

# EXTENDIBLE CHARACTERS AND MONOMIAL GROUPS OF ODD ORDER

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ABSTRACT. Let  $G$  be a finite  $p$ -solvable group, where  $p$  is an odd prime. We establish a connection between extendible irreducible characters of subgroups of  $G$  that lie under monomial characters of  $G$  and nilpotent subgroups of  $G$ . We also provide a way to get “good” extendible irreducible characters inside subgroups of  $G$ . As an application, we show that every normal subgroup  $N$  of a finite monomial odd  $p, q$ -group  $G$ , that has nilpotent length less than or equal to 3, is monomial.

## 1. INTRODUCTION

A finite group  $G$  is called monomial ( $M$ -group) if each of its complex irreducible characters can be induced from some linear character of some subgroup of  $G$ . One of the outstanding questions in the theory of monomial groups is

**Question 1.** *Is a normal subgroup  $N$  of an  $M$ -group  $G$  itself an  $M$ -group?*

It was shown by Dornhoff [5] and independently by Seitz [18] that the answer is yes when  $N$  is a normal Hall subgroup of  $G$ , and conjectured that the answer is always yes. For  $M$ -groups of even order, Dade [3] and van der Waall [19] showed separately that the answer could be no. They constructed an example of a monomial group of order  $7 \cdot 2^9$  which has a normal subgroup of index two that is not monomial. In their common example both  $N$  and  $G/N$  have even order. So Question 1 remains open when  $N$  or  $G/N$  or both have odd order. We remark, that there has been evidence (see for example [8, 10, 13, 16, 17] ) suggesting that the answer to the above question is yes if  $G$  is an odd  $M$ -group.

In [14] we prove

**Theorem A.** *If  $G$  is a monomial group of order  $p^a q^b$ , where  $p$  and  $q$  are odd primes and  $a, b$  are non-negative integers, then any normal subgroup  $N$  of  $G$  is again monomial.*

That is, for monomial odd  $p^a q^b$ -groups  $G$  the answer to Question 1 is yes.

For the proof of Theorem A a special type of reductions was followed. These reductions are based on an observation of M. Isaacs, according to which the Clifford theory for abelian normal subgroups  $L$  of a group  $G$  preserves monomiality of characters (see exercise (6.11) in [7]). Firstly we fix a monomial group  $G$ , a normal subgroup  $N$  of  $G$  and an irreducible character  $\psi \in \text{Irr}(N)$ . This way we form the triple  $T = (G, N, \psi)$ . Now we apply Isaacs observation to normal subgroups  $L$  of  $G$  that are contained in  $N$  and linear characters of  $L$  that lie under  $\psi$ . In particular, if  $L$  is any

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normal subgroup of  $G$  contained in  $N$ , and  $\lambda$  is a linear character of  $L$  lying under  $\psi$ , then we may pass from  $G$  to the stabilizer  $G(\lambda)$  of  $\lambda$  in  $G$ , without losing the monomiality of those irreducible characters of  $G(\lambda)$  that lie above  $\lambda$ . This way we get a new triple  $T_1 = (G(\lambda), N(\lambda), \psi_\lambda)$ , that we call a *linear reduction of  $T$* , where  $\psi_\lambda \in \text{Irr}(N(\lambda))$  is the  $\lambda$ -Clifford correspondent of  $\psi$ , and thus induces  $\psi$ . Clearly  $G(\lambda)$  may not be a monomial group, but every irreducible character of  $G(\lambda)$  lying above  $\lambda$  is still monomial. Repeated applications of the same type of reductions leads to a “minimal” triple  $T' = (G', N', \psi')$ , where  $N' \trianglelefteq G' \leq G$  and  $\psi' \in \text{Irr}(N')$  induces  $\psi$  to  $N$ , and where no more reductions can be performed to  $T'$  (a more detailed analysis on the triples and their reductions is given in Section 3 below). We call  $T'$  a *linear limit* of  $T$ . If  $Z(T')$  is the center of the induced character  $(\psi')^{G'}$ , then there is a unique linear character  $\zeta'$  of  $Z(T')$  lying under  $(\psi')^{G'}$ . We call  $\zeta'$  the *central character* of the triple  $T'$  and  $Z(T')$  the *center* of  $T'$ . Furthermore, Isaacs observation implies (see Proposition 3.17 below) that every irreducible character in  $\text{Irr}(G')$  that lies above  $\zeta'$  is monomial. Also the kernel  $\text{Ker}(\zeta')$  of  $\zeta'$  is a normal subgroup of  $G'$ , while  $\text{Ker}(\zeta') \leq \text{Ker}(\psi')$ . The question partially answered in [14] is

**Question 2.** *Assume that  $N$  is a normal subgroup of an  $M$ -group  $G$ . Let  $\psi$  be any irreducible character of  $N$ . Does there exist a linear limit  $(G', N', \psi')$  of  $(G, N, \psi)$  with the quotient group  $N'/(\text{Ker } \zeta')$  nilpotent?*

It is clear that a positive answer to Question 2 implies a positive answer to Question 1. What we actually prove in [14] is that Question 2 has a positive answer when  $G$  is an odd  $p, q$ -group.

In this paper we publish two of the main tools, Theorems B and D below, needed for the proof of Theorem A, that we think, are interesting in themselves, and provide an explanation of the approach we have used in [14]. As an easy consequence of these two theorems we prove that Question 2 has a positive answer when  $G$  is an odd  $p, q$ -group and  $N$  is a normal subgroup of nilpotent length  $\leq 3$ , (Theorems C and E).

For the general case we used in [14], apart from Theorems B and D, what we called there “triangular sets”. This is quite a complicated machinery. Fortunately E. C. Dade came up with an easier correspondence than the one the triangular sets provide, thus we are able to prove the general case without their use as we will see in a forthcoming paper.

When applying the linear reductions described above, we often reach a situation where  $G$  satisfies the following

**Condition X.**  *$P$  is a normal  $p$ -subgroup of  $G$ , for some odd prime  $p$ , such that its center  $Z(P)$  is maximal among the abelian  $G$ -invariant subgroups of  $P$ . Furthermore,  $\zeta \in \text{Irr}(Z(P))$  is a  $G$ -invariant faithful irreducible character of  $Z(P)$ , and thus  $Z(P)$  is a cyclic central subgroup of  $G$ .*

In particular, suppose that  $P$  is a normal subgroup of  $G$  and  $\alpha \in \text{Irr}(P)$ . Let  $(G', P', \alpha')$  be a linear limit of  $(G, P, \alpha)$ , and assume that  $\zeta'$  is the central character of  $(G', P', \alpha')$ . Then the groups  $G'/\text{Ker}(\zeta')$  and  $P'/\text{Ker}(\zeta')$  satisfy Condition X (see Proposition 3.7 below). Assume further that  $N \trianglelefteq G$  contains  $P$  while  $N/P$  is a  $q$ -group for some prime  $q \neq p$ . In order to answer Question 2 in this special case, it would be enough to show that a  $q$ -Sylow subgroup  $Q'$  of  $G' \cap N'$  satisfies  $[Q', P'] \leq \text{Ker}(\zeta')$ . This is actually true, and it follows from the fact that every irreducible character of  $G'$  lying above  $\zeta'$  is monomial. Theorem B handles this situation.

