be superharmonic in a punctured neighborhood $V \setminus \{\infty\}$ of ∞. Then u can be extended at ∞ so that it becomes superharmonic in V if and only if*

1. $\liminf_{x\to\infty} u(x) \geq 0$, *in case* $n \geq 3$, *and*

2. $\liminf_{x\to\infty} \frac{u(x)}{\log|x|} \geq 0$, *in case* $n = 2$.

Proof:
The proof is a direct application of the definition and Theorem 2.5.

Theorem 4.6 *u is superharmonic in $\overline{\mathbf{R^2}}$ if and only if u is constant.*
If $n \geq 3$, then u is superharmonic in $\overline{\mathbf{R^n}}$ if and only if its restriction to $\mathbf{R^n}$ is a non-negative superharmonic function and $u(\infty) = \liminf_{x\to\infty} u(x)$.

Proof:
The Kelvin Transform $u^*(x^*) = u(x)$ is superharmonic in $\mathbf{R^2} \setminus \{0\}$ and it can be extended at 0 so that it is superharmonic in $\mathbf{R^2}$. Its value at 0 is, necessarily, $u^*(0) = \liminf_{x^*\to 0} u^*(x^*) = \liminf_{x\to\infty} u(x)$.

If we define $u(\infty) = u^*(0)$, then u is lower-semicontinuous in the compact set $\overline{\mathbf{R^2}}$. Therefore, it takes a minimum value in $\overline{\mathbf{R^2}}$.

If this minimum value is taken at a point in $\mathbf{R^2}$, then u is constant in $\mathbf{R^2}$.

Otherwise, u^*, which is superharmonic in $\mathbf{R^2}$, takes its minimum value at 0 and it is constant. Thus, u is also constant in $\mathbf{R^2} \setminus \{0\}$ and, hence, in $\mathbf{R^2}$.

To deal with the case $n \geq 3$, we just apply the Minimum Principle for u in $\mathbf{R^n}$ and Theorem 4.4.

Corollary 4.1 *A superharmonic function in $\mathbf{R^2}$ which is bounded from below is constant.*

4.4 Poisson integrals at ∞

Definition 4.4 *Let f be integrable on $S(0;R)$ with respect to the surface measure. We define the* ***Poisson integral*** *of f in the exterior of $B(0;R)$ by*

$$P_f(x;\infty,R) \;=\; \frac{1}{\omega_{n-1}R}\int_{S(0;R)} \frac{|x|^2 - R^2}{|x-y|^n}\, f(y)\; dS(y)\;, \qquad |x| > R\;.$$

In order to have continuity at ∞, the values that are assigned to the Poisson integral at the point $x = \infty$ are defined (with the help of the Dominated Convergence Theorem) by

$$P_f(\infty;\infty,R) \;=\; \lim_{x\to\infty} P_f(x;\infty,R) \;=\; \begin{cases} 0\;, & \text{if } n \geq 3 \\ \frac{1}{\omega_1 R}\int_{S(0;R)} f(y)\; dS(y)\;, & \text{if } n = 2\;. \end{cases}$$

It is obvious that, if f is integrable on $S(0;R)$, then f^* is integrable on $S(0;\frac{1}{R})$ and trivial calculations result to the formula

$$P_f(x;\infty,R) \;=\; |x^*|^{n-2} P_{f^*}\Big(x^*;0,\frac{1}{R}\Big)$$

for all x with $|x|>R$.

This says that *the Kelvin Transform of the Poisson integral of f in the exterior of $B(0;R)$ is equal to Poisson integral of the Kelvin Transform of f in the symmetric ball $B(0;\frac{1}{R})$.*

Now, the following properties are straightforward, and can be proved either directly, using the usual properties of the Poisson kernel, or using this last formula.

1. *$P_f(\cdot\,;\infty,R)$ is harmonic in $\overline{\mathbf{R^n}}\setminus\overline{B(0;R)}$.*

This is obvious, since this function is the Kelvin Transform of a function harmonic in $B(0;\frac{1}{R})$.

2. *If f is continuous at some $y_0\in S(0;R)$, then*

$$\lim_{x\to y_0,|x|>R} P_f(x;\infty,R) \;=\; f(y_0)\ .$$

Just observe that f^* is continuous at y_0^* and write

$$\begin{aligned}\lim_{x\to y_0,|x|>R} P_f(x;\infty,R) &= \lim_{B(0;\frac{1}{R})\ni x^*\to y_0^*} |x^*|^{n-2} P_{f^*}(x^*;0,\frac{1}{R})\\ &= |y_0^*|^{n-2} f^*(y_0^*) \;=\; f(y_0)\ .\end{aligned}$$

3. Thus, *if f is continuous on $S(0;R)$, then $P_f(\cdot\,;\infty,R)$ is the unique solution to the Problem of Dirichlet in $\overline{\mathbf{R^n}}\setminus\overline{B(0;R)}$ with f as boundary function.*

The uniqueness is proved easily by taking Kelvin transforms and reducing to the uniqueness of the Problem of Dirichlet in $B(0;\frac{1}{R})$.

4.5 The effect of the dimension

Many of the properties of harmonic or superharmonic or subharmonic functions that we have studied continue to hold when the domain of definition contains ∞ as interior point. If this domain is the whole space $\overline{\mathbf{R^n}}$, then most of these properties are trivial due to the Theorems 4.3, 4.5 and Corollary 4.1. If the domain of definition, Ω, misses some point $x_0\in\mathbf{R^n}$, we, then, take the translates $u(\cdot\,+x_0)$ which are defined in the set $\Omega-x_0$ not containing 0 and apply the Kelvin Transform. This reduces our study to the case of a domain of definition, $(\Omega-x_0)^*$, contained in $\mathbf{R^n}$.

We shall describe, now, a difference between the cases $n=2$ and $n\geq 3$ which has already, to a certain degree, appeared in our results.

Take, for example, the Maximum-Minimum Principle for harmonic functions.

The function h is, in case $n \geq 3$, harmonic in the open set $\overline{\mathbf{R^n}} \setminus \overline{B(0;1)}$ with boundary values 1 in $S(0;1)$. One would expect that the function is identically 1 in $\overline{\mathbf{R^n}} \setminus \overline{B(0;1)}$, but it is not. In fact, its value at ∞ is 0, as is the value at ∞ of every harmonic function there.

This is, best, explained using the Kelvin Transform $h^*(x^*) = |x|^{n-2}h(x) = 1$ which is harmonic in the symmetric set $B(0;1)$, has boundary values 1 on $S(0;1)$ and is, indeed, identically 1 in $B(0;1)$.

Thus, in case $n \geq 3$, the "correct" statement of the Maximum-Minimum Principle is:

Let Ω be open in $\overline{\mathbf{R^n}}$ containing ∞ and let $x_0 \notin \Omega$. If u is superharmonic in Ω and

$$\liminf_{\Omega \ni x \to y} |x - x_0|^{n-2} u(x) \geq m$$

for every $y \in \partial\Omega$, then

$$|x - x_0|^{n-2} u(x) \geq m$$

for every $x \in \Omega$.

There are similar statements for subharmonic and harmonic functions.

The situation is simpler when $n = 2$. In this case, the formula of the Kelvin Transform, $u^*(x^*) = u(x)$, does not contain the factor $|x|^{n-2}$ and all results which hold for open subsets of $\mathbf{R^2}$ transfer, without any change, for open sets in $\overline{\mathbf{R^2}}$ containing ∞.

4.6 Dimension 2, in particular

We state, below, the most important of the properties that hold in case $n = 2$ and remark that some of them hold in case $n \geq 3$, also, while some others hold after an appropriate modification, as explained a few lines above. It is left to the interested reader to investigate the case $n \geq 3$.

In all results below the open sets are subsets of $\overline{\mathbf{R^2}}$.

1. *All versions of the Maximum/Minimum Principles are valid.*
2. *Locally uniform limits of harmonic functions are harmonic.*
3. *If Ω_1 and Ω_2 are open, f is meromorphic in Ω_1, $f(\Omega_1) \subseteq \Omega_2$ and u is harmonic or superharmonic or subharmonic in Ω_2, then $u \circ f$ is harmonic or superharmonic or subharmonic, respectively, in Ω_1 (except if f is constant c in some component of Ω and $u(c) = \pm\infty$).*
4. *If u is harmonic (superharmonic) in an open set containing all x with $|x| \geq R$ together with ∞, then the Poisson Formula*

$$u(x) \ = (\geq)\ P_u(x; \infty, R)$$

holds for all x with $|x| > R$.

5. *If u is superharmonic in an open set Ω, B denotes any open disc with $\overline{B} \subseteq \Omega$ (its center may well be ∞) and u_B denotes the function which equals u in $\Omega \setminus B$ and equals the Poisson integral of u in B, then*

1. $u \geq u_B$ *in* Ω,

2. u_B *is superharmonic in* Ω *and*

3. u_B *is harmonic in* B.

6. *Harnack's inequalities hold in general: if* u *is positive and harmonic in the open* Ω *and* K *is a compact subset of* Ω*, then*

$$\frac{1}{C} \leq \frac{u(x)}{u(x')} \leq C$$

for all $x, x' \in K$*, where* C *is a positive constant depending only on* Ω *and* K.
7. *If* $\{u_m\}$ *is an increasing sequence of harmonic functions in the connected open set* Ω*, then, either the* u_m *converge uniformly on compact subsets of* Ω *to some harmonic function in* Ω *or they diverge to* $+\infty$ *uniformly on compact subsets of* Ω.
8. *The minimum of finitely many superharmonic functions is superharmonic.*
9. *Limits of increasing sequences of superharmonic functions in a connected open set are either identically* $+\infty$ *or superharmonic.*
10. *The Perron Process: suppose that* $\mathcal{V}$ *is a non-empty family of subharmonic functions in the connected open set* Ω *so that* $\mathcal{V}$ *contains the maximum of every two of its elements and that it contains* v_B *(see 5 above) for all* $v \in \mathcal{V}$ *and all closed discs* $\overline{B} \subseteq \Omega$.

Then, the upper envelope of $\mathcal{V}$ *is either identically* $+\infty$ *or harmonic in* Ω.

From this we get the corollary
11. *Let* $\mathcal{U}$ *be a non-empty family of superharmonic functions in the open* Ω *having at least one subharmonic minorant. Then the upper envelope of all subharmonic minorants of* $\mathcal{U}$ *is harmonic in* Ω *and it is called the largest harmonic minorant of* $\mathcal{U}$.
12. *For every extended-real-valued* f *defined in* $\partial\Omega$ *the functions* $\overline{H}_f^\Omega$ *and* $\underline{H}_f^\Omega$ *are defined in* Ω*, each of them is, in every connected component of* Ω*, either identically* $+\infty$ *or identically* $-\infty$ *or harmonic and they satisfy* $\underline{H}_f^\Omega \leq \overline{H}_f^\Omega$.

If this inequality is equality in Ω *and the common function is harmonic in* Ω*, we, then, call* f *resolutive, denote this common function by* H_f^Ω *and call it the generalized solution of the Problem of Dirichlet in* Ω *with boundary function* f.
13. We have *Wiener's Theorem: if there is some disc disjoint from the open* Ω*, then every function continuous in* $\partial\Omega$ *is resolutive.*

We just translate so that the disc has center at 0 and, then, apply the Kelvin Transform. Since the resulting open set is bounded, we may apply the original version of Wiener's Theorem.

The resulting functional

$$C(\partial\Omega) \ni f \mapsto H_f^\Omega(x_0) \in \mathbf{C}$$

is linear, non-negative and bounded with norm 1.
14. *For every open* Ω *which is disjoint from some disc and every* $x_0 \in \Omega$ *(even*

∞) the harmonic measure $d\mu_{x_0}^{\Omega}$ is defined in $\partial\Omega$. This is a complete probability measure whose σ-algebra of measurable sets contains $\mathcal{B}(\partial\Omega)$.

Every extended-real-valued f in $\partial\Omega$ is resolutive if and only if it is $d\mu_x^{\Omega}$-integrable for all $x \in \Omega$ and, in this case,

$$H_f^{\Omega}(x) \;=\; \int_{\partial\Omega} f(y)\; d\mu_x^{\Omega}(y)$$

for all $x \in \Omega$.

If $E \subseteq \partial\Omega$, then E is of zero harmonic measure with respect to Ω if and only if there is a non-negative superharmonic function in Ω having limit $+\infty$ at every point of E.

Borel subsets of $\partial\Omega$ of zero harmonic measure with respect to Ω are negligible regarding the assumptions of all versions of the Maximum/Minimum Principle.

15. Regularity of boundary points is defined as originally and we have the basic result that *for any open Ω, the Problem of Dirichlet is solvable for every continuous boundary function if the set is regular.*

The converse is, also, true, if Ω is disjoint from some disc.

16. *A useful sufficient condition for the regularity of a boundary point y_0 of an open set Ω is that there is a continuum containing y_0 and contained in $\overline{\mathbf{R}^2} \setminus \Omega$.*

If, in particular, no component of $\overline{\mathbf{R}^2} \setminus \Omega$ reduces to only one point, then Ω is a regular set.

Chapter 5

Green's Function

5.1 Definition

Definition 5.1 *Suppose that Ω is an open subset of $\mathbf{R^n}$ and let $x_0 \in \Omega$. Consider the family $\mathcal{U}^{\Omega}_{x_0}$ of all functions u with the properties*

1. *u is superharmonic in Ω,*
2. *u is a majorant of $-h_{x_0}$ in Ω.*

*In case this family is non-empty we say that Ω **has a Green's function with respect to the point** x_0 and, if $U^{\Omega}_{x_0}$ is its lower envelope, the function*

$$G^{\Omega}_{x_0} \;=\; h_{x_0} + U^{\Omega}_{x_0}$$

*is called **the Green's function of** Ω **with respect to the point** x_0.*

*In case this family is empty, we say that Ω **has no Green's function with respect to** x_0.*

Observe that $-h_{x_0}$ is a harmonic majorant of itself in every connected component of Ω not containing the point x_0. Thus, the existence of $G^{\Omega}_{x_0}$ is guarranteed in all these components and it is identically 0 there.

For the same reason, if O is the connected component of Ω which contains the point x_0, then the existence of $G^{\Omega}_{x_0}$ in Ω is equivalent to the existence of $G^{O}_{x_0}$ in O and, in this case,

$$G^{\Omega}_{x_0}(x) \;=\; G^{O}_{x_0}(x)\ , \qquad x \in O\ .$$

This remark helps us to reduce the study of the Green's function to the case of *connected* open sets.

Proposition 5.1 *If $\Omega \subseteq \mathbf{R^n}$ has a Green's function, $G^{\Omega}_{x_0}$, with respect to its point x_0, then*

1. *$G^{\Omega}_{x_0} - h_{x_0}$ is harmonic in Ω,*

2. *$G^{\Omega}_{x_0}$ is superharmonic in Ω and harmonic in $\Omega \setminus \{x_0\}$,*

3. *$G^{\Omega}_{x_0}(x) > 0$ for every x in the connected component of Ω containing x_0 and $G^{\Omega}_{x_0}(x) = 0$ for all other $x \in \Omega$.*

Proof:
1. By its definition, $G^{\Omega}_{x_0} - h_{x_0}$ is the least harmonic majorant of the subharmonic function $-h_{x_0}$ in Ω.
2. This is obvious.
3. If O is any connected component of Ω not containing x_0, then $-h_{x_0}$ is, clearly, the least harmonic majorant of itself in O. Therefore, $G^{\Omega}_{x_0} = 0$ identically in O.

If O is the connected component containing x_0, then $G^{\Omega}_{x_0} \geq 0$ everywhere in O. In case $G^{\Omega}_{x_0}(x) = 0$ for at least one $x \in O$, then, by the Minimum Principle, $G^{\Omega}_{x_0} = 0$ identically in O, implying that h_{x_0} is harmonic in O.

5.2 Green's function, the problem of Dirichlet and harmonic measure

Proposition 5.2 *If Ω is any bounded open subset of $\mathbf{R^n}$, then Ω has a Green's function with respect to every $x_0 \in \Omega$ and*

$$G^{\Omega}_{x_0}(x) \ = \ h_{x_0}(x) + H^{\Omega}_{-h_{x_0}}(x) \ = \ h_{x_0}(x) - \int_{\partial\Omega} h_{x_0}(y) \, d\mu^{\Omega}_{x}(y)$$

for all $x \in \Omega$.

Proof:
1. Since Ω is bounded, there is a large enough constant playing the role of a superharmonic majorant of $-h_{x_0}$ in Ω.

Therefore, by definition, Ω has a Green's function with respect to x_0.

Since $-h_{x_0}$ is continuous in $\partial\Omega$, Wiener's Theorem implies that this function is resolutive with respect to Ω. One can see this, directly, as follows.

$-h_{x_0}$ is subharmonic in Ω, bounded from above in Ω and, hence, belongs to $\Psi^{\Omega}_{-h_{x_0}}$. Therefore,

$$-h_{x_0}(x) \ \leq \ \underline{H}^{\Omega}_{-h_{x_0}}(x)$$

for all $x \in \Omega$. This, easily, implies that $\underline{H}^{\Omega}_{-h_{x_0}}$ is harmonic and, by the Maximum-Minimum Principle, bounded from below in Ω. Therefore $\underline{H}^{\Omega}_{-h_{x_0}}$ belongs to $\Phi^{\Omega}_{-h_{x_0}}$ and, thus,

$$\overline{H}^{\Omega}_{-h_{x_0}} \ \leq \ \underline{H}^{\Omega}_{-h_{x_0}}$$

everywhere in Ω, implying that $-h_{x_0}$ is resolutive with respect to Ω.

By the continuity of $-h_{x_0}$ in $\partial\Omega$ and the definition of harmonic measure,

$$H^{\Omega}_{-h_{x_0}}(x) \ = \ -\int_{\partial\Omega} h_{x_0}(y) \, d\mu^{\Omega}_{x}(y)$$

for all $x \in \Omega$.
2. Assume that u belongs to the family $\Phi^{\Omega}_{-h_{x_0}}$, implying

$$\liminf_{\Omega \ni x \to y} \big(u(x) + h_{x_0}(x)\big) \;\geq\; 0$$

for all $y \in \partial\Omega$.

From the Minimum Principle, we have that $u \geq -h_{x_0}$ everywhere in Ω and, hence, u belongs to $\mathcal{U}^{\Omega}_{x_0}$.

If, conversely, u belongs to $\mathcal{U}^{\Omega}_{x_0}$, then it is automatically true that u is bounded from below in Ω and that

$$\liminf_{\Omega \ni x \to y} u(x) \;\geq\; -h_{x_0}(y)$$

for all $y \in \partial\Omega$. Therefore $u \in \Phi^{\Omega}_{-h_{x_0}}$.

Hence, the families $\mathcal{U}^{\Omega}_{x_0}$ and $\Phi^{\Omega}_{-h_{x_0}}$ are identical and, thus,

$$U^{\Omega}_{x_0} \;=\; H^{\Omega}_{-h_{x_0}}$$

everywhere in Ω.

5.3 A few examples

Proposition 5.3 *1. $\mathbf{R^2}$ has no Green's function with respect to any point of it.*

2. If $n \geq 3$, then for every $x_0 \in \mathbf{R^n}$, $G^{\mathbf{R^n}}_{x_0} = h_{x_0}$ in $\mathbf{R^n}$.

Proof:
1. Assume that there is a superharmonic majorant u of $-h_{x_0}$ in $\mathbf{R^2}$.

For the arbitrary ball $B(x_0; R)$, we, then, have $u(x) \geq \log R$ for all x in $S(x_0; R)$ and, by the Minimum Principle,

$$u(x) \;\geq\; \log R$$

in $B(x_0; R)$.

Since R is arbitrary, we get a contradiction.
2. Assume that $n \geq 3$ and let u be any superharmonic majorant of $-h_{x_0}$ in $\mathbf{R^n}$.

For every ball $B(x_0; R)$, we, then, have $u(x) \geq -\frac{1}{R^{n-2}}$ for all $x \in S(x_0; R)$ and, by the Minimum Principle, $u(x) \geq -\frac{1}{R^{n-2}}$ for all $x \in B(x_0; R)$. Therefore,

$$u(x) \;\geq\; 0$$

for all $x \in \mathbf{R^n}$, implying that 0 is the smallest superharmonic majorant of $-h_{x_0}$ in $\mathbf{R^n}$.

We conclude that

$$G^{\mathbf{R^n}}_{x_0}(x) \;=\; h_{x_0}(x)$$

for all $x \in \mathbf{R^n}$.

Example

Take $\Omega = B(x_1; R)$ and any $x_0 \in B(x_1; R)$, $x_0 \neq x_1$ and consider the symmetric $x_0^* = x_1 + \frac{R^2}{|x_0 - x_1|^2}\,(x_0 - x_1)$ of x_0 with respect to $S(x_1; R)$.

If $n \geq 3$, the function $\frac{|x_1 - x_0^*|^{n-2}}{R^{n-2}}\, h_{x_0^*}$ is harmonic in $\mathbf{R^n} \setminus \{x_0^*\}$ and coincides with h_{x_0} in $S(x_1; R)$.

Therefore,

$$G_{x_0}^{B(x_1;R)} \;=\; h_{x_0} - \frac{|x_1 - x_0^*|^{n-2}}{R^{n-2}}\, h_{x_0^*}$$

in $B(x_1; R)$.

If $n = 2$, then, similarly,

$$G_{x_0}^{B(x_1;R)} \;=\; h_{x_0} - h_{x_0^*} - \log\Big(\frac{|x_1 - x_0^*|}{R}\Big)$$

in $B(x_1; R)$.

In case $x_0 = x_1$, then

$$G_{x_1}^{B(x_1;R)} \;=\; h_{x_1} - h_*(R)$$

in $B(x_1; R)$, which can be recognized as the limit of both previous cases as $x_0 \to x_1$.

5.4 Monotonicity

Theorem 5.1 *If the open set $\Omega \subseteq \mathbf{R^n}$ has a Green's function with respect to some $x_0 \in \Omega$ and if Ω' is another open set with*

$$x_0 \in \Omega' \subseteq \Omega\ ,$$

then Ω' has, also, a Green's function with respect to x_0 and

$$G_{x_0}^{\Omega'} \;\leq\; G_{x_0}^{\Omega}$$

everywhere in Ω'.

Proof:

It is clear that every element of $\mathcal{U}_{x_0}^{\Omega}$ belongs to $\mathcal{U}_{x_0}^{\Omega'}$.

Corollary 5.1 *If $n \geq 3$, then every open subset of $\mathbf{R^n}$ has a Green's function in all of its components.*

Theorem 5.2 *Let $\{\Omega_m\}$ be an increasing sequence of open subsets of $\mathbf{R^n}$ with $\Omega = \cup_{m=1}^{+\infty}\Omega_m$ and $x_0 \in \Omega_1$.*

1. *If all Ω_m have a Green's function $G_{x_0}^{\Omega_m}$ with respect to x_0, then either*

$$G_{x_0}^{\Omega_m} \uparrow +\infty$$

everywhere in the component of Ω containing x_0 and Ω has no Green's function with respect to x_0 or, in the opposite case, Ω has a Green's function with respect to x_0 and

$$G_{x_0}^{\Omega_m} \uparrow G_{x_0}^{\Omega}$$

in Ω.

2. *If Ω has a Green's function with respect to x_0, then all Ω_m have a Green's function with respect to the same point and*

$$G_{x_0}^{\Omega_m} \uparrow G_{x_0}^{\Omega}$$

in Ω.

Proof:

It is obvious, from Proposition 5.1(3) and Theorem 5.1, that in every component of Ω which does not contain x_0 the Green's functions of all sets considered are identically 0. Therefore, the proof reduces to the case of a connected Ω.

Assume that all Ω_m have a Green's function with respect to x_0.

Take an arbitrary $\overline{B(x;r)} \subseteq \Omega$. Then, for a large enough m_0, $\overline{B(x;r)} \subseteq \Omega_{m_0}$ and Proposition 5.1(1) and Theorem 5.1 imply that $\{G_{x_0}^{\Omega_m} - h_{x_0}\}_{m=m_0}^{+\infty}$ is an increasing sequence of harmonic functions in $B(x;r)$.

Therefore, by Theorem 1.16, every point of Ω has some neighborhood where the sequence $\{G_{x_0}^{\Omega_m} - h_{x_0}\}$, eventually, either converges to a harmonic function or diverges to $+\infty$. Since Ω is connected, this sequence either converges to a harmonic function everywhere in Ω or diverges to $+\infty$ everywhere in Ω.

In the first case the harmonic limit-function majorizes $-h_{x_0}$ in Ω and, hence, belongs to $\mathcal{U}_{x_0}^{\Omega}$. Therefore Ω has a Green's function with respect to x_0 and the above limit majorizes $G_{x_0}^{\Omega} - h_{x_0}$ in Ω.

From Theorem 5.1, the same limit-function is majorized by $G_{x_0}^{\Omega} - h_{x_0}$ in Ω and we, finally, get

$$G_{x_0}^{\Omega_m} \uparrow G_{x_0}^{\Omega}$$

in Ω.

Conversely, if Ω has a Green's function with respect to x_0, then all Ω_m have a Green's function with respect to x_0 and $G_{x_0}^{\Omega_m} \leq G_{x_0}^{\Omega}$ in Ω_m for all m.

Therefore, the limit of $G_{x_0}^{\Omega_m}$ cannot be identically $+\infty$ in Ω.

5.5 Symmetry

Theorem 5.3 *Let x_0 and x_1 belong to the same component of the open $\Omega \subseteq \mathbf{R^n}$. If Ω has a Green's function with respect to x_0, then it has a Green's function with respect to x_1 and*

$$G_{x_0}^{\Omega}(x_1) \;=\; G_{x_1}^{\Omega}(x_0)\ .$$

Proof:
1. Assume that Ω is bounded. From Proposition 5.2, we have that

$$G_{x_0}^{\Omega}(x) \ = \ h_{x_0}(x) - \int_{\partial\Omega} h_{x_0}(y) \ d\mu_x^{\Omega}(y)$$

for all $x \in \Omega$.

Now, observe, by interchanging differentiations and integration, that the integral is harmonic as a function of x_0 in Ω and, hence, $G_{x_0}^{\Omega}(x_1) - h_{x_0}(x_1)$ is, as a function of x_0, a harmonic majorant of $-h_{x_1}$ in Ω.

Therefore,

$$G_{x_1}^{\Omega}(x_0) \ \leq \ G_{x_0}^{\Omega}(x_1) \ .$$

The reverse inequality is proved symmetrically.

2. If Ω is not bounded, consider the sets $\Omega_m = \Omega \cap B(0;m)$.

Assuming that Ω has a Green's function with respect to x_0, we get that $G_{x_0}^{\Omega}(x_1) < +\infty$.

For large enough m, x_0 and x_1 are both included in the same component of Ω_m and we apply part 1 for Ω_m :

$$G_{x_1}^{\Omega_m}(x_0) \ = \ G_{x_0}^{\Omega_m}(x_1) \ \leq \ G_{x_0}^{\Omega}(x_1) \ < \ +\infty$$

and, thus, the limit of $G_{x_1}^{\Omega_m}$ is not identically $+\infty$ in the component of Ω containing x_1. Theorem 5.2 implies that Ω has a Green's function with respect to x_1 and

$$G_{x_1}^{\Omega}(x_0) \ = \ \lim_{m\to+\infty} G_{x_1}^{\Omega_m}(x_0) \ = \ \lim_{m\to+\infty} G_{x_0}^{\Omega_m}(x_1) \ = \ G_{x_0}^{\Omega}(x_1) \ .$$

Observe that both sides of the equality of Theorem 5.3 are equal to 0, if x_0 and x_1 belong to different components of Ω.

Definition 5.2 *We say that the open set $\Omega \subseteq \mathbf{R^n}$ has a Green's function in any one of its connected components, if it has a Green's function with respect to at least one of the points of that component.*

5.6 Green's function and regularity

Theorem 5.4 *Let $\Omega \subseteq \mathbf{R^n}$ be an open set having a Green's function in every one of its components and $y_0 \in \partial\Omega$. If*

$$\lim_{\Omega \ni x \to y_0} G_z^{\Omega}(x) \ = \ 0$$

for at least one z in every component of Ω, then y_0 is a regular boundary point.

If Ω is bounded and y_0 is a regular boundary point, then the above limit holds for every $z \in \Omega$.

Proof:

Assuming that y_0 is a regular boundary point of the bounded Ω, we get, by Theorem 3.7, that

$$\lim_{\Omega\ni x\to y_0} G_{x_0}^{\Omega}(x) \;=\; h_{x_0}(y_0) - \lim_{\Omega\ni x\to y_0} H_{h_{x_0}}^{\Omega}(x) \;=\; 0\ .$$

Now, let $\lim_{\Omega\ni x\to y_0} G_z^{\Omega}(x) = 0$ for at least one z in every component of Ω. If Ω has finitely many components O_j, $1 \leq j \leq M$, and $z_j \in O_j$ is such that $\lim_{\Omega\ni x\to y_0} G_{z_j}^{\Omega}(x) = 0$, then we form the function u which coincides in each O_j with $G_{z_j}^{\Omega}$.

This u is, obviously, a barrier for Ω at y_0.

If Ω has infinitely many components O_j, $j \in \mathbf{N}$, then we form the positive superharmonic function u in Ω which coincides in each O_j with $\min\big(G_{z_j}^{\Omega}, \frac{1}{j}\big)$.

For arbitrary $\epsilon > 0$ we take $j_0 \geq \frac{1}{\epsilon}$ and we have

$$\lim_{\cup_{j=1}^{j_0-1} O_j\ni x\to y_0} u(x) \;=\; 0$$

and

$$\limsup_{\cup_{j=j_0}^{+\infty} O_j\ni x\to y_0} u(x) \;\leq\; \epsilon\ .$$

Hence,

$$\limsup_{\Omega\ni x\to y_0} u(x) \;\leq\; \epsilon$$

and, since ϵ is arbitrary, u is a barrier for Ω at y_0.

5.7 Extensions of Green's Function

In this section we shall describe two possible extensions of a Green's function in the complement of its domain of definition Ω. The second extension is for general bounded open sets Ω, while the first is for regular bounded open Ω and it is intuitively simpler.

Proposition 5.4 *Suppose that Ω is a bounded regular open set and $x_0 \in \Omega$. Then, the function $G_{x_0}^{\Omega}$, extended as identically 0 in $\mathbf{R}^{\mathbf{n}} \setminus \Omega$, has the following properties.*

1. *It is positive in the component O of Ω containing x_0 and it is identically 0 in $\mathbf{R}^{\mathbf{n}} \setminus O$,*

2. *it is continuous and subharmonic in $\mathbf{R}^{\mathbf{n}} \setminus \{x_0\}$ and*

3. *it is harmonic in $O \setminus \{x_0\}$ and its difference with h_{x_0} is harmonic in O.*

Proof:

All statements are already known, except for the second. The continuity is a corollary of Theorem 5.4 and the subharmonicity is a consequence of the continuity and of the simple fact that, for every $y \in \partial O$, the value of the function is 0 while the area-means over every $B(y; r)$ are, clearly, positive.

Assume, now, that Ω is a bounded open set and $x_0 \in \Omega$. By Proposition 5.2,

$$G^{\Omega}_{x_0}(x) \; = \; h_{x_0}(x) - \int_{\partial\Omega} h_{x_0}(y) \; d\mu^{\Omega}_{x}(y)$$

for all $x \in \Omega$.

Take x in the same connected component O of Ω with x_0 and, since $G^{\Omega}_{x}(x_0) = G^{\Omega}_{x_0}(x)$, we have

$$G^{\Omega}_{x_0}(x) \; = \; h_{x}(x_0) - \int_{\partial\Omega} h_{x}(y) \; d\mu^{\Omega}_{x_0}(y)$$

for all x in the component O of Ω containing x_0.

If $x \in \Omega$ does not belong to O, the left side of the last formula is 0. Since, by Proposition 3.3, $d\mu^{\Omega}_{x_0}$ is supported in ∂O and the function h_x is harmonic in O and continuous in $\overline{O}$, the right side of the last formula is, also, 0.

Observe that this right side is, for the same reason, 0 for every $x \notin \overline{O}$.

Therefore,

$$G^{\Omega}_{x_0}(x) \; = \; h_{x_0}(x) - \int_{\partial\Omega} h_{x}(y) \; d\mu^{\Omega}_{x_0}(y)$$

for all $x \in \Omega$.

By Theorem 2.8, the above integral is, as a function of x, superharmonic in $\mathbf{R^n}$ and harmonic in $\mathbf{R^n} \setminus \partial O$.

Hence, we have proved the

Proposition 5.5 *Suppose that Ω is a bounded open set and $x_0 \in \Omega$. Then the function*

$$h_{x_0}(x) - \int_{\partial\Omega} h_{x}(y) \; d\mu^{\Omega}_{x_0}(y) \; , \qquad x \in \mathbf{R^n} \; ,$$

is an extension of $G^{\Omega}_{x_0}$ with the following properties.

1. *It is positive in the component O of Ω containing x_0 and identically* 0 *in $\mathbf{R^n} \setminus \overline{O}$,*

2. *it is subharmonic in $\mathbf{R^n} \setminus \{x_0\}$ and*

3. *it is harmonic in $O \setminus \{x_0\}$ and its difference with h_{x_0} is harmonic in O.*

5.8 Green's potentials

Now, let Ω be a bounded open set and $d\mu$ be a non-negative Borel measure with compact support in Ω.

Consider the function

$$U_\Omega^{d\mu}(x) \;=\; \int_\Omega G_x^\Omega(y)\;d\mu(y)\;, \qquad x\in\Omega\;.$$

From Proposition 5.2,

$$\begin{aligned} U_\Omega^{d\mu}(x) \;&=\; \int_{supp(d\mu)} h_x(y)\;d\mu(y) - \int_{supp(d\mu)}\int_{\partial\Omega} h_y(z)\;d\mu_x^\Omega(z)\;d\mu(y)\\ &=\; U_h^{d\mu}(x) - \int_{\partial\Omega}\int_{supp(d\mu)} h_z(y)\;d\mu(y)\;d\mu_x^\Omega(z)\;, \end{aligned}$$

where the interchange of integrations is trivial to justify, since $supp(d\mu)$ and $\partial\Omega$ are a positive distance apart. For the same reason, the inner integral defines a continuous function of z in $\partial\Omega$ and, thus, the last term is a harmonic function of x in Ω.

We conclude, by Theorem 2.8 and Theorem 2.15, that

1. $U_\Omega^{d\mu}$ is superharmonic in Ω and harmonic in $\Omega\setminus supp(d\mu)$,
2. $\Delta U_\Omega^{d\mu}=\kappa_n d\mu$ as distributions in Ω and
3. $U_\Omega^{d\mu}\geq 0$ everywhere in Ω.

Now, only assume that the open set Ω has a Green's function in every one of its components.

Consider any open exhaustion $\{\Omega_{(m)}\}$ of Ω and the restrictions $d\mu_{\Omega_{(m)}}$ of the non-negative Borel measure $d\mu$ in Ω.

By Theorem 5.1 and by the previous discussion, the sequence $\{U_{\Omega_{(m+1)}}^{d\mu_{\Omega_{(m)}}}\}_{m=k}^{+\infty}$ is an increasing sequence of superharmonic functions in $\Omega_{(k+1)}$ which are harmonic in $\Omega_{(k+1)}\setminus supp(d\mu)$. Therefore, it either diverges to $+\infty$ everywhere in $\Omega_{(k+1)}$ or it converges to a superharmonic function in $\Omega_{(k+1)}$ which is harmonic in $\Omega_{(k+1)}\setminus supp(d\mu)$. Since k is arbitrary, Theorem 5.2 and the Monotone Convergence Theorem give the next result.

Theorem 5.5 *Let $\Omega\subseteq\mathbf{R^n}$ be an open set with a Green's function in every one of its components and $d\mu$ be a non-negative Borel measure in Ω.*

Assume that $\int_\Omega G_x^\Omega(y)\;d\mu(y)<+\infty$ for at least one x in every connected component of Ω.

Then the function

$$U_\Omega^{d\mu}(x) \;=\; \int_\Omega G_x^\Omega(y)\;d\mu(y)\;, \qquad x\in\Omega\;,$$

is a non-negative superharmonic function in Ω, harmonic in $\Omega\setminus supp(d\mu)$.

Also,

$$\Delta U_{\Omega}^{d\mu} \;=\; \kappa_n d\mu$$

as distributions in Ω

Proof:

Since only the distribution equality remains to be proved, we take any k and observe that, for all $m \geq k$, the difference $U_{\Omega_{(m+1)}}^{d\mu_{\Omega_{(m)}}} - U_{\Omega_{(k+1)}}^{d\mu_{\Omega_{(k)}}}$ is harmonic in $\Omega_{(k)}$. This is true because, by the discussion before the theorem,

$$\Delta\Big(U_{\Omega_{(m+1)}}^{d\mu_{\Omega_{(m)}}} - U_{\Omega_{(k+1)}}^{d\mu_{\Omega_{(k)}}}\Big) \;=\; \kappa_n(d\mu_{\Omega_{(m)}} - d\mu_{\Omega_{(k)}}) \;=\; 0$$

as distributions in $\Omega_{(k)}$.