**Theorem B.** *Assume that  $G$  is a finite  $p$ -solvable group where  $p$  is some odd prime. Let  $P \trianglelefteq G$  be a normal  $p$ -subgroup of  $G$  that along with  $\zeta \in \text{Irr}(Z(P))$  satisfies Condition X. Assume further that  $S \trianglelefteq G$  is a nilpotent normal  $p'$ -subgroup of  $G$  and  $\beta \in \text{Irr}(S)$ . Let  $\chi \in \text{Irr}(G)$  be an irreducible monomial character of  $G$ , that lies above  $\zeta \times \beta$  and satisfies  $\chi(1)_{p'} = \beta(1)$ . If  $Q$  is any  $p'$ -subgroup of  $G$  such that  $PQ \trianglelefteq G$ , then  $Q$  centralizes  $P$ .*

Based on Theorem B we actually show

**Theorem C.** *Assume that  $G$  is a finite monomial group. Assume further that  $G$  has normal subgroups  $M \leq N$  such that  $M$  is nilpotent with odd order and  $N/M$  is nilpotent. Then the answer to both Questions 1 and 2 above is yes.*

Note that in Theorem C the group  $G$  need not be a  $p, q$ -group, not even odd.

Now assume that  $N$  has nilpotent length 3. In particular, assume that  $Q \trianglelefteq M \trianglelefteq N$  are all normal subgroups of  $G$  with  $Q$  being a  $q$ -group,  $M/Q$  a  $p$ -group and  $N/M$  being a  $q$ -group, for two odd primes  $p \neq q$ . By induction we may assume that, after performing the necessary linear reductions, the group  $M$  is nilpotent. So the obstacle this time is of the form  $Q_1 \trianglelefteq P \times Q_1 \trianglelefteq P \times Q \trianglelefteq G$ , where now both  $G, P$  and  $G, Q_1$  satisfy Condition X and in addition, every irreducible character of  $G$  lying above two specific  $G$ -invariant characters  $\alpha \times \beta \in \text{Irr}(P \times Q_1)$  is monomial. Again, in order to answer Question 2 we need to show that  $Q$  centralizes  $P$ . But this time we can't so easily guarantee the existence of a monomial character of  $G$  with the correct degree. Observe that according to Theorem B we need a monomial character of  $G$  whose degree has the  $q$ -part equal to  $\beta(1)$ . If the character  $\beta$  extends to  $G$  then this problem is solved using some basic  $\pi$ -theory. Unfortunately, there is no reason for  $\beta$  to extend, but we can replace him with another "good" one as the following theorem shows.

**Theorem D.** *Let  $P$  be a  $p$ -subgroup, for some odd prime  $p$ , of a finite group  $G$ . Let  $Q_1, Q$  be  $q$ -subgroups of  $G$ , for some odd prime  $q \neq p$ , with  $Q_1 \leq Q$ . Assume that  $P$  normalizes  $Q_1$ , while  $Q$  normalizes the product  $P \cdot Q_1$ . Assume further that  $\beta$  is an irreducible character of  $Q_1$ . Then there exists an irreducible character  $\beta^\nu$  of  $Q_1$  such that*

$$\begin{aligned} P(\beta) &= P(\beta^\nu), \\ Q(\beta) &\leq Q(\beta^\nu) \text{ and } N_Q(P(\beta)) \leq Q(\beta^\nu), \\ &\beta^\nu \text{ extends to } Q(\beta^\nu). \end{aligned}$$

Theorem B along with Theorem D, enables us to prove:

**Theorem E.** *The answer to both Questions 1 and 2 is yes if  $G$  is an odd monomial  $p, q$ -group and  $N$  has nilpotent length 3.*

Section 2 below contains the proof of Theorem B that is the key step for the proof of Theorem C. The proof of Theorem C can be found in Section 4, while in Section 3 we go through the basic definitions and properties of linear limits and we state related theorems, needed for the proof of Theorems C and E. In sections 5 and 6 we prove Theorems D and E, respectively. All the groups of this paper are assumed to be finite. In addition, all the modules have finite dimension. The notation and terminology follows [7], with a few exceptions. That is, we write  $N_M(K)$  or  $N(M \text{ in } K)$  for

the normalizer of  $M$  in  $K$ , whenever  $M, K$  are subgroups of a finite group  $G$ . Also if  $\phi \in \text{Irr}(M)$  we denote by  $K(\phi)$  the stabilizer of  $\phi$  in  $K$ . In addition, we use the terminology of [2] when symplectic modules are concerned. So, if  $\mathcal{F}$  is any finite field of characteristic  $p$ , and  $G$  is any finite group, we say that a finite-dimensional  $\mathcal{F}G$ -module  $\mathcal{B}$  is a *symplectic*  $\mathcal{F}G$ -module if  $\mathcal{B}$  carries a symplectic bilinear form  $\langle \cdot, \cdot \rangle$  that is invariant by  $G$ . For any  $\mathcal{F}G$ -submodule  $\mathcal{S}$  of  $\mathcal{B}$ , we denote by  $\mathcal{S}^\perp := \{t \in \mathcal{B} \mid \langle \mathcal{S}, t \rangle = \{0\}\}$  the *perpendicular*  $\mathcal{F}G$ -submodule to  $\mathcal{S}$ . The  $\mathcal{F}G$  submodule  $\mathcal{S}$  of  $\mathcal{B}$  is called *isotropic* if  $\mathcal{S} \leq \mathcal{S}^\perp$ , and it is called *self-perpendicular* if  $\mathcal{S} = \mathcal{S}^\perp$ . We say that  $\mathcal{B}$  is *anisotropic* if it contains no non-trivial isotropic  $\mathcal{F}G$ -submodules. Furthermore,  $\mathcal{B}$  is called *hyperbolic* if it contains some self-perpendicular  $\mathcal{F}G$ -submodule  $\mathcal{S}$ .

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## 2. PROOF OF THEOREM B

We begin with an equivalent form of Theorem 3.2 in [2].

**Theorem 2.1.** *Suppose that  $\mathcal{F}$  is a finite field of odd characteristic  $p$ , that  $G$  is a finite  $p$ -solvable group, that  $H$  is a subgroup of  $p$ -power index in  $G$ , that  $\mathcal{B}$  is an anisotropic symplectic  $\mathcal{F}G$ -module and that  $\mathcal{S}$  is an  $\mathcal{F}G$ -submodule of  $\mathcal{B}$ . Then the  $G$ -invariant symplectic form on  $\mathcal{B}$  restricts to a  $G$ -invariant symplectic form on  $\mathcal{S}$ . If  $\mathcal{S}$ , with this form, restricts to a hyperbolic symplectic  $\mathcal{F}H$ -module  $\mathcal{S}|_H$ , then  $\mathcal{S} = 0$ .*

*Proof.* Since  $\mathcal{B}$  is symplectic and  $\mathcal{F}G$ -anisotropic, so is its  $\mathcal{F}G$ -submodule  $\mathcal{S}$ . Theorem 3.2 of [2], applied to  $\mathcal{S}$ , tells us that  $\mathcal{S}$  is  $\mathcal{F}G$ -hyperbolic if  $\mathcal{S}|_H$  is  $\mathcal{F}H$ -hyperbolic. In that case  $\mathcal{S}$  is both  $\mathcal{F}G$ -anisotropic and  $\mathcal{F}G$ -hyperbolic. So it must be 0.  $\square$

We can now prove Theorem B

*Proof.* In view of Condition X,  $Z(P)$  is a cyclic central subgroup of  $G$ , and it is maximal among the abelian subgroups of  $P$  that are normal in  $G$ . So every characteristic abelian subgroup of  $P$  is contained in  $Z(P)$  and thus is cyclic. Hence P. Hall's theorem (see Theorem 4.9 in [6]) implies that either  $P$  is an abelian group or it is the central product

$$(2.2a) \quad P = T \odot Z(P),$$

where  $T = \Omega_1(P)$  is an extra special  $p$ -group of exponent  $p$ , and

$$(2.2b) \quad T \cap Z(P) = Z(T).$$

In the case that  $P = Z(P)$  is an abelian group, Theorem B holds trivially, as  $P = Z(P) \leq Z(G)$  is centralized by  $G$ . Thus we may assume that  $P > Z(P)$  and (2.2) holds.