Therefore, by the monotonicity of the sequence, the function $U_{\Omega}^{d\mu} - U_{\Omega_{(k+1)}}^{d\mu_{\Omega_{(k)}}}$ is, also, harmonic in $\Omega_{(k)}$ and, thus,

$$\Delta U_{\Omega}^{d\mu} \;=\; \Delta U_{\Omega_{(k+1)}}^{d\mu_{\Omega_{(k)}}} \;=\; \kappa_n d\mu_{\Omega_{(k)}} \;=\; \kappa_n d\mu$$

as distributions in $\Omega_{(k)}$. Since k is arbitrary, the proof is finished.

Definition 5.3 *Let* $\Omega \subseteq \mathbf{R^n}$ *be an open set with a Green's function in every one of its components and* $d\mu$ *be a non-negative Borel measure in* Ω.

Assume that $\int_\Omega G_x^\Omega(y)\; d\mu(y) < +\infty$ *for at least one* x *in every connected component of* Ω.

Then the superharmonic function

$$U_{\Omega}^{d\mu}(x) \;=\; \int_\Omega G_x^\Omega(y)\; d\mu(y)\;, \qquad x \in \Omega\;,$$

is called the ***Green's potential of*** $d\mu$ ***with respect to*** Ω.

If $d\mu = f\, dm$ *for some non-negative* f *locally integrable in* Ω, *then the Green's potential is, also, denoted by* U_Ω^f *and it is called the* ***Green's potential of*** f ***with respect to*** Ω.

Lemma 5.1 *Let* Ω *be a bounded regular open set,* $d\mu$ *be a non-negative Borel measure in* Ω *defining its Green's potential with respect to* Ω.

Then the largest harmonic minorant of $U_\Omega^{d\mu}$ *in* Ω *is identically* 0 *in* Ω.

Proof:

Without loss of generality, we may assume that Ω is connected.

Consider any $x_0 \in \Omega$ with $U_\Omega^{d\mu}(x_0) < +\infty$ and an open exhaustion $\{\Omega_{(m)}\}$ of Ω. Then,

$$U_\Omega^{d\mu}(x_0) \;=\; \int_{\overline{\Omega_{(m)}}} G_{x_0}^\Omega(y)\; d\mu(y) + \int_{\Omega\setminus\overline{\Omega_{(m)}}} G_{x_0}^\Omega(y)\; d\mu(y)$$

where, by the Monotone Convergence Theorem, the last term tends to 0 as $m \to +\infty$.

Choose m_0 so that

$$\int_{\Omega \setminus \overline{\Omega_{(m_0)}}} G^{\Omega}_{x_0}(y)\ d\mu(y) \ < \ \epsilon \ .$$

Let $d\mu_{m_0}$ be the restriction of $d\mu$ on $\overline{\Omega_{(m_0)}}$ and $d\nu_{m_0}$ be the restriction of $d\mu$ on $\Omega \setminus \overline{\Omega_{(m_0)}}$.

We, first of all, have

$$U^{d\nu_{m_0}}_{\Omega}(x_0) \ < \ \epsilon \ .$$

We, then, observe that, for each $y \in \overline{\Omega_{(m_0)}}$, the function G^{Ω}_{y} is positive and harmonic in $\Omega \setminus \overline{\Omega_{(m_0)}}$, and, by Proposition 5.4, it can become continuous in $\overline{\Omega} \setminus \overline{\Omega_{(m_0)}}$ with values 0 everywhere in $\partial\Omega$. Therefore, there is some large $k > m_0$ so that $G^{\Omega}_{y}(x) < \epsilon$ for some particular value of $y \in \overline{\Omega_{(m_0)}}$ and all $x \in \Omega \setminus \Omega_{(k)}$.

From Harnack's inequalities we get that there is a constant $C > 0$ so that

$$G^{\Omega}_{y}(x) \ = \ G^{\Omega}_{x}(y) \ < \ C\epsilon$$

for *all* $y \in \overline{\Omega_{(m_0)}}$ and all $x \in \Omega \setminus \Omega_{(k)}$.

This implies

$$U^{d\mu_{m_0}}_{\Omega}(x) \ < \ C d\mu(\Omega_{(m_0)})\ \epsilon$$

for all $x \in \Omega \setminus \Omega_{(k)}$.

Now, let u be the largest harmonic minorant of $U^{d\mu}_{\Omega}$ in Ω. Since the identically 0 function is a harmonic minorant of $U^{d\mu}_{\Omega}$ in Ω, we have

$$u \ \geq \ 0$$

in Ω.

By the continuity of $U^{d\mu_{m_0}}_{\Omega}$ in $\partial\Omega_{(k)}$,

$$H^{\Omega_{(k)}}_{U^{d\mu_{m_0}}_{\Omega}}(x_0) \ = \ \int_{\partial\Omega_{(k)}} U^{d\mu_{m_0}}_{\Omega}(x)\ d\mu^{\Omega_{(k)}}_{x_0}(x) \ .$$

Also, by the superharmonicity of $U^{d\nu_{m_0}}_{\Omega}$ in $\Omega \supseteq \Omega_{(k)}$,

$$H^{\Omega_{(k)}}_{U^{d\nu_{m_0}}_{\Omega}}(x_0) \ \leq \ U^{d\nu_{m_0}}_{\Omega}(x_0) \ .$$

Hence,

$$\begin{aligned}
u(x_0) &= H^{\Omega_{(k)}}_{u}(x_0) \ \leq \ H^{\Omega_{(k)}}_{U^{d\mu}_{\Omega}}(x_0) \\
&= H^{\Omega_{(k)}}_{U^{d\mu_{m_0}}_{\Omega}}(x_0) + H^{\Omega_{(k)}}_{U^{d\nu_{m_0}}_{\Omega}}(x_0) \\
&\leq \int_{\partial\Omega_{(k)}} U^{d\mu_{m_0}}_{\Omega}(x)\ d\mu^{\Omega_{(k)}}_{x_0}(x) + U^{d\nu_{m_0}}_{\Omega}(x_0) \\
&\leq C d\mu(\Omega_{(m_0)})\ \epsilon + \epsilon \ .
\end{aligned}$$

Since ϵ is arbitrary, we get $u(x_0) = 0$ and, from the Maximum-Minimum Principle, u is identically 0 in Ω.

5.9 The Decomposition Theorem of F. Riesz

Theorem 5.6 *(F. Riesz Decomposition) Let $\Omega \subseteq \mathbf{R^n}$ be any open set. Suppose that there is a superharmonic function u in Ω which is not harmonic in any component of Ω and let $d\mu = \frac{1}{\kappa_n}\Delta u$ be the associated non-negative Borel measure in Ω. Suppose, also, that u has a subharmonic minorant in Ω.*

Then Ω has a Green's function in each of its components, the Green's potential of $d\mu$ is defined in Ω and

$$u \;=\; U_{\Omega}^{d\mu} + u^*$$

everywhere in Ω, where u^* is the largest harmonic minorant of u in Ω.

Proof:

Let $\{\Omega_{(m)}\}$ be an open exhaustion of Ω all of whose terms are regular. In fact, it is easy to see that the usual construction, given in section 0.1.1, produces $\Omega_{(m)}$ which satisfy the ball-criterion at every one of their boundary points.

If $d\mu_m$ is the restriction of $d\mu$ in $\Omega_{(m)}$, then, by Theorem 2.17,

$$u \;=\; U_h^{d\mu_m} + w_m$$

in $\Omega_{(m)}$, where w_m is a harmonic function in $\Omega_{(m)}$. Therefore,

$$u \;=\; U_{\Omega_{(m)}}^{d\mu_m} + v_m$$

in $\Omega_{(m)}$, where v_m is another harmonic function in $\Omega_{(m)}$. To see this, we observe that, by Theorems 2.15 and 5.5, the functions $U_h^{d\mu_m}$ and $U_{\Omega_{(m)}}^{d\mu_m}$ have the same distributional Laplacian in $\Omega_{(m)}$ and, hence, by Theorem 1.20, they differ by a function harmonic in $\Omega_{(m)}$.

If u^* is the largest harmonic minorant of u in Ω, we have

$$u^* - v_m \;\le\; U_{\Omega_{(m)}}^{d\mu_m}$$

in $\Omega_{(m)}$ and, since the largest harmonic minorant of $U_{\Omega_{(m)}}^{d\mu_m}$ in $\Omega_{(m)}$ is, by Lemma 5.1, identically 0,

$$v_m \;\ge\; u^*$$

in $\Omega_{(m)}$, whence

$$U_{\Omega_{(m)}}^{d\mu_m} \;\le\; u - u^*$$

in $\Omega_{(m)}$.

Taking any x_0 with $u(x_0) < +\infty$, by Theorem 5.2 and the Monotone Convergence Theorem,

$$\int_{\Omega} \lim_{m\to+\infty} G_{x_0}^{\Omega_{(m)}}(x)\; d\mu(x) \;\le\; u(x_0) - u^*(x_0) \;<\; +\infty \;.$$

This, by Theorem 5.2 again, implies that Ω has a Green's function in its component which contains x_0, and hence in every one of its components.

Also,

$$U_\Omega^{d\mu} \;\leq\; u - u^*$$

in Ω and we get that $U_\Omega^{d\mu}$ is well defined as a superharmonic function in Ω.

The functions v_m decrease towards some harmonic function v in Ω with

$$u \;=\; U_\Omega^{d\mu} + v$$

in Ω and, hence,

$$u^* \;\leq\; v$$

in Ω.

On the other hand, from $u = U_\Omega^{d\mu} + v \geq v$, we get

$$u^* \;\geq\; v$$

in Ω and we conclude

$$v \;=\; u^*$$

in Ω, finishing the proof.

Theorem 5.7 *Let $\Omega \subseteq \mathbf{R^n}$ be any open set with a Green's function in each of its components. Then the following are equivalent.*

1. *u is superharmonic in Ω with largest harmonic minorant identically 0 in Ω.*

2. *u is the Green's potential with respect to Ω of some (unique) non-negative Borel measure in Ω.*

Proof:

That 1 implies 2 is just a consequence of Theorem 5.6.

The only thing that we have to prove is that, if $U_\Omega^{d\mu}$ is a Green's potential, then its largest harmonic minorant in Ω is the constant 0.

If u^* is the largest harmonic minorant of $U_\Omega^{d\mu}$ in Ω, then, by Theorem 5.6,

$$U_\Omega^{d\mu} \;=\; U_\Omega^{d\nu} + u^*$$

in Ω, where $d\nu = \frac{1}{\kappa_n}\Delta U_\Omega^{d\mu} = d\mu$.

Thus, $u^* = 0$ in Ω.

5.10 Green's function and harmonic measure

Lemma 5.2 *Suppose that $0 < d < 1$ and ϕ is a function defined in $S(0;1)$, integrable with respect to $d\sigma$ with the properties*

1. *$\phi = 0$ in $S(0;1) \setminus B(e_1;d)$,*

2. $|\phi(y)| \leq |y-e_1|^2$ *for all* y *in* $S(0;1) \cap B(e_1;d)$,

where $e_1 = (1,0,\dots,0)$.

Then,

$$|\overrightarrow{grad\, P_\phi(\cdot\, ; 0,1)}(te_1)| \ \leq \ C(n)d$$

for all t *with* $1-d<t<1$*, where* $C(n)$ *depends only on the dimension.*

Proof:

We have that

$$P_\phi(x;0,1) \ = \ \frac{1}{\omega_{n-1}} \int_{S(0;1)} \frac{1-|x|^2}{|y-x|^n} \ \phi(y) \ d\sigma(y) \ .$$

An easy calculation gives

$$\frac{\partial}{\partial x_j} \frac{1-|x|^2}{|y-x|^n} \ = \ -\frac{2x_j}{|y-x|^n} + n(y_j - x_j)\frac{1-|x|^2}{|y-x|^{n+2}} \ .$$

Therefore, if $x = te_1$ and $2 \leq j \leq n$,

$$\frac{\partial}{\partial x_j} \frac{1-|\cdot|^2}{|y-\cdot|^n}(te_1) \ = \ ny_j \frac{1-t^2}{|y-te_1|^{n+2}}$$

and

$$\begin{aligned}
\Big|\frac{\partial P_\phi(\cdot\, ;0,1)}{\partial x_j}(te_1)\Big| \ &\leq \ \frac{n}{\omega_{n-1}} \int_{S(0;1)\cap B(e_1;d)} |y_j| \frac{1-t^2}{|y-te_1|^{n+2}} \ |y-e_1|^2 \ d\sigma(y) \\
&\leq \ \frac{2nd(1-t)}{\omega_{n-1}} \int_{V_1} \frac{|y-e_1|^2}{|y-te_1|^{n+2}} \ d\sigma(y) \\
&+ \ \frac{2nd(1-t)}{\omega_{n-1}} \int_{V_2} \frac{|y-e_1|^2}{|y-te_1|^{n+2}} \ d\sigma(y) \ ,
\end{aligned}$$

where

$$V_1 \ = \ \{y \in S(0;1) : |y-e_1| < 1-t\}$$

and

$$V_2 \ = \ \{y \in S(0;1) : 1-t \leq |y-e_1| < d\} \ .$$

The first integral is

$$\leq \ \frac{C}{(1-t)^{n+2}} \int_{V_1} |y-e_1|^2 \ d\sigma(y) \ \leq \ \frac{C(n)}{1-t} \ ,$$

while, the second integral is

$$\leq \ C \int_{V_2} \frac{1}{|y-e_1|^n} \ d\sigma(y) \ \leq \ \frac{C(n)}{1-t} \ .$$

We, thus, get

$$\Big|\frac{\partial P_\phi(\cdot\ ;0,1)}{\partial x_j}(te_1)\Big| \ \le\ C(n)d$$

for $2 \le j \le n$.

If $j = 1$,

$$\frac{\partial}{\partial x_1}\frac{1-|\cdot|^2}{|y-\cdot|^n}(te_1) \ =\ -\frac{2t}{|y-te_1|^n} + n(y_1-t)\frac{1-t^2}{|y-te_1|^{n+2}}\ .$$

The absolute value of the second term is $\le nd\frac{1-t^2}{|y-te_1|^{n+2}}$ for all $y \in S(0;1)\cap B(e_1;d)$ and the previous argument applies, word for word, to show that its integral is $\le C(n)d$.

The absolute value of the first term is $\le \frac{2}{|y-te_1|^n}$ and, hence,

$$\begin{aligned}
\Big|\frac{\partial P_\phi(\cdot\ ;0,1)}{\partial x_1}(te_1)\Big| &\le \frac{2n}{\omega_{n-1}}\int_{S(0;1)\cap B(e_1;d)}\frac{|y-e_1|^2}{|y-te_1|^n}\,d\sigma(y) + C(n)d\\
&= \frac{2n}{\omega_{n-1}}\int_{V_1}\frac{|y-e_1|^2}{|y-te_1|^n}\,d\sigma(y)\\
&+ \frac{2n}{\omega_{n-1}}\int_{V_2}\frac{|y-e_1|^2}{|y-te_1|^n}\,d\sigma(y) + C(n)d\\
&\le \frac{C(n)}{(1-t)^n}\int_{V_1}|y-e_1|^2\,d\sigma(y)\\
&+ C(n)\int_{V_2}\frac{1}{|y-e_1|^{n-2}}\,d\sigma(y) + C(n)d\\
&\le C(n)d\ .
\end{aligned}$$

The above estimates of $\Big|\frac{\partial P_\phi(\cdot\ ;0,1)}{\partial x_j}(te_1)\Big|$ for $1 \le j \le n$ conclude the proof.

Lemma 5.3 *Suppose that $\{\phi_m\}$ is a sequence of functions integrable in $S(0;1)$ with respect to $d\sigma$ and that $\phi_m \to \phi$ in $L^1(S(0;1),d\sigma)$.*

If $0 < d < 1$ and $\phi_m = 0$ identically in $S(0;1)\cap B(e_1;d)$ for all m, then

$$\overrightarrow{grad\,P_{\phi_m}(\cdot\ ;0,1)} \ \to\ \overrightarrow{grad\,P_\phi(\cdot\ ;0,1)}$$

uniformly in $B(e_1;\frac{d}{2})$.

All these gradients at points of $S(0;1)\cap B(e_1;d)$ are normal to $S(0;1)$.

Proof:

Since all ϕ_m are 0 in $S(0;1)\cap B(e_1;d)$, we can easily show that, firstly, all $P_{\phi_m}(\cdot\ ;0,1)$ are harmonic in the open set $\mathbf{R^n}\setminus\big(S(0;1)\setminus B(e_1;d)\big)$ and, secondly,

$$P_{\phi_m}(\cdot\ ;0,1) \ \to\ P_\phi(\cdot\ ;0,1)$$

uniformly on compact subsets of $\mathbf{R^n}\setminus\big(S(0;1)\setminus B(e_1;d)\big)$.

This implies the first statement and the second is due to the fact that $S(0;1)\cap B(e_1;d)$ is, by Theorem 1.9, a common level surface of all $P_{\phi_m}(\cdot\,;0,1)$ and of $P_\phi(\cdot\,;0,1)$.

Theorem 5.8 *Suppose that u is harmonic in the open set Ω, $\Sigma\subseteq\partial\Omega$ is open relative to $\partial\Omega$ and Ω is C^2 at every point of Σ.*

Let

$$\lim_{\Omega\ni x\to y} u(x) \;=\; 0$$

for all $y\in\Sigma$.

Then, $\overrightarrow{grad\,u}$ can be continuously extended in $\Omega\cup\Sigma$ and at each point of Σ it is normal to $\partial\Omega$.

If, also, there is an open $V\supseteq\Sigma$ so that $u>0$ in $V\cap\Omega$, then $\overrightarrow{grad\,u}$ is non-zero at every point of Σ and has the direction towards Ω.

Proof:

Take an arbitrary $x_0\in\Sigma$ and let $\delta>0$ be small enough so that $B(x_0;\delta)\cap\partial\Omega\subseteq\Sigma$ and, also, so that there is some defining function $\phi\in C^2(B(x_0;\delta))$ for Ω. Because of continuity, we may assume, taking a smaller δ if necessary, that for some constants $M_0,m_0>0$, $\left|\overrightarrow{grad\,\phi}(y)\right|\geq m_0$ for all $y\in B(x_0;\delta)\cap\partial\Omega$ and $\max_{x\in B(x_0;\delta),|\alpha|=2}|D^\alpha\phi(x)|\leq M_0$.

Regarding the last statement of the theorem, we may, also, assume that δ is small enough so that $u>0$ in $B(x_0;\delta)\cap\Omega$.

Considering only y in $B(x_0;\frac{1}{3}\delta)$ and looking at the discussion at the end of section 0.1.2, we see that there is a fixed radius $r_0=\min\left(\frac{m_0}{2C_nM_0},\frac{1}{6}\delta\right)$ so that, for every $y\in B(x_0;\frac{1}{3}\delta)$, there are two open balls b_+ and b_- with common radius r_0 and mutually tangent at the point y so that

$$b_+\;\subseteq\;B(x_0;\delta)\cap\Omega\,,\qquad b_-\;\subseteq\;B(x_0;\delta)\setminus\overline{\Omega}\,.$$

It is obvious that the open ball B_- , which has the same center as b_- and radius $R_0=3r_0$, contains the ball b_+ and is contained in $B(x_0;\delta)$.

Now, consider the function F continuous in the closed ring $\overline{B_-}\setminus b_-$, harmonic in its interior $B_-\setminus\overline{b_-}$, identically 1 in ∂B_- and identically 0 in ∂b_- .

If M is an upper bound for u in $B(x_0;\delta)\cap\Omega$, applying the Maximum-Minimum Principle to the functions $MF\pm u$ in $B_-\cap\Omega$, we find

$$|u(z)|\;\leq\;MF(z)$$

for all $z\in B_-\cap\Omega$. By explicitly writing the formula of F, we see that

$$F(z)\;\leq\;K_0|z-y|^2$$

for all $z\in\partial b_+$, where K_0 is a constant depending only on the fixed r_0.

Therefore,

$$|u(z)|\;\leq\;MK_0|z-y|^2$$

for all $z \in \partial b_+$.

Consider, now, an arbitrary sequence $\{x_m\}$ in $B(x_0;\delta)\cap\Omega$ with $x_m \to y$.

Let y_m be a point in $\partial\Omega$ of minimum distance from x_m.

Then,

$$|y_m - y| \;\leq\; |y_m - x_m| + |y - x_m| \;\leq\; 2|y - x_m| \;\to\; 0 \ ,$$

implying that $y_m \to y$. We may, thus, assume that all y_m are contained in $B(x_0;\frac{1}{3}\delta)$ and, hence, we may construct the balls $b_{+,m}$, $b_{-,m}$ and $B_{-,m}$ corresponding to y_m, whose radii r_0 and R_0 do not depend on m.

It is easy to see that these balls converge towards the balls b_+, b_- and B_- , respectively.

Since $|y_m - x_m| \leq |y - x_m| \to 0$, it is, also, easy to see that, for large enough m, x_m belongs to the radius of $b_{+,m}$ which goes through y_m.

Take $0 < d < 1$ and apply Lemma 5.2 to the ball $b_{+,m}$, after the appropriate dilation and translation.

If $\gamma_{d,m} = \partial b_{+,m} \cap B(y_m;r_0 d)$ and v_m, w_m are the restrictions of u on $\gamma_{d,m}$ and $\partial b_{+,m}\setminus\gamma_{d,m}$, respectively, then

$$u(x_m) \;=\; P_u(x_m;b_{+,m}) \;=\; P_{v_m}(x_m;b_{+,m}) + P_{w_m}(x_m;b_{+,m})$$

and

$$\overrightarrow{grad\,u}(x_m) \;=\; \overrightarrow{grad\,P_{v_m}(\cdot\ ;b_{+,m})}(x_m) + \overrightarrow{grad\,P_{w_m}(\cdot\ ;b_{+,m})}(x_m)\ .$$

From Lemma 5.2, we get that, if k,m are so large that $|x_k - y_k| < r_0 d$ and $|x_m - y_m| < r_0 d$, then

$$\begin{aligned}&\big|\overrightarrow{grad\,u}(x_k) - \overrightarrow{grad\,u}(x_m)\big| \;\leq\; 2CMK_0 r_0 d\\ &+\big|\overrightarrow{grad\,P_{w_k}(\cdot\ ;b_{+,k})}(x_k) - \overrightarrow{grad\,P_{w_m}(\cdot\ ;b_{+,m})}(x_m)\big|\ .\end{aligned}$$

By Lemma 5.3, the convergence of the balls and the uniform continuity of u, we have that there is a vector $\overrightarrow{v_d}$, normal to $\partial\Omega$ at y, so that

$$\overrightarrow{grad\,P_{w_m}(\cdot\ ;b_{+,m})}(x_m) \;\to\; \overrightarrow{v_d}\ .$$

Therefore,

$$\limsup_{k,m\to+\infty}\big|\overrightarrow{grad\,u}(x_k) - \overrightarrow{grad\,u}(x_m)\big| \;\leq\; 2CMK_0 r_0 d$$

and, since d is arbitrary, $\{\overrightarrow{grad\,u}(x_m)\}$ converges to some vector $\overrightarrow{v}$.

Thus,

$$\big|\overrightarrow{grad\,P_{w_m}(\cdot\ ;b_{1,m})}(x_m) - \overrightarrow{v}\big| \;\leq\; CMK_0 r_0 d + \big|\overrightarrow{grad\,u}(x_m) - \overrightarrow{v}\big|\ .$$

This implies that

$$|\overrightarrow{v_d} - \overrightarrow{v}| \;\leq\; CMK_0 r_0 d$$

and, finally, that $\overrightarrow{v}$ is normal to $\partial\Omega$ at y.

It is easy to see (combining two sequences into a single sequence) that $\overrightarrow{v}$ does not depend upon the sequence $\{x_m\}$ and we conclude that, defining

$$\overrightarrow{grad\, u}(y) \;=\; \overrightarrow{v}\ ,$$

$\overrightarrow{grad\, u}$ becomes continuous in $\Omega \cup \{y\}$.

The continuous dependence of $\overrightarrow{v}$ upon $y \in B(x_0; \frac{1}{3}\delta) \cap \partial\Omega$ is clear and, thus, $\overrightarrow{grad\, u}$ is continuously extended in $\Omega \cup \big(B(x_0; \frac{1}{3}\delta) \cap \partial\Omega\big)$ and, since $x_0 \in \Sigma$ is arbitrary, $\overrightarrow{grad\, u}$ is continuously extended in $\Omega \cup \Sigma$.

Now, suppose that $u > 0$ in $B(x_0; \delta) \cap \Omega$.

As before, consider $y \in B(x_0; \frac{1}{3}\delta) \cap \partial\Omega$, the corresponding ball b_+ and another open ball b'_+ with the same center as b_+ and radius equal to $\frac{1}{2}r_0$. It is obvious that the open ring $b_+ \setminus \overline{b'_+}$ is contained in $B(x_0; \delta) \cap \Omega$ and is tangent to $\partial\Omega$ at y.

If $\overrightarrow{\eta}(y)$ is the unit vector which is normal to $\partial\Omega$ at y and in the direction towards the exterior of Ω, then the line l containing this vector contains, also, the center of b_+.

Consider the function G which is continuous in $\overline{b_+} \setminus b'_+$, harmonic in $b_+ \setminus \overline{b'_+}$, identically 1 in $\partial b'_+$ and identically 0 in ∂b_+ .

If $m > 0$ is a lower bound of u in $\partial b'_+$, then, by the Maximum-Minimum Principle,

$$u \;\geq\; mG$$

everywhere in $b_+ \setminus \overline{b'_+}$ and, since $G(y) = u(y) = 0$,

$$\begin{aligned} 0 \;&>\; m \lim_{l \ni x \to y} \frac{G(y) - G(x)}{|x-y|} \;\geq\; \lim_{l \ni x \to y} \frac{u(y) - u(x)}{|x-y|} \\ &=\; \lim_{l \ni x \to y} \overrightarrow{grad\, u}(x') \cdot \overrightarrow{\eta}(y) \;=\; \overrightarrow{grad\, u}(y) \cdot \overrightarrow{\eta}(y)\ , \end{aligned}$$

where $x' = x'(x)$ is a point of the segment $[x, y]$.

Therefore,

$$\overrightarrow{grad\, u}(y) \;\neq\; 0$$

and $\overrightarrow{grad\, u}(y)$ is in the direction opposite to $\overrightarrow{\eta}(y)$.

We present two proofs of the next result. The first is more straightforward and its main ingredient is Green's Formula. (There is only an unpleasant technical detail in this proof, which is left to the interested reader to deal with.) The idea in the second proof is that a certain kernel associated to an open set Ω with C^2-boundary behaves like the Poisson kernel associated to a ball. (All details in this proof are, actually, presented.)

Theorem 5.9 *Let Ω be a connected bounded open set with C^2-boundary. Suppose, also, that $x_0 \in \Omega$, $\overrightarrow{\eta}$ is the continuous unit vector field which is normal to $\partial\Omega$ and in the direction towards the exterior of Ω and dS is the surface measure in $\partial\Omega$.*

Then,

$$d\mu_{x_0}^{\Omega} \;=\; \frac{1}{\kappa_n}\,\frac{\partial G_{x_0}^{\Omega}}{\partial \eta}\,dS\ .$$

Moreover, $d\mu_{x_0}^{\Omega}$ and dS are mutually absolutely continuous.

First proof:

By Proposition 3.4, Ω is regular and, by Proposition 5.4, $G_{x_0}^{\Omega}$ can be considered continuous in $\overline{\Omega}\setminus\{x_0\}$ and identically 0 in $\partial\Omega$. By Theorem 5.8, $\overrightarrow{grad\, G_{x_0}^{\Omega}}$ can be continuously extended in $\overline{\Omega}\setminus\{x_0\}$.

Since $G_{x_0}^{\Omega} > 0$ in Ω, by the same theorem,

$$\frac{\partial G_{x_0}^{\Omega}}{\partial \eta}(y) \;=\; \overrightarrow{grad\, G_{x_0}^{\Omega}}(y)\cdot\overrightarrow{\eta}(y) \;<\; 0$$

for every $y \in \partial\Omega$.

Consider the open set

$$\Omega^{\epsilon} \;=\; \{x \in \Omega : G_{x_0}^{\Omega}(x) > \epsilon\}\ .$$

This has the following properties.

1. $\overline{\Omega^{\epsilon}} = \{x \in \Omega : G_{x_0}^{\Omega}(x) \geq \epsilon\} \subseteq \Omega$.

In fact, if $x_m \to x$ and $G_{x_0}^{\Omega}(x_m) > \epsilon$ for all m, then, by the continuity in $\overline{\Omega}\setminus\{x_0\}$, we get $G_{x_0}^{\Omega}(x) \geq \epsilon$.

If, conversely, $G_{x_0}^{\Omega}(x) = \epsilon$, then in every neighborhood of x there are points where $G_{x_0}^{\Omega}$ becomes larger than ϵ and points where it becomes smaller than ϵ and, hence, x is in the boundary of Ω^{ϵ}. Otherwise, by the Maximum-Minimum Principle, $G_{x_0}^{\Omega}$ would be constant in a neighborhood of x, and, by Theorem 1.10, it would be constant in $\Omega\setminus\{x_0\}$, something impossible.

2. $\partial\Omega^{\epsilon} = \{x \in \Omega : G_{x_0}^{\Omega}(x) = \epsilon\}$.

This was proved in the previous paragraph.

3. Ω^{ϵ} is connected and contains x_0.

That Ω^{ϵ} contains x_0 is clear. If O is a component of Ω^{ϵ} not containing x_0, then, by 1 and 2, $G_{x_0}^{\Omega} = \epsilon$ identically in ∂O. By the Maximum-Minimum Principle, $G_{x_0}^{\Omega}$ is identically ϵ in O and, hence, by Theorem 1.10, in $\Omega\setminus\{x_0\}$, which is impossible.

4. If ϵ is small enough, Ω^{ϵ} has C^{∞}-boundary.

By the continuity of $\left|\overrightarrow{grad\, G_{x_0}^{\Omega}}\right|$ and its non-vanishing in $\partial\Omega$, there is some $\delta > 0$ so that $\overrightarrow{grad\, G_{x_0}^{\Omega}}(x) \neq 0$ for all $x \in \Omega$ with $d(x,\partial\Omega) < \delta$. Now, $G_{x_0}^{\Omega}$ has a positive minimum value in the compact set $\{x \in \Omega : d(x,\partial\Omega) \geq \delta\}$. Hence, if ϵ is small enough, $G_{x_0}^{\Omega}(x) = \epsilon$ implies $d(x,\partial\Omega) < \delta$ and, thus, $\overrightarrow{grad\, G_{x_0}^{\Omega}}(x) \neq 0$.

Therefore, the function $G_{x_0}^{\Omega}$ is a C^{∞} defining function for Ω^{ϵ} in a neighborhood of every one of its boundary points.

Now, fix a small ϵ so that property 4 holds.

Considering an arbitrary u harmonic in Ω, since $G^{\Omega}_{x_0} - h_{x_0}$ is harmonic in Ω, we get, by Green's formula,

$$\begin{aligned}&\frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon} u(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y) - \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon}\frac{\partial u}{\partial\eta}(y)G^{\Omega}_{x_0}(y)\ dS(y)\\ &= \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon} u(y)\frac{\partial h_{x_0}}{\partial\eta}(y)\ dS(y) - \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon}\frac{\partial u}{\partial\eta}(y)h_{x_0}(y)\ dS(y)\ .\end{aligned}$$

Here, $\overrightarrow{\eta}$ is the continuous unit vector field normal to $\partial\Omega^\epsilon$ and in the direction towards the exterior of Ω^ϵ.

Therefore, first by Theorem 1.7 and then by Theorem 1.6,

$$\begin{aligned}u(x_0) &= \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon} u(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y) - \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon}\frac{\partial u}{\partial\eta}(y)G^{\Omega}_{x_0}(y)\ dS(y)\\ &= \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon} u(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y) - \epsilon\ \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon}\frac{\partial u}{\partial\eta}(y)\ dS(y)\\ &= \frac{1}{\kappa_n}\int_{\partial\Omega^\epsilon} u(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y)\ .\end{aligned}$$

Assume, now, that u is continuous in $\overline{\Omega}$. By the continuity of u, of $G^{\Omega}_{x_0}$ and of $\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}$ in $\overline{\Omega}\setminus\{x_0\}$, we find, when $\epsilon\to 0$, that

$$u(x_0) = \frac{1}{\kappa_n}\int_{\partial\Omega} u(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y)\ .$$

The proof of this is quite technical and the main idea is in the discussion in paragraph 4 of section 0.1.2. There is no actual need to see the details.

If f is any function continuous in $\partial\Omega$, then, by the regularity of Ω, the function $u = H^{\Omega}_f$ is harmonic in Ω and, extended as f in $\partial\Omega$, is continuous in $\overline{\Omega}$.

Hence,

$$H^{\Omega}_f(x_0) = \frac{1}{\kappa_n}\int_{\partial\Omega} f(y)\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}(y)\ dS(y)\ ,$$

implying that

$$d\mu^{\Omega}_{x_0} = \frac{1}{\kappa_n}\ \frac{\partial G^{\Omega}_{x_0}}{\partial\eta}\ dS$$

in $\partial\Omega$.

Since $\frac{\partial G^{\Omega}_{x_0}}{\partial\eta}$ is continuous and negative in $\partial\Omega$, there are two constants C_1, C_2 so that

$$0 < C_1 \le \frac{1}{\kappa_n}\ \frac{\partial G^{\Omega}_{x_0}}{\partial\eta} \le C_2$$

everywhere in $\partial\Omega$.

We conclude that $d\mu^{\Omega}_{x_0}$ and dS are mutually absolutely continuous.

Second proof:

The idea of this second proof is to show that the kernel $\frac{1}{\kappa_n}\,\frac{\partial G_x^\Omega}{\partial\eta}$ behaves like the Poisson kernel. Namely, it has the following four properties.

1. $\frac{1}{\kappa_n}\,\frac{\partial G_x^\Omega}{\partial\eta} > 0$ everywhere in $\partial\Omega$.
2. $\frac{\partial G_x^\Omega}{\partial\eta}(y)$ is, for every $y \in \partial\Omega$, a harmonic function of x in Ω.
3. $\frac{1}{\kappa_n}\int_{\partial\Omega}\frac{\partial G_x^\Omega}{\partial\eta}(y)\,dS(y) = 1$.
4. If V is any neighborhood of $y_0 \in \partial\Omega$, then $\lim_{\Omega\ni x\to y_0}\frac{1}{\kappa_n}\,\frac{\partial G_x^\Omega}{\partial\eta}(y) = 0$, uniformly for $y \in \partial\Omega \setminus V$.

We have already proved the first property and the third is an immediate application of Theorem 1.7 with $u = 1$.

The second property can be proved as follows. Fix $y \in \partial\Omega$, an open exhaustion $\{\Omega_{(k)}\}$ of Ω, a compact subset K of $\Omega_{(k)}$ and a sequence $\{x_m\}$ in $\Omega \setminus \Omega_{(k)}$ converging to y over the line containing $\overrightarrow{\eta}(y)$.

For a fixed $x' \in K$,

$$\frac{G_{x_m}^\Omega(x')}{|x_m - y|} \;=\; \frac{G_{x'}^\Omega(x_m)}{|x_m - y|} \;\to\; -\,\frac{\partial G_{x'}^\Omega}{\partial\eta}(y)$$

as $m \to +\infty$. In particular, the sequence $\left\{\frac{G_{x_m}^\Omega(x')}{|x_m-y|}\right\}$ is bounded and, since every $\frac{G_{x_m}^\Omega}{|x_m-y|}$ is a positive harmonic function in $\Omega_{(k)}$, Harnack's Inequalities imply that this sequence of harmonic functions is uniformly bounded in K. Since K is arbitrary, from Theorem 1.18, we get that there is some subsequence converging to some function harmonic in $\Omega_{(k)}$. But, for every $x \in \Omega_{(k)}$,

$$\frac{G_{x_m}^\Omega(x)}{|x_m - y|} \;\to\; -\,\frac{\partial G_x^\Omega}{\partial\eta}(y)$$

implying that $\frac{\partial G_{(\cdot)}^\Omega}{\partial\eta}(y)$ is harmonic in $\Omega_{(k)}$ and, hence, in Ω.

For the fourth property and for $V = B(y_0; R)$, consider the compact set $K = \overline{\Omega} \cap S(y_0; \frac{1}{2}R)$ and the open set $U = \{x : d(x; K) < \frac{1}{4}R\}$. We, also, define the set $\Omega' = \Omega \cup U$, which is, also, a connected open set.