Since  $\chi$  lies above  $\zeta \times \beta \in \text{Irr}(Z(P) \times S)$ , Clifford's theorem implies the existence of a unique irreducible character  $\Psi$  of  $G(\beta) = G(\zeta \times \beta)$ , that also lies above  $\zeta \times \beta$  and induces  $\chi$  in  $G$ .

Furthermore, the hypothesis that  $\chi(1)_{p'} = \beta(1)$ , implies that  $|G : G(\beta)|$  is a power of  $p$ , since  $|G : G(\beta)|$  divides  $\chi(1)/\beta(1)$ . So we get

$$(2.3) \quad \Psi(1)_{p'} = (\Psi^G(1))_{p'} = \chi(1)_{p'} = \beta(1).$$

By hypothesis,  $\chi$  is a monomial character. Furthermore,  $Z(P) \times S$  is a nilpotent normal subgroup of  $G$ . Hence we can apply Theorem 3.1 in [17]. We conclude that  $\Psi \in \text{Irr}(G(\zeta \times \beta))$  is also monomial. Therefore there exists a subgroup  $H$  of  $G(\beta)$ , and a linear character  $\lambda \in \text{Lin}(H)$  that induces  $\Psi = \lambda^{G(\beta)}$ . Clearly  $\lambda$  also induces  $\chi = \lambda^G$ .

The product  $HS$  forms a subgroup of  $G$ . Furthermore,

**Claim 1.**  $|G : HS|$  is a power of  $p$ , and  $(\lambda^{HS})|_S = \beta$ .

*Proof.* Clearly Clifford's theorem implies

$$\Psi|_S = m \cdot \beta,$$

for some integer  $m \geq 0$ . Hence  $\deg(\Psi) = m \deg(\beta)$ . By (2.3) we have  $\Psi(1)_{p'} = \beta(1)$ . Thus  $m$  is a power of  $p$ . As  $H \leq HS \leq G(\beta)$ , the induced character  $\lambda^{HS}$  lies in  $\text{Irr}(HS)$  and induces  $(\lambda^{HS})^{G(\beta)} = \lambda^{G(\beta)} = \Psi$ . So

$$\deg(\lambda^{HS}) \cdot |G(\beta) : HS| = \deg(\Psi) = m \deg(\beta).$$

Clifford's theorem also implies that  $\lambda^{HS}|_S = r\beta$ , for some integer  $r$ . As  $\deg(\lambda^{HS}) = |HS : H| = |S : H \cap S|$  we get that both  $\deg(\lambda^{HS}|_S)$  and  $r$  are  $p'$ -numbers. But

$$r \deg(\beta) \cdot |G(\beta) : HS| = \deg(\lambda^{HS}) \cdot |G(\beta) : HS| = m \deg(\beta),$$

with  $m$  a  $p$ -number. Hence  $r = 1$ , while  $|G(\beta) : HS|$  is a power of  $p$ . This, along with the fact that  $G(\beta)$  has  $p$ -power index in  $G$ , completes the proof of the claim.  $\square$

The fact that  $\lambda \in \text{Lin}(H)$  induces irreducibly to  $G$  implies that the center,  $Z(G)$ , of  $G$  is a subgroup of  $H$ . This, along with the fact that  $Z(P) \leq Z(G)$ , implies

$$(2.4) \quad Z(P) \leq Z(G) \leq H.$$

Let  $Q$  be any  $p'$ -subgroup of  $G$  that satisfies  $N := PQ \trianglelefteq G$ . In order to show that  $Q$  centralizes  $P$ , we can, without loss, assume that  $S$  is a subgroup of  $Q$ , or else we may work with the  $p'$ -group  $QS$  that also satisfies  $P(QS) = (PQ)S \trianglelefteq G$ . Let  $E := [P, Q]$ . Then  $E$  is a characteristic subgroup of  $N$  and thus a normal subgroup of  $G$ . Furthermore  $S$  centralizes  $E$ , since  $E$  is a subgroup of  $P$ . Even more, we have

**Claim 2.**  $E = [P, Q]$  is an abelian group.

*Proof.* Suppose not. Then  $E$  is a non-abelian normal subgroup of  $G$  contained in  $P = T \cdot Z(P)$ . As  $Z(P) \leq Z(G)$  (by (2.4)), we have  $E = [P, Q] = [T, Q] \leq T$ , where  $T = \Omega_1(P)$ . Furthermore,  $Z(E)$  is an abelian normal subgroup of  $G$ , contained in  $T \leq P$ . Hence  $Z(E)$  is contained in  $T \cap Z(P) = Z(T)$ . As  $E$  is non-abelian and  $Z(T)$  has order  $p$ , we conclude that  $Z(E) = Z(T) \leq Z(P) \leq Z(G)$ . Therefore  $E = [T, Q]$  is an extra special subgroup of  $T$  of exponent  $p$ , and its center is central in  $G$ . Hence the group  $E$  satisfies condition (4.3a) in [2]. In addition,  $Q$  is a  $p'$ -subgroup of  $G$  such that  $QE$  is normal in  $G$  (as  $P = [P, Q]C_P(Q)$  and thus  $G = EN_G(Q)$ ). Since  $P$  is a  $p$ -group, the

commutator subgroup  $[E, Q] = [[P, Q], Q]$  coincides with  $E = [P, Q]$ . Hence (4.3b) in [2] holds with  $Q$  here, in the place of  $K$  there.

As the index of  $HS$  in  $G$  is a power of  $p$ , and  $PQ = N$  is a normal subgroup of  $G$ , we conclude that  $HS$  contains a  $p'$ -Hall subgroup of  $PQ$ . Hence  $HS$  contains a  $P$ -conjugate of  $Q$ . Therefore, we may replace  $H$  and  $\lambda$  by some  $P$ -conjugates, and assume that  $HS$  contains  $Q$ . (Observe that because  $P \leq G(\beta)$ , the  $P$ -conjugate of  $H$  is still a subgroup of  $G(\beta)$  while the corresponding  $P$ -conjugate of  $\lambda$  induces  $\Psi$  in  $G(\beta)$ .)

The subgroup  $H \cap (E \times S)$  of  $E \times S$  is equal to  $(H \cap E) \times (H \cap S)$ , since  $|E|$  and  $|S|$  are relatively prime. This implies that

$$HS \cap (E \times S) = (H \cap (E \times S)) \cdot S = (H \cap E) \times S.$$

Hence  $H \cap E = HS \cap E$ . Thus  $H \cap E$  is a normal subgroup of  $HS$ . Furthermore, the restriction  $\lambda|_{H \cap E}$  of  $\lambda$  to  $H \cap E$  is a linear character of  $H \cap E$  that is clearly  $H$ -invariant. It is also  $S$ -invariant, as  $S$  centralizes  $E \geq H \cap E$ . Hence  $\lambda|_{H \cap E}$  is  $HS$ -invariant. We conclude that the restriction of the irreducible character  $\lambda^{HS}$  of  $HS$  to  $H \cap E$  is a multiple of the linear character  $\lambda|_{H \cap E}$ . Of course the irreducible character  $\lambda^{HS}$  of  $HS$  induces irreducibly to  $\chi \in \text{Irr}(G)$ , and lies above a non-trivial character of  $Z(E)$  (as  $Z(E) \leq Z(P)$  and  $\zeta \in \text{Irr}(Z(P))$  is faithful). Hence we can apply Lemma (4.4) and its Corollary (4.8) of [2], using  $HS$  here in the place of  $H$  there, and  $\lambda^{HS}$  here in the place of  $\phi$  there. We conclude that  $HS \cap E = H \cap E$  is a maximal abelian subgroup of  $E$ .