The parts of Ω and Ω' in $B(y_0; \frac{1}{4}R)$ coincide and y_0, being a regular boundary point of Ω, is, also, a regular boundary point of Ω'. Therefore, for every fixed $z' \in K$,

$$\lim_{\Omega\ni x\to y_0} G_x^{\Omega'}(z') \;=\; \lim_{\Omega\ni x\to y_0} G_{z'}^{\Omega'}(x) \;=\; 0$$

and, applying Harnack's Inequalities to the compact subset K of Ω',

$$\lim_{\Omega\ni x\to y_0} G_x^{\Omega'}(z) \;=\; 0\ ,$$

uniformly for $z \in K$. Since $\Omega \subseteq \Omega'$, Theorem 5.1 implies that

$$\lim_{\Omega\ni x\to y_0} G_x^{\Omega}(z) \;=\; 0\ ,$$

uniformly for $z \in K$. Finally, by the Maximum-Minimum Principle,

$$\lim_{\Omega \ni x \to y_0} G_x^\Omega(z) \;=\; 0 \;,$$

uniformly for $z \in \overline{\Omega} \setminus B(y_0; \frac{1}{2}R)$.

Now, for an arbitrary $y \in \partial\Omega \setminus V$ there is some open ball b_- contained in $\mathbf{R^n} \setminus \overline{\Omega}$ and having y in its boundary. It was proved in the first part of the proof of Theorem 5.8 that, if y is contained in a small enough neighborhood of any boundary point, then the radius of b_- can be considered bounded from below by a positive constant. Covering the compact set $\partial\Omega \setminus V$ by finitely many such neighborhoods, we conclude that there is some fixed r so that the ball b_-, corresponding to the arbitrary $y \in \partial\Omega \setminus V$, has radius r. Since b_- can be taken smaller, if necessary, we may assume that $r < \frac{1}{6}R$.

Together with b_-, we, also, consider the open ball B_- which has the same center as b_- and radius $2r$ and it is easy to see that $\overline{\Omega} \cap \overline{B_-} \subseteq \overline{\Omega} \setminus B(y_0; \frac{1}{2}R)$ for all $y \in \partial\Omega \setminus V$.

Let F be the function which is continuous in the closed ring $\overline{B_-} \setminus b_-$, harmonic in the open ring $B_- \setminus \overline{b_-}$, identically 1 in ∂B_- and identically 0 in ∂b_- . It is, then, easy to calculate the number

$$\frac{\partial F}{\partial \eta}(y) \;,$$

where $\overrightarrow{\eta}(y)$ is the unit vector normal to ∂b_- at y and, at the same time, normal to $\partial\Omega$ at y and directed towards the exterior of Ω. This number is negative and depends only on r and, hence, not on $y \in \partial\Omega \setminus V$.

Take, now, an arbitrary ϵ and let $x \in \Omega$ be close enough to y_0 so that

$$G_x^\Omega(z) \;\le\; \epsilon$$

for every $z \in \overline{\Omega} \setminus B(y_0; \frac{1}{2}R)$. From the Maximum-Minimum Principle, we find that

$$G_x^\Omega(z) \;\le\; \epsilon\, F(z)$$

for all $z \in \overline{\Omega} \cap \overline{B_-}$. Since $G_x^\Omega(y) = F(y) = 0$,

$$0 \;\ge\; \frac{\partial G_x^\Omega}{\partial \eta}(y) \;\ge\; \epsilon\; \frac{\partial F}{\partial \eta}(y)$$

for all $y \in \partial\Omega \setminus V$. This finishes the proof of property 4.

Now, take an arbitrary $f \in C(\partial\Omega)$ and consider the function

$$u(x) \;=\; \frac{1}{\kappa_n} \int_{\partial\Omega} \frac{\partial G_x^\Omega}{\partial \eta}(y) f(y)\; dS(y) \;, \qquad x \in \Omega \;.$$

Take an arbitrary $\epsilon > 0$ and a small neighborhood V of $y_0 \in \partial\Omega$ so that $|f(y) - f(y_0)| < \epsilon$ for all $y \in \partial\Omega \cap V$. From properties 1, 3 and 4 of the kernel

$\frac{1}{\kappa_n}\frac{\partial G_x^\Omega}{\partial \eta}$, we get

$$
\begin{aligned}
\limsup_{\Omega\ni x\to y_0} |u(x) - f(y_0)| &\le \limsup_{\Omega\ni x\to y_0} \frac{1}{\kappa_n}\int_{\partial\Omega} \frac{\partial G_x^\Omega}{\partial\eta}(y)|f(y)-f(y_0)|\, dS(y) \\
&\le \limsup_{\Omega\ni x\to y_0} \frac{1}{\kappa_n}\int_{\partial\Omega\cap V} \frac{\partial G_x^\Omega}{\partial\eta}(y)|f(y)-f(y_0)|\, dS(y) \\
&\quad + \limsup_{\Omega\ni x\to y_0} \frac{1}{\kappa_n}\int_{\partial\Omega\setminus V} \frac{\partial G_x^\Omega}{\partial\eta}(y)|f(y)-f(y_0)|\, dS(y) \\
&\le \epsilon\, \frac{1}{\kappa_n}\int_{\partial\Omega\cap V} \frac{\partial G_x^\Omega}{\partial\eta}(y)\, dS(y) \\
&\le \epsilon\ .
\end{aligned}
$$

This implies that

$$\lim_{\Omega\ni x\to y_0} u(x) \ = \ f(y_0)$$

for all $y_0 \in \partial\Omega$. Now, this, together with property 2 of the kernel, says that u is the solution of the Problem of Dirichlet in Ω with boundary function f. Therefore, for every $f \in C(\partial\Omega)$,

$$H_f^\Omega(x) \ = \ \frac{1}{\kappa_n}\int_{\partial\Omega} \frac{\partial G_x^\Omega}{\partial\eta}(y) f(y)\, dS(y)$$

for all $x \in \Omega$, completing the second proof.

5.11 ∞ as interior point. Mainly, $n = 2$

We shall, now, consider an open set Ω which contains ∞.

By Theorems 4.4 and 4.6, potential theory in $\overline{\mathbf{R^n}}$ is a triviality. We, therefore, assume that Ω is not identical to $\overline{\mathbf{R^n}}$ and, hence,

$$X \notin \Omega\ ,$$

for some $X \in \mathbf{R^n}$.

Now, if $x_0 \in \Omega \cap \mathbf{R^n}$, we consider the function

$$h_{x_0,X}(x) \ = \ h_{x_0}(x) - h_X(x) \ = \ \begin{cases} \log \frac{|x-X|}{|x-x_0|}\ , & \text{if } n = 2 \\ \frac{1}{|x-x_0|^{n-2}} - \frac{1}{|x-X|^{n-2}}\ , & \text{if } n \ge 3\ . \end{cases}$$

This is harmonic in $\overline{\mathbf{R^n}} \setminus \{x_0, X\}$ and, hence, in $\Omega \setminus \{x_0\}$.

Similarly, if $x_0 = \infty$, we consider the function

$$h_{\infty,X}(x) \ = \ h_\infty(x - X) \ = \ \begin{cases} \log|x - X|\ , & \text{if } n = 2 \\ \frac{1}{|x-X|^{n-2}}\ , & \text{if } n \ge 3\ , \end{cases}$$

which is harmonic in $\overline{\mathbf{R^n}} \setminus \{\infty, X\}$ and, hence, in $\Omega \setminus \{\infty\}$.

In any case, $h_{x_0,X}$ is superharmonic in Ω.

We consider the family $\mathcal{U}^{\Omega}_{x_0,X}$ of all superharmonic majorants of the subharmonic function $-h_{x_0,X}$ in Ω and, if this family is non-empty, the smallest harmonic majorant $U^{\Omega}_{x_0,X}$.

We, now, define the function

$$G^{\Omega}_{x_0} \; = \; h_{x_0,X} + U^{\Omega}_{x_0,X} \; .$$

If we take another point $X' \notin \Omega$, then the function $h_{x_0,X} - h_{x_0,X'}$ is harmonic in Ω. Therefore, the function u is a superharmonic majorant of $-h_{x_0,X}$ in Ω if and only if the function $u + h_{x_0,X} - h_{x_0,X'}$ is a superharmonic majorant of $-h_{x_0,X'}$ in Ω. This implies that

$$U^{\Omega}_{x_0,X'} \; = \; U^{\Omega}_{x_0,X} + h_{x_0,X} - h_{x_0,X'}$$

in Ω and, hence, the function $G^{\Omega}_{x_0}$, defined above, does not depend upon the choice of $X \notin \Omega$.

The function $G^{\Omega}_{x_0}$, if it exists, is called the Green's function of Ω with respect to x_0.

Observe that, if $n \geq 3$, then $h_{\infty,X}$ is harmonic in Ω and, hence, $G^{\Omega}_{\infty} = 0$ identically in Ω.

In the following we shall avoid certain complications arising in case $n \geq 3$ (and described, to some extent, in the previous chapter) and we shall concentrate on the case $n = 2$. In this case all results in this chapter extend in a straightforward manner. We, briefly, describe the situation.

In all that follows, Ω is an open subset of $\overline{\mathbf{R^2}}$ with $\infty \in \Omega$ and $X \notin \Omega$.

1. *The function $G^{\Omega}_{x_0}$, if it exists, is superharmonic in Ω and harmonic in $\Omega \setminus \{x_0\}$, it is positive in the component of Ω containing x_0 and it is identically* 0 *in every other component of Ω.*

Also, the function $G^{\Omega}_{x_0} - h_{x_0,X}$ is harmonic in Ω.

2. *If some disc in $\mathbf{R^2}$ is disjoint from Ω, then Ω has a Green's function with respect to every $x_0 \in \Omega$ and*

$$G^{\Omega}_{x_0}(x) \; = \; h_{x_0,X}(x) + H^{\Omega}_{-h_{x_0,X}}(x) \; = \; h_{x_0,X}(x) - \int_{\partial\Omega} h_{x_0,X}(y) \, d\mu^{\Omega}_x(y)$$

for all $x \in \Omega$.

3. *The open set $\overline{\mathbf{R^2}} \setminus \{X\}$ has no Green's function with respect to any one of its points.*

4. *If $\Omega = \overline{\mathbf{R^2}} \setminus \overline{B(X;R)}$, then*

$$G^{\Omega}_{x_0} \; = \; h_{x_0} - h_{x_0^*} + \log R \; ,$$

if $x_0 \in \Omega \setminus \{\infty\}$ and x_0^* is the symmetric of x_0 with respect to $S(X;R)$, and

$$G^{\Omega}_{\infty} \; = \; h_{\infty,X} - \log R \; .$$

5. *If $x_0 \in \Omega' \subseteq \Omega$ and Ω has a Green's function with respect to x_0, then Ω' has, also, a Green's function with respect to x_0 and*

$$G_{x_0}^{\Omega'} \ \leq \ G_{x_0}^{\Omega}$$

in Ω'.
6. *If $\Omega_m \uparrow \Omega$, $x_0 \in \Omega_1$ and all Ω_m have a Green's function with respect to x_0, then, either $G_{x_0}^{\Omega_m} \uparrow +\infty$ in the component of Ω containing x_0 and Ω has no Green's function with respect to x_0, or Ω has a Green's function with respect to x_0 and*

$$G_{x_0}^{\Omega_m} \ \uparrow \ G_{x_0}^{\Omega}$$

in Ω.
7. *If x_0 and x_1 are in the same component of Ω, then Ω has a Green's function with respect to x_0 if and only it has a Green's function with respect to x_1 and, in this case,*

$$G_{x_0}^{\Omega}(x_1) \ = \ G_{x_1}^{\Omega}(x_0) \ .$$

We say that Ω has a Green's function in one of its components, if it has a Green's function with respect to at least one x_0 in this component.
8. *Let $y_0 \in \partial\Omega$. If*

$$\lim_{\Omega \ni z \to y_0} G_x^{\Omega}(z) \ = \ 0$$

for at least one x in every component of Ω, then y_0 is a regular boundary point of Ω.

If there is some disc disjoint from Ω and y_0 is a regular boundary point, then

$$\lim_{\Omega \ni z \to y_0} G_x^{\Omega}(z) \ = \ 0$$

for all $x \in \Omega$.
9. *If Ω is regular and disjoint from some disc, $x_0 \in \Omega$ and O is the component of Ω containing x_0, then $G_{x_0}^{\Omega}$ extended as identically 0 in $\overline{\mathbf{R^2}} \setminus \Omega$ is subharmonic in $\overline{\mathbf{R^2}} \setminus \{x_0\}$ and identically 0 in $\overline{\mathbf{R^2}} \setminus O$.*
10. *If there is some disc disjoint from Ω, $x_0 \in \Omega$ and O is the component of Ω containing x_0, then the function*

$$h_{x_0,X}(x) - \int_{\partial\Omega} h_{x,X}(y) \ d\mu_{x_0}^{\Omega}(y) \ , \qquad x \in \overline{\mathbf{R^2}} \ ,$$

is an extension of $G_{x_0}^{\Omega}$ in $\overline{\mathbf{R^2}}$ which is subharmonic in $\overline{\mathbf{R^2}} \setminus \{x_0\}$ and identically 0 in $\overline{\mathbf{R^2}} \setminus \overline{O}$.
11. *If the connected Ω is disjoint from some disc and has C^2-boundary, $x_0 \in \Omega$, $\overrightarrow{\eta}$ is the continuous unit vector field which is normal to $\partial\Omega$ directed towards the exterior of Ω and dS is the surface measure in $\partial\Omega$, then,*

$$d\mu_{x_0}^{\Omega} \ = \ \frac{1}{\kappa_n} \frac{\partial G_{x_0}^{\Omega}}{\partial \eta} \ dS \ .$$

Moreover, $d\mu_{x_0}^{\Omega}$ and dS are mutually absolutely continuous.

Chapter 6

Potentials

6.1 Definitions

We shall consider the following two types of kernels.

Definition 6.1 ***Kernels of first type.***
Let

$$h_*(r) \;=\; \begin{cases} \log \frac{1}{r} \;, & \textit{if } n = 2 \\ \frac{1}{r^{n-2}} \;, & \textit{if } n \geq 3 \;. \end{cases}$$

Take any non-constant increasing convex function H defined in $(-\infty, +\infty)$, in case $n = 2$, or in $(0, +\infty)$, in case $n \geq 3$, and define

$$K_*(r) \;=\; H\big(h_*(r)\big) \;, \qquad 0 < r < +\infty \;.$$

Hence, K_* is continuous and decreasing in $(0, +\infty)$ with $\lim_{r \to 0+} K_*(r) = +\infty$ and we, next, define the kernel

$$K(x, y) \;=\; K_*(|x - y|) \;=\; H\big(h_*(|x - y|)\big) \;=\; H\big(h(x - y)\big)$$

for all $x, y \in \mathbf{R^n}$.

We postulate the following rules.

1. $\int_0^1 K_*(r) r^{n-1} \, dr < +\infty$.
2. $\lim_{r \to +\infty} K_*(r) \leq 0$, *in case* $n = 2$, *or* $\lim_{r \to +\infty} K_*(r) = 0$, *in case* $n \geq 3$.
3. *If* $K_* > 0$ *in* $(0, +\infty)$, *then* $\limsup_{r \to +\infty} \frac{K_*(r-1)}{K_*(r)} < +\infty$.

Kernels of second type.

If Ω is an open subset of $\mathbf{R^n}$ with a Green's function in every one of its connected components, we consider the kernel

$$G^\Omega(x, y) \;=\; G^\Omega_x(y) \;=\; G^\Omega_y(x)$$

for all $x, y \in \Omega$.

Comments 1. Observe that, in case $n \geq 3$, all our kernels are non-negative. Also, observe that all kernels are symmetric: $K(x, y) = K(y, x)$.
2. The third rule is not needed if in the theory, which we shall develop, we restrict to the consideration of measures with compact support.

Examples
1. The **Riesz kernel of order** α is defined by

$$K_\alpha(x, y) \;=\; K_{\alpha *}(|x - y|) \;=\; \frac{1}{|x - y|^{n-\alpha}} \;,$$

where $0 < \alpha < n$.

If $0 < \alpha < 2$, in case $n = 2$, or $0 < \alpha \leq 2$, in case $n \geq 3$, then K_α is of first type.
2. The **classical kernels**, i.e. the Newtonian, in case $n \geq 3$, and the logarithmic, in case $n = 2$, are of first type.

Definition 6.2 ***K-potential for a kernel of second type.***
Let $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its components and $d\mu$ be a non-negative Borel measure in Ω. The corresponding kernel is $K = G^\Omega$ and we have already defined the Green's potential

$$U_K^{d\mu}(x) \;=\; U_\Omega^{d\mu}(x) \;=\; \int_\Omega G^\Omega(x, y)\, d\mu(y) \;, \qquad x \in \Omega \;,$$

only in case this is finite for at least one x in each component of Ω.

If $d\mu$ is a locally finite complex Borel measure in Ω, we define $U_K^{d\mu}$ by the same formula (and linearity) only when $U_K^{|d\mu|}(x)$ is finite for at least one x in each component of Ω.

Under these assumptions, we say that ***the K-potential is well-defined****.*

We know, from last chapter, that, if K is of second type, the K-potential of a non-negative Borel measure in the associated Ω is (if it is well-defined) superharmonic in Ω and harmonic outside the support of the measure. Therefore, the K-potential of a locally finite Borel measure in Ω is (if it is well-defined) finite almost everywhere in Ω and is a linear combination of (four) superharmonic functions.

Definition 6.3 ***K-potential for a kernel of first type.***
If $d\mu$ is a non-negative Borel measure in $\mathbf{R^n}$ and K is a non-negative kernel of first type, we define the K-potential of $d\mu$ by

$$U_K^{d\mu}(x) \;=\; \int_{\mathbf{R^n}} K(x, y)\, d\mu(y) \;, \qquad x \in \mathbf{R^n} \;,$$

only when this is finite for at least one $x \in \mathbf{R^n}$.

If $d\mu$ is a locally finite complex Borel measure in $\mathbf{R^n}$ (and K is non-negative), we define $U_K^{d\mu}$ by the same formula (and linearity), only when $U_K^{|d\mu|}(x)$ is finite for at least one $x \in \mathbf{R^n}$.

If K is of variable sign and $d\mu$ is a locally finite complex (non-negative, in particular) Borel measure in $\mathbf{R^n}$, we define $U_K^{d\mu}$ as before, but only if $d\mu$ is compactly supported.

*Under these assumptions, we say that **the K-potential is well-defined**.*

It is obvious that, if K is a non-negative kernel of first type and $d\mu$ is a non-negative Borel measure, then the K-potential is defined everywhere as either a non-negative number or as $+\infty$. Proposition 6.1, which will be proved in a moment, describes the situation more clearly. If $d\mu$ is a locally finite complex Borel measure with real values (and K is non-negative), then its K-potential is defined at those points where not both $U_K^{d\mu^+}$ and $U_K^{d\mu^-}$ take the value $+\infty$. A similar comment can be made for a general locally finite complex Borel measure.

6.2 Potentials of non-negative Borel measures

Lemma 6.1 *If K is a kernel of first type, then, for every $y \in \mathbf{R^n}$, $K(\cdot\ ,y)$ is continuous and subharmonic in $\mathbf{R^n} \setminus \{y\}$.*

Proof:
It is clear, since h is harmonic in $\mathbf{R^n}\setminus\{0\}$ and H is convex in an open interval containing the values of h.

Proposition 6.1 *1. If $d\mu$ is a non-negative Borel measure in $\mathbf{R^n}$, K is of first type and $U_K^{d\mu}$ is well-defined, then this K-potential is continuous and subharmonic in $\mathbf{R^n} \setminus supp(d\mu)$.*

If, in particular $K = h$, then the h-potential is superharmonic in $\mathbf{R^n}$ and harmonic in $\mathbf{R^n} \setminus supp(d\mu)$.
2. If $d\mu$ is a non-negative Borel measure in $\Omega \subseteq \mathbf{R^n}$, which has a Green's function in all its components, and $U_\Omega^{d\mu}$ is well-defined, then this K-potential is superharmonic in Ω and harmonic in $\Omega \setminus supp(d\mu)$.

Proof:
The case of second type is just Theorem 5.5 and the case $K = h$ is only Theorem 2.8. Hence, assume that K is of the first type and, to begin with, that K is non-negative.

By definition, $U_K^{d\mu}(x_0) < +\infty$ for some $x_0 \in \mathbf{R^n}$.

Therefore, for an arbitrary $\epsilon > 0$,

$$\int_{\{y:|y|>R\}} K(x_0, y)\ d\mu(y)\ \leq\ \epsilon\ ,$$

for all large R. Take any $x \in \mathbf{R^n} \setminus supp(d\mu)$ and, by the third rule on K, find a constant $C > 0$ and R so large that, besides the previous inequality,

$$K(x, y)\ \leq\ CK(x_0, y)$$

for all y with $|y| > R$, is, also, true.

This implies,

$$\int_{\{y:|y|>R\}} K(x,y)\ d\mu(y)\ \le\ C\epsilon\ .$$

Since $K(x,\cdot\ )$ is bounded on $supp(d\mu)$,

$$U_K^{d\mu}(x)\ \le\ \int_{\{y:|y|\le R\}} K(x,y)\ d\mu(y) + C\epsilon\ <\ +\infty\ .$$

If, now, $x_m \to x$, then $K(x_m,\cdot\ ) \to K(x,\cdot\ )$ uniformly in $\overline{B(0;R)} \cap supp(d\mu)$ and, thus,

$$\limsup_{m\to+\infty} \left| U_K^{d\mu}(x_m) - U_K^{d\mu}(x)\right|\ \le\ 2C\epsilon\ ,$$

proving the continuity of $U_K^{d\mu}$ at x.

From Lemma 6.1, for every $\overline{B(x;r)} \subseteq \mathbf{R^n} \setminus supp(d\mu)$,

$$\mathcal{M}^r_{U_K^{d\mu}}(x)\ =\ \int_{supp(d\mu)} \mathcal{M}^r_{K(\cdot\ ,y)}(x)\ d\mu(y)\ \ge\ \int_{supp(d\mu)} K(x,y)\ d\mu(y)\ =\ U_K^{d\mu}(x)$$

and $U_K^{d\mu}$ is subharmonic in $\mathbf{R^n} \setminus supp(d\mu)$.

If K is of variable sign, then, by definition, $d\mu$ is supported in a compact set and we may choose the R above so that $\overline{B(0;R)}$ contains the support of $d\mu$. The proof of continuity of the K-potential in $\mathbf{R^2} \setminus supp(d\mu)$ is, now, easier, since there is no "tail"-term in the integral.

Comment: Suppose that K is a non-negative kernel of first type and $d\mu$ is a non-negative Borel measure in $\mathbf{R^n}$. If $U_K^{d\mu}(x_0) < +\infty$ for some x_0 (the condition for the K-potential to be well-defined), then the finiteness of the K-potential at any other x depends only on the behaviour of $d\mu$ in a neighborhood of x.

In fact, for all large R, we have that $\int_{\{y:|y|>R\}} K(x_0,y)\ d\mu(y) < 1$. As in the last proof, there is some $C > 0$ and some large R, so that, besides the last inequality, we also have $K(x,y) \le CK(x_0,y)$ for all y with $|y| > R$. This implies that $\int_{\{y:|y|>R\}} K(x,y)\ d\mu(y) < C$ and, thus, the finiteness of the K-potential depends on the restriction of $d\mu$ in $\overline{B(0;R)}$.

The same comment is valid for the Green's potentials.

In fact, suppose that $\Omega \subseteq \mathbf{R^n}$ has a Green's function in all its components and take an $\Omega_{(m)}$ from some open exhaustion of Ω so that it contains x and x_0 (assumed to be in the same component of Ω) and $\int_{\Omega\setminus\overline{\Omega_{(m)}}} G^\Omega(x_0,y)\ d\mu(y) < 1$. By Harnack's Inequalities, there exists a $C > 0$ so that $G^\Omega(x,y) \le CG^\Omega(x_0,y)$ for all $y \in \Omega \setminus \overline{\Omega_{(m)}}$, implying $\int_{\Omega\setminus\overline{\Omega_{(m)}}} G^\Omega(x,y)\ d\mu(y) < C$.

Proposition 6.2 *(Lower-semicontinuity in the space-variable)*
1. If $d\mu$ is a non-negative Borel measure in $\mathbf{R^n}$, K is of first type and $U_K^{d\mu}$ is well-defined, then this K-potential is lower-semicontinuous in $\mathbf{R^n}$.
2. If $d\mu$ is a non-negative Borel measure in $\Omega \subseteq \mathbf{R^n}$, which has a Green's function in all its components, and $U_\Omega^{d\mu}$ is well-defined, then this K-potential is lower-semicontinuous in Ω.

Proof:
This is a simple application of Fatou's Lemma.

Proposition 6.3 *(Lower-semicontinuity in measure)*
1. Let K be of first type, $\{d\mu_k\}$ be a sequence of non-negative Borel measures with $\liminf_{k\to+\infty} U_K^{d\mu_k}(x) < +\infty$ for at least one x. If this sequence converges weakly on compact sets to some non-negative Borel measure $d\mu$, then $d\mu$ has a well-defined K-potential and

$$\liminf_{k\to+\infty} U_K^{d\mu_k}(x) \;\geq\; U_K^{d\mu}(x)$$

for every $x \in \mathbf{R^n}$.

If the kernel is of variable sign, we, also, assume that all $d\mu_k$ are supported in a common compact subset of $\mathbf{R^n}$.
2. Let $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its components and $\{d\mu_k\}$ be a sequence of non-negative Borel measures in Ω with $\liminf_{k\to+\infty} U_\Omega^{d\mu_k}(x) < +\infty$ for at least one x in each component of Ω. If the sequence converges weakly on compact subsets of Ω to some non-negative Borel measure $d\mu$ in Ω, then $d\mu$ has a well-defined Green's potential and

$$\liminf_{k\to+\infty} U_\Omega^{d\mu_k}(x) \;\geq\; U_\Omega^{d\mu}(x)$$

for every $x \in \Omega$.

Proof:
1. Assume that K is of first type and non-negative and consider the truncated kernel

$$K_N(x,y) \;=\; \min(K(x,y), N)$$

for all $x, y \in \mathbf{R^n}$.

For every x, $K_N(x,\ \cdot)$ is continuous in $\mathbf{R^n}$ and, taking an arbitrary $R > 0$,

$$\liminf_{k\to+\infty} U_K^{d\mu_k}(x) \;\geq\; \liminf_{k\to+\infty} \int_{\overline{B(0;R)}} K_N(x,y)\, d\mu_k(y) \;=\; \int_{\overline{B(0;R)}} K_N(x,y)\, d\mu(y)\ .$$

Now, letting $R \to +\infty$ and then $N \to +\infty$, we conclude the proof in this case.

If K is of variable sign, then we repeat the same proof, replacing the arbitrary ball with a single compact set $F \subseteq \mathbf{R^2}$ so that all $d\mu_k$ are supported in F.
2. We consider, again, the truncated kernels G_N^Ω and any open exhaustion $\{\Omega_{(m)}\}$ of Ω.

Then, as before,

$$\liminf_{k\to+\infty} U_\Omega^{d\mu_k}(x) \;\geq\; \liminf_{k\to+\infty} \int_{\overline{\Omega_{(m)}}} G_N^\Omega(x,y)\, d\mu_k(y) \;=\; \int_{\overline{\Omega_{(m)}}} G_N^\Omega(x,y)\, d\mu(y)$$

and the proof is finished, by letting $m \to +\infty$ and $N \to +\infty$.

6.3 The maximum principle for potentials

Lemma 6.2 *Let Ω be a regular bounded open set and $d\mu$ be a non-negative Borel measure supported in a compact subset of Ω. Then, $\lim_{\Omega\ni x\to y} U_\Omega^{d\mu}(x) = 0$, for all $y \in \partial\Omega$.*

Proof:
For fixed $z \in supp(d\mu)$ and $y \in \partial\Omega$, we have that $\lim_{\Omega\ni x\to y} G_z^\Omega(x) = 0$.
It is easy to see, by Harnack's Inequalities, that $\lim_{\Omega\ni x\to y} G_z^\Omega(x) = 0$ uniformly for $z \in supp(d\mu)$ and the proof is, now, clear.

Theorem 6.1 *(The Maximum Principle)*
1. Let K be of first type and $d\mu$ be a non-negative Borel measure in $\mathbf{R^n}$. If $\alpha > 0$ and $U_K^{d\mu}(x) \le \alpha$ for all $x \in supp(d\mu)$, then, $U_K^{d\mu} \le \alpha$ everywhere in $\mathbf{R^n}$.
2. Let $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its components and $d\mu$ be a non-negative Borel measure in Ω. If $\alpha > 0$ and $U_\Omega^{d\mu}(x) \le \alpha$ for all $x \in supp(d\mu)$, then, $U_\Omega^{d\mu} \le \alpha$ everywhere in Ω.

Proof:
1. Let K be of first type and suppose that $d\mu$ is compactly supported. We shall consider, for any $N > 0$, the truncated kernel

$$K^N(x,y) = \begin{cases} K(x,y) \ , & \text{if } K(x,y) \le N \\ 0 \ , & \text{if } K(x,y) > N \end{cases}$$

and the function

$$U_{K^N}^{d\mu}(x) = \int_{\mathbf{R^n}} K^N(x,y)\, d\mu(y) \ , \qquad x \in \mathbf{R^n} \ .$$

Then, $K^N \uparrow K$ and,

$$U_{K^N}^{d\mu}(x) \uparrow U_K^{d\mu}(x)$$

for all x.
Applying Egoroff's Theorem, we find that, for every $\epsilon > 0$, there is a closed set $F \subseteq supp(d\mu)$ with $d\mu(F) > d\mu(\mathbf{R^n}) - \epsilon$ and

$$U_{K^N}^{d\mu} \uparrow U^{d\mu}$$

uniformly in F.
Consider, also, $d\mu_F$, the restriction of $d\mu$ in F.
Then, for every x,

$$\begin{aligned} U^{d\mu_F}(x) - U_{K^N}^{d\mu_F}(x) &= \int_{\{y:K(x,y)>N\}} K(x,y)\, d\mu_F(y) \\ &\le \int_{\{y:K(x,y)>N\}} K(x,y)\, d\mu(y) = U_K^{d\mu}(x) - U_{K^N}^{d\mu}(x) \end{aligned}$$

and, thus,

$$U_{K^N}^{d\mu_F} \uparrow U_K^{d\mu_F}$$

uniformly in F.

Therefore, for every $\delta > 0$, if N is large enough, we have

$$\int_{\{y:K(x,y)>N\}} K(x,y)\ d\mu_F(y) \ \leq\ \delta$$

for all $x \in F$.

Fix, now, $x \in F$ and take any $\{x_m\}$ converging to x. Then,

$$\begin{aligned}
\limsup_{m\to+\infty} U_K^{d\mu_F}(x_m) &\leq \limsup_{m\to+\infty} \int_{\{y:K(x_m,y)>N\}} K(x_m,y)\ d\mu_F(y) \\
&\quad + \limsup_{m\to+\infty} \int_{\mathbf{R^n}} K^N(x_m,y)\ d\mu_F(y) \\
&= \limsup_{m\to+\infty} \int_{\{y:K(x_m,y)>N\}} K(x_m,y)\ d\mu_F(y) \\
&\quad + \int_{\mathbf{R^n}} K^N(x,y)\ d\mu_F(y)\ .
\end{aligned}$$

It is, geometrically, clear that there is some number M, depending only on the dimension n, with the property that, for every z, we can find closed convex cones $\Gamma_1^z, \ldots, \Gamma_M^z$ with the same vertex z and each having an opening of $\frac{\pi}{6}$ from its axis of symmetry so that

$$\cup_{k=1}^M \Gamma_k^z \ =\ \mathbf{R^n}\ .$$

Now, if $y \in \Gamma_k^z \cap F$ and ξ_k^z is a closest point of $\Gamma_k^z \cap F$ from z, then

$$|z-y| \ \geq\ |\xi_k^z - y|\ .$$

Applying this to every $z = x_m$,

$$\begin{aligned}
\int_{\{y:K(x_m,y)>N\}} K(x_m,y)d\mu_F(y) &\leq \sum_{k=1}^M \int_{\{y:K(x_m,y)>N\}\cap\Gamma_k^{x_m}} K(x_m,y)d\mu_F(y) \\
&\leq \sum_{k=1}^M \int_{\{y:K(x_m,y)>N\}\cap\Gamma_k^{x_m}} K(\xi_k^{x_m},y)d\mu_F(y) \\
&\leq \sum_{k=1}^M \int_{\{y:K(\xi_k^{x_m},y)>N\}} K(\xi_k^{x_m},y)d\mu_F(y) \\
&\leq M\delta\ .
\end{aligned}$$

Therefore, we have that

$$\limsup_{m\to+\infty} U_K^{d\mu_F}(x_m) \ \leq\ M\delta + \int_{\{y:K(x,y)\leq N\}} K(x,y)\ d\mu_F(y)\ .$$

Now, let $N \to +\infty$ and, then, $\delta \to 0$ and get

$$\limsup_{m\to+\infty} U_K^{d\mu_F}(x_m) \ \le\ U_K^{d\mu_F}(x) \ \le\ U_K^{d\mu}(x) - m\epsilon \ \le\ \alpha - m\epsilon \ ,$$

where $m = \min\big(0, \min_{z,y\in supp(d\mu)} K(z,y)\big)$.

Combining the last result with Proposition 6.2, we get that $U_K^{d\mu_F}$ is continuous at every $x \in F$ and, by Proposition 6.1, it is continuous in $\mathbf{R^n}$.

From the second rule on our kernels, we, also, have that

$$\limsup_{x\to\infty} U_K^{d\mu_F}(x) \ \le\ 0 \ .$$

By the subharmonicity of $U_K^{d\mu_F}$ in $\mathbf{R^n} \setminus F$ and the Maximum Principle, we get

$$U_K^{d\mu_F} \ \le\ \alpha - m\epsilon$$

everywhere in $\mathbf{R^n}$.

Now, let $x \notin supp(d\mu)$ and let $\rho > 0$ be the distance of x from $supp(d\mu)$. Then,

$$\begin{aligned} U_K^{d\mu}(x) &= \int_F K(x,y)\ d\mu(y) + \int_{supp(d\mu)\setminus F} K(x,y)\ d\mu(y) \\ &\le \alpha - m\epsilon + K_*(\rho)\epsilon \end{aligned}$$

and, since ϵ is arbitrary,

$$U_K^{d\mu}(x) \ \le\ \alpha$$

for all x.

If $d\mu$ is not compactly supported, in which case K is, necessarily, non-negative, we consider the restrictions $d\mu_m$ in $\overline{B(0;m)}$ and we have $U_K^{d\mu_m}(x) \le U_K^{d\mu}(x) \le \alpha$ for all $x \in supp(d\mu)$ and, hence, for all $x \in supp(d\mu_m)$. From what we proved up to now, $U_K^{d\mu_m}(x) \le \alpha$ everywhere in $\mathbf{R^n}$ and letting $m \to +\infty$, we finish the proof in this case.

2. Let Ω have a Green's function in all its components and $d\mu$ be a non-negative Borel measure with compact support contained in Ω. From the first part of the proof, we know that there is a compact subset F of $supp(d\mu)$ with $d\mu(F) \ge d\mu(\mathbf{R^n}) - \epsilon$ so that the h-potential $U_h^{d\mu_F}$ is continuous in $\mathbf{R^n}$. We know, from Theorems 2.15 and 5.5, that $U_\Omega^{d\mu_F}$ and $U_h^{d\mu_F}$ have the same distributional derivative in Ω and, hence, they differ by a function harmonic in Ω. This implies that $U_\Omega^{d\mu_F}$ is continuous in Ω.

Assume, for the moment, that Ω is a regular bounded set. From Lemma 6.2,

$$\lim_{\Omega\ni x\to y} U_\Omega^{d\mu_F}(x) \ =\ 0$$

for all $y \in \partial\Omega$. Since, also,

$$\lim_{\Omega\ni x\to y} U_\Omega^{d\mu_F}(x) \ =\ U_\Omega^{d\mu_F}(y) \ \le\ U_\Omega^{d\mu}(y) \ \le\ \alpha$$

for all $y \in F$, we get, by the Maximum Principle, that

$$U_{\Omega}^{d\mu_F}(x) \;\leq\; \alpha$$

for all $x \in \Omega \setminus F$ and, hence, for all $x \in \Omega$.

For an arbitrary $x \in \Omega \setminus supp(d\mu)$, denote $M = \sup_{y\in supp(d\mu)} G^{\Omega}(x,y) < +\infty$. Then,

$$U_{\Omega}^{d\mu}(x) \;\leq\; U_{\Omega}^{d\mu_F}(x) + M\epsilon \;\leq\; \alpha + M\epsilon$$

and, thus,

$$U_{\Omega}^{d\mu}(x) \;\leq\; \alpha$$

for all $x \in \Omega \setminus F$ and, thus, for all $x \in \Omega$.