Let  $\bar{P} := P/Z(P)$ . Then  $\bar{P}$  is a symplectic  $\mathbb{Z}_p G$ -module, since it affords the  $G$ -invariant symplectic bilinear form  $c$  defined as  $c(\bar{x}, \bar{y}) = \zeta([x, y])$ , for all  $x, y \in P$  (where  $\bar{x}$  and  $\bar{y}$  are the images of  $x, y$  in  $\bar{P}$ ). According to the hypotheses of the theorem,  $Z(P)$  is the maximal abelian  $G$ -invariant subgroup of  $P$ . Hence  $\bar{P}$  is an anisotropic  $\mathbb{Z}_p G$ -module. If  $\bar{E}$  is the image of  $E$  in  $\bar{P}$ , i.e.,  $\bar{E} \cong E/Z(E)$ , then  $\bar{E}$  is a symplectic  $\mathbb{Z}_p G$ -submodule of  $\bar{P}$ , as  $E$  is normal in  $G$ . Furthermore,  $\bar{E}$  is  $\mathbb{Z}_p HS$ -hyperbolic as  $HS \cap E$  is a maximal abelian  $HS$ -invariant subgroup of  $E$ . Since the index  $[G : HS]$  is a power of  $p$ , Theorem 2.1 forces  $\bar{E}$  to be trivial. Hence  $E = Z(E)$  is abelian, and the claim follows.  $\square$

Now  $E = [P, Q]$  is an abelian subgroup of  $P$  normal in  $G$ . According to the hypotheses of the theorem,  $Z(P)$  is a maximal such subgroup of  $P$ . Therefore  $1 \leq [P, Q] \leq Z(P) \leq Z(G)$ . So  $Q$  centralizes  $[P, Q]$ , which implies that  $[P, Q, Q] = 1$  and thus  $[P, Q] = [P, Q, Q] = 1$ .

This completes the proof of the theorem.  $\square$

An immediate consequence is

**Corollary 2.5.** *Assume that a finite  $p$ -solvable group  $G$  satisfies Condition X for its normal  $p$ -subgroup  $P$  and the character  $\zeta \in \text{Irr}(Z(P))$ . Assume further that  $\chi$  is a monomial character of  $G$  that lies above  $\zeta$  and its degree is a power of  $p$ . If  $Q$  is any  $p'$ -subgroup of  $G$  so that  $PQ \trianglelefteq G$  then  $Q$  centralizes  $P$ .*

## 3. LINEAR LIMITS

In this section we give the basic definitions and properties of ordered triples and their linear limits. A more detailed approach on the subject that actually contains most of the results that follow, can be found in [4].

We denote by  $\mathfrak{T}$  the family of all ordered triples  $(G, N, \psi)$ , where  $G$  is a finite group,  $N$  is a normal subgroup of  $G$  and  $\psi$  is an irreducible character of  $N$ . Until the end of the section we fix an element  $T = (G, N, \psi)$  of  $\mathfrak{T}$ . We define the *center*  $Z(T)$  of  $T$  to be the center of the induced character  $\psi^G$ , and the *central character*  $\zeta^{(T)}$  to be the unique linear character of  $Z(T)$  lying under  $\psi^G$ . Then  $Z(T)$  is a normal subgroup of  $G$  contained in  $N$ , while  $\zeta^{(T)}$  is a  $G$ -invariant linear character of  $Z(T)$ . So the restrictions of both  $\psi$  and  $\psi^G$  to  $Z(T)$  are multiples of  $\zeta^{(T)}$ . Even more,  $Z(T)$  is fully characterized by Proposition 2.3 in [4], that we partly restate here.

**Proposition 3.1.** *The center  $Z(T)$  is the largest normal subgroup  $L$  of  $G$  contained in  $N$  such that  $\psi \in \text{Irr}(N)$  lies over some  $G$ -invariant linear character  $\lambda$  of  $L$ . Any other such  $L$  is a subgroup of  $Z(T)$ , and the corresponding  $\lambda$  is the restriction of  $\zeta^{(T)}$  to  $L$ , while the restriction of  $\psi$  to  $L$  equals  $\psi(1)\lambda$ .*

We also define the *kernel*  $\text{Ker}(T)$  to be the kernel of  $\psi^G$ . So  $\text{Ker}(T)$  is  $\text{Core}_G(\text{Ker}(\psi))$ , and thus is contained in  $\text{Ker}(\psi)$ . In addition,  $\text{Ker}(T) = \text{Ker}(\zeta^{(T)})$  (see the second section in [4] for a detailed analysis of character triples).

A *subtriple* of  $T$  is any triple  $T' = (G', N', \psi')$  contained in  $\mathfrak{T}$ , with  $G'$  being a subgroup of  $G$ , and  $\psi'$  being any irreducible character of  $N' = G' \cap N$  lying under  $\psi$ . We write  $T' \leq T$  to denote that  $T'$  is a subtriple of  $T$ . If there exists some normal subgroup  $L$  of  $G$  contained in  $N$  and a linear character  $\lambda \in \text{Lin}(L)$  lying under  $\psi$ , then the stabilizer  $G' = G(\lambda)$  of  $\lambda$  in  $G$  is a subgroup of  $G$ , while  $N' = N \cap G'$  is the stabilizer  $N' = N(\lambda)$  of  $\lambda$  in  $N$ . Furthermore, the  $\lambda$ -Clifford correspondent  $\psi'$  of  $\psi$  in  $N'$ , is the unique irreducible character of  $N'$  lying above  $\lambda$  and inducing  $\psi$ . So the triple  $T(\lambda) = (G', N', \psi')$  is a subtriple of  $T$ . By a *linear reduction* of  $T$  we mean any subtriple  $T' \leq T$  of the form  $T' = T(\lambda)$ , for some  $L$  and  $\lambda$  satisfying the above conditions. A linear reduction  $T'$  is called *proper* if  $T' \neq T$ . If the only possible linear reduction of  $T$  is  $T$  itself, then  $T$  is called *linearly irreducible*. A subtriple  $T'$  of  $T$  is called a *multilinear reduction* of  $T$  if there is some finite chain  $T_0, T_1, \dots, T_n$  of subtriples  $T_i \leq T$ , starting with  $T_0 = T$  and ending with  $T_n = T'$ , such that each  $T_i$  is a linear reduction of  $T_{i-1}$ , for all  $i = 1, 2, \dots, n$ . A *linear limit* of  $T$  is a linearly irreducible multilinear reduction of  $T$ , i.e., it is a multilinear reduction of  $T$  that has no proper linear reductions.