To drop the assumption of regularity, take an open exhaustion $\{\Omega_{(m)}\}$ of Ω consisting of regular sets and large m so that $supp(d\mu) \subseteq \Omega_{(m)}$. Since $U_{\Omega_{(m)}}^{d\mu} \leq U_{\Omega}^{d\mu} \leq \alpha$ in $supp(d\mu)$ and $\Omega_{(m)}$ is regular, we get

$$U_{\Omega_{(m)}}^{d\mu}(x) \;\leq\; \alpha$$

for all $x \in \Omega_{(m)}$. By Theorem 5.2,

$$U_{\Omega}^{d\mu}(x) \;\leq\; \alpha$$

for all $x \in \Omega$.

Finally, if $d\mu$ does not have compact support in Ω, we consider the restrictions $d\mu_m$ in the terms $\Omega_{(m)}$ of some open exhaustion of Ω and we conclude the proof in the same manner as in the previous paragraph.

6.4 The continuity principle for potentials

Proposition 6.4 *1. If K is a kernel of first type and $d\mu$ is a non-negative Borel measure with compact support, there is some closed $F \subseteq supp(d\mu)$ with arbitrarily small $d\mu(\mathbf{R^n} \setminus F)$ so that $U_K^{d\mu_F}$ is continuous in $\mathbf{R^n}$.*
2. If $\Omega \subseteq \mathbf{R^n}$ has a Green's function in all its components and $d\mu$ is a non-negative Borel measure with compact support in Ω, there is some closed $F \subseteq supp(d\mu)$ with arbitrarily small $d\mu(\Omega \setminus F)$ so that $U_{\Omega}^{d\mu_F}$ is continuous in Ω.

Proof:
It is, actually, part of the proof of Theorem 6.1.

Theorem 6.2 *(The Continuity Principle)*
1. If K is a kernel of first type, $d\mu$ is a non-negative Borel measure with compact support and $U_K^{d\mu}$, restricted to $supp(d\mu)$, is continuous in $supp(d\mu)$, then it is continuous in $\mathbf{R^n}$.
2. If $\Omega \subseteq \mathbf{R^n}$ has a Green's function in all its components, $d\mu$ is a non-negative Borel measure with compact support in Ω and $U_{\Omega}^{d\mu}$, restricted to $supp(d\mu)$, is continuous in $supp(d\mu)$, then it is continuous in Ω.

Proof:
1. Following the argument in the proof of the Theorem 6.1, we have that $U^{d\mu}_{K^N} \uparrow U^{d\mu}_{K}$, that each $U^{d\mu}_{K^N}$ is continuous everywhere and that $U^{d\mu}_{K}$, restricted to $supp(d\mu)$, is continuous in $supp(d\mu)$.

By Dini's Theorem,

$$U^{d\mu}_{K^N} \ \uparrow \ U^{d\mu}_{K}$$

uniformly in the compact $supp(d\mu)$.

Therefore, in the same proof, we do not need to reduce $supp(d\mu)$ to any subset F. The final result there is that $U^{d\mu}_{K}$ is continuous everywhere.
2. Since $U^{d\mu}_{\Omega}$ and $U^{d\mu}_{h}$ differ by a harmonic function in Ω, the proof in this case is straightforward, by applying the first part to $U^{d\mu}_{h}$.

Chapter 7

Energy

7.1 Definitions

Definition 7.1 ***K-energy for kernels of second type.***
Let $\Omega \subseteq \mathbf{R^n}$ be an open set with a Green's function in all its components and $d\mu_1$ and $d\mu_2$ two non-negative Borel measures in Ω. We define the ***mutual K-energy*** *or* ***mutual Ω-energy*** *of $d\mu_1$ and $d\mu_2$ by*

$$I_K(d\mu_1, d\mu_2) \;=\; I_\Omega(d\mu_1, d\mu_2) \;=\; \int_\Omega \int_\Omega G^\Omega(x,y)\; d\mu_1(x)\; \overline{d\mu_2}(y)\ ,$$

which is either a non-negative number or $+\infty$.

If $d\mu_1$ and $d\mu_2$ are two locally finite complex Borel measures in Ω, we define their mutual Ω-energy by the same formula (and linearity), but only when $I_K(|d\mu_1|, |d\mu_2|) < +\infty$.

In case $d\mu_1 = d\mu_2 = d\mu$, we call ***K-energy*** *or* ***Ω-energy*** *of $d\mu$ the*

$$I_\Omega(d\mu) \;=\; I_\Omega(d\mu, d\mu)\ .$$

Definition 7.2 ***K-energy for kernels of first type.***
Let K be a non-negative kernel of first type and $d\mu_1$ and $d\mu_2$ two non-negative Borel measures in $\mathbf{R^n}$. We define the ***mutual K-energy*** *of $d\mu_1$ and $d\mu_2$ by*

$$I_K(d\mu_1, d\mu_2) \;=\; \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} K(x,y)\; d\mu_1(x)\; \overline{d\mu_2}(y)\ .$$

If $d\mu_1$ and $d\mu_2$ are two locally finite complex Borel measures in $\mathbf{R^n}$, we define their mutual K-energy by the same formula (and linearity), but only when $I_K(|d\mu_1|, |d\mu_2|) < +\infty$.

In case $d\mu_1 = d\mu_2 = d\mu$, we call ***K-energy*** *of $d\mu$ the*

$$I_K(d\mu) \;=\; I_K(d\mu, d\mu)\ .$$

If K is of variable sign, we define the mutual K-energy and K-energy as before, but only for compactly supported Borel measures.

It is clear, by the Theorem of Fubini, that if $I_K(|d\mu_1|, |d\mu_2|) < +\infty$, then $U_K^{d\mu_1}(x)$ is a finite number for almost every x with respect to $d\mu_2$ and vice-versa.

Example.

$$I_K(d\delta_a, d\delta_b) \;=\; K(a,b)$$

for all types of kernels and all a and b. In particular, the K-energy of a Dirac mass is always $+\infty$.

Proposition 7.1 *(Lower-semicontinuity of energy in measure)*
1. Let K be of first type and $\{d\mu_m^1\}$ and $\{d\mu_m^2\}$ be two sequences of non-negative Borel measures in $\mathbf{R^n}$ converging to the non-negative Borel measures $d\mu^1$ and $d\mu^2$, respectively, weakly on compact sets. Then,

$$\liminf_{m\to+\infty} I_K(d\mu_m^1, d\mu_m^2) \;\geq\; I_K(d\mu^1, d\mu^2)\ .$$

In case the kernel K is of variable sign, we, also, assume that all $d\mu_m^1$ and $d\mu_m^2$ are supported in a common compact subset of $\mathbf{R^n}$.
2. Let $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its components and $\{d\mu_m^1\}$ and $\{d\mu_m^2\}$ be two sequences of non-negative Borel measures in Ω converging to the non-negative Borel measures $d\mu^1$ and $d\mu^2$, respectively, weakly on compact subsets of Ω. Then

$$\liminf_{m\to+\infty} I_\Omega(d\mu_m^1, d\mu_m^2) \;\geq\; I_\Omega(d\mu^1, d\mu^2)\ .$$

Proof:
1. Assume that K is of first type and non-negative and consider the truncated kernel

$$K_N(x,y) \;=\; \min(K(x,y), N)$$

for all $x, y \in \mathbf{R^n}$.

Now, K_N is continuous in $\mathbf{R^n} \times \mathbf{R^n}$ and, taking an arbitrary $R > 0$,

$$\begin{aligned}
\liminf_{m\to+\infty} I_K(d\mu_m^1, d\mu_m^2) \;&\geq\; \liminf_{m\to+\infty} \int_{B(0;R)} \int_{B(0;R)} K_N(x,y)\ d\mu_m^1(x)\ d\mu_m^2(y) \\
&=\; \int_{B(0;R)} \int_{B(0;R)} K_N(x,y)\ d\mu^1(x)\ d\mu^2(y)\ ,
\end{aligned}$$

since the product measures $d\mu_m^1 \times d\mu_m^2$ converge to $d\mu^1 \times d\mu^2$ weakly on compact sets in $\mathbf{R^n} \times \mathbf{R^n}$.

Now, by the Monotone Convergence Theorem, letting $N \to +\infty$ and then $R \to +\infty$, we conclude the proof in this case.

If K is of variable sign, then we repeat the same proof, replacing the arbitrary ball with a single compact set F so that all $d\mu_m$ are supported in F.
2. Now, assuming that $K = G^\Omega$, consider, besides K_N, any open exhaustion $\{\Omega_{(k)}\}$ of Ω.

Then, as before,

$$\begin{aligned}\liminf_{m\to+\infty} I_\Omega(d\mu^1_m, d\mu^2_m) &\geq \liminf_{m\to+\infty} \int_{\overline{\Omega_{(k)}}} \int_{\overline{\Omega_{(k)}}} K_N(x,y)\ d\mu^1_m(x)\ d\mu^2_m(y) \\ &= \int_{\overline{\Omega_{(k)}}} \int_{\overline{\Omega_{(k)}}} K_N(x,y)\ d\mu^1(x)\ d\mu^2(y)\end{aligned}$$

and the proof is finished, by letting $k \to +\infty$ and $N \to +\infty$.

7.2 Representation of energy: Green's kernel

Let Ω be any open set in $\mathbf{R^n}$ with a Green's function in all its components. In case $n \geq 3$ and $\Omega = \mathbf{R^n}$, the kernel $G_x^{\mathbf{R^n}}$ coincides with the Newtonian kernel h_x.

The basis of all results in this section is the following simple

Lemma 7.1 *Suppose that Ω is a bounded open set with C^2-boundary. Let g_1 and g_2 be two non-negative functions in $\mathcal{D}(\Omega)$.*

Then

$$I_\Omega(g_1\, dm, g_2\, dm) = -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{g_1}}(x) \cdot \overrightarrow{grad\, U_\Omega^{g_2}}(x)\ dm(x)\ .$$

Proof:

By Theorems 2.18 and 5.5, the potential $U_\Omega^{g_i}$ is in $C^\infty(\Omega)$ and

$$\Delta U_\Omega^{g_i} = \kappa_n g_i$$

as distributions in Ω.

Since both sides in this equation are continuous functions, it holds in the classical sense everywhere in Ω.

We, also, get that $U_\Omega^{g_i}$ is harmonic in the open set $\Omega \setminus supp(g_i)$. Since Ω is a regular set, by Lemma 6.2, $U_\Omega^{g_i}$ can be considered continuous in $\overline{\Omega}$ and identically 0 in $\partial\Omega$.

Therefore, by Theorem 5.8, $\overrightarrow{grad\, U_\Omega^{g_i}}$ extends continuously in $\overline{\Omega}$.

Now, applying Green's Formula,

$$\begin{aligned}I_\Omega(g_1\, dm, g_2\, dm) &= \int_\Omega \int_\Omega G^\Omega(x,y) g_1(x)\ dm(x)\ g_2(y)\ dm(y) \\ &= \int_\Omega U_\Omega^{g_1}(y) g_2(y)\ dm(y) \\ &= \frac{1}{\kappa_n} \int_\Omega U_\Omega^{g_1}(y) \Delta U_\Omega^{g_2}(y)\ dm(y) \\ &= -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{g_1}}(y) \cdot \overrightarrow{grad\, U_\Omega^{g_2}}(y)\ dm(y) \\ &\quad + \frac{1}{\kappa_n} \int_{\partial\Omega} U_\Omega^{g_1}(z) \frac{\partial U_\Omega^{g_2}}{\partial \eta}(z)\ dS(z) \\ &= -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{g_1}}(y) \cdot \overrightarrow{grad\, U_\Omega^{g_2}}(y)\ dm(y)\ ,\end{aligned}$$

$\overrightarrow{\eta}$ being the continuous unit vector field normal to $\partial\Omega$ and directed towards the exterior of Ω.

Remark

Observe that, under the assumptions of Lemma 7.1, the same application of Green's Theorem implies

$$\int_\Omega \int_\Omega G^\Omega(x,y) g_1(x)\ dm(x)\ g_2(y)\ dm(y)$$
$$= -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{g_1}}(y) \cdot \overrightarrow{grad\, U_{\Omega'}^{g_2}}(y)\ dm(y)\ ,$$

whenever $\Omega' \supseteq \Omega$ is another bounded open set with C^2-boundary.

Proposition 7.2 *Suppose that the open set $\Omega \subseteq \mathbf{R^n}$ has a Green's function in all its components and that $d\mu$ is a non-negative Borel measure in Ω with $I_\Omega(d\mu) < +\infty$.*

Then $U_\Omega^{d\mu}$ is a superharmonic function in Ω and it is finite almost everywhere with respect to $d\mu$.

Proof:

If $d\mu$ is the zero measure in any of the components of Ω, then the result is obvious in those components. Otherwise, the result is only an application of Fubini's Theorem.

Proposition 7.3 *If the open set $\Omega \subseteq \mathbf{R^n}$ has a Green's function in all its components and $d\mu_1$, $d\mu_2$ are two non-negative Borel measures in Ω with $I_\Omega(d\mu_1) < +\infty$ and $I_\Omega(d\mu_2) < +\infty$, then*

$$I_\Omega(d\mu_1, d\mu_2)\ =\ -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x) \cdot \overrightarrow{grad\, U_\Omega^{d\mu_2}}(x)\ dm(x)\ <\ +\infty\ .$$

We, also, have that $U_\Omega^{d\mu_1}$ is finite almost everywhere with respect to $d\mu_2$ and vice-versa.

Proof:

1. Assume, first, that both $d\mu_i$ are supported in a compact subset A of Ω and that Ω is bounded with C^2-boundary.

Each $U_\Omega^{d\mu_i}$ is superharmonic in Ω, harmonic in $\Omega \setminus A$ and, since Ω is regular, it can, by Lemma 6.2, be considered continuous in $\mathbf{R^n} \setminus A$ and identically 0 in $\mathbf{R^n} \setminus \Omega$.

Consider any approximation to the identity $\{\Phi_\delta : \delta > 0\}$ and the convolution

$$U_\Omega^{d\mu_i} * \Phi_\delta(x)\ =\ \int_{\mathbf{R^n}} U_\Omega^{d\mu_i}(x-y)\Phi_\delta(y)\ dm(y)\ , \qquad x \in \Omega_\delta\ .$$

Then, the functions $U_\Omega^{d\mu_i} * \Phi_\delta$ are in $C^\infty(\Omega_\delta)$ and, by Theorem 2.10, they are superharmonic in Ω_δ and

$$U_\Omega^{d\mu_i} * \Phi_\delta\ \uparrow\ U_\Omega^{d\mu_i}$$

in Ω as $\delta \downarrow 0$.

Taking $\delta < \frac{1}{3}d(A, \partial\Omega)$, we know, from the beginning of section 1.3, that $U_\Omega^{d\mu_i} * \Phi_\delta = U_\Omega^{d\mu_i}$ in $\big(\Omega \setminus A\big)_\delta$. Therefore, the function

$$v_{i,\delta} \;=\; \begin{cases} U_\Omega^{d\mu_i} * \Phi_\delta \;, & \text{if } x \in \Omega_\delta \\ U_\Omega^{d\mu_i} \;, & \text{if } x \in \Omega \setminus \Omega_\delta \;, \end{cases}$$

is in $C^\infty(\Omega)$, is superharmonic in Ω, harmonic in $\Omega \setminus A$ and can be considered continuous in $\mathbf{R^n} \setminus A$ and identically 0 in $\mathbf{R^n} \setminus \Omega$. Also, $v_{i,\delta} \uparrow U_\Omega^{d\mu_i}$ in $\mathbf{R^n}$ as $\delta \downarrow 0$.

Consider, also, the convolution

$$d\mu_i * \Phi_\delta(x) \;=\; \int_{\mathbf{R^n}} \Phi_\delta(x-y)\; d\mu_i(y) \;, \qquad x \in \mathbf{R^n} \;.$$

The non-negative functions $d\mu_i * \Phi_\delta$ are in $C^\infty(\mathbf{R^n})$ and they are supported in $A_\delta = A + \overline{B(0;\delta)}$, a compact subset of Ω.

Therefore, the functions $U_\Omega^{d\mu_i * \Phi_\delta}$ are superharmonic in Ω, harmonic in $\Omega \setminus A_\delta$ and, by Theorem 2.18, they are in $C^\infty(\Omega)$. By Lemma 6.2, they can be considered continuous in $\mathbf{R^n}$ and identically 0 in $\mathbf{R^n} \setminus \Omega$.

Employing the informal notation for distributions,

$$\Delta\big(U_\Omega^{d\mu_i} * \Phi_\delta\big) \;=\; \Delta U_\Omega^{d\mu_i} * \Phi_\delta \;=\; \kappa_n d\mu_i * \Phi_\delta \;=\; \Delta U_\Omega^{d\mu_i * \Phi_\delta}$$

in Ω_δ.

Therefore, the functions $v_{i,\delta}$ and $U_\Omega^{d\mu_i * \Phi_\delta}$ differ by a function harmonic in Ω_δ and, hence, in Ω. Both functions are continuous in $\overline{\Omega}$ and identically 0 in $\partial\Omega$. By the Maximum-Minimum Principle,

$$v_{i,\delta} \;=\; U_\Omega^{d\mu_i * \Phi_\delta}$$

identically in Ω.

Apply, now, Lemma 7.1 to the functions $g_i = d\mu_i * \Phi_{\delta_i}$, for $\delta_i < \frac{1}{3}d(A, \partial\Omega)$, to get

$$\begin{aligned} &\int_\Omega U_\Omega^{d\mu_1 * \Phi_{\delta_1}}(x) d\mu_2 * \Phi_{\delta_2}(x)\; dm(x) \\ &= \; -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1 * \Phi_{\delta_1}}}(x) \cdot \overrightarrow{grad\, U_\Omega^{d\mu_2 * \Phi_{\delta_2}}}(x)\; dm(x) \\ &= \; -\frac{1}{\kappa_n} \int_\Omega \overrightarrow{grad\, v_{1,\delta_1}}(x) \cdot \overrightarrow{grad\, v_{2,\delta_2}}(x)\; dm(x) \;. \end{aligned}$$

Since $U_\Omega^{d\mu_1 * \Phi_{\delta_1}} = v_{1,\delta_1} \uparrow U_\Omega^{d\mu_1}$ in Ω as $\delta_1 \downarrow 0$, the left side of the last equality tends to

$$\int_\Omega U_\Omega^{d\mu_1}(x) d\mu_2 * \Phi_{\delta_2}(x)\; dm(x)$$

as $\delta_1 \to 0$.

Regarding the right side, it is true, by Theorem 2.18, that $U_\Omega^{d\mu_i}$ has partial derivatives at almost every point in Ω which are locally integrable in Ω. From Theorem 5.8, we have that $\frac{\partial U_\Omega^{d\mu_i}}{\partial x_j}$ can be continuously extended in $\overline{\Omega} \setminus A$ and, hence, if we further extend it as identically 0 in $\mathbf{R^n} \setminus \overline{\Omega}$, it becomes a function in $L^1(\mathbf{R^n})$. Applying Lemma 0.4, we find that there is a sequence $\{\delta_i\}$ so that $\delta_i \to 0$ and $\frac{\partial U_\Omega^{d\mu_i}}{\partial x_j} * \Phi_{\delta_i} \to \frac{\partial U_\Omega^{d\mu_i}}{\partial x_j}$ almost everywhere in $\mathbf{R^n}$. Since, $\frac{\partial v_{i,\delta_i}}{\partial x_j} = \frac{\partial\left(U_\Omega^{d\mu_i} * \Phi_{\delta_i}\right)}{\partial x_j} = \frac{\partial U_\Omega^{d\mu_i}}{\partial x_j} * \Phi_{\delta_i}$ in Ω_{δ_i}, we get

$$\frac{\partial v_{i,\delta_i}}{\partial x_j} \ \to \ \frac{\partial U_\Omega^{d\mu_i}}{\partial x_j}$$

almost everywhere in Ω as $\delta_i \to 0$.

From Lemma 7.1,

$$\begin{aligned}
-\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, v_{i,\delta_i}}(x)\right|^2 dm(x) \ &= \ \int_\Omega U_\Omega^{d\mu_i * \Phi_{\delta_i}}(x) d\mu_i * \Phi_{\delta_i}(x)\ dm(x) \\
&\le \ \int_\Omega U_\Omega^{d\mu_i}(x) d\mu_i * \Phi_{\delta_i}(x)\ dm(x) \\
&= \ \int_\Omega U_\Omega^{d\mu_i * \Phi_{\delta_i}}(x)\ d\mu_i(x) \\
&\le \ \int_\Omega U_\Omega^{d\mu_i}(x)\ d\mu_i(x) \\
&= \ I_\Omega(d\mu_i) \ < \ +\infty \ .
\end{aligned}$$

From our last two results, we conclude that there is some sequence of δ_i's so that, for all j,

$$\frac{\partial v_{i,\delta_i}}{\partial x_j} \ \to \ \frac{\partial U_\Omega^{d\mu_i}}{\partial x_j}$$

weakly in $L^2(\Omega)$.

Letting this sequence of δ_1's tend to 0,

$$\begin{aligned}
&\int_\Omega U_\Omega^{d\mu_1}(x) d\mu_2 * \Phi_{\delta_2}(x)\ dm(x) \\
&= \ -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x) \cdot \overrightarrow{grad\, v_{2,\delta_2}}(x)\ dm(x) \ .
\end{aligned}$$

The left side is equal to $\int_\Omega U_\Omega^{d\mu_2 * \Phi_{\delta_2}}(x)\ d\mu_1(x)$ and, following the same procedure with $\delta_2 \to 0$, we find

$$I_\Omega(d\mu_1, d\mu_2) \ = \ -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x) \cdot \overrightarrow{grad\, U_\Omega^{d\mu_2}}(x)\ dm(x) \ .$$

2. Modifying slightly the proof of part 1 and using the Remark after Lemma 7.1, we, easily, prove that, under the same assumptions,

$$\int_\Omega\int_\Omega G^\Omega(x,y)\ d\mu_1(x)\ d\mu_2(y) \ = \ -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x) \cdot \overrightarrow{grad\, U_{\Omega'}^{d\mu_2}}(x)\ dm(x) \ ,$$

whenever $\Omega' \supseteq \Omega$ is another bounded open set with C^2-boundary.
3. Now, assume that Ω has a Green's function in all its connected components and that $d\mu_1$ and $d\mu_2$ are two non-negative Borel measures supported in a compact subset A of Ω.

Consider any open exhaustion $\{\Omega_{(m)}\}$ of Ω, each $\Omega_{(m)}$ having C^2-boundary and containing A.

From part 2, we have that

$$I_{\Omega_{(m)}}(d\mu_1, d\mu_2) \;=\; -\frac{1}{\kappa_n}\int_{\Omega_{(m)}} \overrightarrow{grad\, U^{d\mu_1}_{\Omega_{(m)}}}(x)\cdot \overrightarrow{grad\, U^{d\mu_2}_{\Omega_{(m')}}}(x)\; dm(x)\;,$$

for every m, m' with $m \le m'$.

Since $G^{\Omega_{(m)}} \uparrow G^{\Omega}$ in Ω, the right side has $I_\Omega(d\mu_1, d\mu_2)$ as its limit when $m \to +\infty$.

We, also, have that $U^{d\mu_i}_{\Omega_{(m)}} = U^{d\mu_i}_h + v_{m,i}$, where $v_{m,i}$ is harmonic in $\Omega_{(m)}$. Since $U^{d\mu_i}_{\Omega_{(m)}} \uparrow U^{d\mu_i}_{\Omega}$ as $m \to +\infty$, we get that $\{v_{m,i}\}$ increases towards a harmonic function v_i in Ω as $m \to +\infty$ and, hence, $U^{d\mu_i}_{\Omega} = U^{d\mu_i}_h + v_i$ in Ω. By Theorems 1.16 and 1.17, $\overrightarrow{grad\, v_{m,i}} \to \overrightarrow{grad\, v_i}$ everywhere in Ω as $m \to +\infty$, implying

$$\overrightarrow{grad\, U^{d\mu_i}_{\Omega_{(m)}}} \;\to\; \overrightarrow{grad\, U^{d\mu_i}_{\Omega}}$$

almost everywhere in Ω as $m \to +\infty$.

From part 1, we get

$$-\frac{1}{\kappa_n}\int_{\Omega_{(m)}} \big|\overrightarrow{grad\, U^{d\mu_i}_{\Omega_{(m)}}}(x)\big|^2\, dm(x) \;=\; \int\!\!\int G^{\Omega_{(m)}}(x,y)\, d\mu_i(x)\, d\mu_i(y) \;\le\; I_\Omega(d\mu_i)\,.$$

Hence, there is some sequence of m's so that

$$\overrightarrow{grad\, U^{d\mu_1}_{\Omega_{(m)}}} \;\to\; \overrightarrow{grad\, U^{d\mu_1}_{\Omega}}$$

weakly in $L^2(\Omega)$ and some sequence of m''s so that

$$\overrightarrow{grad\, U^{d\mu_2}_{\Omega_{(m')}}} \;\to\; \overrightarrow{grad\, U^{d\mu_2}_{\Omega}}$$

weakly in $L^2(\Omega)$.

We, now, let, first, $m' \to +\infty$ and, then, $m \to +\infty$ through these sequences and get

$$I_\Omega(d\mu_1, d\mu_2) \;=\; -\frac{1}{\kappa_n}\int_{\Omega} \overrightarrow{grad\, U^{d\mu_1}_{\Omega}}(x)\cdot \overrightarrow{grad\, U^{d\mu_2}_{\Omega}}(x)\; dm(x)\;.$$

4. Consider, finally, the general case.

Take the measures $d\mu_{m,i}$, which are the restrictions of $d\mu_i$ to the terms of an open exhaustion $\{\Omega_{(m)}\}$ of Ω.

By the assumption $I_\Omega(d\mu_i) < +\infty$ and Proposition 7.2, it is implied that $U^{d\mu_i}_{\Omega}$ and all $U^{d\mu_{m,i}}_{\Omega}$ are superharmonic in Ω.

Therefore, for any $k > m$, $U_\Omega^{d\mu_{k,i}} - U_\Omega^{d\mu_{m,i}}$ is harmonic in $\Omega_{(m)}$ and

$$U_\Omega^{d\mu_{k,i}} - U_\Omega^{d\mu_{m,i}} \ \uparrow\ U_\Omega^{d\mu_i} - U_\Omega^{d\mu_{m,i}}$$

in $\Omega_{(m)}$ as $k \to +\infty$. The last function is harmonic in $\Omega_{(m)}$ and, from Theorems 1.16 and 1.17,

$$\overrightarrow{grad\big(U_\Omega^{d\mu_{k,i}} - U_\Omega^{d\mu_{m,i}}\big)} \ \to\ \overrightarrow{grad\big(U_\Omega^{d\mu_i} - U_\Omega^{d\mu_{m,i}}\big)}$$

in $\Omega_{(m)}$ as $k \to +\infty$.

Since $U_\Omega^{d\mu_{m,i}}$ has partial derivatives almost everywhere in Ω, we get that

$$\overrightarrow{grad\, U_\Omega^{d\mu_{k,i}}} \ \to\ \overrightarrow{grad\, U_\Omega^{d\mu_i}}$$

almost everywhere in $\Omega_{(m)}$ and, since m is arbitrary, the last limit holds almost everywhere in Ω.

From the equality of part 3,

$$-\frac{1}{\kappa_n}\int_\Omega \big|\overrightarrow{grad\, U_\Omega^{d\mu_{m,i}}}(x)\big|^2\ dm(x) \ =\ I_\Omega(d\mu_{m,i}) \ \le\ I_\Omega(d\mu_i)\ ,$$

there is a sequence of m's so that

$$U_\Omega^{d\mu_{m,i}} \ \to\ U_\Omega^{d\mu_i}$$

weakly in $L^2(\Omega)$.

If in the equality of part 3,

$$\begin{aligned} &\int\!\!\int G^\Omega(x,y)\ d\mu_{m_1,1}(x)\ d\mu_{m_2,2}(y) \\ &=\ -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_{m_1,1}}}(x)\cdot \overrightarrow{grad\, U_\Omega^{d\mu_{m_2,2}}}(x)\ dm(x)\ , \end{aligned}$$

we let, first, m_1 and, then, m_2 tend to $+\infty$, we find

$$\int\!\!\int G^\Omega(x,y)\ d\mu_1(x)\ d\mu_2(y) \ =\ -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x)\cdot \overrightarrow{grad\, U_\Omega^{d\mu_2}}(x)\ dm(x)\ .$$

Regarding the last statement of the theorem, observe that, from the Cauchy-Schwartz inequality,

$$I_\Omega(d\mu_1, d\mu_2)^2 \ \le\ I_\Omega(d\mu_1) I_\Omega(d\mu_2) \ <\ +\infty$$

and, hence, from Fubini's Theorem, $U_\Omega^{d\mu_1}$ is finite almost everywhere with respect to $d\mu_2$ and vice-versa.

7.3 Measures of finite energy: Green's kernel

Definition 7.3 *Let the open $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its connected components. Then,*

$$\mathcal{W}_\Omega \;=\; \{d\mu : d\mu \text{ is a locally finite measure in } \Omega \text{ with } I_\Omega(|d\mu|) < +\infty\}$$

is called ***the space of measures in Ω of finite Ω-energy****.*

Theorem 7.1 *Let the open $\Omega \subseteq \mathbf{R^n}$ have a Green's function in all its connected components. Then, for all $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_\Omega$ the bilinear form $I_\Omega(d\mu_1, d\mu_2)$ is a complex number and*

$$I_\Omega(d\mu_1, d\mu_2) \;=\; -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x)\cdot \overrightarrow{grad\, U_\Omega^{d\mu_2}}(x)\; dm(x)\;.$$

Under this bilinear form, $\mathcal{W}_\Omega$ becomes an inner product space.

Proof:
If $d\mu_1$ and $d\mu_2$ are two locally finite complex Borel measures in Ω of finite Ω-energy, then, by Proposition 7.3 and the Cauchy-Schwartz inequality,

$$\begin{aligned}
I_\Omega(|d\mu_1 + d\mu_2|) \;&\le\; I_\Omega(|d\mu_1| + |d\mu_2|)\\
&=\; I_\Omega(|d\mu_1|) + 2I_\Omega(|d\mu_1|, |d\mu_2|) + I_\Omega(|d\mu_2|)\\
&=\; -\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, U_\Omega^{|d\mu_1|}}(x)\right|^2\; dm(x)\\
&\quad\; -\frac{2}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{|d\mu_1|}}(x)\cdot \overrightarrow{grad\, U_\Omega^{|d\mu_2|}}(x)\; dm(x)\\
&\quad\; -\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, U_\Omega^{|d\mu_2|}}(x)\right|^2\; dm(x)\\
&\le\; I_\Omega(|d\mu_1|) + 2I_\Omega(|d\mu_1|)^{\frac{1}{2}} I_\Omega(|d\mu_2|)^{\frac{1}{2}} + I_\Omega(|d\mu_2|)\\
&<\; +\infty\;.
\end{aligned}$$

Thus, $\mathcal{W}_\Omega$ is a linear space.

From Fubini's Theorem and from $I_\Omega(|d\mu_1|, |d\mu_2|) \le I_\Omega(|d\mu_1|)^{\frac{1}{2}} I_\Omega(|d\mu_2|)^{\frac{1}{2}} < +\infty$, we get that $I_\Omega(d\mu_1, d\mu_2)$ is a complex number for every $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_\Omega$.

If $d\mu$ is a locally finite Borel measure in Ω with real values and $I_\Omega(|d\mu|) < +\infty$, then its non-negative and non-positive variations, $d\mu^+$ and $d\mu^-$, satisfy $I_\Omega(d\mu^+) < +\infty$ and $I_\Omega(d\mu^-) < +\infty$. Proposition 7.3 implies that

$$\begin{aligned}
I_\Omega(d\mu) \;&=\; I_\Omega(d\mu^+) - 2I_\Omega(d\mu^+, d\mu^-) + I_\Omega(d\mu^-)\\
&=\; -\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, U_\Omega^{d\mu^+}}(x)\right|^2\; dm(x)\\
&\quad\; +\frac{2}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu^+}}(x)\cdot \overrightarrow{grad\, U_\Omega^{d\mu^-}}(x)\; dm(x)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, U_\Omega^{d\mu^-}}(x)\right|^2\, dm(x) \\
= \quad & -\frac{1}{\kappa_n}\int_\Omega \left|\overrightarrow{grad\, U_\Omega^{d\mu}}(x)\right|^2\, dm(x)\ .
\end{aligned}
$$

If $d\mu_1$ and $d\mu_2$ are two locally finite Borel measures in $\mathcal{W}_\Omega$ with real values, then, applying the last equality to $d\mu_1 + d\mu_2$, we get

$$
I_\Omega(d\mu_1, d\mu_2) \;=\; -\frac{1}{\kappa_n}\int_\Omega \overrightarrow{grad\, U_\Omega^{d\mu_1}}(x)\cdot \overrightarrow{grad\, U_\Omega^{d\mu_2}}(x)\; dm(x)\ .
$$

By linearity, this extends to hold for all measures in $\mathcal{W}_\Omega$ ($d\mu_2$ being replaced by $\overline{d\mu_2}$.)

Thus, the bilinear form $I_\Omega(d\mu_1, d\mu_2)$ is a complex number for all $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_\Omega$ and, using the representation of the last formula, we get that

$$
I_\Omega(d\mu) \geq 0 \qquad \text{for all non-negative } d\mu \in \mathcal{W}_\Omega\ .
$$

Assume, now, that, for some $d\mu \in \mathcal{W}_\Omega$,

$$
I_\Omega(d\mu) \;=\; 0\ .
$$

Then $\overrightarrow{grad\, U_\Omega^{d\mu}}(x) = 0$ for almost every $x \in \Omega$ and, since $U_\Omega^{d\mu}$ is, by Theorem 2.18, absolutely continuous on almost every line parallel to any of the principal axes, we easily get that $U_\Omega^{d\mu}$ is equal to some constant c almost everywhere in Ω. Since this function is superharmonic, it is identically equal to c in Ω.

Finally,

$$
\kappa_n d\mu \;=\; \Delta U_\Omega^{d\mu} \;=\; 0
$$

in Ω in the sense of distributions and, hence, $d\mu$ is the zero measure.

Therefore, the bilinear form is an inner product.

7.4 Representation of energy: kernels of first type

If K is any non-negative kernel of first type, then

$$
\lim_{r\to+\infty} K_*(r) \;=\; 0\ .
$$

Lemma 7.2 *Suppose that K is a kernel of first type with the property that, for some $R > 0$,*

$$
K(x) \;=\; 0
$$

for all x with $|x| \geq R$.

Then $K \in L^1(\mathbf{R^n})$ and

$$
\widehat{K}(\xi) \;>\; 0
$$

for all $\xi \in \mathbf{R^n}$*. There is, also, a constant* $C > 0$ *so that*

$$\widehat{K}(\xi) \ \geq \ \frac{C}{1+|\xi|^2}$$

for all $\xi \in \mathbf{R^n}$.

Proof:

The kernel is non-negative and, by the first property of kernels of first type, we have

$$\int_{\mathbf{R^n}} K(x)\ dm(x) \ = \ \omega_{n-1} \int_0^R K_*(r) r^{n-1}\ dr \ < \ +\infty \ .$$

Now,

$$\begin{aligned} \widehat{K}(\xi) &= \int_{\mathbf{R^n}} e^{-2\pi i x\cdot\xi} K(x)\ dm(x) \\ &= \int_0^{+\infty} K_*(r) \int_{S^{n-1}} e^{-2\pi i r x'\cdot\xi}\ d\sigma(x')\, r^{n-1}\ dr \\ &= \int_0^{+\infty} K_*(r) J(r|\xi|) r^{n-1}\ dr \ , \end{aligned}$$

where we use the notation $x' = \frac{x}{r} = (x_1', \ldots, x_n')$ with $r = |x|$ and

$$J(r) \ = \ \int_{S^{n-1}} e^{-2\pi i r x_1'}\ d\sigma(x')\ , \qquad r \geq 0\ .$$

Finally, writing $x' = (x_1', x'')$ with $x'' = (1 - x_1'^2)^{\frac{1}{2}} y$, $y \in S^{n-2}$, and using $d\sigma(x') = (1 - x_1'^2)^{\frac{n-2}{2}}\ d\sigma(y)\ dx_1'$, we get

$$\begin{aligned} J(r) &= \int_{-1}^{1} \int_{S^{n-2}} e^{-2\pi i r x_1'} (1 - x_1'^2)^{\frac{n-2}{2}}\ d\sigma(y)\ dx_1' \\ &= 2\omega_{n-2} \int_0^{\frac{\pi}{2}} \cos(2\pi r \cos\phi) \sin^{n-2}\phi\ d\phi \end{aligned}$$

and J has the following properties:

1. $J(0) = \omega_{n-1}$ and $J'(0) = 0$.
2. $J(r) < \omega_{n-1}$ for all $r > 0$.
3. $J(r) = -\frac{1}{4\pi^2}\big(J''(r) + \frac{n-1}{r} J'(r)\big)$ for all $r \geq 0$.
4. $\lim_{r\to+\infty} J(r) = 0$.