If  $T' = T(\lambda)$  is a linear reduction of  $T$  then without loss (see Proposition 2.18 in [4]) we may assume that  $\lambda$  is a linear character of some normal subgroup  $L$  of  $G$  satisfying  $Z(T) \leq L \leq N$ . In addition,  $\lambda$  not only lies under  $\psi$  but also lies above the central character  $\zeta^{(T)}$  of  $T$ . Furthermore, for any multilinear reduction  $T'$  of  $T$  we have  $Z(T) \leq Z(T')$  while  $\zeta^{(T)}$  is the restriction of  $\zeta^{(T')}$  to  $Z(T)$ , see equation 2.24 in [4]. In addition  $\text{Ker}(T) = \text{Ker}(\zeta^{(T)}) \leq \text{Ker}(\zeta^{(T')}) = \text{Ker}(T')$ .

The first remarks follow easily from the above definitions.

**Remark 3.2.** If  $T' = (G', N', \psi')$  is a multilinear reduction of  $T = (G, N, \psi)$ , then any linear limit  $T''$  of  $T'$  is also a linear limit of  $T$ . In addition,  $Z(T'') \geq Z(T') \geq Z(T)$  and  $\text{Ker}(T'') \geq \text{Ker}(T') \geq \text{Ker}(T)$ .

**Remark 3.3.** Assume that  $H$  is a subgroup of  $G$  with  $N \leq H \leq G$ . Let  $S$  be the ordered triple  $S = (H, N, \psi)$ . If  $T' = (G', N', \psi')$  is a linear limit of  $T$  then  $S' = (G' \cap H, N', \psi')$  is a multilinear reduction of  $S$  with  $Z(T') \leq Z(S')$ . So we can get a linear limit  $S'' = (H'', N'', \psi'')$  of  $S'$ , and thus of  $S$ , with  $N'' \leq N'$  and  $Z(T') \leq Z(S') \leq Z(S'')$ . Furthermore, the central character  $\zeta^{(T')}$  of  $T'$  is the restriction of  $\zeta^{(S')}$ , and thus of  $\zeta^{(S'')}$ .

**Remark 3.4.** Let  $T' = (G', N', \psi')$  be a linear limit of  $T = (G, N, \psi)$ . If  $B$  is a subgroup of  $G$  that centralizes  $N$ , (that is  $B \leq C_G(N)$ ) then  $B \leq G'$ .

Now assume that  $N \leq H \leq G$ , while the irreducible character  $\theta \in \text{Irr}(H)$  of  $H$  lies above  $\psi \in \text{Irr}(N)$ . If  $T' = (G', N', \psi')$  is a multilinear reduction of  $T$ , then we form the group  $H' = H \cap G'$  that we call the  $T'$ -reduction of  $H$ . Since  $T'$  is a multilinear reduction of  $T$ , there is a chain of linear subtriples  $T_0, T_1, \dots, T_n$  of  $T$ , starting with  $T_0 = T$  and ending with  $T_n = T'$  such that  $T_i$  is a linear reduction of  $T_{i-1}$ , for all  $i = 1, \dots, n$ . So  $T_i = T_{i-1}(\lambda_i)$ , where  $\lambda_i$  is a linear character of some normal subgroup  $L_i$  of  $G_{i-1}$  with  $Z(T_{i-1}) \leq L_i \leq N_{i-1}$ . Furthermore,  $\psi_i \in \text{Irr}(N_i)$  is the  $\lambda_i$ -Clifford correspondent of  $\psi_{i-1} \in \text{Irr}(N_{i-1})$ . Because  $\theta \in \text{Irr}(H)$  lies above  $\psi$ , there is also a finite chain of irreducible characters  $\theta_i$  of  $H_i = G_i \cap H$ , starting with  $\theta_0 = \theta$  and ending with  $\theta_n = \theta'$ , such that  $\theta_i$  is the unique  $\lambda_i$ -Clifford correspondent of  $\theta_{i-1}$ , for all  $i = 0, 1, \dots, n$ . So  $\theta_i$  lies above  $\psi_i$ , for all  $i = 0, 1, \dots, n$ . We call  $\theta'$  the  $T'$ -reduction of  $\theta$ . It is clear from the definition of  $\theta'$  that it lies above  $\psi'$  and induces  $\theta$  in  $H$ . Actually  $\theta'$  is the unique irreducible character of  $H'$  with these properties by

**Remark 3.5.** Assume that  $N \leq H \leq G$ , and let  $T' = (G', N', \psi')$  be any linear reduction of  $T$ . If  $\phi$  is an irreducible character of  $H' = H \cap G'$  that lies above  $\psi'$  then  $\phi$  is the  $T'$ -reduction of the irreducible character  $\phi^H$  of  $H$ .

*Proof.* According to [4] induction is a bijection between the irreducible characters of  $H'$  lying above  $\psi'$  and those of  $H$  lying above  $\psi$ . Hence  $\phi^H$  is an irreducible character of  $H$  and its  $T'$ -reduction is  $\phi$ .  $\square$

If, in addition,  $H$  is a normal subgroup of  $G$ , then we clearly have

**Remark 3.6.** Assume that  $N \leq H$  are normal subgroups of  $G$ , while  $\theta \in \text{Irr}(H|\psi)$ . So we can form the triple  $S = (G, H, \theta)$ . Then  $Z(S) \geq Z(T)$  and  $\text{Ker}(S) \geq \text{Ker}(T)$ . If  $T' = (G', N', \psi')$  is a linear limit of  $T$ ,  $H' = H \cap G'$  and  $\theta' \in \text{Irr}(H')$  is the  $T'$ -reduction of  $\theta$ , then the triple  $S' = (G', H', \theta')$  is a multilinear reduction of  $S = (G, H, \theta)$ .

The following is Proposition 2.21 in [4].

**Proposition 3.7.** *Let  $T' = (G', N', \psi')$  be a linear limit of  $T$ . Then  $Z(T')/\text{Ker}(T')$  is the center of  $N'/\text{Ker}(T')$  and is a cyclic group, which affords a faithful  $G'/\text{Ker}(T')$ -invariant linear character that inflates to  $\zeta^{(T')} \in \text{Irr}(Z(T'))$ . Furthermore,  $Z(T')/\text{Ker}(T')$  is maximal among the abelian normal subgroups of  $G'/\text{Ker}(T')$  contained in  $N'/\text{Ker}(T')$ .*



If  $N$  is a nilpotent group then we can easily see

**Remark 3.8.** Assume that  $N$  is a nilpotent group. Let  $p_1, \dots, p_n$  be the distinct primes dividing  $|N|$ . Then  $N = N_1 \times \dots \times N_n$ , where  $N_i$  is the  $p_i$ -Sylow subgroup of  $N$ , for each  $i = 1, \dots, n$ . Let  $\psi = \psi_1 \times \dots \times \psi_n$  with  $\psi_i \in \text{Irr}(N_i)$  for  $i = 1, \dots, n$ , be the corresponding factorization of  $\psi$ . We write  $T_i$  for the triples  $T_i = (G, N_i, \psi_i)$ , for all  $i = 1, \dots, n$ . If  $T'_i = (G'_i, N'_i, \psi'_i)$  is a linear limit of  $T_i$  for all such  $i$ , then the group  $G' = G'_1 \cap \dots \cap G'_n$ , its normal subgroup  $N' = N'_1 \times \dots \times N'_n$  and the irreducible character  $\psi' = \psi'_1 \times \dots \times \psi'_n$  of  $N'$  form a linear limit  $T' = (G', N', \psi')$  of  $T$ . Furthermore, if  $Z'_i$  is the center  $Z(T'_i)$  of  $T'_i$ , and  $\zeta'_i \in \text{Irr}(Z'_i)$  the corresponding central character of  $T'_i$ , for all  $i = 1, \dots, n$ , then  $Z' = Z'_1 \times \dots \times Z'_n$  is the center  $Z(T')$  of  $T'$ , while  $\zeta' = \zeta'_1 \times \dots \times \zeta'_n$  is the central character of  $T'$ . In addition, the kernel of  $T'$  satisfies,  $\text{Ker}(T') = \text{Ker}(T'_1) \times \dots \times \text{Ker}(T'_n)$ .

Assume now that  $N/Z(T)$  is an abelian group. Then we can introduce an alternating bilinear form (see section 5 in [4])  $c$  from  $N/Z(T) \times N/Z(T)$  to the multiplicative group  $\mathbb{C}^\times$  of complex numbers, defined as

$$(3.9) \quad c(\bar{x}, \bar{y}) = \zeta^{(T)}([x, y]),$$

for all elements  $\bar{x}, \bar{y}$  of  $N/Z(T)$ , where  $x$  and  $y$  are pre-images of  $\bar{x}$  and  $\bar{y}$  respectively, in  $N$ . The action of  $G$  on  $N$  via conjugation makes  $c$  a  $G/N$ -invariant bilinear form. As in the introduction we define the *perpendicular subgroup*  $B^\perp$  for any subgroup  $B \leq N/Z(T)$ , to be

$$(3.10) \quad B^\perp = \{\bar{x} \in N/Z(T) \mid c(B, \bar{x}) = 1\}.$$

We also call any subgroup  $B$  of  $N/Z(T)$  *isotropic* if  $c(B, B) = 1$ , and we say that  $N/Z(T)$  is  *$G/N$ -anisotropic* if 1 is the only  $G/N$ -invariant isotropic subgroup of  $N/Z(T)$ . So the form  $c$  is non-singular if and only if the perpendicular subgroup of  $N/Z(T)$  equals 1, and in this case  $N/Z(T)$  becomes a symplectic  $G/N$ -group. Because the perpendicular subgroup of  $N/Z(T)$  is a  $G/N$ -invariant isotropic subgroup of  $N/Z(T)$ , if  $N/Z(T)$  is anisotropic as a  $G/N$ -group then it is also symplectic. Also, Proposition 5.2 and Proposition 5.8 in [4] imply

**Proposition 3.11.** *If  $N/Z(T)$  is a nilpotent group and  $T$  is linearly irreducible, then  $N/Z(T)$  is an abelian anisotropic  $G/N$ -group.*

This, along with Proposition 5.9 in [4], implies

**Proposition 3.12.** *Assume that  $T' = (G', N', \psi')$  is a linear limit of  $T$ , where  $N'/Z(T')$  is a nilpotent group. Then  $\psi'$  vanishes on  $N' - Z(T')$ , and is a multiple of  $\zeta^{(T')}$  on  $Z(T')$ . Hence  $G'(\psi') = G'$ .*

Assume now that the abelian factor group  $N/Z(T)$  is a symplectic  $G/N$ -group, i.e., assume that the form  $c$  defined above is non-singular. (As we noted above, if  $T$  is linearly irreducible then  $N/Z(T)$  is a symplectic  $G/N$ -group.) Then the following proposition suggests another way to look at linear limits of  $T$ .

**Proposition 3.13.** *Assume that  $N/Z(T)$  is an abelian and symplectic  $G/N$ -group. Assume further that  $T' = (G', N', \psi')$  is a linear limit of  $T$ . Then  $G = G'N$  and  $G' \cap N = N'$ . Thus inclusion  $G' \hookrightarrow G$  induces an isomorphism of  $G'/N'$  onto  $G/N$ . This isomorphism turns  $N'/Z(T')$  into a symplectic  $G/N$ -group. In addition,  $N'/Z(T')$  is naturally isomorphic, as a symplectic  $G/N$ -group,*

to the factor group  $L^\perp/L$ , where  $L$  is maximal among the  $G/N$ -invariant isotropic subgroups of  $N/Z(T)$ , and  $L^\perp$  is the perpendicular subgroup to  $L$  with respect to the bilinear form  $c$ . Furthermore,  $L = Z(T')/Z(T)$  while  $L^\perp = N'/Z(T)$ .

*Proof.* This is Proposition 5.23 in [4], combined with the earlier results of Propositions 5.18 and 5.22 in [4].  $\square$

**Proposition 3.14.** *Assume that  $T' = (G', N', \psi')$  is a multilinear reduction of  $T = (G, N, \psi)$ . If  $Z' = Z(T')$  is the center of  $T'$  and  $\zeta'$  the central character of  $T'$ , then  $G' = G(\zeta')$  and  $\psi'$  is the unique irreducible character of  $N' = N(\zeta')$  lying above  $\zeta'$  and inducing  $\psi$ . If in addition  $T'$  is a linear limit of  $T$  and  $N'/Z(T')$  is nilpotent then  $\psi'$  is the only character in  $\text{Irr}(N'|\zeta')$ .*

*Proof.* Let  $T(\lambda) = T_1 = (G_1, N_1, \psi_1)$  be a linear reduction of  $T$ , where  $\lambda$  is a linear character of a normal subgroup  $L$  of  $G$  contained in  $N$ . Then  $L$  is contained in the center  $Z(T_1)$  of  $T_1$ , see Proposition 2.3 in [4]. Hence  $L \leq Z(T_1) \leq N$ . Furthermore, the same proposition in [4] implies that  $\lambda$  is a restriction of the  $G_1$ -invariant linear character  $\zeta^{(T_1)}$  of  $Z(T_1)$ . Since  $L \trianglelefteq G$  we have

$$G(\zeta^{(T_1)}) \leq G(\lambda) = G_1 = G_1(\zeta^{(T_1)}) \leq G(\zeta^{(T_1)}).$$

Therefore,  $G_1 = G(\zeta^{(T_1)})$  which in turn implies that  $N_1 = N(\zeta^{(T_1)})$ . Furthermore, since  $\psi_1$  is the  $\lambda$ -Clifford correspondent of  $\psi$ , while  $\zeta^{(T_1)}$  is a  $G_1$ -invariant irreducible character of  $Z(T_1) \leq N_1$  lying above  $\lambda$  and under  $\psi_1$ , we conclude that  $\psi_1$  is the unique irreducible character of  $N_1$  that lies above  $\zeta^{(T_1)}$  and induces  $\psi$ . Hence the first part of the proposition holds for a linear reduction. A linear limit  $T'$  of  $T$  is a series of linear reductions with starting triple  $T = T_0$  and ending triple  $T'$ . Furthermore,  $Z(T) = Z(T_0) \leq Z(T_1) \leq \dots \leq Z(T_n) = Z(T')$ . Hence the first part of the proposition follows.

Assume now that  $T'$  is a linear limit of  $T$  while  $N'/Z(T')$  is a nilpotent group. Then Proposition 3.12 implies that  $\psi'$  is fully ramified with respect to  $N'/Z(T')$ . Hence, see Lemma 2.6 in [11], the unique irreducible character of  $N'$  lying above  $\zeta^{(T')}$  is  $\psi'$ . This completes the proof of the proposition.  $\square$

We conclude this section by proving that linear limits preserve monomial characters (see Proposition 3.17 below). Its proof is based on the following lemma, that is actually the exercise (6.11) in [7].