The first three properties are trivial to prove. The fourth is an application of Theorem 0.13 to $J(r) = \omega_{n-2} \int_{-1}^{1} e^{-2\pi i r x_1'} (1 - x_1'^2)^{\frac{n-2}{2}}\ dx_1'$.

Therefore,

$$
\begin{aligned}
\widehat{K}(\xi) &= -\frac{1}{4\pi^2|\xi|^n}\int_0^{+\infty} K_*\Big(\frac{r}{|\xi|}\Big)\big(J''(r)+\frac{n-1}{r}J'(r)\big)r^{n-1}\,dr\\
&= -\frac{1}{4\pi^2|\xi|^n}\int_0^{+\infty} K_*\Big(\frac{r}{|\xi|}\Big)\frac{d}{dr}\big(J'(r)r^{n-1}\big)\,dr\\
&= \frac{1}{4\pi^2|\xi|^{n+1}}\int_0^{+\infty} {K_*}'\Big(\frac{r}{|\xi|}\Big)J'(r)r^{n-1}\,dr\\
&= \frac{1}{4\pi^2|\xi|^2}\int_0^{+\infty}\big(\omega_{n-1}-J(r)\big)\,d\Big({K_*}'\Big(\frac{r}{|\xi|}\Big)\Big(\frac{r}{|\xi|}\Big)^{n-1}\Big)
\end{aligned}
$$

where the integrations by parts are easy to justify.

But, if we set $s = h_*(r)$ and $K_*(r) = H(s)$, then, in case $n \geq 3$, the last integral becomes

$$
\begin{aligned}
&(n-2)\int_0^{+\infty}\big(\omega_{n-1}-J(h_*^{-1}(s))\big)\,d\big(H'(|\xi|^{n-2}s)\big)\\
\geq\ &(n-2)\int_0^{1}\big(\omega_{n-1}-J(h_*^{-1}(s))\big)\,d\big(H'(|\xi|^{n-2}s)\big)\\
\geq\ &(n-2)\min_{1\leq r<+\infty}\big(\omega_{n-1}-J(r)\big)H'(|\xi|^{n-2})
\end{aligned}
$$

and, in case $n = 2$, it becomes

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\big(\omega_1-J(h_*^{-1}(s))\big)\,d\big(H'(s+\log|\xi|)\big)\\
\geq\ &\int_{-\infty}^{0}\big(\omega_1-J(h_*^{-1}(s))\big)\,d\big(H'(s+\log|\xi|)\big)\\
\geq\ &\min_{1\leq r<+\infty}\big(\omega_1-J(r)\big)H'(\log|\xi|)\ .
\end{aligned}
$$

Considering R large enough so that $H'(R^{n-2}) > 0$ or $H'(\log R) > 0$, respectively, then, for all ξ with $|\xi| \geq R$, we have

$$
\widehat{K}(\xi) \ \geq\ \frac{C}{|\xi|^2}\ .
$$

In any case, both integrals $(n-2)\int_0^{+\infty}\big(\omega_{n-1}-J(h_*^{-1}(s))\big)\,d\big(H'(|\xi|^{n-2}s)\big)$ and $\int_{-\infty}^{+\infty}\big(\omega_1-J(h_*^{-1}(s))\big)\,d\big(H'(s+\log|\xi|)\big)$ are positive for all $\xi \neq 0$. Hence, $\widehat{K}(\xi) > 0$ for all $\xi \neq 0$. Since K is non-negative, $\widehat{K}(0) = \int_{\mathbf{R^n}} K(x)\,dm(x) > 0$.

The proof is concluded by the continuity of $\widehat{K}$ in $\overline{B(0;R)}$.

Lemma 7.3 *Suppose that K is a non-negative kernel of first type with the property that, for some $R > 0$,*

$$
K_*(r) \ =\ 0
$$

for all $r \geq R$.

If $d\mu_1$ and $d\mu_2$ are any two compactly supported non-negative Borel measures in $\mathbf{R^n}$ with $I_K(d\mu_1) < +\infty$ and $I_K(d\mu_2) < +\infty$, then

$$I_K(d\mu_1, d\mu_2) \;=\; \int_{\mathbf{R^n}} \widehat{K}(\xi)\widehat{d\mu_1}(\xi)\overline{\widehat{d\mu_2}(\xi)}\; dm(\xi)\;.$$

Therefore, $I_K(d\mu_1, d\mu_2) < +\infty$ and $U_K^{d\mu_1}$ is finite almost everywhere with respect to $d\mu_2$ and vice-versa.

Proof:

Suppose that $A_j = supp(d\mu_j)$. By Proposition 6.4, we can find restrictions $d\mu_{1,\epsilon_1}$ and $d\mu_{2,\epsilon_2}$ of $d\mu_1$ and $d\mu_2$ in compact sets $A_{1,\epsilon_1} \subseteq A_1$ and $A_{2,\epsilon_2} \subseteq A_2$, respectively, so that, for both $j = 1, 2$,

$$\|d\mu_j - d\mu_{j,\epsilon_j}\| \;\to\; 0$$

as $\epsilon_j \to 0$ and each $U_K^{d\mu_{j,\epsilon_j}}$ is a continuous potential.

We may, also, assume that $A_{j,\epsilon_j} \uparrow A_j$ as $\epsilon_j \downarrow 0$, as we may easily see by looking in the proof of Theorem 6.1. Hence,

$$I_K(d\mu_{1,\epsilon_1}, d\mu_{2,\epsilon_2}) \;\uparrow\; I_K(d\mu_1, d\mu_2)$$

as $\epsilon_j \downarrow 0$.

Now introduce the functions

$$\phi_k(x) \;=\; k^{\frac{n}{2}} e^{-\pi k|x|^2}$$

for all $k > 0$ and all $x \in \mathbf{R^n}$. Then,

$$\int_{\mathbf{R^n}} \phi_k(x)\; dm(x) \;=\; 1$$

and

$$\widehat{\phi_k}(\xi) \;=\; e^{-\pi\frac{|\xi|^2}{k}}$$

for all $\xi \in \mathbf{R^n}$. Now, both functions ϕ_k and $U_K^{d\mu_{j,\epsilon_j}} = K * d\mu_{j,\epsilon_j}$ are in $L^1(\mathbf{R^n})$ and, hence,

$$U_K^{d\mu_{j,\epsilon_j}} * \phi_k \;\in\; L^1(\mathbf{R^n})\;.$$

Furthermore, since $\big(U_K^{d\mu_{j,\epsilon_j}}\big)\widehat{\ } \in L^\infty(\mathbf{R^n})$ and $\widehat{\phi_k} \in L^1(\mathbf{R^n})$, we get

$$\big(U_K^{d\mu_{j,\epsilon_j}} * \phi_k\big)\widehat{\ } \;=\; \big(U_K^{d\mu_{j,\epsilon_j}}\big)\widehat{\ }\,\widehat{\phi_k} \;\in\; L^1(\mathbf{R^n})\;.$$

Hence, by the Inversion Formula,

$$U_K^{d\mu_{j,\epsilon_j}} * \phi_k(x) \;=\; \int_{\mathbf{R^n}} e^{2\pi i x\cdot\xi}\big(U_K^{d\mu_{j,\epsilon_j}}\big)\widehat{\ }(\xi)\;\widehat{\phi_k}(\xi)\; dm(\xi)$$

and

$$
\begin{aligned}
&\int_{\mathbf{R^n}} U_K^{d\mu_{1,\epsilon_1}} * \phi_k(x)\ d\mu_{2,\epsilon_2}(x) \\
&= \int_{\mathbf{R^n}} \int_{\mathbf{R^n}} e^{2\pi i x\cdot\xi} \big(U_K^{d\mu_{1,\epsilon_1}}\big)\widehat{\ }(\xi)\widehat{\phi_k}(\xi)\ dm(\xi)\ d\mu_{2,\epsilon_2}(x) \\
&= \int_{\mathbf{R^n}} \big(U_K^{d\mu_{1,\epsilon_1}}\big)\widehat{\ }(\xi)\widehat{\phi_k}(\xi)\overline{\widehat{d\mu_{2,\epsilon_2}}(\xi)}\ dm(\xi) \\
&= \int_{\mathbf{R^n}} e^{-\pi\frac{|\xi|^2}{k}}\widehat{K}(\xi)\widehat{d\mu_{1,\epsilon_1}}(\xi)\overline{\widehat{d\mu_{2,\epsilon_2}}(\xi)}\ dm(\xi)\ .
\end{aligned}
$$

In the same way, for each $j = 1, 2$,

$$
\int_{\mathbf{R^n}} U_K^{d\mu_{j,\epsilon_j}} * \phi_k(x)\ d\mu_{j,\epsilon_j}(x) \ = \ \int_{\mathbf{R^n}} e^{-\pi\frac{|\xi|^2}{k}}\widehat{K}(\xi)\big|\widehat{d\mu_{j,\epsilon_j}}(\xi)\big|^2\ dm(\xi)\ .
$$

By the continuity of $U_K^{d\mu_{j,\epsilon_j}}$ and the Maximum Principle for K-potentials we have that $U_K^{d\mu_{j,\epsilon_j}}$ is, also, bounded and, hence,

$$
U_K^{d\mu_{j,\epsilon_j}} * \phi_k \ \to \ U_K^{d\mu_{j,\epsilon_j}}
$$

uniformly on compact sets. Therefore, the left side of the last equality tends to $\int_{\mathbf{R^n}} U_K^{d\mu_{j,\epsilon_j}}(x)\ d\mu_{j,\epsilon_j}(x)$ as $k \to +\infty$. By the Monotone Convergence Theorem,

$$
\int_{\mathbf{R^n}} \widehat{K}(\xi)\big|\widehat{d\mu_{j,\epsilon_j}}(\xi)\big|^2\ dm(\xi) \ = \ \int_{\mathbf{R^n}} U_K^{d\mu_{j,\epsilon_j}}(x)\ d\mu_{j,\epsilon_j}(x) \ \le \ I_K(d\mu_j)\ .
$$

By the Cauchy-Schwartz inequality,

$$
\int_{\mathbf{R^n}} \widehat{K}(\xi)\big|\widehat{d\mu_{1,\epsilon_1}}(\xi)\overline{\widehat{d\mu_{2,\epsilon_2}}(\xi)}\big|\ dm(\xi) \ < \ +\infty
$$

and we conclude that

$$
I_K(d\mu_{1,\epsilon_1}, d\mu_{2,\epsilon_2}) \ = \ \int_{\mathbf{R^n}} \widehat{K}(\xi)\widehat{d\mu_{1,\epsilon_1}}(\xi)\overline{\widehat{d\mu_{2,\epsilon_2}}(\xi)}\ dm(\xi)\ .
$$

Since $\widehat{d\mu_{j,\epsilon_j}}(\xi) \to \widehat{d\mu_j}(\xi)$ for all ξ as $\epsilon_j \to 0$ we may, now, take a decreasing sequence of ϵ_j's so that

$$
\widehat{d\mu_{j,\epsilon_j}}(\xi) \ \to \ \widehat{d\mu_j}(\xi)
$$

weakly in $L^2(\widehat{K}dm)$. Letting, first, $\epsilon_1 \to 0$ and, then, $\epsilon_2 \to 0$ through the appropriate sequences, we find from the last equality,

$$
I_K(d\mu_1, d\mu_2) \ = \ \int_{\mathbf{R^n}} \widehat{K}(\xi)\widehat{d\mu_1}(\xi)\overline{\widehat{d\mu_2}(\xi)}\ dm(\xi)\ .
$$

The last statement of the theorem is, just, an application of Cauchy-Schwartz inequality and the Theorem of Fubini.

Proposition 7.4 *Suppose that K is a kernel of first type with the property that, for some $R > 0$,*

$$K_*(r) = 0$$

for all $r \geq R$.

If $d\mu$ is a locally finite complex Borel measure, let $d\mu_r$ be its restriction in the ball $B(0;r)$.

If $I_K(|d\mu|) < +\infty$, then the definition

$$\widehat{d\mu} = \lim_{r\to+\infty} \widehat{d\mu_r}$$

is justified as a limit in the space $L^2(\widehat{K}dm)$.

If $I_K(|d\mu_1|) < +\infty$ and $I_K(|d\mu_2|) < +\infty$, then

$$I_K(d\mu_1, d\mu_2) = \int_{\mathbf{R^n}} \widehat{d\mu_1}(\xi)\overline{\widehat{d\mu_2}(\xi)}\widehat{K}(\xi)\ dm(\xi)\ .$$

Proof:
1. Assume that $d\mu$ is a non-negative Borel measure. By Lemma 7.3, for every r, r' with $r < r'$, we have

$$I_K(d\mu_{r'} - d\mu_r) = \int_{\mathbf{R^n}} \left|\widehat{d\mu_{r'}}(\xi) - \widehat{d\mu_r}(\xi)\right|^2 \widehat{K}(\xi)\ dm(\xi)\ .$$

On the other hand, we can easily prove that

$$I_K(d\mu_{r'} - d\mu_r) \leq I_K(d\mu_{r'}) - I_K(d\mu_r)\ .$$

From the last two relations, since $I_K(d\mu_r) \uparrow I_K(d\mu)$, we see that, if $I_K(d\mu) < +\infty$, then $\widehat{d\mu} = \lim_{r\to+\infty} \widehat{d\mu_r}$ exists as a limit in $L^2(\widehat{K}dm)$.

Now, if $d\mu_1$ and $d\mu_2$ are non-negative Borel measures with $I_K(d\mu_1) < +\infty$ and $I_K(d\mu_2) < +\infty$, we apply Lemma 7.3 to their restrictions in $B(0;r)$ and let $r \to +\infty$ to get

$$I_K(d\mu_1, d\mu_2) = \int_{\mathbf{R^n}} \widehat{d\mu_1}(\xi)\overline{\widehat{d\mu_2}(\xi)}\widehat{K}(\xi)\ dm(\xi)\ .$$

2. Now, let $d\mu$ be a locally finite complex Borel measure with $I_K(|d\mu|) < +\infty$. Then the non-negative and non-positive parts of the real and imaginary parts of $d\mu$ all have finite K-energy. From part 1, we get that $\widehat{d\mu} = \lim_{r\to+\infty} \widehat{d\mu_r}$ exists as a limit in $L^2(\widehat{K}dm)$ and that

$$I_K(d\mu_1, d\mu_2) = \int_{\mathbf{R^n}} \widehat{d\mu_1}(\xi)\overline{\widehat{d\mu_2}(\xi)}\widehat{K}(\xi)\ dm(\xi)\ ,$$

for all locally finite complex Borel measures $d\mu_1$ and $d\mu_2$ with $I_K(|d\mu_1|) < +\infty$ and $I_K(|d\mu_2|) < +\infty$.

7.5 Measures of finite energy: kernels of first type

Definition 7.4 *Suppose that K is a non-negative kernel of first type and let*

$$\mathcal{W}_K = \{d\mu : d\mu \text{ is a locally finite Borel measure in } R^n \text{ and } I_K(|d\mu|) < +\infty\} .$$

*$\mathcal{W}_K$ is called **the space of measures of finite K-energy**.*

Proposition 7.5 *Suppose that K is a non-negative kernel of first type and let $d\mu_1$ and $d\mu_2$ be any two non-negative Borel measures with $I_K(d\mu_1) < +\infty$ and $I_K(d\mu_2) < +\infty$. Then,*

$$I_K(d\mu_1, d\mu_2) \le \big(I_K(d\mu_1)\big)^{\frac{1}{2}}\big(I_K(d\mu_2)\big)^{\frac{1}{2}}$$

and $U_K^{d\mu_1}$ is finite almost everywhere with respect to $d\mu_2$ and vice-versa.

Proof:

If the measures are supported in a compact set and K vanishes for all large enough values of its argument, then the result is an application of the last Lemma.

Assume, now, that $d\mu_1$ and $d\mu_2$ are supported in a compact set but that $K(x) > 0$ for all $x \in \mathbf{R^n}$. (Observe that, if $K(x_0) = 0$ for some x_0, then $K(x) = 0$ for all x with $|x| \ge |x_0|$.)

Consider, for every $\delta > 0$, the kernel

$$K_\delta(x,y) = \begin{cases} K(x,y) - \delta \ , & \text{if } K(x,y) \ge \delta \\ 0 \ , & \text{if } K(x,y) \le \delta \ . \end{cases}$$

Since $\lim_{r\to+\infty} K_*(r) = 0$, we see that K_δ is a non-negative kernel of first type and that $K_\delta(x) = 0$ for all large enough $|x|$ and, hence, Lemma 7.3 applies to K_δ.

Therefore,

$$I_{K_\delta}(d\mu_1, d\mu_2) \le \big(I_{K_\delta}(d\mu_1)\big)^{\frac{1}{2}}\big(I_{K_\delta}(d\mu_2)\big)^{\frac{1}{2}} .$$

Since $K_\delta \uparrow K$ as $\delta \to 0$, by the Monotone Convergence Theorem or by Fatou's Lemma, we get

$$I_K(d\mu_1, d\mu_2) \le \big(I_K(d\mu_1)\big)^{\frac{1}{2}}\big(I_K(d\mu_2)\big)^{\frac{1}{2}} .$$

If the measures are not supported in a compact set, then apply the result to the restrictions of the measures on the balls $B(0; R)$ and let $R \to +\infty$.

Theorem 7.2 *Suppose that K is any non-negative kernel of first type.*

Then, $\mathcal{W}_K$ is a linear space. Also, for any $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_K$, $I_K(d\mu_1, d\mu_2)$ is well-defined as a complex number and, under this bilinear form, $\mathcal{W}_K$ becomes an inner product space

Also, if $d\mu \in \mathcal{W}_K$, then $U_K^{d\mu}$ is finite almost everywhere with respect to all $d\nu \in \mathcal{W}_K$.

Proof:
If $d\mu_1 \in \mathcal{W}_K$ and $d\mu_2 \in \mathcal{W}_K$, then, by the last Proposition,

$$\begin{aligned} I_K(|d\mu_1 + d\mu_2|) &\leq I_K(|d\mu_1| + |d\mu_2|) \\ &= I_K(|d\mu_1|) + 2I_K(|d\mu_1|, |d\mu_2|) + I_K(|d\mu_2|) \\ &\leq I_K(|d\mu_1|) + 2\big(I_K(|d\mu_1|)\big)^{\frac{1}{2}}\big(I_K(|d\mu_2|)\big)^{\frac{1}{2}} + I_K(|d\mu_2|) \\ &< +\infty \ . \end{aligned}$$

Therefore, $\mathcal{W}_K$ is a linear space.

By Fubini's Theorem and from $I_K(|d\mu_1|, |d\mu_2|) < +\infty$, we, also, have that $U_K^{d\mu_1}$ is finite almost everywhere with respect to $d\mu_2$ and vice-versa and that $I_K(d\mu_1, d\mu_2)$ is well-defined as a complex number.

It is obvious that $I_K(\cdot\ ,\ \cdot)$ is a bilinear form and we, only, have to prove that it is positive definite.

Now, take an arbitrary locally finite complex Borel measure $d\mu$ in $\mathbf{R^n}$ with real values and with $I_K(|d\mu|) < +\infty$ and consider its non-negative and non-positive variations $d\mu^+$ and $d\mu^-$.

Then, we, also, have $I_K(d\mu^+) < +\infty$ and $I_K(d\mu^-) < +\infty$ and, hence,

$$\begin{aligned} I_K(d\mu) &= I_K(d\mu^+) - 2I_K(d\mu^+, d\mu^-) + I_K(d\mu^-) \\ &\geq I_K(d\mu^+) - 2\big(I_K(d\mu^+)\big)^{\frac{1}{2}}\big(I_K(d\mu^-)\big)^{\frac{1}{2}} + I_K(d\mu^-) \\ &\geq 0 \ . \end{aligned}$$

If $d\mu$ is a locally finite complex Borel measure in $\mathbf{R^n}$ with $I_K(|d\mu|) < +\infty$ and $d\mu_1$, $d\mu_2$ are its real and imaginary parts, then

$$I_K(d\mu) = I_K(d\mu_1) + I_K(d\mu_2) \geq 0 \ .$$

By the standard argument, we can prove, now, that

$$\big(I_K(d\mu_1, d\mu_2)\big)^2 \leq I_K(d\mu_1) I_K(d\mu_2)$$

for all $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_K$.

Now, let $d\mu$ be a locally finite complex Borel measure in $\mathbf{R^n}$ with $I_K(|d\mu|) < +\infty$ and assume that

$$I_K(d\mu) = 0 \ .$$

If $K_*(r) > 0$ for all $r > 0$ and $K_* = H \circ h_*$, then there exists some t_0 with $H'(t_0) > 0$. We define

$$H_1(t) = \begin{cases} \frac{1}{2}\big(H(t) - H'(t_0)(t - t_0) - H(t_0)\big) \ , & \text{if } t > t_0 \\ 0 \ , & \text{if } t \leq t_0 \end{cases}$$

and

$$H_2 \;=\; H - H_1 \;.$$

We consider, next,

$$K_{1*} \;=\; H_1 \circ h_* \;, \qquad K_{2*} \;=\; H_2 \circ h_* \;.$$

The corresponding kernels K_1 and K_2 are non-negative and of first type and K_1 has the property that

$$K_{1*}(r) \;=\; 0$$

for all $r \geq R = h^{-1}(t_0)$.

Now, for each $i = 1, 2$,

$$I_{K_i}(|d\mu|) \;\leq\; I_K(|d\mu|) \;<\; +\infty \;.$$

Also,

$$I_{K_1}(d\mu) + I_{K_2}(d\mu) \;=\; I_K(d\mu)$$

and, since both terms in the left side are non-negative, we get

$$I_{K_1}(d\mu) \;=\; 0 \;.$$

Hence, we may assume that our kernel K has the property that, for some R,

$$K_*(r) \;=\; 0$$

for all $r \geq R$.

Consider, now, an arbitrary $\phi \in \mathcal{D}(\mathbf{R^n})$ and the function

$$f \;=\; \frac{\widehat{\phi}}{\widehat{K}} \;.$$

Lemma 7.2 together with Corollary 0.1 easily give that f belongs to $\mathcal{S}(\mathbf{R^n})$. Theorem 0.18 implies that there is a $g \in \mathcal{S}(\mathbf{R^n})$ so that $\widehat{g} = f$ and, hence,

$$g * K \;=\; \phi \;.$$

Since $K \in L^1(\mathbf{R^n})$ and $g \in L^1(\mathbf{R^n}) \cap L^\infty(\mathbf{R^n})$, it is immediate that

$$I_K(|g|) \;<\; +\infty \;.$$

Thus,

$$\begin{aligned} \Big| \int_{\mathbf{R^n}} \phi(x)\; d\mu(x) \Big| \;&=\; \Big| \int_{\mathbf{R^n}} K * g(x)\; d\mu(x) \Big| \\ &=\; |I_K(d\mu, \overline{g})| \;\leq\; \big(I_K(d\mu)\big)^{\frac{1}{2}} \big(I_K(\overline{g})\big)^{\frac{1}{2}} \;=\; 0 \;. \end{aligned}$$

Therefore, $d\mu$ is the zero measure.

In the case of $\mathbf{R^2}$ and of kernels of first type and of variable sign the measures considered are supported in compact sets.

Definition 7.5 *Assume that K is a kernel of first type and of variable sign. We define*

$$\mathcal{W}_K^0 \;=\; \{d\mu : d\mu \text{ is a compactly supported complex Borel measure with } I_K(|d\mu|) < +\infty \text{ and } d\mu(\mathbf{R^2}) = 0\}\ .$$

Now, we may extend the results about non-negative kernels. For example, here is the central result.

Theorem 7.3 *Assume that K is a kernel of first type and of variable sign.*

Then, $\mathcal{W}_K^0$ is a linear space. Also, for any $d\mu_1$ and $d\mu_2$ in $\mathcal{W}_K^0$, $I_K(d\mu_1, d\mu_2)$ is well-defined as a complex number and, under this bilinear form, $\mathcal{W}_K^0$ becomes an inner product space.

Moreover, if $d\mu \in \mathcal{W}_K^0$, then $U_K^{d\mu}$ is finite almost everywhere with respect to all $d\nu \in \mathcal{W}_K^0$.

Proof:

If $d\mu_1$ and $d\mu_2$ are in $\mathcal{W}_K^0$, we consider a large enough R so that the two measures are supported in the disc $B(0;R)$ and then consider the modified kernel

$$K_R(x,y) \;=\; \begin{cases} K(x,y) - K_*(2R)\ , & \text{if } |x-y| \le 2R \\ 0\ , & \text{if } |x-y| > 2R\ . \end{cases}$$

The new kernel is non-negative and of first type with the property that it vanishes for all large enough values of its argument.

We, also, have that, for all $x \in B(0;R)$,

$$U_K^{d\mu_1}(x) \;=\; \int_{B(0;R)} K(x,y)\ d\mu_1(y) \;=\; \int_{B(0;R)} K_R(x,y)\ d\mu_1(y) \;=\; U_{K_R}^{d\mu_1}(x)$$

and, thus,

$$\begin{aligned} I_K(d\mu_1, d\mu_2) &= \int_{B(0;R)} U_K^{d\mu_1}(x)\ d\mu_2(x) \\ &= \int_{B(0;R)} U_{K_R}^{d\mu_1}(x)\ d\mu_2(x) \\ &= I_{K_R}(d\mu_1, d\mu_2)\ . \end{aligned}$$

Applying, now, Theorem 7.2 to the kernel K_R, we see that $I_K(d\mu_1, d\mu_2)$ is well-defined as a complex number, that

$$I_K(d\mu_1) \;\ge\; 0$$

and that $I_K(d\mu_1) = 0$ if and only if $d\mu_1$ is the zero measure.

In fact, either by Lemma 7.3 or by Proposition 7.4, we have that

$$I_K(d\mu) \;=\; \int_{\mathbf{R^2}} \big|\widehat{d\mu}(\xi)\big|^2 \widehat{K_R}(\xi)\ dm(x)\ ,$$

for all $d\mu \in \mathcal{W}_K^0$ which are supported in $B(0;R)$.

There is another special result for the logarithmic kernel in $\mathbf{R^2}$.

Theorem 7.4 *If $K(x,y) = \log \frac{1}{|x-y|} = h_*(|x-y|)$, then the space of all complex Borel measures $d\mu$ in $\mathbf{R^2}$ which are supported in the unit disc $B(0;1)$ and satisfy $I_h(|d\mu|) < +\infty$ becomes an inner product space under the bilinear form $I_h(d\mu_1, d\mu_2)$.*

Proof:
Consider the measure $d\sigma$ on S^1 and its h-potential

$$U_h^{d\sigma}(x) \;=\; \int_{S^1} \log \frac{1}{|x-y|}\, d\sigma(y) \;=\; \begin{cases} \log \frac{1}{|x|}\ , & \text{if } |x| > 1 \\ 0\ , & \text{if } |x| \le 1\ . \end{cases}$$

Now, for every $d\mu_1$ and $d\mu_2$ with $I_h(|d\mu_1|) < +\infty$ and $I_h(|d\mu_2|) < +\infty$, we define the measures

$$d\nu_i \;=\; d\mu_i - \frac{d\mu_i(\mathbf{R^2})}{2\pi} d\sigma\ .$$

Then, both $d\nu_i$ belong to $\mathcal{W}_h^0$ and, hence, Theorem 7.3 applies to them.
Now, observe that

$$\begin{aligned} I_h(d\mu_1, d\mu_2) \;&=\; I_h(d\nu_1, d\nu_2) + \frac{d\mu_1(\mathbf{R^2})}{2\pi} I_h(d\sigma, d\nu_2) \\ &\quad + \frac{\overline{d\mu_2(\mathbf{R^2})}}{2\pi} I_h(d\nu_1, d\sigma) + \frac{d\mu_1(\mathbf{R^2})}{2\pi}\frac{\overline{d\mu_2(\mathbf{R^2})}}{2\pi} I_h(d\sigma) \\ &=\; I_h(d\nu_1, d\nu_2) + \frac{d\mu_1(\mathbf{R^2})}{2\pi} \int_{\mathbf{R^2}} U_h^{d\sigma}(x)\, \overline{d\nu_2}(x) \\ &\quad + \frac{\overline{d\mu_2(\mathbf{R^2})}}{2\pi} \int_{\mathbf{R^2}} \overline{U_h^{d\sigma}(x)}\, d\nu_1(x) \\ &\quad + \frac{d\mu_1(\mathbf{R^2})}{2\pi}\frac{\overline{d\mu_2(\mathbf{R^2})}}{2\pi} \int_{\mathbf{R^2}} U_h^{d\sigma}(x)\, d\sigma(x) \\ &=\; I_h(d\nu_1, d\nu_2)\ . \end{aligned}$$

Chapter 8

Capacity

8.1 Definitions

Let K be a kernel of either the first or the second type.
Remark: *Whenever, in this chapter, $K = G^{\Omega}$ is a kernel of second type, we shall understand that the set Ω has a Green's function in all its components and that all sets to be considered are subsets of Ω. Also, all measures are Borel measures in Ω.*

If the kernel is of first type, then all sets are subsets of $\mathbf{R^n}$ and all measures are Borel measures in $\mathbf{R^n}$.

Definition 8.1 *Let E be a compact set. By Γ_K^E we denote the family of all non-negative Borel measures $d\mu$ which are supported in E and satisfy*

$$U_K^{d\mu} \ \le \ 1$$

everywhere in E. Therefore, by the Maximum Principle of potentials, $U_K^{d\mu} \le 1$ everywhere in $\mathbf{R^n}$, in case K is of first type, or in Ω, in case $K = G^{\Omega}$ is of second type.

Definition 8.2 *We define the K**-capacity** of the compact E by*

$$C_K(E) \ = \ \sup_{d\mu \in \Gamma_K^E} d\mu(E) \ .$$

*We, also, define the **inner** K**-capacity** of the set A by*

$$C_K^i(A) \ = \ \sup\{C_K(E) : E \text{ is a compact subset of } A\} \ .$$

*Finally, the **outer** K**-capacity** of a set A is defined by*

$$C_K^o(A) \ = \ \inf\{C_K^i(O) : O \text{ is open with } A \subseteq O\} \ .$$

It is almost obvious that $C_K^i(A) \le C_K^o(A)$ for all A.

Definition 8.3 *A set A is called K-**capacitable** if $C_K^i(A) = C_K^o(A)$.*

The proof of the next result is trivial.

Proposition 8.1 *1. If E_1 and E_2 are compact sets with $E_1 \subseteq E_2$, then $\Gamma_K^{E_1} \subseteq \Gamma_K^{E_2}$.*

2. If E_1 and E_2 are compact sets with $E_1 \subseteq E_2$, then $C_K(E_1) \leq C_K(E_2)$.

3. If $A_1 \subseteq A_2$, then $C_K^i(A_1) \leq C_K^i(A_2)$ and $C_K^o(A_1) \leq C_K^o(A_2)$.

The proof of the next result is based on Proposition 8.1(3) and is, also, trivial.

Proposition 8.2 *All open sets O are K-capacitable:*

$$C_K^i(O) \;=\; C_K^o(O) \;.$$

Proposition 8.3 *Every compact set E is K-capacitable and*

$$C_K^i(E) \;=\; C_K^o(E) \;=\; C_K(E) \;.$$

Proof:

The equality $C_K^i(E) = C_K(E)$ is clear and comes from Proposition 8.1(2).

From the definitions, the inequality $C_K(E) \leq C_K^o(E)$ is, also, clear and for the rest of the proof we may assume that $C_K(E) < +\infty$.

Consider the sets

$$E^\delta \;=\; \{x : d(x,E) \leq \delta\}$$

and

$$O^\delta \;=\; \{x : d(x,E) < \delta\} \;,$$

where δ is small enough so that $E^\delta \subseteq \Omega$ in case $K = G^\Omega$.

The sets E^δ are compact, the sets O^δ are open and

$$E \;\subseteq\; O^\delta \;\subseteq\; E^\delta$$

and

$$E^\delta \;\downarrow\; E$$

as $\delta \downarrow 0$.

Therefore, $C_K(E^\delta)$ is decreasing as $\delta \downarrow 0$ and let

$$C_K(E^\delta) \;\downarrow\; \alpha$$

as $\delta \downarrow 0$, where α is such that

$$C_K(E) \;\leq\; \alpha \;\leq\; +\infty \;.$$

Assume that $C_K(E) < \alpha$ and consider any β with

$$C_K(E) \;<\; \beta \;<\; \alpha \;.$$

Then, we can find non-negative Borel measures $d\mu^\delta$ supported in E^δ so that

$$\beta \;\leq\; d\mu^\delta(E^\delta)$$

and

$$U_K^{d\mu^\delta} \;\leq\; 1$$

everywhere in $\mathbf{R^n}$, in case K is of first type, or in Ω, in case $K = G^\Omega$.

Deviding $d\mu^\delta$ by an appropriate positive number we may assume, if necessary, that

$$d\mu^\delta(E^\delta) \;\leq\; \beta + 1 \;<\; +\infty \;.$$

Hence, there is some sequence of δ's tending to 0 so that

$$d\mu^\delta \;\to\; d\mu$$

weakly on compact sets, where $d\mu$ is a non-negative Borel measure.

It is clear that $d\mu$ is supported in E and that

$$d\mu(E) \;=\; \lim d\mu^\delta(E^\delta) \;\geq\; \beta \;>\; C_K(E)$$

as $\delta \to 0$ through this sequence.

Because of the lower-semicontinuity of potentials in measure,

$$U_K^{d\mu} \;\leq\; \liminf_{\delta\to 0} U_K^{d\mu^\delta} \;\leq\; 1$$

everywhere.

This is a contradiction to the definition of $C_K(E)$ and, thus,

$$C_K(E^\delta) \;\downarrow\; C_K(E)$$

as $\delta \downarrow 0$.

Now,

$$C_K(E) \;=\; C_K^i(E) \;\leq\; C_K^i(O^\delta) \;\leq\; C_K^i(E^\delta) \;=\; C_K(E^\delta) \;,$$

implying that

$$C_K(E) \;=\; C_K^o(E) \;.$$

We may, now, write,

Definition 8.4 *If A is K-capacitable, we define its K**-capacity** by*

$$C_K(A) \;=\; C_K^i(A) \;=\; C_K^o(A) \;.$$

It can be proved that all Borel sets are K-capacitable and, even more, that all "analytic" sets are K-capacitable. We shall not work in this direction and we shall have only in mind that all compact and all open sets are K-capacitable.

Lemma 8.1 *Assume that $C^i_K(A) = 0$ and $d\mu$ is any non-negative Borel measure supported in a compact set with $U^{d\mu}_K$ being bounded from above in the set A. Then $d\mu(E) = 0$ for every compact $E \subseteq A$.*

Proof:
It is clear that we may suppose that A is bounded.

Assume that $d\mu(E) > 0$ for some compact $E \subseteq A$ and consider the restriction $d\mu_E$ of $d\mu$ in E.

Then $U^{d\mu_E}_K$ is, also, bounded from above in the set $A \supseteq E$ and, hence,

$$U^{d\mu_E}_K \ \leq \ M$$

everywhere, for some $M > 0$. Now, define

$$d\nu \ = \ \frac{1}{M} d\mu_E \ .$$

Then, $d\nu \in \Gamma^E_K$ and

$$d\nu(E) \ > \ 0 \ .$$

Therefore,

$$C^i_K(A) \ \geq \ C_K(E) \ \geq \ d\nu(E) \ > \ 0 \ .$$

Lemma 8.2 *If all A_m are Borel sets and $C^i_K(A_m) = 0$ for all m, then*

$$C^i_K(\cup_{m=1}^{+\infty} A_m) \ = \ 0 \ .$$

Proof:
Take an arbitrary non-negative Borel measure supported in a compact subset of $\cup_{m=1}^{+\infty} A_m$ with

$$U^{d\mu}_K \ \leq \ 1$$

everywhere.

By Lemma 8.1,

$$d\mu(A_m) \ = \ 0$$

for every m and, thus,

$$d\mu(\cup_{m=1}^{+\infty} A_m) \ = \ 0 \ .$$

By the definition,

$$C^i_K(\cup_{m=1}^{+\infty} A_m) \ = \ 0 \ .$$

Proposition 8.4 *Suppose that $A = \cup_{k=1}^{+\infty} A_k$.*
1. If K is a non-negative kernel, then

$$C^o_K(A) \ \leq \ \sum_{k=1}^{+\infty} C^o_K(A_k) \ .$$

2. If K is of variable sign and $D = diam(A)$, then

$$\frac{C^o_K(A)}{1 + C^o_K(A) K_*(D)^-} \ \leq \ \sum_{k=1}^{+\infty} C^o_K(A_k) \ ,$$

where $K_*(D)^- = -\min(0, K_*(D))$.

Proof:
1. It is enough to assume that $C_K^o(A_k) < +\infty$ for all k and, then, we take open $U_k \subseteq \mathbf{R^n}$ so that $A_k \subseteq U_k$ and $C_K(U_k) \leq C_K^o(A_k) + \frac{\epsilon}{2^k}$.

For an arbitrary compact $E \subseteq \cup_{k=1}^{+\infty} U_k$ we take N so that $E \subseteq \cup_{k=1}^{N} U_k$ and, then, write $E = \cup_{k=1}^{N} E_k$, where each E_k is a compact subset of U_k.

Now, we take $d\mu \in \Gamma_K^E$ and observe that $d\mu_{E_k} \in \Gamma_K^{E_k}$ for every k. Therefore,

$$d\mu(E) \ \leq \ \sum_{k=1}^{N} d\mu(E_k) \ \leq \ \sum_{k=1}^{N} C_K(E_k) \ \leq \ \sum_{k=1}^{+\infty} C_K(U_k) \ \leq \ \sum_{k=1}^{+\infty} C_K^o(A_k) + \epsilon \ .$$

This implies

$$C_K^o(A) \ \leq \ C_K(\cup_{k=1}^{+\infty} U_k) \ \leq \ \sum_{k=1}^{+\infty} C_K^o(A_k) + \epsilon$$

and concludes the proof.
2. We repeat the same argument observing that, if $d\mu \in \Gamma_K^E$, then

$$U_K^{d\mu_{E_k}}(x) \ \leq \ U_K^{d\mu}(x) - K_*(D) d\mu(E) \ \leq \ 1 + K_*(D)^- d\mu(E)$$

for every $x \in E_k$.

The rest of the proof remains unchanged.

The proof of the next result is an immediate consequence of Proposition 8.4.

Theorem 8.1 *Suppose that $C_K(A_k) = 0$ for all k.*
1. If K is a non-negative kernel then, $C_K(\cup_{k=1}^{+\infty} A_k) = 0$.
2. If K is of variable sign and $\cup_{k=1}^{+\infty} A_k$ is bounded, then $C_K(\cup_{k=1}^{+\infty} A_k) = 0$.

Definition 8.5 *We say that some property holds* ***quasi-almost everywhere*** *or* ***q-a.e.*** *in a set A, if it holds everywhere in A except in a subset of A of zero K-capacity.*

8.2 Equilibrium measures

Theorem 8.2 *Suppose E is a compact set. Then the extremal problem*

$$\gamma_K(E) \ = \ \inf I_K(d\mu)$$

over all probability Borel measures (i.e. non-negative Borel measures with total mass equal to 1) supported in E has a solution $d\mu_0$.

This satisfies

1. *$U_K^{d\mu_0} \leq \gamma_K(E)$ everywhere,*
2. *$U_K^{d\mu_0} = \gamma_K(E)$ quasi-almost everywhere in E.*

Proof:

Assume, first, that

$$\gamma_K(E) \ < \ +\infty \ .$$

Consider a sequence of probability Borel measures $d\mu_m$ supported in E so that

$$I_K(d\mu_m) \ \to \ \gamma_K(E) \ .$$

There is some subsequence converging weakly in E to some probability Borel measure $d\mu_0$ supported in E.

By the lower-semicontinuity of energy in measure,

$$\gamma_K(E) \ \leq \ I_K(d\mu_0) \ \leq \ \liminf_{m\to+\infty} I_K(d\mu_m) \ = \ \gamma_K(E) \ .$$

Thus, $d\mu_0$ is a solution of the extremal problem.

Claim: $U_K^{d\mu_0} \geq \gamma_K(E)$ q-a.e. in E.

To prove this, consider $\epsilon > 0$ and assume that the compact set

$$E^\epsilon \ = \ \{x \in E : U_K^{d\mu_0}(x) \leq \gamma_K(E) - \epsilon\}$$

has positive K-capacity.

By definition, there is some non-negative Borel measure $d\tau$ supported in E^ϵ with

$$d\tau(E^\epsilon) \ > \ 0$$

and

$$U_K^{d\tau} \ \leq \ 1$$

everywhere.

Define, now, the probability measure

$$d\sigma \ = \ \frac{1}{d\tau(E^\epsilon)} d\tau$$

supported in E^ϵ, satisfying

$$U_K^{d\sigma} \ \leq \ M \ = \ \frac{1}{d\tau(E^\epsilon)}$$

everywhere.

We apply, now, a *variational argument*, considering $0 < \delta < 1$ and defining

$$d\mu_\delta \ = \ (1-\delta) d\mu_0 + \delta d\sigma \ .$$

Then $d\mu_\delta$ is a probability measure supported in E and, hence,

$$\begin{aligned}
\gamma_K(E) \ &\leq \ I_K(d\mu_\delta) \\
&= \ (1-\delta)^2 I_K(d\mu_0) + 2\delta(1-\delta) \int_{E^\epsilon} U_K^{d\mu_0}(x) \ d\sigma(x) + \delta^2 I_K(d\sigma) \\
&= \ \gamma_K(E) - 2\delta I_K(d\mu_0) + 2\delta \int_{E^\epsilon} U_K^{d\mu_0}(x) \ d\sigma(x) \\
&\quad + \delta^2 \Big(I_K(d\sigma) + I_K(d\mu_0) - 2 \int_{E^\epsilon} U_K^{d\mu_0}(x) \ d\sigma(x) \Big) \ .
\end{aligned}$$

Thus,

$$0 \ \le \ -2\delta\epsilon + \delta^2\Big(I_K(d\sigma) + I_K(d\mu_0) - 2\int_{E^\epsilon} U_K^{d\mu_0}(x)\ d\sigma(x)\Big)\ .$$

This is, clearly, absurd for small $\delta > 0$ and, thus,

$$C_K(E^\epsilon) \ = \ 0\ .$$

Since

$$\{x \in E : U_K^{d\mu_0}(x) < \gamma(K)\} \ = \ \cup_{m=1}^{+\infty} E^{\frac{1}{m}}\ ,$$

Theorem 8.1 implies the claim.

Claim: $U_K^{d\mu_0} \le \gamma_K(E)$ everywhere.

Consider an arbitrary $x \in E$ and assume that

$$U_K^{d\mu_0}(x) \ > \ \gamma_K(E)\ .$$

Take an arbitrary γ_1 so that

$$U_K^{d\mu_0}(x) \ > \ \gamma_1 \ > \ \gamma_K(E)$$

and, by lower-semicontinuity,

$$U_K^{d\mu_0} \ > \ \gamma_1$$

everywhere in some open neighborhood $B(x; r_x)$ of x.

The Borel set $\{x \in E : U_K^{d\mu_0} < \gamma_K(E)\}$ has zero K-capacity and, by Lemma 8.1,

$$d\mu_0\big(\{x \in E : U_K^{d\mu_0} < \gamma_K(E)\}\big) \ = \ 0\ .$$

Then,

$$I_K(d\mu_0) \ \ge \ \gamma_1 d\mu_0(E \cap B(x; r_x)) + \gamma_K(E) d\mu_0(E \setminus B(x; r_x))\ .$$

By $I_K(d\mu_0) = \gamma_K(E)$, it is implied that

$$d\mu_0(E \cap B(x; r_x)) \ = \ 0\ .$$

Consider the open set

$$O \ = \ \cup\big\{B(x; r_x) : x \in E \text{ and } U_K^{d\mu_0}(x) > \gamma_K(E)\big\}\ .$$

If F is any compact subset of O, then we can cover F by finitely many balls of the form $B(x; r_x)$ with $x \in E$ and $U_K^{d\mu_0}(x) > \gamma_K(E)$ and, hence, $d\mu_0(F) = 0$. Since F is arbitrary,

$$d\mu_0(O) \ = \ 0\ .$$

Therefore, $d\mu_0$ is supported in the compact set $E \setminus O$ and

$$U_K^{d\mu_0} \ \le \ \gamma_K(E)$$

in this set. By the Maximum Principle of potentials, the claim is clear and, together with the first claim, the proof of the theorem in case $\gamma_K(E) < +\infty$ is complete.

Assume, now, that $\gamma_K(E) = +\infty$.

It is, then, obvious that Γ_K^E contains only the zero measure and, hence, $C_K(E) = 0$.

Therefore, every probability Borel measure supported in E (for example, any $d\delta_x$ with $x \in E$) is a solution of the extremal problem.

Theorem 8.3 *If E is a compact set and $\gamma_K(E) < +\infty$, then the solution of the extremal problem of Theorem 8.2 is unique.*

Proof:

If $d\mu_0$ and $d\mu_0'$ are two solutions of the extremal problem, then $U_K^{d\mu_0} = \gamma_K(E)$ and $U_K^{d\mu_0'} = \gamma_K(E)$ quasi-almost everywhere in E and $U_K^{d\mu_0} \le \gamma_K(E)$ and $U_K^{d\mu_0'} \le \gamma_K(E)$ everywhere.

By Lemma 8.2,

$$d\mu_0'\big(\{x \in E : U_K^{d\mu_0} < \gamma_K(E)\}\big) \;=\; 0 \;,$$

implying

$$\begin{aligned} I_K(d\mu_0 - d\mu_0') &= I_K(d\mu_0) - 2\int_E U_K^{d\mu_0}(x)\; d\mu_0'(x) + I_K(d\mu_0') \\ &= \gamma_K(E) - 2\gamma_K(E) + \gamma_K(E) \\ &= 0 \;. \end{aligned}$$

By Theorems 7.1, 7.2 and 7.3, we get

$$d\mu_0 \;=\; d\mu_0' \;.$$

Definition 8.6 *If E is any compact set, then the quantity*

$$\gamma_K(E) \;=\; \inf I_K(d\mu)$$

*over all probability Borel measures which are supported in E, is called the K-**energy** of E.*

Definition 8.7 *If E is a compact set with $\gamma_K(E) < +\infty$, then the unique probability Borel measure $d\mu_0$ with*

$$I_K(d\mu_0) \;=\; \gamma_K(E)$$

*is called the K-**equilibrium measure** of E.*

Theorem 8.4 *If E is compact, then $\gamma_K(E) = +\infty$ if and only if $C_K(E) = 0$.*

Proof:

Assume that $C_K(E) > 0$ and take a non-negative Borel measure $d\mu$ supported in E with $d\mu(E) > 0$ and

$$U_K^{d\mu} \;\leq\; 1$$

everywhere.

Then $d\nu = \frac{1}{d\mu(E)} d\mu$ is a probability Borel measure supported in E with

$$I_K(d\nu) \;=\; \int_E U_K^{d\nu}(x)\; d\nu(x) \;\leq\; \frac{1}{d\mu(E)} \;<\; +\infty\ .$$

Thus, $\gamma_K(E) < +\infty$.

Assume, conversely, that $\gamma_K(E) < +\infty$ and $d\mu_0$ is the K-equilibrium measure of E.

If $0 < \gamma_K(E) < +\infty$, then

$$d\mu \;=\; \frac{1}{\gamma_K(E)} d\mu_0$$

is in Γ_K^E and

$$d\mu(E) \;=\; \frac{1}{\gamma_K(E)} \;>\; 0\ .$$

Therefore, $C_K(E) > 0$.

If $\gamma_K(E) \leq 0$, then $d\mu_0 \in \Gamma_K^E$ and

$$C_K(E) \;\geq\; d\mu_0(E) \;=\; 1\ .$$

Proposition 8.5 *If K is a non-negative kernel, then*

$$\gamma_K(E) \;>\; 0$$

for every compact set E.

This is, also, true if K is the logarithmic kernel in $\mathbf{R^2}$ and $E \subseteq B(0;1)$.

Proof:

Let $\gamma_K(E) < +\infty$ and consider the K-equilibrium measure $d\mu_0$ of E.

Then, by Theorems 7.1, 7.2 and 7.4,

$$\gamma_K(E) \;=\; I_K(d\mu_0) \;>\; 0\ .$$

Consider, now, for any compact set E, the three extremal problems

(I) $\gamma_K(E) = \inf I_K(d\mu)$ *over all probability Borel measures $d\mu$ supported in E,*

(II) $A = \inf d\mu(\mathbf{R^n})$ *over all non-negative Borel measures $d\mu$ supported in a compact set with $U_K^{d\mu} \geq 1$ q-a.e. in E,*

(III) $C_K(E) = \sup d\mu(E)$ *over all non-negative Borel measures $d\mu$ supported in E with $U_K^{d\mu} \leq 1$ everywhere in E.*

Theorem 8.5 *Suppose that E is a compact set with $\gamma_K(E) > 0$. Then, the three extremal problems (I), (II) and (III) are equivalent in the sense*

$$\frac{1}{\gamma_K(E)} \;=\; A \;=\; C_K(E)\;.$$

In case $0 < \gamma_K(E) < +\infty$, the problem (I) has a unique solution, the K-equilibrium measure $d\mu_0$ of E, and (III) has, also, a unique solution, the measure $\frac{1}{\gamma_K(E)}\,d\mu_0$. The same measure is, also, a solution of problem (II) whose solution, though, may not be unique.

If $\gamma_K(E) \le 0$, then the problem (I) has, again, the K-equilibrium measure of E as its unique solution, but

$$A \;=\; C_K(E) \;=\; +\infty$$

and the problems (II) and (III) have no solution.

Proof:
1. Suppose that $\gamma_K(E) = +\infty$.

By Theorem 8.4, $C_K(E) = 0$. Considering the zero measure, we find that $A = 0$.

2. Now, let $0 < \gamma_K(E) < +\infty$.

Consider the K-equilibrium measure $d\mu_0$ of E, which is the unique solution of problem (I), and define

$$d\mu_1 \;=\; \frac{1}{\gamma_K(E)}\,d\mu_0\;.$$

Then, $d\mu_1 \in \Gamma_K^E$ and, thus,

$$C_K(E) \;\ge\; d\mu_1(E) \;=\; \frac{1}{\gamma_K(E)}\;.$$

On the other hand, let $d\nu$ be any measure qualifying for problem (III).

By Lemma 8.1, $d\nu\big(\{x \in E : U_K^{d\mu_1}(x) < 1\}\big) = 0$ and, hence,

$$\frac{1}{\gamma_K(E)} \;=\; d\mu_1(E) \;\ge\; \int_E U_K^{d\nu}(x)\,d\mu_1(x) \;=\; \int_E U_K^{d\mu_1}(x)\,d\nu(x) \;\ge\; d\nu(E)\;.$$

Therefore,

$$\frac{1}{\gamma_K(E)} \;\ge\; C_K(E)$$

and, hence,

$$\frac{1}{\gamma_K(E)} \;=\; C_K(E)\;.$$

Furthermore, $d\mu_1$ is a solution of problem (III).

Suppose, now, that $d\nu$ is another solution of problem (III). Then,

$$\begin{aligned} I_K(d\mu_1 - d\nu) &= I_K(d\mu_1) - 2\int_E U_K^{d\mu_1}(x)\ d\nu(x) + I_K(d\nu) \\ &\leq C_K(E) - 2C_K(E) + C_K(E) \\ &= 0\ . \end{aligned}$$

By Theorems 7.1, 7.2 and 7.3, $d\nu = d\mu_1$.

The measure $d\mu_1$ qualifies for problem (II) and, thus,

$$A \ \leq\ C_K(E)\ .$$

Let $d\nu$ be any measure qualifying for problem (II). Then, by Lemma 8.1,

$$C_K(E) \ \leq\ \int_E U_K^{d\nu}(x)\ d\mu_1(x) \ =\ \int_{\mathbf{R^n}} U_K^{d\mu_1}(x)\ d\nu(x) \ \leq\ d\nu(\mathbf{R^n})\ .$$

Therefore,

$$C_K(E) \ \leq\ A\ ,$$

and the proof is complete in case $\gamma_K(E) > 0$.

Now, assume that $\gamma_K(E) \leq 0$.

If we consider the K-equilibrium measure, $d\mu_0$, of E, then, for every $\alpha > 0$, the measure $\alpha d\mu_0$ belongs to Γ_K^E and, thus,

$$C_K(E) \ \geq\ \alpha d\mu_0(E) \ =\ \alpha\ .$$

Since α is arbitrary,

$$C_K(E) \ =\ +\infty$$

and the problem (III) cannot have a solution.

On the other hand, suppose that there is some $d\nu$ qualifying for problem (II).

Then, by Lemma 8.1,

$$1 \ \leq\ \int_E U_K^{d\nu}(x)\ d\mu_0(x) \ =\ \int_{\mathbf{R^n}} U_K^{d\mu_0}(x)\ d\nu(x) \ \leq\ \gamma_K(E) d\nu(\mathbf{R^n})\ .$$

Since $0 \leq d\nu(\mathbf{R^n}) < +\infty$, this is impossible and, hence, there is no measure qualifying for problem (II). Thus,

$$A \ =\ +\infty\ .$$

Example

Let $E = S^1$ in $\mathbf{R^2}$ and $K_*(r) = \log^+ \frac{3}{r}$.

By symmetry and uniqueness, the K-equilibrium measure of E must be

$$d\mu_0 \ =\ \frac{1}{2\pi} d\sigma$$

and

$$U_K^{d\mu_0}(x) \;=\; \frac{1}{2\pi}\int_{S^1} K(x,y)\; d\sigma(y)\ .$$

If $|x| < 2$, then

$$U_K^{d\mu_0}(x) \;=\; \log 3 + \frac{1}{2\pi}\int_{S^1} \log\frac{1}{|x-y|}\; d\sigma(y) \;=\; \log 3 - \log^+ |x|\ .$$

Therefore,

$$I_K(d\mu_0) \;=\; \log 3$$

and

$$C_K(E) \;=\; \frac{1}{\log 3}\ .$$

Consider the measures

$$d\mu_1 \;=\; \frac{1}{\gamma_K(E)}\; d\mu_0 \;=\; \frac{1}{2\pi \log 3}\; d\sigma$$

and

$$d\nu \;=\; \frac{1}{\log 3}\; d\delta_0\ .$$

Then,

$$U_K^{d\nu}(x) \;=\; \frac{1}{\log 3}\log^+\frac{3}{|x|} \;=\; 1$$

when $|x| = 1$.

Hence, $d\mu_1$ and $d\nu$ are two solutions of problem (II).

Example

If A is a countable set, then $C_K^i(A) = 0$ with respect to any kernel.

By Lemma 8.2, it is enough to consider the case $A = \{x_0\}$ and, since all non-negative measures supported in $\{x_0\}$ are of the form

$$\alpha d\delta_{x_0}\ , \qquad \alpha \geq 0\ ,$$

we, then, have

$$U_K^{\alpha d\delta_{x_0}} \;=\; \alpha K(x, x_0)\ .$$

This is bounded from above in $\{x_0\}$ only if $\alpha = 0$ and, hence, the only measure in $\Gamma_K^{\{x_0\}}$ is the zero measure.

This implies that

$$C_K(\{x_0\}) \;=\; 0\ .$$

Theorem 8.1 implies that $C_K(A) = 0$ for every countable A and every non-negative K and, also, for every bounded and countable A and every K of variable sign.

Example

If the compact set E has positive Lebesgue measure, then $C_K(E) > 0$ for all kernels.

This is true because dm, restricted in E, defines a potential which is bounded from above.

8.3 Transfinite diameter

Theorem 8.6 *If E is a compact set, define*

$$M_m = \sup_{x_1,\ldots,x_m \in E} \inf_{x\in E} \frac{1}{m}\sum_{j=1}^{m} K(x,x_j)$$

and

$$D_m = \inf_{x_1,\ldots,x_m\in E} \frac{2}{m(m-1)} \sum_{1\le i<j\le m} K(x_i,x_j) \ .$$

Then,

$$\lim_{m\to+\infty} M_m = \lim_{m\to+\infty} D_m = \gamma_K(E) \ .$$

Proof:

We divide the proof into steps.

Step 1.

Since K is continuous in $(\mathbf{R^n}\times\mathbf{R^n})\setminus\{(x,x): x\in\mathbf{R^n}\}$, if K is of first type, or in $(\Omega\times\Omega)\setminus\{(x,x): x\in\Omega\}$, if $K=G^\Omega$ is of second type, it is easy to see that there exist $x_1,\ldots,x_{m+1}\in E$ so that

$$D_{m+1} = \frac{2}{(m+1)m} \sum_{1\le i<j\le m+1} K(x_i,x_j) \ .$$

Rewrite

$$D_{m+1} = \frac{2}{m+1}\Big(\frac{1}{m}\sum_{j=2}^{m+1} K(x_1,x_j) + \sum_{i=2}^{m}\frac{1}{m}\sum_{j=i+1}^{m+1} K(x_i,x_j)\Big) \ .$$

Observe that the second sum in the last equality does not depend on x_1 and, hence, x_1 minimizes $\frac{1}{m}\sum_{j=2}^{m+1} K(x,x_j)$.

Therefore,

$$\frac{1}{m}\sum_{j=2}^{m+1} K(x_1,x_j) \le M_m \ .$$

Similarly,

$$\frac{1}{m}\sum_{j\ne i} K(x_i,x_j) \le M_m$$

for all i and, thus,

$$D_{m+1} \le \frac{1}{m+1}(M_m+\cdots+M_m) = M_m$$

for all m.

Step 2.

If $\gamma_K(E) < +\infty$, consider the equilibrium measure $d\mu_0$ of E. Then, for all $x_1, \ldots, x_m \in E$,

$$\inf_{x\in E} \frac{1}{m}\sum_{j=1}^{m} K(x,x_j) \ \leq \ \frac{1}{m}\sum_{j=1}^{m}\int_E K(x,x_j)\ d\mu_0(x) \ \leq \ \gamma_K(E)$$

and, hence,

$$M_m \ \leq \ \gamma_K(E)$$

for all m.

This is, also, true in case $\gamma_K(E) = +\infty$.

Step 3.

Consider $x_1, \ldots, x_m \in E$ so that

$$D_m \ = \ \frac{1}{m(m-1)}\sum_{i\neq j} K(x_i,x_j)$$

and define the probability Borel measure

$$d\nu_m \ = \ \sum_{j=1}^{m}\frac{1}{m}\ d\delta_{x_j} \ .$$

Consider, also, the trunkated kernel

$$K_N(x,y) \ = \ \min(K(x,y), N) \ .$$

Then,

$$\begin{aligned}\int_E\int_E K_N(x,y)\ d\nu_m(x)\ d\nu_m(y) \ &= \ \frac{1}{m}\sum_{j=1}^{m}\int_E K_N(x,x_j)\ d\nu_m(x) \\ &\leq \ \frac{1}{m}\sum_{j=1}^{m}\Big(\frac{1}{m}\sum_{i\neq j} K(x_i,x_j)+\frac{N}{m}\Big) \\ &\leq \ \frac{1}{m(m-1)}\sum_{i\neq j} K(x_i,x_j)+\frac{N}{m} \ .\end{aligned}$$

Choose any subsequence $\{m_k\}$ of m's so that

1. $D_{m_k} \to \liminf_{m\to+\infty} D_m$,
2. $d\nu_{m_k} \to d\nu$ weakly in E

as $k \to +\infty$, where $d\nu$ is a Borel probability measure supported in E.

Then, by the the weak convergence $d\nu_{m_k} \times d\nu_{m_k} \to d\nu \times d\nu$ in $E \times E$ and the continuity of K_N in $E \times E$,

$$\begin{aligned}\int_E\int_E K_N(x,y)\ d\nu(x)\ d\nu(y) \ &= \ \lim_{k\to+\infty}\int_E\int_E K_N(x,y)\ d\nu_{m_k}(x)\ d\nu_{m_k}(y) \\ &\leq \ \liminf_{m\to+\infty} D_m \ .\end{aligned}$$

Letting $N \to +\infty$, we find

$$\gamma_K(E) \ \leq \ I_K(d\nu) \ \leq \ \liminf_{m\to+\infty} D_m \ .$$

After these three steps we have

$$\gamma_K(E) \ \leq \ \liminf_{m\to+\infty} D_m \ \leq \ \limsup_{m\to+\infty} D_m \ \leq \ \limsup_{m\to+\infty} M_m \ \leq \ \gamma_K(E)$$

and

$$\gamma_K(E) \ \leq \ \liminf_{m\to+\infty} D_m \ \leq \ \liminf_{m\to+\infty} M_m \ \leq \ \limsup_{m\to+\infty} M_m \ \leq \ \gamma_K(E) \ .$$

The proof is, clearly, complete.

Proposition 8.6 *If E is a compact set with $\gamma_K(E) < +\infty$, $x_1, \ldots, x_m \in E$ are such that $D_m = \frac{1}{m(m-1)} \sum_{i\neq j} K(x_i, x_j)$ and $d\nu_m = \sum_{j=1}^m \frac{1}{m} d\delta_{x_j}$, then*

$$d\nu_m \ \to \ d\mu_0$$

weakly in E, where $d\mu_0$ is the K-equilibrium measure of E.

Proof:
Since $D_m \to \gamma_K(E)$, in Step 3 of the proof of the Theorem 8.6 we proved that, if $d\nu$ is any limit point of $d\nu_m$ in the weak sense in E, then,

$$\gamma_K(E) \ \leq \ I_K(d\nu) \ \leq \ \liminf_{m\to+\infty} D_m \ = \ \gamma_K(E) \ .$$

Thus,

$$\gamma_K(E) \ = \ I_K(d\nu)$$

and, by the uniqueness of the K-equilibrium measure of E,

$$d\nu \ = \ d\mu_0 \ .$$

Definition 8.8 *If E is a compact set, then the number*

$$d_K(E) \ = \ e^{-\gamma_K(E)}$$

*is called **the K-transfinite diameter** of E.*

Proposition 8.7 *Let K be any kernel of first type, E and E' be two compact sets and $f : E \to E'$ be surjective and a contraction.*
Then, $d_K(E') \leq d_K(E)$.

Proof:
We have to prove that $\gamma_K(E) \leq \gamma_K(E')$.

Consider points $x'_1, \ldots, x'_m \in E'$ so that the quantity D'_m, corresponding to E', is

$$D'_m \ = \ \frac{1}{m(m-1)} \sum_{i \neq j} K(x'_i, x'_j) \ .$$

Consider, also, points $x_1, \ldots, x_m \in E$ so that

$$f(x_j) \ = \ x'_j \ .$$

Since f is a contraction and K depends upon the distance of its two arguments,

$$D_m \ \leq \ \frac{1}{m(m-1)} \sum_{i \neq j} K(x_i, x_j) \ \leq \ \frac{1}{m(m-1)} \sum_{i \neq j} K(x'_i, x'_j) \ = \ D'_m \ .$$

We finish the proof, by letting $m \to +\infty$.

Example
If K is of first type, then rigid motions preserve K-transfinite diameter and K-energy. It is, also, trivial to see through the definition that they preserve K-capacity.

8.4 The Theorem of Evans

Theorem 8.7 *(Evans) Suppose that E is a compact set with $\gamma_K(E) = +\infty$ or, equivalently, $C_K(E) = 0$. Then, there exists a probability Borel measure $d\mu$ supported in E so that*

$$U_K^{d\mu} \ = \ +\infty$$

identically in E.

Proof:
Consider points $x_1, \ldots, x_m \in E$ so that

$$\inf_{x \in E} \frac{1}{m} \sum_{j=1}^{m} K(x, x_j) \ \geq \ \frac{1}{2} M_m$$

and the probability Borel measure

$$d\mu_m \ = \ \frac{1}{m} \sum_{j=1}^{m} d\delta_{x_j} \ .$$

which is supported in E.

Since $M_m \to +\infty$, there is a sequence $\{m_k\}$ so that

$$\int_E K(x, y) \ d\mu_{m_k}(y) \ \geq \ 2^k$$

for all $x \in E$.

Now, construct the probability Borel measure

$$d\mu \;=\; \sum_{k=1}^{+\infty} \frac{1}{2^k} d\mu_{m_k}$$

which is, also, supported in E.

Then,

$$U_K^{d\mu}(x) \;=\; \sum_{k=1}^{+\infty} \frac{1}{2^k} U_K^{d\mu_{m_k}}(x) \;=\; +\infty$$

for all $x \in E$.

Theorem 8.8 *Suppose that $d\mu$ is a non-negative Borel measure supported in a compact set and that*

$$U_K^{d\mu} \;=\; +\infty$$

identically in some set A.

Then, the bounded set A is K-capacitable and

$$C_K(A) \;=\; 0 \;.$$

Proof:

Consider the open sets

$$O_m \;=\; \{x : U_K^{d\mu}(x) > m\} \;,$$

an arbitrary compact $E \subseteq O_m$ and a non-negative Borel measure $d\nu \in \Gamma_K^E$.

Then,

$$d\nu(E) \;\leq\; \frac{1}{m} \int_E U_K^{d\mu}(x)\; d\nu(x) \;=\; \frac{1}{m} \int_{\mathbf{R^n}} U_K^{d\nu}(x)\; d\mu(x) \;\leq\; \frac{1}{m}\; d\mu(\mathbf{R^n}) \;.$$

Hence, $C_K(E) \leq \frac{1}{m}\; d\mu(\mathbf{R^n})$ and, thus,

$$C_K(O_m) \;\leq\; \frac{1}{m}\; d\mu(\mathbf{R^n}) \;.$$

Therefore,

$$0 \;\leq\; C_K^i(A) \;\leq\; C_K^o(A) \;\leq\; C_K(O_m) \;\leq\; \frac{1}{m}\; d\mu(\mathbf{R^n})$$

for all m, implying

$$C_K^i(A) \;=\; C_K^o(A) \;=\; 0 \;.$$

Thus, Theorem 8.7 appears as a *partial* converse to Theorem 8.8: if the *compact* set A satisfies $C_K(A) = 0$, then it is the $+\infty$-set of the K-potential of a compactly supported non-negative Borel measure.

In section 9.7 this matter will be fully explored in the case $K = h$.

8.5 Kernels of variable sign

As Theorem 8.5 shows, if E is a compact set and $\gamma_K(E) > 0$, then $C_K(E) = \frac{1}{\gamma_K(E)}$. But, if $\gamma_K(E) \le 0$, then $C_K(E) = +\infty$. Proposition 8.5 shows that the situation is simple when the kernel is non-negative and, especially, when the dimension is $n \ge 3$.

To get around the complication arising in case of a kernel of variable sign when the dimension is $n = 2$, we may use another definition of capacity.

Definition 8.9 *If E is a compact set, the **Robin-K-capacity** of E is defined by*

$$C_{K,R}(E) \;=\; e^{-\gamma_K(E)} \;=\; d_K(E) \;.$$

Observe that, if $\gamma_K(E) > 0$, then

$$C_{K,R}(E) \;=\; e^{-\frac{1}{C_K(E)}} \;,$$

while, if $\gamma_K(E) \le 0$, then

$$C_{K,R}(E) \;\ge\; 1 \qquad \text{and} \qquad C_K(E) \;=\; +\infty \;.$$

Therefore, the compact sets of zero K-capacity are the same as the compact sets of zero Robin-K-capacity.

Also, if $\{E_m\}$ is a sequence of compact sets, then $C_K(E_m) \to 0$ if and only if $C_{K,R}(E_m) \to 0$.

We may give the following definitions.

Definition 8.10 *For every set A its **inner Robin-K-capacity** is defined by*

$$C^i_{K,R}(A) \;=\; \sup\{C_{K,R}(E) : E \textit{ is a compact subset of } A\}$$

*and its **outer Robin-K-capacity** is defined by*

$$C^o_{K,R}(A) \;=\; \inf\{C^i_{K,R}(O) : O \textit{ is an open set with } O \supseteq A\}$$

The inequality

$$C^i_{K,R}(A) \;\le\; C^o_{K,R}(A)$$

is obvious, while the proof of the next result is trivial and is based on the definitions.

Proposition 8.8 *1. If E_1 and E_2 are compact sets with $E_1 \subseteq E_2$, then $\gamma_K(E_2) \le \gamma_K(E_1)$ and $C_{K,R}(E_1) \le C_{K,R}(E_2)$.*

2. If $A_1 \subseteq A_2$, then $C^i_{K,R}(A_1) \le C^i_{K,R}(A_2)$ and $C^o_{K,R}(A_1) \le C^o_{K,R}(A_2)$.

From Proposition 8.8(2) we get

Proposition 8.9 *For every open set O,*

$$C^i_{K,R}(O) \;=\; C^o_{K,R}(O) \;.$$

Proposition 8.10 *For every compact set E,*

$$C^i_{K,R}(E) \;=\; C^o_{K,R}(E) \;=\; C_{K,R}(E)\;.$$

Proof:

From Proposition 8.8(1) we have

$$C^i_{K,R}(E) \;=\; C_{K,R}(E)$$

and from Proposition 8.8(2),

$$C_{K,R}(E) \;\le\; C^o_{K,R}(E)\;.$$

Assume, now, that $C_{K,R}(E) < +\infty$ and consider the compact sets

$$E^\delta \;=\; \{x : d(x,E) \le \delta\}$$

and the open sets

$$O^\delta \;=\; \{x : d(x,E) < \delta\}\;,$$

where δ is small enough so that $E^\delta \subseteq \Omega$ in case $K = G^\Omega$.

These satisfy

$$E \;\subseteq\; O^\delta \;\subseteq\; E^\delta\;.$$

As δ decreases $\gamma_K(E^\delta)$ increases and let

$$\gamma_K(E^\delta) \;\uparrow\; \alpha$$

for some $\alpha \le \gamma_K(E)$.

Suppose that $\alpha < \gamma_K(E)$ and consider the K-equilibrium measures $d\mu_0^\delta$ of E^δ.

Then, there is some sequence δ_k so that

$$d\mu_0^{\delta_k} \;\to\; d\mu$$

weakly in E, for some probability Borel measure $d\mu$ supported in E. From Proposition 6.3,

$$\gamma_K(E) \;\le\; I_K(d\mu) \;\le\; \liminf_{k\to+\infty} I_K(d\mu_0^{\delta_k}) \;=\; \alpha\;.$$

We, thus, get a contradiction and, hence,

$$\gamma_K(E^\delta) \;\uparrow\; \gamma_K(E)\;.$$

Therefore,

$$C_{K,R}(E^\delta) \;\downarrow\; C_{K,R}(E)\;,$$

from which we, immediately, get that

$$C^i_{K,R}(O^\delta) \;\downarrow\; C_{K,R}(E)$$

and, finally,

$$C^o_{K,R}(E) \;=\; C_{K,R}(E)\;.$$

Definition 8.11 *The set A is called* ***Robin-K-capacitable****, if $C^i_{K,R}(A) = C^o_{K,R}(A)$ and we, then, define its* ***Robin-K-capacity*** *by*

$$C_{K,R}(A) \;=\; C^i_{K,R}(A) \;=\; C^o_{K,R}(A)\ .$$

The proof of the following is trivial.

Proposition 8.11 *If $C^i_K(A) < +\infty$ or, equivalently, if $C^i_{K,R}(A) < 1$, then the set A is K-capacitable if and only if it is Robin-K-capacitable and, in this case,*

$$C_{K,R}(A) \;=\; e^{-\frac{1}{C_K(A)}}\ .$$

If $C^i_K(A) = +\infty$ or, equivalently, if $C^i_{K,R}(A) \geq 1$, then the set A is K-capacitable and $C_K(A) = +\infty$.

Chapter 9

The Classical Kernels

In this chapter we study the particular case of the so-called *classical kernels*: the Newtonian and the logarithmic kernels and, also, the Green's kernel related to an open set with a Green's function in all its components.

9.1 Extension through sets of zero capacity

Theorem 9.1 *Suppose that $\Omega \subseteq \mathbf{R^n}$ is a bounded open set and E is a compact subset of Ω. Let $K = h$ be the Newtonian or the logarithmic kernel or $K = G^{\Omega}$.*

Then $\gamma_K(E) = +\infty$ or, equivalently $C_K(E) = 0$, if and only if every function which is harmonic and bounded in $\Omega \setminus E$ can be extended in E so that it becomes harmonic in Ω.

Proof:

Suppose that $\gamma_K(E) = +\infty$ and consider an open set Ω_1 with C^1-boundary so that

$$E \ \subseteq\ \Omega_1 \ \subseteq\ \Omega$$

and $H_u^{\Omega_1}$, the solution of the problem of Dirichlet in Ω_1 with the restriction of u in $\partial\Omega_1$ as boundary function.

The third example in section 8.2 implies that $\partial E = E$. Therefore, by Theorem 8.7 and Proposition 6.1, there is a superharmonic function v in Ω_1 so that

$$v \ \geq\ 0$$

everywhere in $\Omega_1 \setminus E$ and

$$\lim_{\Omega_1 \setminus E \ni x \to y} v(x) \ =\ +\infty$$

for all $y \in E$.

From Theorem 3.4, we have that E is of zero harmonic measure with respect to $\Omega_1 \setminus E$ as a subset of its boundary.

Now, $H_u^{\Omega_1} - u$ is harmonic and bounded in $\Omega_1 \setminus E$. Also, by Proposition 3.6 and Theorem 3.7,

$$\lim_{\Omega_1 \setminus E \ni x \to y} \left(H_u^{\Omega_1} - u \right) \; = \; 0$$

for all $y \in \partial\Omega_1$.

By Theorem 3.5,

$$u \; = \; H_u^{\Omega_1}$$

identically in $\Omega_1 \setminus E$.