**Lemma 3.15.** *Let  $B$  be a normal subgroup of a finite group  $G$  and  $\gamma$  be a linear character of  $B$ . Assume further that  $\chi \in \text{Irr}(G|\gamma)$  is an irreducible character of  $G$  lying above  $\gamma$ . If  $\chi_\gamma \in \text{Irr}(G(\gamma))$  is the  $\gamma$ -Clifford correspondent of  $\chi$  in the stabilizer  $G(\gamma)$  of  $\gamma$  in  $G$ , then  $\chi$  is monomial if and only if  $\chi_\gamma$  is monomial.*

*Proof.* It is clear that if  $\chi_\gamma$  is monomial then  $\chi$  is monomial, as  $\chi_\gamma$  induces  $\chi$  in  $G$ .

So we assume that  $\chi$  is a monomial character, and we will show that  $\chi_\gamma$  is also monomial. Let  $K = \text{Ker}(\chi)$ . Of course  $K \trianglelefteq G$ . It is clear that  $\chi_\gamma$  is monomial if and only if the irreducible character  $\chi_\gamma/K$  of the factor group  $G(\gamma)/K$  that inflates to  $\chi_\gamma$ , is monomial. Hence it suffices to prove the lemma in the case of a faithful irreducible character  $\chi$ , as we can pass to the quotient groups  $G/K$  and  $(BK)/K$ . So in the rest of the proof we assume that  $K = 1$ .

Clifford's Theorem implies that the restriction  $\chi|_B$  of  $\chi$  to  $B$  is a sum of  $G$ -conjugates of  $\gamma$ . Thus  $1 = \text{Ker}(\chi|_B) = \bigcap_{s \in G/G(\gamma)} (\text{Ker}(\gamma^s))$ . But the derived group  $[B, B]$  of  $B$  is contained in the kernel of  $\gamma^s$  for every  $s \in G$ , as  $\gamma$  is linear. Thus  $[B, B] \leq \text{Ker}(\chi|_B) = 1$ . So  $B$  is abelian.

We can now follow the hint of problem 6.11 in [7]. As  $\chi$  is monomial, there exists  $H \leq G$  and  $\lambda \in \text{Lin}(H)$  with  $\chi = \lambda^G$ . Thus the irreducible character  $\lambda^{HB}$  of  $HB$  lies above a  $G$ -conjugate  $\gamma^s$  of  $\gamma$ , where  $s \in G$ . As the  $G$ -conjugate  $\lambda^{s^{-1}} \in \text{Lin}(H^{s^{-1}})$  of  $\lambda$  also induces  $\chi$ , we can replace  $H$  by  $H^{s^{-1}}$  and  $\lambda$  by  $\lambda^{s^{-1}}$ . This way  $\lambda^{HB}$  is replaced by  $(\lambda^{s^{-1}})^{H^{s^{-1}}B} = (\lambda^{HB})^{s^{-1}}$ , which lies above  $\gamma$ .

According to Mackey's Theorem

$$(3.16) \quad \lambda^{HB}|_B = (\lambda|_{H \cap B})^B.$$

As  $B$  is abelian, the right hand side of (3.16) equals the sum of  $|B : H \cap B|$  distinct character extensions of  $\lambda|_{H \cap B}$  to  $B$ , each one appearing with multiplicity one. Thus every irreducible constituent of  $\lambda^{HB}|_B$  appears with multiplicity one. This, along with Clifford's Theorem, (as  $\lambda^{HB}$  lies above  $\gamma$ ), implies that

$$\lambda^{HB}|_B = e \cdot \sum_{s \in S} \gamma^s = \sum_{s \in S} \gamma^s,$$

where  $S$  is a family of representatives for the cosets  $H(\gamma)Bs$  of  $H(\gamma)B = (HB)(\gamma)$  in  $HB$ , and  $e$  is a positive integer. Furthermore, Clifford's Theorem implies the existence of an irreducible character  $\theta \in \text{Irr}((HB)(\gamma))$  lying above  $\gamma$  and inducing  $\lambda^{HB}$ . The fact that  $e = 1$  implies that  $\theta|_B = \gamma$ , i.e.,  $\theta \in \text{Irr}((HB)(\gamma))$  is an extension of  $\gamma \in \text{Irr}(B)$  to  $(HB)(\gamma)$ . Thus  $\theta \in \text{Lin}((HB)(\gamma)|\gamma)$  induces  $\lambda^{HB}$ . Hence  $\theta^G = \chi$ , as  $\lambda$  induces  $\chi$ . Therefore,  $\theta^{G(\gamma)}$  is an irreducible character of  $G(\gamma)$  lying above  $\gamma$  and inducing  $\chi$ . As the  $\gamma$ -Clifford correspondent  $\chi_\gamma$  of  $\chi$  is unique, we conclude that  $\theta^{G(\gamma)} = \chi_\gamma$ . Hence  $\chi_\gamma$  is induced from the linear character  $\theta$ , and thus is monomial.

This completes the proof of the lemma in the case of an abelian  $B$ . So the lemma follows.  $\square$

The above lemma implies

**Proposition 3.17.** *Assume that  $T' = (G', N', \psi')$  is a linear limit of  $T$ . Assume further that  $\chi' \in \text{Irr}(G'|\psi')$  is the  $T'$ -reduction of  $\chi \in \text{Irr}(G|\psi)$ . Then  $\chi$  is monomial if and only if  $\chi'$  is monomial. In particular, if  $G$  is a monomial group, then every irreducible character of  $G'$  that lies above  $\psi'$  is monomial.*

*Proof.* Since  $T'$  is a linear limit of  $T$ , there exists some chain  $T_0 = T \geq T_1 \geq \dots \geq T_n = T'$  of linear subtriples of  $T$ , such that  $T_i$  is a linear reduction of  $T_{i-1}$ , for all  $i = 1, \dots, n$ . So  $T_i = T_{i-1}(\lambda_i)$ , where  $\lambda_i$  is a linear character of some normal subgroup  $L_i$  of  $G_{i-1}$  with  $Z(T_{i-1}) \leq L_i \leq N_{i-1}$ . If  $\chi$  is an irreducible character of  $G$  lying above  $\psi$ , and thus above  $\zeta^T$ , then there is also a finite chain of irreducible characters  $\chi_i \in \text{Irr}(G_i)$  starting with  $\chi_0 = \chi$  and ending with  $\chi_n = \chi'$ , such that  $\chi_i$  is the unique  $\lambda_i$ -Clifford correspondent of  $\chi_{i-1}$ . Hence Lemma 3.15 implies that  $\chi_i$  is monomial if and only if  $\chi_{i-1}$  is monomial. We conclude that  $\chi_0 = \chi$  is monomial if and only if  $\chi' = \chi_n$  is monomial.