Therefore, the extension of u which is harmonic in Ω is

$$\begin{cases} u \; , & \text{in } \Omega \setminus E \\ H_u^{\Omega_1} & \text{in } E \; . \end{cases}$$

Now, assume that $\gamma_K(E) < +\infty$ and let $d\mu_0$ be the K-equilibrium measure of E and $U_K^{d\mu_0}$ the corresponding K-potential. By Theorem 8.2 and Proposition 6.1, this is a bounded harmonic function in $\Omega \setminus E$. If there is a function u harmonic in Ω with $u = U_K^{d\mu_0}$ everywhere in $\Omega \setminus E$, then the superharmonicity of $U_K^{d\mu_0}$ in Ω and the Minimum Principle for superharmonic functions imply that $u = U_K^{d\mu_0}$ everywhere in Ω.

Thus, $-\kappa_n d\mu_0 = \Delta U_K^{d\mu_0} = 0$ in Ω, contradicting that $d\mu_0$ is a probability Borel measure.

9.2 Sets of zero harmonic measure

Definition 9.1 *A compact set $E \subseteq \mathbf{R^n}$ is said to be* ***of zero harmonic measure****, if, for every R with $E \subseteq B(0;R)$, ∂E is of zero harmonic measure with respect to the open set $B(0;R) \setminus E$.*

Lemma 9.1 *Suppose that $\Omega \subseteq \mathbf{R^n}$ is a bounded open set and E is a compact subset of $\partial\Omega$. If E is of zero harmonic measure, then $\partial E = E$ is of zero harmonic measure with respect to Ω.*

Proof:

Consider a large enough R so that $\overline{\Omega} \subseteq B(0;R)$ and an arbitrary $x_0 \in \Omega$. Then

$$d\mu_{x_0}^{B(0;R)\setminus E}(E) \; = \; 0 \; .$$

If

$$u \; \in \; \Phi_{\chi_E}^{B(0;R)\setminus E} \; ,$$

then, by the Minimum Principle for superharmonic functions, we get that $u \geq 0$ in $B(0;R) \setminus E$ and, hence,

$$u \; \in \; \Phi_{\chi_E}^{\Omega\setminus E} \; .$$

This implies that

$$d\mu_{x_0}^{B(0;R)\setminus E}(E) \; \geq \; d\mu_{x_0}^{\Omega\setminus E}(E)$$

and, thus,
$$d\mu_{x_0}^{\Omega\setminus E}(E) \;=\; 0 \;.$$

Lemma 9.1 implies that a compact set $E \subseteq \mathbf{R^n}$ is of zero harmonic measure if and only if ∂E is of zero harmonic measure with respect to every bounded open set Ω with $\partial E \subseteq \partial\Omega$.

Theorem 9.2 *Suppose that E is a compact set in $\mathbf{R^n}$. Then E is of zero harmonic measure if and only if $C_h(E) = 0$.*

proof:
1. Let $C_h(E) = 0$ or, equivalently, $\gamma_h(E) = +\infty$.

We showed in the proof of Theorem 9.1 that, for every $B(0;R)$ containing E, ∂E is of zero harmonic measure with respect to $B(0;R)\setminus E$ and, hence, that E is of zero harmonic measure.

2. Suppose, conversely, that E is of zero harmonic measure.

If $\gamma_h(E) < +\infty$, and $d\mu_0$ is the h-equilibrium measure of E, then, for all R with $E \subseteq B(0;R)$, $U_h^{d\mu_0}$ is a bounded harmonic function in $B(0;R)\setminus E$. This is true due to Theorem 8.2 and the superharmonicity of $U_h^{d\mu_0}$.

By Theorem 3.5,
$$U_h^{d\mu_0}(x) \;\le\; M(R) \;=\; \max_{y\in S(0;R)} U_h^{d\mu_0}(y)$$
for all $x \in B(0;R)\setminus E$.

If $R \to +\infty$ and $n = 2$, then $M(R) \to -\infty$ and we get a contradiction, since $U_h^{d\mu_0}$ results to be identically $-\infty$ in $\mathbf{R^2}\setminus E$.

If $R \to +\infty$ and $n \ge 3$, then $M(R) \to 0$ and we get that
$$U_h^{d\mu_0} \;=\; 0$$
in $\mathbf{R^n}\setminus E$ and, by the Minimum Principle for superharmonic functions, in $\mathbf{R^n}$.

Therefore, $d\mu_0$ is the zero measure and we, again, arrive at a contradiction.

9.3 The set of irregular boundary points

Lemma 9.2 *Suppose that Ω is a bounded open set, $x_0 \in \Omega$ and $\lambda > 0$.*

If $\Omega^\lambda = \{x \in \Omega : G_{x_0}^{\Omega}(x) > \lambda\}$, then Ω^λ is an open connected set containing x_0 and
$$d\mu_{x_0}^{\Omega^\lambda}(\partial\Omega^\lambda \cap \partial\Omega) \;=\; 0 \;.$$

Moreover, the function
$$G_{x_0}^{\Omega}(x) - \lambda \;, \qquad x \in \Omega^\lambda \;,$$
is the Green's function of Ω^λ with respect to x_0.

Proof:
1. That Ω^λ is open and contains x_0 is clear.

Suppose that Ω^λ has a connected component O not containing x_0. By definition, $G^\Omega_{x_0} - h_{x_0}$ is the least harmonic majorant of $-h_{x_0}$ in Ω.

Consider the function

$$u(x) \;=\; \begin{cases} \lambda - h_{x_0} \;, & \text{if } x \in O \\ G^\Omega_{x_0} - h_{x_0} \;, & \text{if } x \in \Omega \setminus O \;. \end{cases}$$

This is, obviously, a majorant of $-h_{x_0}$ in Ω and it is easy to see that it is superharmonic in Ω. Therefore, $u \geq G^\Omega_{x_0} - h_{x_0}$ everywhere in Ω and this is false in O.

2. Now, let

$$v \;\in\; \Psi^{\Omega^\lambda}_{\chi_{\partial\Omega^\lambda \cap \partial\Omega}}$$

and consider the function

$$v_1 \;=\; \max(v, 0) \;.$$

Then,

1. v_1 is subharmonic in Ω^λ,
2. v_1 is bounded from above in Ω^λ,
3. $\lim_{\Omega^\lambda \ni x \to y} v_1(x) = 0$, if $y \in \partial\Omega^\lambda \setminus (\partial\Omega^\lambda \cap \partial\Omega) = \partial\Omega^\lambda \cap \Omega$ and
4. $\limsup_{\Omega^\lambda \ni x \to y} v_1(x) \leq 1$, if $y \in \partial\Omega^\lambda \cap \partial\Omega$.

Hence, the function

$$V(x) \;=\; G^\Omega_{x_0}(x) - \lambda v_1(x) \;, \qquad x \in \Omega^\lambda \;,$$

is superharmonic in Ω^λ with

$$\liminf_{\Omega^\lambda \ni x \to y} V(x) \;\geq\; 0$$

for all $y \in \partial\Omega^\lambda$ and, by the Minimum Principle for superharmonic functions,

$$V \;\geq\; 0$$

in Ω^λ.

If, now, we define

$$V(x) \;=\; G^\Omega_{x_0}(x) \;, \qquad x \in \Omega \setminus \Omega^\lambda \;,$$

then V is superharmonic and non-negative in Ω and, hence, $V - h_{x_0}$ is a superharmonic majorant of $-h_{x_0}$ in Ω.

Therefore,

$$V \;\geq\; G^\Omega_{x_0}$$

in Ω, implying that

$$v_1 \;=\; 0$$

in Ω^λ and, hence,

$$v \;\leq\; 0$$

in Ω^λ. Therefore,

$$d\mu_{x_0}^{\Omega^\lambda}(\partial\Omega^\lambda \cap \partial\Omega) \;=\; H^{\Omega^\lambda}_{\chi_{\partial\Omega^\lambda\cap\partial\Omega}}(x_0) \;=\; 0\;.$$

3. Consider, now, a small $\delta > 0$ so that $\overline{B(x_0;\delta)} \subseteq \Omega$ and the number $M = \max_{x\in S(x_0;\delta)} G_{x_0}^{\Omega}(x)$.

The set Ω^M is, as we showed in part 1, connected and, hence, it is contained in $B(x_0;\delta)$. Therefore $G_{x_0}^{\Omega}$ is bounded in $\Omega \setminus B(x_0;\delta)$. For the same reason, $G_{x_0}^{\Omega^\lambda}$ is bounded away from some neighborhood of x_0.

4. The function $G_{x_0}^{\Omega} - \lambda - h_{x_0}$ is a harmonic majorant of $-h_{x_0}$ in Ω^λ and, hence,

$$G_{x_0}^{\Omega} - \lambda \;\geq\; G_{x_0}^{\Omega^\lambda}$$

in Ω^λ.

The function

$$G_{x_0}^{\Omega^\lambda} - G_{x_0}^{\Omega} + \lambda$$

is harmonic in Ω^λ and, by the continuity of $G_{x_0}^{\Omega}$ at all points of $\partial\Omega^\lambda \cap \Omega$,

$$\liminf_{\Omega^\lambda \ni x \to y} \big(G_{x_0}^{\Omega^\lambda}(x) - G_{x_0}^{\Omega}(x) + \lambda\big) \;\geq\; 0$$

for all $y \in \partial\Omega^\lambda \cap \Omega$.

From part 3, we have that this function is bounded in Ω^λ and, from part 2, that $\partial\Omega^\lambda \cap \partial\Omega$ is of zero harmonic measure as a subset of $\partial\Omega^\lambda$. We, thus, get by Theorem 3.5 that,

$$G_{x_0}^{\Omega} - \lambda \;\leq\; G_{x_0}^{\Omega^\lambda}$$

in Ω^λ.

Theorem 9.3 *If Ω is a bounded open set, then the set of its irregular boundary points is of zero h-capacity and, hence, of zero harmonic measure with respect to Ω.*

Proof:

Let $x_0 \in \Omega$ and consider an arbitrary $\lambda > 0$ and the set

$$\Omega^\lambda \;=\; \{x \in \Omega : G_{x_0}^{\Omega}(x) > \lambda\}\;.$$

If $A_\lambda = \partial\Omega^\lambda \cap \partial\Omega$, we shall prove that $C_h(A_\lambda) = 0$.

On the contrary, suppose $\gamma_h(A_\lambda) < +\infty$ and let $d\mu_0$ be the h-equilibrium measure of A_λ.

Then,

1. $U_h^{d\mu_0} \leq \gamma_h(A_\lambda)$ everywhere and
2. $U_h^{d\mu_0} = \gamma_h(A_\lambda)$ quasi-almost everywhere in A_λ.

Now, consider some μ with $0 < \mu < \lambda$. By Lemma 9.2, $G_{x_0}^{\Omega} - \mu$ is the Green's function of Ω^μ with respect to x_0.

By Proposition 5.5, the function

$$h_{x_0}(x) - \int_{\partial\Omega^\mu} h_x(y)\ d\mu_{x_0}^{\Omega^\mu}(y)\ , \qquad x \in \mathbf{R}^\mathbf{n} \setminus \{x_0\}\ ,$$

is a subharmonic extension of $G_{x_0}^{\Omega} - \mu$ in $\mathbf{R}^\mathbf{n} \setminus \{x_0\}$ and, by upper-semicontinuity, for all $z \in A_\lambda$,

$$h_{x_0}(z) - \int_{\partial\Omega^\mu} h_z(y)\ d\mu_{x_0}^{\Omega^\mu}(y)\ \geq\ \limsup_{\Omega^\lambda \ni x \to z} G_{x_0}^{\Omega}(x) - \mu\ \geq\ \lambda - \mu\ >\ 0\ .$$

Therefore,

$$\begin{aligned} \lambda - \mu &\leq \int_{A_\lambda} \left(h_{x_0}(z) - \int_{\partial\Omega^\mu} h_z(y)\ d\mu_{x_0}^{\Omega^\mu}(y) \right) d\mu_0(z) \\ &= U_h^{d\mu_0}(x_0) - \int_{\partial\Omega^\mu} U_h^{d\mu_0}(y)\ d\mu_{x_0}^{\Omega^\mu}(y)\ . \end{aligned}$$

Every point of $\Omega^\mu \cap \Omega$ is a point of continuity of $G_{x_0}^{\Omega^\mu}$. Hence, by Theorem 5.4, every such point is a regular boundary point of Ω^μ. Therefore, by Theorem 3.7, the function

$$U_h^{d\mu_0}(x) - \int_{\partial\Omega^\mu} U_h^{d\mu_0}(y)\ d\mu_x^{\Omega^\mu}(y)\ , \qquad x \in \Omega^\mu\ ,$$

which is harmonic and bounded in Ω^μ, has zero boundary limits at all points of $\partial\Omega^\mu \setminus A_\mu$. Since A_μ is of zero harmonic measure with respect to the set Ω^μ,

$$U_h^{d\mu_0}(x) - \int_{\partial\Omega^\mu} U_h^{d\mu_0}(y)\ d\mu_x^{\Omega^\mu}(y)\ =\ 0$$

for all $x \in \Omega^\mu$.

Applying this to $x = x_0$, we get a contradiction.

Therefore,

$$C_h(A_\lambda)\ =\ 0\ .$$

By Theorem 5.4, the set of irregular boundary points of Ω is the union of $A_{\frac{1}{m}}$ for all $m \in \mathbf{N}$. Theorem 8.1 concludes the proof of the first statement.

The set of irregular boundary points of Ω is a Borel set. By Theorem 9.2 and Lemma 9.1, all its compact subsets are of zero harmonic measure with respect to Ω and, thus, the set itself is of zero harmonic measure with respect to Ω.

9.4 The support of the equilibrium measure

Proposition 9.1 *Suppose that $E \subseteq \mathbf{R^n}$ is a compact set with $\gamma_h(E) < +\infty$.*

Then, the h-equilibrium measure $d\mu_0$ of E is supported in the, so-called, ***outer boundary*** *of E; namely, the set ∂O, where O is the unbounded connected component of $\Omega = \mathbf{R^n} \setminus E$.*

Proof:

Since

$$U_h^{d\mu_0} \;=\; \gamma_h(E)$$

quasi-almost everywhere in E, we get, by the last example in section 8.2, that

$$U_h^{d\mu_0} \;=\; \gamma_h(E)$$

almost everywhere in E.

If O' is a component of Ω different from O, then $U_h^{d\mu_0}$ is harmonic and bounded in O' and has boundary limits equal to $\gamma_h(E)$ in $\partial O'$ except in a subset of $\partial O'$ of zero harmonic measure. Therefore,

$$U_h^{d\mu_0} \;=\; \gamma_h(E)$$

everywhere in O'.

We conclude that

$$U_h^{d\mu_0} \;=\; \gamma_h(E)$$

almost everywhere, and, hence, everywhere in the interior of $\mathbf{R^n} \setminus O$. This implies that

$$-\kappa_n d\mu_0 \;=\; \Delta U_h^{d\mu_0} \;=\; 0$$

in the interior of $\mathbf{R^n} \setminus O$.

Example.

Let $E = \overline{B(x_0;R)}$.

Then, the h-equilibrium measure $d\mu_0$ of $\overline{B(x_0;R)}$ is supported in $S(x_0;R)$. By the rotation invariance of $\gamma_h\big(\overline{B(x_0;R)}\big) = I_h(d\mu_0)$ and the uniqueness of the h-equilibrium measure, we get that $d\mu_0$ must be rotation invariant and, hence,

$$d\mu_0 \;=\; \frac{1}{\omega_{n-1}R^{n-1}}\, dS$$

is the normalized surface area measure of $S(x_0;R)$.

Thus,

$$\begin{aligned}
\gamma_h\big(\overline{B(x_0;R)}\big) \;&=\; U_h^{d\mu_0}(x_0) \;=\; \int_{S(x_0;R)} h^*(R)\; d\mu_0(y) \\
&=\; h^*(R) \;=\; \begin{cases} \log\frac{1}{R}\,, & \text{if } n=2 \\ \frac{1}{R^{n-2}}\,, & \text{if } n \geq 3\,. \end{cases}
\end{aligned}$$

Example.

Let $E = \overline{B(x_0; r, R)}$ with $0 < r < R$.

Then, the h-equilibrium measure $d\mu_0$ of $\overline{B(x_0; R)}$ is supported in $S(x_0; R)$ and, exactly as before, we get that

$$\gamma_h\left(\overline{B(x_0; r, R)}\right) = h^*(R) \ .$$

Proposition 9.2 *Suppose that E is a compact set with $\gamma_h(E) < +\infty$ and let the open $\Omega \subseteq \mathbf{R^n}$ contain E.*

Then, $y \in \partial E$ is a regular boundary point of $\Omega \setminus E$ if and only if

$$\lim_{\Omega \setminus E \ni x \to y} U_h^{d\mu_0}(x) = \gamma_h(E) \ ,$$

where $d\mu_0$ is the h-equilibrium measure of E.

Proof:

Since the notion of regularity has a local character, we may assume that Ω is equal to some large enough ball $B(0; R)$.

Let V be the generalized solution of the Dirichlet problem in $B(0; R) \setminus E$ with boundary function

$$\begin{cases} \gamma_h(E) \ , & \text{if } y \in \partial E \\ U_h^{d\mu_0}(y) \ , & \text{if } y \in S(0; R) \ . \end{cases}$$

If $y \in \partial E$ is a regular boundary point of $B(0; R) \setminus E$, then, by Theorem 3.7, $\lim_{B(0;R) \setminus E \ni x \to y} V(x) = \gamma_h(E)$.

If, conversely, $\lim_{B(0;R) \setminus E \ni x \to y} V(x) = \gamma_h(E)$, then the function $\gamma_h(E) - V$ is a barrier at y with respect to $B(0; R) \setminus E$.

V and $U_h^{d\mu_0}$ are bounded and harmonic in $B(0; R) \setminus E$ and they have the same boundary limits at all points of $S(0; R)$.

V has boundary limit equal to $\gamma_h(E)$ at every $y \in \partial E$, except at every irregular y. But, by Theorem 9.3, the set of irregular boundary points of $B(0; R) \setminus E$ is of zero harmonic measure with respect to $B(0; R) \setminus E$.

$U_h^{d\mu_0}$ has, also, boundary limit $\gamma_h(E)$ at every $y \in \partial E$, except at every $y \in \partial E$ with $U_h^{d\mu_0} < \gamma_h(E)$. But, by Theorem 8.2, all these y's belong to a Borel set of zero h-capacity and, hence, of zero harmonic measure with respect to $B(0; R) \setminus E$.

By Theorem 3.5,

$$U_h^{d\mu_0} = V$$

in $B(0; R) \setminus E$ and the proof is, now, clear.

9.5 Capacity and conformal mapping

We shall, now, study, *in the framework of potential theory*, a fundamental subject of complex analysis, namely the existence of conformal mapping between simply connected open sets.

Lemma 9.3 *If I is any compact linear segment in $\mathbf{R^2}$, then $C_h(I) > 0$.*

Proof:

If $d\mu$ is the linear Lebesgue measure on I, then we easily estimate

$$I_h(d\mu) \ < \ +\infty \ .$$

Proposition 9.3 *If E is any continuum in $\mathbf{R^2}$, then $C_h(E) > 0$.*

Proof:

Let $a, b \in E$ with $a \neq b$ and consider the segment $I = [a, b]$.

If

$$Pr : E \ \to \ E'$$

is the orthogonal projection of E on the line containing I, then this function is a contraction and $E' = Pr(E) \supseteq I$. From Proposition 8.7,

$$d_h(E) \ \geq \ d_h(E') \ \geq \ d_h(I) \ > \ 0 \ .$$

The following is just an extension of the corresponding definition for subsets of $\mathbf{R^2}$.

Definition 9.2 *An open set $\Omega \subseteq \overline{\mathbf{R^2}}$ is called simply-connected if it is connected and $\overline{\mathbf{R^2}} \setminus \Omega$ is, also, connected.*

It is clear that symmetry with respect to any circle preserves the property of simple-connectedness.

Through the Kelvin Transform and in view of Proposition 9.3, the first part of the next result is identical to the well known theorem of complex analysis. The proof which is presented here is not the standard proof presented in the elementary graduate courses of function theory. It is based on the existence of the Green's function and, thus, the proof is reduced to a maximization problem, exactly as the standard proof.

Theorem 9.4 *(The Riemann Mapping Theorem) Suppose that Ω is a simply-connected open subset of $\overline{\mathbf{R^2}}$ with $\infty \in \Omega$ and E is its (compact connected) complement. Then, there is a conformal mapping of Ω onto $B(0;1)$, i.e. a function*

$$\phi : \Omega \ \to \ B(0;1) \ ,$$

which is one-to-one in Ω, onto $B(0;1)$ and holomorphic in Ω, if and only if

$$C_h(E) \ > \ 0 \ .$$

In this case we can arrange it so that, also, $\phi(\infty) = 0$ and, then necessarily,

$$|\phi'(\infty)| \ = \ e^{-\gamma_h(E)} \ = \ d_h(E) \ = \ C_{h,R}(E)$$

and

$$G_\infty^\Omega(x) \ = \ \log \frac{1}{|\phi(x)|} \ , \qquad x \in \Omega \ ,$$

is the Green's function of Ω with respect to ∞.

Proof: 1. Suppose that

$$\phi : \Omega \; \to \; B(0;1)$$

is one-to-one, onto, conformal and that $\phi(\infty) = 0$.

If $E = \{a\}$ consists of only one point, then the Kelvin Transform ϕ^* of ϕ with respect to any circle centered at a is bounded and holomorphic in $\mathbf{R^2}$. Both its real and its imaginary parts are harmonic functions bounded in $\mathbf{R^2}$ and, hence, by the Theorem of Picard, ϕ^* and, therefore, ϕ is a constant function. Similarly, if E is empty, then the restriction of ϕ in $\mathbf{R^2}$ is bounded and holomorphic in $\mathbf{R^2}$ and, hence, is constant.

We conclude that E has more than one point and, since E is a continuum, Proposition 9.3 implies that $C_h(E) > 0$.

2. Suppose, conversely, that $C_h(E) > 0$ and consider the h-equilibrium measure $d\mu_0$ of E and the h-potential

$$U_h^{d\mu_0}(x) \;=\; \int_E \log \frac{1}{|x-y|}\, d\mu_0(y)\ , \qquad x \in \mathbf{R^2}\ .$$

After a translation, we may suppose that $0 \in E$.

By Theorem 8.2, the function

$$U(x) \;=\; \gamma_h(E) - U_h^{d\mu_0}(x) \;=\; \log|x| + \gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y)$$

is non-negative in $\Omega \setminus \{\infty\}$.

Hence,

$$\gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y) \;\geq\; -\log|x| \;=\; -h_\infty(x)$$

for all $x \in \Omega\setminus\{\infty\}$ and we observe that the left side of this inequality is harmonic in $\Omega \setminus \{\infty\}$ and that its limit at ∞ is the finite number $\gamma_h(E)$. By Theorem 4.1, it is a harmonic majorant of $-h_\infty$ in Ω. Therefore, Ω has a Green's function with respect to ∞ and, moreover,

$$G_\infty^\Omega \;\leq\; U$$

in Ω.

We shall, in fact, prove that $G_\infty^\Omega = U$ in Ω.

Since, by Proposition 3.5, all points of ∂E are regular boundary points of Ω, Proposition 9.2 implies that

$$\lim_{\Omega \ni x \to y} U_h^{d\mu_0}(x) \;=\; \gamma_h(E)$$

for every y in ∂E. We may observe that in this particular situation, where all points of ∂E are regular boundary points of Ω, the proof of Proposition 9.2 simplifies, as it does not need Theorem 9.3.

Now, let u be any harmonic majorant of $-h_\infty$ in Ω and consider the function

$$V(x) \;=\; -u(x) + \gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y) \ , \qquad x \in \Omega \setminus \{\infty\} \ .$$

V is harmonic in Ω and

$$V(x) \;\leq\; \gamma_h(E) - U_h^{d\mu_0}(x)$$

for all $x \in \Omega \setminus \{\infty\}$. Hence,

$$\limsup_{\Omega \ni x \to y} V(x) \;\leq\; 0$$

for all $y \in \partial E$. We conclude that

$$V \;\leq\; 0$$

in Ω. Therefore,

$$\gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y)$$

is the smallest harmonic majorant of $-h_\infty$ in Ω and, finally,

$$G_\infty^\Omega(x) \;=\; \gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y) + h_\infty(x) \;=\; \gamma_h(E) - U_h^{d\mu_0}(x) \ , \qquad x \in \Omega \ .$$

3. We, now, define

$$h(x) \;=\; \gamma_h(E) - \int_E \log \frac{|x|}{|x-y|}\, d\mu_0(y) \ , \qquad x \in \Omega \ .$$

We have, already, seen that h is the smallest harmonic majorant of $-h_\infty$ in Ω and that $G_\infty^\Omega(x) = \log|x| + h(x)\,,\ x \in \Omega$.

Since Ω is simply connected, there is, by Theorem 1.4 through the Kelvin Transform, a harmonic conjugate g of h in Ω.

Consider, also, the many-valued function

$$F_\infty^\Omega(x) \;=\; \arg(x) + g(x) \ , \qquad x \in \Omega \setminus \{\infty\} \ ,$$

and the single-valued

$$\phi(x) \;=\; e^{-G_\infty^\Omega(x) - iF_\infty^\Omega(x)} \;=\; \frac{1}{x}\, e^{-h(x) - ig(x)} \ , \qquad x \in \Omega \setminus \{\infty\} \ ,$$

which, by Theorem 4.1, is analytic in Ω, since the right side is analytic in Ω with value 0 at ∞.

4. We consider, for every $\lambda > 0$, the open sets

$$\Omega^\lambda \;=\; \{x \in \Omega : G_\infty^\Omega(x) > \lambda\} \ .$$

If $x \in \Omega$ and $G_\infty^\Omega(x) = \lambda$, then in any $B(x;r) \subseteq \Omega \setminus \{\infty\}$ there are points of Ω^λ. Otherwise, by the Maximum-Minimum Principle, G_∞^Ω would be constant in $B(x;r)$ and, by Theorem 1.10, in the connected $\Omega \setminus \{\infty\}$. Hence, x belongs to $\partial\Omega^\lambda$.

Conversely, if x belongs to $\partial\Omega^\lambda$, then $x \in \Omega$. Otherwise, $x \in \partial E$ and x would be a regular boundary point of Ω, implying that $\lim_{\Omega \ni z \to x} G_\infty^\Omega(z) = 0$. But $\lim_{\Omega^\lambda \ni z \to x} G_\infty^\Omega(z) \geq \lambda$, resulting to a contradiction. Thus,

$$\partial\Omega^\lambda \;=\; \{x \in \Omega : G_\infty^\Omega(x) = \lambda\} \;=\; \{x \in \Omega : |\phi(x)| = e^{-\lambda}\} \;.$$

Since ϕ' is holomorphic in Ω, its zeros (the critical points of ϕ) are at most countably many. Therefore, the set of critical values,

$$\{w : w = \phi(x) \text{ for some } x \in \Omega \text{ with } \phi'(x) = 0\} \;,$$

is at most countable. Furthermore, the set

$$\Lambda \;=\; \{\lambda > 0 : e^{-\lambda} = |w| \text{ for some critical value } w \text{ of } \phi\}$$

is at most countable.

If $\lambda > 0$ and $\lambda \notin \Lambda$, then the function $e^{-2\lambda} - |\phi|^2$ is a C^∞-defining function of Ω^λ at all its boundary points.

Now, Ω^λ is an open subset of Ω with C^1-boundary, ϕ is holomorphic in Ω and $\phi(\partial\Omega^\lambda) \subseteq S(0;e^{-\lambda})$.

By Theorem 0.7 applied through the Kelvin Transform, for every value $y \in B(0;e^{-\lambda})$ the total multiplicity of all solutions of the equation $\phi(x) = y$, $x \in \Omega^\lambda$, is equal to the multiplicity of the only solution, ∞, of $\phi(x) = 0$. Since

$$|\phi'(\infty)| \;=\; \lim_{x\to\infty} |x\phi(x)| \;=\; e^{h(\infty)} \;=\; e^{-\gamma_h(E)} \;\neq\; 0 \;,$$

this multiplicity is exactly 1. Hence, ϕ is a bijection of Ω^λ onto $B(0;e^{-\lambda})$.

Considering a sequence of λ's in $\mathbf{R}^+ \setminus \Lambda$ converging to $0+$, we conclude that ϕ is a bijection of Ω onto $B(0;1)$.

5. Suppose, now, that $\psi : \Omega \to B(0;1)$ is another conformal mapping with $\psi(\infty) = 0$.

We can prove that

$$\lim_{\Omega \ni x \to y} |\psi(x)| \;=\; 1$$

for all $y \in \partial\Omega$. In fact, let $\{x_n\}$ be some sequence in Ω with $x_n \to y \in \partial\Omega$ and $\psi(x_n) \to w_0$ for some $w_0 \in B(0;1)$. Take $x_0 \in \Omega$ so that $\psi(x_0) = w_0$ and consider a small disc $\overline{B(x_0;\delta)} \subseteq \Omega$. Since ψ is one-to-one, $w_0 \notin \psi\big(S(x_0;\delta)\big)$ and, by Theorem 0.7, every w which is close enough to w_0 can be written as $w = \psi(x)$ for some $x \in B(x_0;\delta)$. But, eventually, all $w_n = \psi(x_n)$ are close to w_0 while $x_n \notin B(x_0;\delta)$. This is impossible, since ψ is one-to-one.

Now, since ψ is holomorphic at ∞ and $\psi(\infty) = 0$, we have that $\psi'(\infty) = \lim_{x\to\infty} x\psi(x) \in \mathbf{C}$ and, therefore, the function

$$\log\frac{1}{|\psi(x)|} - \log|x| \;, \qquad x \in \Omega \;,$$

is harmonic in Ω.

This implies that the function

$$\log\frac{1}{|\phi(x)|}-\log\frac{1}{|\psi(x)|} \;=\; G_\infty^\Omega(x)-\log\frac{1}{|\psi(x)|}\;, \qquad x\in\Omega\;,$$

is harmonic in Ω. Its boundary limits are all 0 and, hence, it is identically 0 in Ω.

Now, it is clear that

$$|\psi'(\infty)| \;=\; e^{-\gamma_h(E)}$$

and

$$G_\infty^\Omega(x) \;=\; \log\frac{1}{|\psi(x)|}\;, \qquad x\in\Omega\;.$$

Example

Consider the line segment $I=[-l,l]$.

If $\Omega=\overline{\mathbf{R^2}}\setminus I$, then the conformal mapping $\phi:\Omega\to B(0;1)$ is the inverse mapping of

$$x \;=\; \frac{l}{2}\Big(y+\frac{1}{y}\Big)\;.$$

Therefore,

$$C_{h,R}(I) \;=\; d_h(I) \;=\; |\phi'(\infty)| \;=\; \lim_{x\to\infty}|x\phi(x)| \;=\; \lim_{y\to 0}\Big|y\frac{l}{2}\Big(y+\frac{1}{y}\Big)\Big| \;=\; \frac{l}{2} \;=\; \frac{|I|}{4}$$

and, thus,

$$\gamma_h(I) \;=\; \log\frac{4}{|I|}\;.$$

9.6 Capacity and Green's function in $\overline{\mathbf{R^2}}$

We shall state and prove a characterization of all open subsets of $\mathbf{R^2}$ which have a Green's function in each of their components.

We remember that the problem in $\mathbf{R^n}$ is completely solved, if $n\geq 3$, since, in this case, all open sets have a Green's function.

We, also, observe that, if the open set Ω has more than one components, then it has a Green's function in every one of its components. This is true because, in this case, every component is disjoint from some ball.

Hence, it is no loss of generality to assume that the open set Ω is connected. It is equivalent, through an application of the Kelvin transform, to consider the case of a connected open set Ω containing ∞.

Theorem 9.5 *Suppose that the open connected set $\Omega\subseteq\overline{\mathbf{R^2}}$ contains ∞ and let E be the (compact) complement of Ω.*

Then, Ω has a Green's function if and only if $C_h(E)>0$.

If $C_h(E) > 0$, then

$$G_\infty^\Omega(x) \ = \ \gamma_h(E) - U_h^{d\mu_0}(x) \ , \qquad x \in \Omega \ ,$$

where $d\mu_0$ is the h-eqilibrium measure of E.

Proof:
If $\Omega = \overline{\mathbf{R^2}}$, then we, already, know that Ω has no Green's function and, also, that $E = \emptyset$ has zero logarithmic capacity.

Hence, we assume that there is at least one point not in Ω and, through a translation, we may assume that $0 \notin \Omega$.
1. Suppose, now, that $C_h(E) > 0$ or, equivalently, that

$$\gamma_h(E) \ < \ +\infty \ .$$

Consider the h-equilibrium measure $d\mu_0$ of E and the h-potential

$$U_h^{d\mu_0}(x) \ = \ \int_E \log \frac{1}{|x-y|} \ d\mu_0(y) \ , \qquad x \in \mathbf{R^2} \ .$$

By Theorem 8.2, the function

$$U(x) \ = \ \gamma_h(E) - U_h^{d\mu_0}(x) \ = \ \log|x| + \gamma_h(E) - \int_E \log \frac{|x|}{|x-y|} \ d\mu_0(y)$$

is non-negative in $\Omega \setminus \{\infty\}$.
Hence,

$$\gamma_h(E) - \int_E \log \frac{|x|}{|x-y|} \ d\mu_0(y) \ \geq \ -\log|x| \ = \ -h_\infty(x)$$

for all $x \in \Omega \setminus \{\infty\}$ and we observe that the right side of this inequality is harmonic in $\Omega \setminus \{\infty\}$ and that its limit at ∞ is the finite number $\gamma_h(E)$. By Proposition 4.7, it is a harmonic majorant of $-h_\infty$ in Ω and, therefore, Ω has a Green's function with respect to ∞ and, moreover,

$$G_\infty^\Omega \ \leq \ U$$

in Ω.
2. If Ω is a regular set, then we repeat part 2 of the proof of Theorem 9.4 and conclude that $G_\infty^\Omega = U$ in Ω.

In general, since E is capacitable, we may consider a sequence of open sets O_m so that $E \subseteq O_m \subseteq \{x : d(x,E) < \frac{1}{m}\}$, $C_h(O_m) \downarrow C_h(E)$ and $\overline{O_{m+1}} \subseteq O_m$ for all m. We take, next, open sets N_m with C^1-boundary so that $\overline{O_{m+1}} \subseteq N_m \subseteq \overline{N_m} \subseteq O_m$ and let $\Omega_m = \overline{\mathbf{R^2}} \setminus \overline{N_m}$. Then, $C_h(\overline{N_m}) \downarrow C_h(E)$ and, by the previous discussion,

$$G_\infty^{\Omega_m} \ = \ \gamma_h(\overline{N_m}) - U_h^{d\mu_{0,m}}$$

everywhere in Ω_m, where $d\mu_{0,m}$ is the h-equilibrium measure of $\overline{N_m}$.

Since $\Omega_m \uparrow \Omega$, Theorem 5.2 implies that $G_\infty^{\Omega_m} \uparrow G_\infty^\Omega$ everywhere in Ω. Therefore,

$$G_\infty^\Omega(x) \;=\; \gamma_h(E) - \lim_{m\to+\infty} U_h^{d\mu_{0,m}}(x)$$

for all $x \in \Omega$.

If we fix an $x \in \Omega$ and, then, take an m_0 so that $\frac{1}{m_0} < \frac{1}{2} d(x,E)$, we have that all measures $d\mu_{0,m}$, $m \geq m_0$, are supported in the compact set $\{z : d(z,E) \leq \frac{1}{2} d(x,E)\}$. Taking any subsequence $d\mu_{0,m_k}$ weakly converging in this compact set to some probability Borel measure $d\nu$, we easily get that $d\nu$ is supported in E. By Proposition 7.1,

$$I_h(d\nu) \;\leq\; \liminf_{k\to+\infty} I_h(d\mu_{0,m_k}) \;=\; \liminf_{k\to+\infty} \gamma_h(\overline{N_{m_k}}) \;=\; \gamma_h(E)\ .$$

By Theorem 8.2, $I_h(d\nu) = \gamma_h(E) = I_h(d\mu_0)$ and, by Theorem 8.3, $d\nu = d\mu_0$. Hence, every weakly convergent subsequence of $\{d\mu_{0,m}\}$ has $d\mu_0$ as weak limit and this implies that $\{d\mu_{0,m}\}$ converges weakly in $\{z : d(z,E) \leq \frac{1}{2} d(x,E)\}$ to $d\mu_0$.

Since the function h_x is continuous in $\{z : d(z,E) \leq \frac{1}{2} d(x,E)\}$,

$$U_h^{d\mu_{0,m}}(x) \;\to\; U_h^{d\mu_0}(x)\ .$$

Thus,

$$G_\infty^\Omega(x) \;=\; \gamma_h(E) - U_h^{d\mu_0}(x)$$

for every $x \in \Omega$.

3. Assume, conversely, that $C_h(E) = 0$.

If $-h_\infty$ has a harmonic majorant h in Ω, then, for every R which is large enough so that $E \subseteq B(0;R)$, we have that $h(\cdot) + \log|\cdot|$ is harmonic and bounded from below in $B(0;R)\setminus E$ and

$$h(y) + \log|y| \;\geq m(R) + \log R$$

for all $y \in S(0;R)$, where $m(R) = \min_{S(0;R)} h$.

Since, by Theorem 9.2, $\partial E = E$ is of zero harmonic measure with respect to $B(0;R)\setminus E$, we get from Theorem 3.5 that, for every $x \in B(0;R)\setminus E$,

$$h(x) + \log|x| \;\geq\; m(R) + \log R\ .$$

Now, since $\lim_{R\to+\infty} m(R) = |h(\infty)|$, letting $R \to +\infty$, we get

$$h(x) \;=\; +\infty\ .$$

We, thus, arrive at a contradiction and, hence, Ω has no Green's function.

9.7 Polar sets and the Theorem of Evans

Definition 9.3 *A set $A \subseteq \mathbf{R^n}$ is called **locally polar** if for every $x \in A$ there is some $B(x;r)$ and a function u superharmonic in $B(x;r)$ so that*

$$u(x) \;=\; +\infty\ , \qquad x \in A \cap B(x;r)\ .$$

Lemma 9.4 *Suppose that u is superharmonic in $B(x_0;R)$ and let $0 < r < R$. Then, there exists a function U superharmonic in $\mathbf{R^n}$ so that*

$$U(x) \;=\; u(x) \;, \qquad x \in B(x_0;r) \;.$$

If $n \geq 3$, we may also have U be bounded from below in $\mathbf{R^n}$.

Proof:
Take r_1, R_1 so that $r < r_1 < R_1 < R$ and cover the ring $\overline{B(x_0;r_1,R_1)}$ by finitely many open balls $B_1, \ldots, B_N$ all of which are contained in $B(x_0;r,R)$.

Then, the function $v = \min(u_{B_1}, \ldots, u_{B_N})$ is superharmonic in $B(x_0;R)$, bounded in $\overline{B(x_0;r_1,R_1)}$ and satisfies

$$v(x) \;=\; u(x) \;, \qquad x \in B(x_0;r) \;.$$

If $M = \sup_{x \in \overline{B(x_0;r_1,R_1)}} v(x)$ and $m = \inf_{x \in \overline{B(x_0;r_1,R_1)}} v(x)$, find $a \in \mathbf{R}^+$ and $b \in \mathbf{R}$ so that $ah_*(r_1) + b > M$ and $ah_*(R_1) + b < m$.

Now, it is easy to see that the function

$$U(x) \;=\; \begin{cases} v(x) \;, & \text{if } x \in B(x_0;r_1) \\ \min(v(x), ah(x) + b) \;, & \text{if } x \in \overline{B(x_0;r_1,R_1)} \\ ah(x) + b \;, & \text{if } x \notin \overline{B(x_0;R_1)} \;, \end{cases}$$

concludes the proof.

Proposition 9.4 *Let $A \subseteq \mathbf{R^n}$ be locally polar and let $x_0 \notin A$. Then, there exists a function u superharmonic in $\mathbf{R^n}$ so that $u(x_0) < +\infty$ and*

$$u(x) \;=\; +\infty \;, \qquad x \in A \;.$$

If $n \geq 3$, we may also have $u > 0$ everywhere in $\mathbf{R^n}$.

Proof:
We, first, observe that $A \neq \mathbf{R^n}$. In fact, if we take an arbitrary $x \in A$, a ball $B(x;R)$ and a u superharmonic in $B(x;R)$ which is $= +\infty$ identically in $A \cap B(x;R)$, then u must be finite almost everywhere in $B(x;R)$. Therefore there is at least one point of this ball not belonging to A. (Continuing this argument, we may, easily, show that A has zero Lebesgue measure.)

Fix $x_0 \notin A$ and for each $x \in A$ we consider a $B(x;R_x)$ not containing x_0 and a u_x superharmonic in $B(x;R_x)$ so that $u = +\infty$ identically in $A \cap B(x;R_x)$. We, then, take $r_x < R_x$ and Lemma 9.4 provides us with a U_x superharmonic in $\mathbf{R^n}$ which is $= +\infty$ identically in $A \cap B(x;r_x)$ and with $U_x(x_0) < +\infty$. In case $n \geq 3$, we may also suppose that $U_x > 0$ everywhere in $\mathbf{R^n}$.

We may replace each $B(x;r_x)$ by a smaller open ball containing x and having rational radius and rational center. We enumerate these countably many balls and we have, now, constructed: a sequence $\{B_k\}$ of open balls covering A and a sequence $\{U_k\}$ of functions superharmonic in $\mathbf{R^n}$ with every U_k being

identically $+\infty$ in $A \cap B_k$ and, in case $n \geq 3$, positive everywhere in $\mathbf{R^n}$. Also: $U_k(x_0) < +\infty$.
Case 1: $n \geq 3$.

For each k we take $\lambda_k > 0$ so that $\sum_{k=1}^{+\infty} \lambda_k U_k(x_0) < +\infty$. Then, the function

$$u \;=\; \sum_{k=1}^{+\infty} \lambda_k U_k$$

is superharmonic and positive in $\mathbf{R^n}$ and is identically $+\infty$ in A.
Case 2: $n = 2$.

For each k we set $m_k = \min_{x \in \overline{B(x_0;k)}} U_k(x)$ and we find $\lambda_k > 0$ so that $\sum_{k=1}^{+\infty} \lambda_k (U_k(x_0) - m_k) < +\infty$. Then, the function

$$u \;=\; \sum_{k=1}^{+\infty} \lambda_k (U_k - m_k)$$

is superharmonic in $\mathbf{R^n}$ and is identically $+\infty$ in A. The reason for both is that the terms of the series are, eventually, non-negative in every large ball $B(0;R)$.

Definition 9.4 *Suppose that $A \subseteq \mathbf{R^n}$. The set A is called* ***polar*** *if there is a function u superharmonic in $\mathbf{R^n}$ so that*

$$u(x) \;=\; +\infty \;, \qquad x \in A \;.$$

Thus, Proposition 9.4 says that a set A is polar if and only if it is locally polar.

Observe that the Theorem of Evans implies that every compact $E \subseteq \mathbf{R^n}$ with $C_h(E) = 0$ is polar.

Theorem 9.6 *Suppose that $A \subseteq \mathbf{R^n}$.*
1. If $C_h(A) = 0$, then A is polar.
2. If $n \geq 3$, then the converse of 1 is, also, true.
If $n = 2$, then the converse of 1 is true, provided that A is bounded.

Proof:
1. Suppose that $C_h(A) = 0$ and let A be bounded. Consider a bounded open set $O \subseteq \mathbf{R^n}$ with $A \subseteq O$ and $C_h(O) < \epsilon < 1$. Now, take any compact exhaustion $\{K_{(m)}\}$ of O. It is true that $C_h(K_{(m)}) \uparrow C_h(O)$. From Theorem 8.5, we, also, have that $\gamma_h(K_{(m)}) = \frac{1}{C_h(K_{(m)})} \downarrow \frac{1}{C_h(O)}$.

For each m consider the h-equilibrium measure $d\mu_m$ of $K_{(m)}$ which, after Proposition 9.1, is supported in $\partial K_{(m)}$. Then, $U_h^{d\mu_m} = \gamma_h(K_{(m)})$ identically in the interior of $K_{(m)}$ and $U_h^{d\mu_m} \leq \gamma_h(K_{(m)})$ everywhere in $\mathbf{R^n}$.

Replacing, if necessary, by a subsequence, assume that $d\mu_m \to d\mu$ weakly in the compact set $\overline{O}$, where $d\mu$ is a probability Borel measure in $\overline{O}$.

For an arbitrary $x \in O$, take m_0 so that x is in the interior of $K_{(m_0)}$. Now, since h_x is continuous in $\overline{O} \setminus int(K_{(m_0)})$ and since all $d\mu_m$, $m \geq m_0$, are supported in this compact set, we get $U_h^{d\mu}(x) = \lim_{m\to+\infty} U_h^{d\mu_m}(x) = \frac{1}{C_h(O)}$. Hence

$$U_h^{d\mu}(x) \;=\; \frac{1}{C_h(O)} \;\geq\; \frac{1}{\epsilon}\;, \qquad x \in O\;.$$

We, thus, find a decreasing sequence of bounded open sets $\{O_k\}$ with $A \subseteq O_k$ and a sequence of probability Borel measures $\{d\mu_k\}$, where each $d\mu_k$ is supported in $\overline{O_k}$ and

$$U_h^{d\mu_k}(x) \;\geq\; 2^k\;, \qquad x \in A\;.$$

Now, consider the probability Borel measure

$$d\mu \;=\; \sum_{k=1}^{+\infty} \frac{1}{2^k}\, d\mu_k\;.$$

It is obvious that

$$U_h^{d\mu} \;=\; +\infty$$

everywhere in A and A is polar.

If A is unbounded, for each $x \in A$ we consider a $B(x;r)$ and, since $A \cap B(x;r)$ is bounded with zero h-capacity, we have that it is a polar set. A is, thus, locally polar and, hence, polar.

2. Let A be bounded and let u be superharmonic in $\mathbf{R^n}$ so that $u = +\infty$ identically in A. By Theorem 2.17, there exists a compactly supported probability measure $d\mu$ so that

$$U_h^{d\mu} \;=\; +\infty$$

everywhere in A.

For arbitrary $k > 0$, consider the bounded open set

$$O \;=\; \{x : U_h^{d\mu}(x) > k\}$$

and any compact $E \subseteq O$ with $C_h(E) > 0$.

If $d\mu_0$ is the h-equilibrium measure of E, then

$$\frac{1}{C_h(E)} \;=\; \gamma_h(E) \;\geq\; \int_{supp(d\mu)} U_h^{d\mu_0}(x)\; d\mu(x) \;=\; \int_E U_h^{d\mu}(x)\; d\mu_0(x) \;\geq\; k\;.$$

Hence,

$$C_h^o(A) \;\leq\; C_h(O) \;\leq\; \frac{1}{k}$$

and, since k is arbitrary, $C_h(A) = 0$.

If A is unbounded and $n \geq 3$, we consider, for each $k \in \mathbf{N}$, the polar sets $A_k = A \cap B(0;k)$ which, by the previous argument, have $C_h(A_k) = 0$ and, then, use Theorem 8.1.

Theorem 9.7 *Suppose that $E \subseteq \mathbf{R^n}$ is compact with $C_h(E) > 0$ and $x_0 \in \partial E$ is a non-regular boundary point of $\mathbf{R^n} \setminus E$.*
1. There is a function u superharmonic in $\mathbf{R^n}$ so that $u(x) = \gamma_h(E)$ for every $x \in E$ and $u \le \gamma_h(E)$ everywhere in $\mathbf{R^n}$.
2. There is a function v superharmonic in $\mathbf{R^n}$ so that $v(x) = \gamma_h(E)$ for every $x \in E \setminus \{x_0\}$, $v(x_0) < \gamma_h(E)$ and $v \le \gamma_h(E)$ everywhere in $\mathbf{R^n}$.

Proof:
1. Let $d\mu_0$ be the h-equilibrium measure of E. Then, $U_h^{d\mu_0} \le \gamma_h(E)$ everywhere in $\mathbf{R^n}$ and the set $A = \{x \in E : U_h^{d\mu_0}(x) < \gamma_h(E)\}$ is polar.

We take u_0 superharmonic in $\mathbf{R^n}$ so that $u_0 > 0$ in E and $u_0 = +\infty$ in A.

Then, the function $u = \min(U_h^{d\mu_0} + u_0, \gamma_h(E))$ satisfies the properties in the statement.
2. By Proposition 9.2, we have that

$$U_h^{d\mu_0}(x_0) \;\le\; \liminf_{x \to x_0} U_h^{d\mu_0}(x) \;\le\; \liminf_{E \not\ni x \to x_0} U_h^{d\mu_0}(x) \;<\; \gamma_h(E) \;.$$

Now, $A \setminus \{x_0\}$ is, also, polar and, by Proposition 9.4, there is a u_1 superharmonic in $\mathbf{R^n}$ with $u_1 > 0$ in E, $u_1 = +\infty$ in $A \setminus \{x_0\}$ and $u_1(x_0) < +\infty$. Replacing, if necessary, u_1 by a small positive multiple of it, we may, also, suppose that

$$U_h^{d\mu_0}(x_0) + u_1(x_0) \;<\; \gamma_h(E) \;.$$

The function we want is $v = \min(U_h^{d\mu_0} + u_1, \gamma_h(E))$.

9.8 The theorem of Wiener

The next result is a characterization of the regular boundary points of an open set.

We remark that, if $n \ge 3$, then for every compact set $E \subseteq \mathbf{R^n}$, we have $\gamma_h(E) > 0$ and, hence, $C_h(E) = \frac{1}{\gamma_h(E)}$. If $n = 2$ and $0 < \rho < 1$, then, for every compact set $E \subseteq \overline{B(x_0;\rho)}$, we have $\gamma_h(E) \ge \gamma_h(\overline{B(x_0;\rho)}) = h_*(\rho) > 0$ and, again, $C_h(E) = \frac{1}{\gamma_h(E)}$.

Theorem 9.8 *(N. Wiener) Let Ω be an open set, $x_0 \in \partial\Omega \cap \mathbf{R^n}$ and*

$$E_k^\rho \;=\; \{x \notin \Omega : \rho^{k+1} \le |x - x_0| \le \rho^k\} \;,$$

for all $k \in \mathbf{N}$, where ρ is any number with $0 < \rho < 1$.

Then, x_0 is a regular boundary point of Ω if and only if

$$\sum_{k=1}^{+\infty} h_*(\rho^k) C_h(E_k^\rho) \;=\; \sum_{k=1}^{+\infty} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)} \;=\; +\infty \;.$$

Proof:
1. At first, we observe that, for every pair of ρ and ρ' in $(0,1)$, the two equalities $\sum_{k=1}^{+\infty} h_*(\rho^k) C_h(E_k^\rho) = +\infty$ and $\sum_{k=1}^{+\infty} h_*(\rho'^k) C_h(E_k^{\rho'}) = +\infty$ are equivalent.

We suppose that $0 < \rho' < \rho < 1$ and it is trivial to see that every $E_k^{\rho'}$ is contained in a finite number $m(\rho,\rho')$ of consecutive E_l^ρ's, where $m(\rho,\rho')$ depends only upon ρ and ρ'. Proposition 8.4 implies that $C_h(E_k^{\rho'})$ is not more than the sum of these consecutive $C_h(E_l^\rho)$'s. (In case $n=2$ it may be necessary to drop the first few k's so that all ring domains are contained in the disc $B(0;\frac{1}{2})$.) It is, also, trivial to see that the quantity $h_*(\rho'^k)$ is comparable to the correspoding $h_*(\rho^l)$'s. This, of course, means that the quotients are bounded both from above and from below by two constants depending only upon the number $m(\rho,\rho')$. From all this it is obvious that

$$\sum_{k=1}^{+\infty} h_*(\rho'^k) C_h(E_k^{\rho'}) \;\leq\; C(\rho,\rho') \sum_{l=1}^{+\infty} h_*(\rho^l) C_h(E_l^\rho) \; .$$

(Since consecutive $E_k^{\rho'}$'s may have one common E_l^ρ used to cover them, each term of the series in the right side of the above inequality is counted at most twice.)

It is also obvious that every E_k^ρ is contained in the union of at most two consecutive $E_l^{\rho'}$'s and, hence

$$\sum_{k=1}^{+\infty} h_*(\rho^k) C_h(E_k^\rho) \;\leq\; C'(\rho,\rho') \sum_{l=1}^{+\infty} h_*(\rho'^l) C_h(E_l^{\rho'}) \; ,$$

where each term of the series in the right side of the above inequality is counted at most $m(\rho,\rho')$ times.
2. Suppose that

$$\sum_{k=1}^{+\infty} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)} \;=\; +\infty$$

and consider the h-equilibrium measure $d\mu_k$ of E_k^ρ and its h-potential

$$U_h^{d\mu_k}(x) \;=\; \int_{E_k^\rho} h_x(y)\, d\mu_k(y) \; , \qquad x \in \mathbf{R^n} \; .$$

We know that

$$U_h^{d\mu_k}(x) \;\leq\; \gamma_h(E_k^\rho)$$

for every $x \in \mathbf{R^n}$ and we shall estimate $U_h^{d\mu_k}$ on every

$$A_m^\rho \;=\; \{x : \rho^{m+1} \leq |x - x_0| \leq \rho^m\} \; .$$

If $x \in A_m^\rho$, then

$$U_h^{d\mu_k}(x) \;\leq\; \begin{cases} h_*(\rho^{k+1} - \rho^m) \; , & \text{if } k < m-1 \\ \gamma_h(E_k^\rho) \; , & \text{if } m-1 \leq k \leq m+1 \\ h_*(\rho^{m+1} - \rho^k) \; , & \text{if } m+1 < k \; . \end{cases}$$

Moreover,

$$h_*(\rho^k) \;\le\; U_h^{d\mu_k}(x_0)\ .$$

Given an arbitrary $\epsilon > 0$ and an arbitrary $N \in \mathbf{N}$, there is a smallest $M \ge N$ so that

$$\frac{1}{\epsilon} \;\le\; \sum_{k=N}^{M} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)}\ .$$

By the minimality of M,

$$\sum_{k=N}^{M} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)} \;<\; \frac{1}{\epsilon} + \frac{h_*(\rho^M)}{\gamma_h(E_M^\rho)} \;\le\; \frac{1}{\epsilon} + 1\ .$$

Therefore, we have arbitrarily large integers N and corresponding $M \ge N$ so that

$$\frac{1}{\epsilon} \;\le\; \sum_{k=N}^{M} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)} \;\le\; \frac{1}{\epsilon} + 1\ .$$

Define, now,

$$U \;=\; \epsilon \sum_{k=N}^{M} \frac{1}{\gamma_h(E_k^\rho)}\, U_h^{d\mu_k}\ .$$

Then,

$$U(x_0) \;\ge\; \epsilon \sum_{k=N}^{M} \frac{h_*(\rho^k)}{\gamma_h(E_k^\rho)} \;\ge\; 1\ .$$

Besides the parameters ρ, N, M we introduce, now, an integer $l \in \mathbf{N}$ having in mind the following. If $n = 2$, then we fix $\rho = \frac{1}{2}$ and $l = 1$ and N will depend upon ϵ in a manner that will be made precise in a moment. If $n \ge 3$, all these parameters will depend upon ϵ and we shall shortly see how.

If $x \in A_m$, trivial estimates show that

$$\begin{aligned}
U(x) \;&\le\; \epsilon \sum_{N \le k \le M,\, k < m-l} \frac{h_*(\rho^{k+1} - \rho^m)}{\gamma_h(E_k^\rho)} + \epsilon \sum_{N \le k \le M,\, m-l \le k \le m+l} 1 \\
&\quad + \epsilon \sum_{N \le k \le M,\, m+l < k} \frac{h_*(\rho^{m+1} - \rho^k)}{\gamma_h(E_k^\rho)} \\
&\le\; \begin{cases} 3\epsilon + (1+\epsilon)\big(1 + \frac{2}{N}\big)\ , & \text{if } n = 2 \\ (2l+1)\epsilon + (1+\epsilon)\big(\frac{1}{\rho^{n-2}} + \rho^{l(n-2)}\big)\frac{1}{(1-\rho^l)^{n-2}}\ , & \text{if } n \ge 3. \end{cases}
\end{aligned}$$

If $N \ge m + l + 1$, then in the estimate of $U(x)$ only the third sum exists and, hence,

$$U(x) \;\le\; \begin{cases} \epsilon \sum_{k=N}^{M} \frac{(m+2)\log 2}{\gamma_h(E_k^\rho)} \;\le\; (1+\epsilon)\frac{m+2}{N} \;\le\; \frac{6}{N \log 2} \log \frac{1}{|x - x_0|}\ , & \text{if } n = 2, \\ (1+\epsilon)\frac{\rho^{l(n-2)}}{(1-\rho^l)^{n-2}}\ , & \text{if } n \ge 3. \end{cases}$$

3. To prove that x_0 is a regular boundary point of Ω we shall examine the function

$$u \;=\; H^{\Omega\cap B(x_0;\rho)}_{1-|\cdot\;-x_0|}$$

in $\Omega \cap B(x_0;\rho)$.

Since the function $1-|\cdot\;-x_0|$ is superharmonic, we have that $u \le 1-|\cdot\;-x_0|$ in $\Omega\cap B(x_0;\rho)$. Therefore, to prove that $1-u$ is a barrier of Ω at x_0, it is enough to prove that

$$\liminf_{\Omega\ni x\to x_0} u(x) \;\ge\; 1\;.$$

We shall compare u with the function

$$V \;=\; \frac{1}{U(x_0)}\,U\;.$$

We consider the case $n=2$ first.

Then $V(x_0)=1$ and we take $N \ge \frac{6}{\epsilon}$.

1. $V(x) \le 1+\frac{14}{3}\epsilon$, if $x \in B(x_0;\frac{1}{2})\setminus\{x_0\}$ and

2. $V(x) \le \frac{12}{N}\log\frac{1}{|x-x_0|} \le 2\epsilon\frac{1-|x-x_0|}{|x-x_0|}$, if $x\in A_m$ and $m+2\le N$.

Trivial estimates show that the function $\frac{1}{1+10\epsilon}V$, which is harmonic in $\Omega\cap B(x_0;\frac{1}{2})$, satisfies

$$\frac{1}{1+10\epsilon}V \;\le\; 1-|\cdot\;-x_0|$$

there. Therefore,

$$\frac{1}{1+10\epsilon}V \;\le\; u$$

in $\Omega\cap B(x_0;\frac{1}{2})$, implying that

$$\liminf_{\Omega\ni x\to x_0} u(x) \;\ge\; \frac{1}{1+10\epsilon}\;.$$

Since ϵ is arbitrary, we find

$$\lim_{\Omega\ni x\to x_0} u(x) \;\ge\; 1\;.$$

Now, let $n\ge 3$.

We take $l=[\epsilon^{-\frac{2}{3}}]$ and $\rho = 1-\epsilon^{\frac{1}{3}}$.

We, again, have $V(x_0)=1$ and, for a constant C depending only upon n,

1. $V(x)\le 1+C\epsilon^{\frac{1}{3}}$, if $x\in B(x_0;\rho)\setminus\{x_0\}$ and

2. $V(x)\le C\exp(-(n-2)\epsilon^{-\frac{1}{3}})$, if $x\in A_m$ and $m+l+1\le N$.

Now, the function $V - 2C\epsilon^{\frac{1}{3}}$ is harmonic in $\Omega \cap B(x_0;\rho)$ and satisfies

$$V(x) - 2C\epsilon^{\frac{1}{3}} \ \le\ 1 - |x - x_0|$$

for every $x \in \Omega \cap B(x_0;\rho)$, provided that we take $N \ge C'\epsilon^{-\frac{2}{3}}$ for some constant C' depending only upon n.

Indeed, if $|x - x_0| \le C\epsilon^{\frac{1}{3}}$, then this inequality is immediate from 1. above. If $|x - x_0| \ge C\epsilon^{\frac{1}{3}}$, then $x \in A_m$ for some $m \le C''\epsilon^{-\frac{1}{3}} \log\bigl(\frac{1}{C\epsilon^{\frac{1}{3}}}\bigr)$. Therefore, the already stated choice of N together with 2. finish the proof of the inequality.

This, now, implies that

$$V(x) - 2C\epsilon^{\frac{1}{3}} \ \le\ u(x)$$

for every $x \in \Omega \cap B(x_0;\rho)$ and

$$\liminf_{\Omega \ni x \to x_0} u(x) \ \ge\ 1 - 2C\epsilon^{\frac{1}{3}} \ .$$

Since ϵ is arbitrary, we find

$$\lim_{\Omega \ni x \to x_0} u(x) \ \ge\ 1 \ .$$

This proves that x_0 is a regular boundary point of Ω. 4. Consider, now, for all n the particular value $\rho = \frac{1}{2}$ and the sets $E_k = E_k^{\frac{1}{2}}$.

Suppose that $\sum_{k=1}^{+\infty} \frac{h_*(\frac{1}{2^k})}{\gamma_h(E_k)} < +\infty$ and let $K \ge 2$ be such that

$$\sum_{k=K}^{+\infty} \frac{h_*(\frac{1}{2^{k+1}})}{\gamma_h(E_k)} \ <\ 1 \ .$$

We define the bounded open set

$$\widetilde{\Omega} \ =\ B(x_0;\frac{1}{2}) \setminus \{x_0\} \setminus \cup_{k=K}^{+\infty} E_k \ .$$

Since $\widetilde{\Omega} \cap B(x_0;\frac{1}{2^K}) \ =\ \Omega \cap B(x_0;\frac{1}{2^K})$, to prove that x_0 is not a regular boundary point of Ω it is enough to prove that it is not a regular boundary point of $\widetilde{\Omega}$.

For each k with $\gamma_h(E_k) < +\infty$, let $d\mu_k$ be the h-equilibrium measure of E_k and let $f(x) = 1 - 2|x - x_0|$ for all $x \in \partial\widetilde{\Omega}$ so that

1. $0 \le f \le 1$ in $\partial\widetilde{\Omega}$,

2. $f(x_0) = 1$ and

3. $f = 0$ in $\partial\widetilde{\Omega} \setminus \{x_0\} \setminus \bigcup_{k=0}^{+\infty} E_k$.

We define the function

$$V \;=\; \sum_{k=K}^{+\infty} \max_{\partial\widetilde{\Omega}\cap E_k} f \frac{1}{\gamma_h(E_k)}\, U_h^{d\mu_k}\;,$$

where we simply omit all terms with $\gamma_h(E_k) < +\infty$.

V is a superharmonic function which is harmonic in $\widetilde{\Omega}$ and non-negative and bounded from above in $\overline{B(x_0;\frac{1}{2})}$, since, for all $m \geq 1$ and all $x \in A_m = A_m^{\frac{1}{2}}$,

$$\begin{aligned} V(x) \;&\leq\; \sum_{K\leq k, k\leq m-2} \frac{h_*(\frac{1}{2^{k+1}}-\frac{1}{2^m})}{\gamma_h(E_k)} + \sum_{K\leq k, m-1\leq k\leq m+1} 1 \\ &\quad + \sum_{K\leq k, m+2\leq k} \frac{h_*(\frac{1}{2^{m+1}}-\frac{1}{2^k})}{\gamma_h(E_k)} \\ &\leq\; \sum_{K\leq k, k\leq m-2} \frac{h_*(\frac{1}{2^{k+2}})}{\gamma_h(E_k)} + 3 + \sum_{K\leq k, m+2\leq k} \frac{h_*(\frac{1}{2^k})}{\gamma_h(E_k)} \\ &\leq\; 3 + \sum_{k=K}^{+\infty} \frac{h_*(\frac{1}{2^{k+2}})}{\gamma_h(E_k)} < +\infty\;. \end{aligned}$$

Moreover,

$$V(x_0) \;\leq\; \sum_{k=K}^{+\infty} \frac{h_*(\frac{1}{2^{k+1}})}{\gamma_h(E_k)} \;<\; 1\;.$$

Now, let v be subharmonic and bounded from above in $\widetilde{\Omega}$ with

$$\limsup_{\widetilde{\Omega}\ni x\to y} v(x) \;\leq\; f(y)$$

for all $y \in \partial\widetilde{\Omega}$.

Since all terms in the sum defining V are non-negative in $\overline{B(x_0;\frac{1}{2})}$ and since $U_h^{d\mu_k} = \gamma_h(E_k)$ quasi-almost everywhere in E_k, it is clear that

$$\limsup_{\widetilde{\Omega}\ni x\to y} v(x) \;\leq\; V(y) \;\leq\; \liminf_{\widetilde{\Omega}\ni x\to y} V(x)$$

for all $y \in \partial\widetilde{\Omega}\setminus\{x_0\}$ except for a boundary subset of at most zero harmonic measure with respect to $\widetilde{\Omega}$. Theorem 3.5 implies that

$$v \;\leq\; V$$

in $\widetilde{\Omega}$.

Therefore,

$$H_f^{\widetilde{\Omega}} \;\leq\; V$$

in $\widetilde{\Omega}$.

We assume that x_0 is a regular boundary point of $\widetilde{\Omega}$ and we shall arrive at a contradiction.

From $\lim_{\widetilde{\Omega}\ni x\to x_0} H_f^{\widetilde{\Omega}}(x) = f(x_0) = 1$ we get $1 \leq \liminf_{\widetilde{\Omega}\ni x\to x_0} V(x)$. This, immediately, implies

$$1 \;\leq\; \liminf_{k\to+\infty} \frac{1}{m(A_k\setminus E_k)} \int_{A_k\setminus E_k} V(x)\; dm(x)\;.$$

By the superharmonicity of V,

$$\lim_{k\to+\infty} \frac{1}{m(A_k)} \int_{A_k} V(x)\; dm(x) \;=\; V(x_0)\;.$$

The last two relations together with the

$$\begin{aligned}
\frac{1}{m(A_k)} \int_{A_k} V(x)\; dm(x) \;&\geq\; \frac{1}{m(A_k)} \int_{A_k\setminus E_k} V(x)\; dm(x) \\
&=\; \left(1 - \frac{m(E_k)}{m(A_k)}\right) \frac{1}{m(A_k\setminus E_k)} \int_{A_k\setminus E_k} V(x)\; dm(x)
\end{aligned}$$

imply that

$$\liminf_{k\to+\infty} \frac{m(E_k)}{m(A_k)} \;\geq\; 1 - V(x_0) \;>\; 0\;.$$

By the definition of capacity,

$$\begin{aligned}
\gamma_h(E_k) \;&\leq\; \frac{1}{m(E_k)^2} \int_{E_k}\int_{E_k} h_*(|x-y|)\; dm(x)dm(y) \\
&\leq\; \frac{1}{m(E_k)^2} \int_{A_k}\int_{A_k} h_*(|x-y|)\; dm(x)dm(y) \\
&=\; \begin{cases} \frac{1}{2^{k(n+2)}m(E_k)^2} \int_{A_0}\int_{A_0} h_*(|x-y|)\; dm(x)dm(y)\;, & \text{if } n \geq 3 \\ \frac{1}{16^k m(E_k)^2} \int_{A_0}\int_{A_0} h_*(|x-y|)\; dm(x)dm(y) & \\ +\frac{k\log 2}{16^k m(E_k)^2} m(A_0)^2\;, & \text{if } n = 2\;. \end{cases} \\
&\leq\; C\frac{h_*(\frac{1}{2^k})}{2^{2kn} m(E_k)^2}\;,
\end{aligned}$$

for some constant $C > 0$ depending only upon n.

Therefore, for some other constant C' depending only upon n,

$$\frac{m(E_k)^2}{m(A_k)^2} \;\leq\; C'\frac{h_*(\frac{1}{2^k})}{\gamma_h(E_k)}\;.$$

This contradicts the convergence of the series $\sum_{k=1}^{+\infty} \frac{h_*(\frac{1}{2^k})}{\gamma_h(E_k)}$.

9.9 Thin Sets

Definition 9.5 *A set $E \subseteq \mathbf{R^n}$ is called* ***thin at*** $x_0 \in \mathbf{R^n}$*, if either x_0 is not an accumulation point of E or x_0 is an accumulation point of E and there exists a superharmonic function u in $\mathbf{R^n}$ so that $u(x_0) < \liminf_{E\setminus\{x_0\}\ni x\to x_0} u(x)$.*

Theorem 9.9 *Suppose $E \subseteq \mathbf{R^n}$ and $x_0 \in \mathbf{R^n}$. Let*

$$E_k \;=\; \{x \in E : \frac{1}{2^{k+1}} \leq |x - x_0| \leq \frac{1}{2^k}\} \;,$$

for all $k \in \mathbf{N}$. Then, the following are equivalent:
1. E is thin at x_0.
2. $\sum_{k=1}^{+\infty} h_*(\frac{1}{2^k}) C_h^o(E_k) < +\infty$.

Proof:
If x_0 is not an accumulation point of E, then E is, automatically, thin at x_0 and the series in 2 converges, since, then, E_k is empty for all large k. We assume, therefore, that x_0 is an accumulation point of E.

Suppose that the series in 2 converges. For every $k \in \mathbf{N}$, we take open sets O_k so that

$$E_k \;\subseteq\; O_k \;\subseteq\; B\big(x_0; \frac{1}{2^{k+2}}, \frac{1}{2^{k-1}}\big)$$

and

$$\sum_{k=1}^{+\infty} h_*(\frac{1}{2^k}) C_h(O_k) \;<\; +\infty \;.$$

We know, from the proof of Theorem 9.6, that there exist probability Borel measures $d\mu_k$ supported in $\overline{O_k}$ so that

$$U_h^{d\mu_k}(x) \;=\; \frac{1}{C_h(O_k)}$$

for every $x \in O_k$. It is easy to see that

$$U_h^{d\mu_k}(x_0) \;\leq\; h_*(\frac{1}{2^{k+2}}) \;.$$

We, now, take $K \geq 2$ and define the function

$$u_0 \;=\; \sum_{k=K}^{+\infty} C_h(O_k) U_h^{d\mu_k} \;.$$

Since $K \geq 2$, all terms in this series are non-negative in $B(x_0; \frac{1}{2})$ and we, also, have that the series converges at x_0. Hence, u_0 is superharmonic in $B(x_0; \frac{1}{2})$ and

$$u_0(x) \;\geq\; 1$$

for every $x \in E \cap B(x_0; 0, \frac{1}{2^K})$. Taking K large enough we, also, have $u_0(x_0) < 1$. Thus,

$$u_0(x_0) \;<\; \liminf_{E\setminus\{x_0\}\ni x\to x_0} u_0(x) \ .$$

Applying Lemma 9.4, we prove that E is thin at x_0.

Now, suppose that E is thin at x_0 and take u superharmonic in $\mathbf{R^n}$ so that

$$u(x_0) \;<\; \liminf_{E\setminus\{x_0\}\ni x\to x_0} u(x) \ .$$

Consider λ so that

$$u(x_0) \;<\; \lambda \;<\; \liminf_{E\setminus\{x_0\}\ni x\to x_0} u(x)$$

and K large enough in order to have

$$u(x) \;>\; \lambda$$

for every $x \in E \cap B(x_0; 0, \frac{1}{2^{K-1}})$.

Define

$$O \;=\; \{x \in \mathbf{R^n} : u(x) > \lambda\}$$

and

$$O_k \;=\; O \cap B(x_0; \frac{1}{2^{k+2}}, \frac{1}{2^{k-1}}) \ .$$

Assume that

$$\sum_{k=1}^{+\infty} h_*(\frac{1}{2^k}) C_h^o(E_k) \;=\; +\infty$$

and, hence,

$$\sum_{k=1}^{+\infty} h_*(\frac{1}{2^k}) C_h(O_k) \;=\; +\infty \ .$$

Take, for each $k \geq K$, compact sets $F_k \subseteq O_k$ so that

$$\sum_{k=K}^{+\infty} h_*(\frac{1}{2^k}) C_h(F_k) \;=\; +\infty \ .$$

Consider the compact set

$$F \;=\; \{x_0\} \cup \bigcup_{k=K}^{+\infty} F_k$$

and its complement $\Omega = \mathbf{R^n} \setminus F$.

By Theorem 9.8, x_0 is a regular boundary point of Ω.

On the other hand, the superharmonic u is $\geq \lambda$ everywhere in F. If we consider the function

$$f(x) \;=\; \begin{cases} \lambda \;, & \text{if } x \in \partial\Omega \cap \overline{B(x_0; \frac{1}{2^{K+1}})} \;, \\ m \;, & \text{if } x \in \Omega \cap S(x_0; \frac{1}{2^K}) \;, \\ 2^{K+1}(m-\lambda)|x-x_0| + 2\lambda - m \;, & \text{if } x \in \partial\Omega \cap \overline{B(x_0; \frac{1}{2^{K+1}}, \frac{1}{2^K})} \;, \end{cases}$$

where $m = \min_{y \in S(x_0; \frac{1}{2^K})} u(y)$, then f is a continuous boundary function of the open set $\Omega_K = \Omega \cap B(x_0; \frac{1}{2^K})$ and

$$u \;\geq\; f$$

everywhere in $\partial\Omega_K$ except at the point x_0. Since $\{x_0\}$ is of zero harmonic measure with respect to Ω_K, this implies that

$$u \;\geq\; H_f^{\Omega_K}$$

everywhere in Ω_K. Therefore,

$$\lambda \;=\; f(x_0) \;=\; \lim_{\Omega_K \ni x \to x_0} H_f^{\Omega_K}(x) \;\leq\; \liminf_{\Omega_K \ni x \to x_0} u(x) \;=\; u(x_0)$$

and we get a contradiction.