The rest of the proposition follows from Remark 3.5.  $\square$

## 4. PROOF OF THEOREM C

We begin with a straightforward lemma

**Lemma 4.1.** *Assume that  $N$  is a finite solvable group and let  $M$  be a nilpotent normal subgroup of  $N$  whose quotient group  $N/M$  is also nilpotent. Assume further that every  $p$ -Sylow subgroup of  $N$  centralizes the  $q$ -Sylow subgroup of  $M$ , for all primes  $p \neq q$  where  $p \mid |N|$  and  $q \mid |M|$ . Then  $N$  is also nilpotent.*

*Proof.* Let  $P, Q$  be a  $p$ - and a  $q$ -Sylow subgroup of  $N$ , for two distinct primes  $p$  and  $q$ . Let  $x \in P$  and  $y \in Q$  be two elements of  $P$  and  $Q$  respectively. It is enough to show that  $[x, y] = 1$ . Because  $N/M$  is nilpotent while  $(PM)/M$  and  $(QM)/M$  are a  $p$ - and a  $q$ -Sylow subgroup, respectively, of  $N/M$ , we have  $x \cdot y = y \cdot x \cdot m$ , for some  $m \in M$ . Hence  $y^{-1} \cdot x \cdot y = x \cdot m_p \cdot m_{p'}$ , where  $m = m_p \times m_{p'}$  is the decomposition of  $m$  to its  $p$ -part  $m_p$  and its  $p'$ -part  $m_{p'}$ . So  $x^y \cdot m_{p'}^{-1} = x \cdot m_p$ . The right hand side of the last equation is an element of  $P$ , and thus has order a power of  $p$ . On the other hand  $x^y$  is an element of some  $p$ -Sylow subgroup of  $N$ . So  $x^y$  commutes with  $m_{p'}$  by hypothesis. Therefore the order of  $x^y \cdot m_{p'}$  can be a power of  $p$  only if  $m_{p'} = 1$ . Similarly we have  $m_{q'} = 1$ , where  $m = m_q \times m_{q'}$  is the decomposition of  $m$  to its  $q$ - and  $q'$ -parts. We conclude that  $m = 1$ , and the lemma follows.  $\square$

The following is the main tool for the proof of Theorem C

**Lemma 4.2.** *Assume that  $G$  is a monomial finite group. Assume further that  $G$  has normal subgroups  $M \leq N$  such that  $M$  is nilpotent with odd order and  $N/M$  is nilpotent. If  $\phi$  is any irreducible character of  $M$ , and  $T$  is the triple  $T = (G, M, \phi)$ , then there exists a linear limit  $T' = (G', M', \phi')$  of  $T$  so that the factor group  $(G' \cap N) / \text{Ker}(T')$  is nilpotent.*

*Proof.* Let  $p_1, \dots, p_k$  be the distinct primes dividing  $|N|$ . Let  $\{H_i\}_{i=1}^k$  be a Sylow system of  $N$ . So  $H_i$  is a  $p_i$ -Hall subgroup of  $N$ . Furthermore,  $\cap_{j \neq i} H_j = Q_i$  is a  $p_i$ -Sylow subgroup of  $N$ . We also write  $M_i$  for the  $p_i$ -Sylow subgroup of  $M$  (some could be trivial, and by hypothesis those that are not have odd order). So  $M_i$  is a normal subgroup of  $G$ , for all  $i = 1, \dots, k$ , and  $M = M_1 \times M_2 \times \dots \times M_k$ . Note that  $M \trianglelefteq H_i M_i$  while the quotient group  $(H_i M_i)/M$  is the  $p_i$ -Hall subgroup of  $N/M$ . Hence  $(H_i M_i)/M$  is a characteristic subgroup of  $N/M$  and thus a normal subgroup of  $G/M$ . So  $H_i M_i$  is a normal subgroup of  $G$ .

Clearly the irreducible character  $\phi$  of  $M$  can be written as  $\phi = \phi_1 \times \dots \times \phi_k$ , where  $\phi_i \in \text{Irr}(M_i)$  for all  $i = 1, \dots, k$ . For every arbitrary but fixed  $i = 1, \dots, k$ , we form the triple  $T_i = (G, M_i, \phi_i)$ . Let  $T'_i = (G'_i, M'_i, \phi'_i)$  be a linear limit of  $T_i$ . We write  $K'_i = \text{Ker}(T'_i)$  for the kernel of  $T'_i$ , and  $Z'_i = Z(T'_i)$  for the center of  $T'_i$ . If  $\zeta'_i \in \text{Lin}(Z'_i)$  is the central character of  $T'_i$ , then according to Proposition 3.14, we have  $G'_i = G_i(\zeta'_i)$ .

For all  $j \neq i$  the  $p_j$ -Sylow subgroup  $M_j$  of  $M$  is a subgroup of  $G'_i$ , since it centralizes  $M_i$  (see Remark 3.4). Hence the group  $L'_i = M \cap G'_i$  is a normal nilpotent subgroup of  $G'_i$  whose  $p_i$ -Sylow subgroup is  $M'_i$  and whose  $p_j$ -Sylow subgroup, for any  $j \neq i$ , equals  $M_j$ . So

$$L'_i = M_1 \times \dots \times M_{i-1} \times M'_i \times M_{i+1} \times \dots \times M_k.$$

Let  $N'_i$  be  $N \cap G'_i = N(\zeta'_i)$  and  $H'_i$  be a  $p'_i$ -Hall subgroup of  $N'_i$ . Then  $(H'_i \cdot M'_i)/L'_i$  is the  $p'_i$ -Hall subgroup of the nilpotent factor group  $N'_i/L'_i$ , and thus  $H'_i \cdot M'_i = H'_i \cdot L'_i$  is a normal subgroup of  $G'_i$ .

According to Proposition 3.7, the group  $Z'_i/K'_i$  is the center of the  $p_i$ -group  $M'_i/K'_i$ . If  $\zeta'_i/K'_i$  is the unique character of the quotient group  $Z'_i/K'_i$  that inflates to  $\zeta'_i$ , then the same proposition implies that for all  $i = 1, \dots, k$  the groups  $M'_i/K'_i, G'_i/K'_i$  and the character  $\zeta'_i/K'_i$  satisfy Condition X. In addition, the group  $H'_i K'_i/K'_i$  is a  $p'_i$ -Hall subgroup of  $N'_i/K'_i$  and its product with  $M'_i/K'_i$  is a normal subgroup of  $G'_i/K'_i$ . According to Proposition 3.17 every irreducible character of  $G'_i$  that lies above  $\phi'_i$  is monomial. But  $\phi'_i$  is the only character of  $M'_i$  lying above  $\zeta'_i$ , by Proposition 3.14. So every irreducible character of  $G'_i$  lying above  $\zeta'_i$  is monomial.

Hence every irreducible character of  $G'_i/K'_i$  that lies above  $\zeta'_i/K'_i$  is monomial.

The character  $\zeta'_i/K'_i$  is a  $G'_i/K'_i$ -invariant  $p_i$ -special character of  $Z'_i/K'_i$ , (one could see [9] for the basic definitions of  $\pi$ -special characters). Hence there exists an irreducible  $p_i$ -special character of  $G'_i/K'_i$  that lies above  $\zeta'_i/K'_i$ . Therefore, that character is monomial and the  $p'_i$ -part of its degree is 1. We can now apply Corollary 2.5 to the groups  $G'_i/K'_i, M'_i/K'_i$  and  $(H'_i K'_i)/K'_i$ , for all  $i = 1, \dots, k$ . We conclude that  $(H'_i K'_i)/K'_i$  centralizes  $M'_i/K'_i$  for all such  $i$ . Hence the commutator subgroup  $[H'_i K'_i, M'_i]$  lies inside  $K'_i$ . Therefore,

$$(4.3) \quad [H'_i, M'_i] \leq K'_i, \text{ for all } i = 1, \dots, k,$$

where  $H'_i$  is any  $p'_i$ -Hall subgroup of  $N'_i$ .

Let  $G' = G'_1 \cap G'_2 \cap \dots \cap G'_k$ . Then  $G' = G(\zeta'_1, \dots, \zeta'_k)$ . We also define  $N' = N \cap G' = N(\zeta'_1, \dots, \zeta'_k)$  and  $M' = M \cap G' = M(\zeta'_1, \dots, \zeta'_k)$ . Of course  $M' = M'_1 \times \dots \times M'_k$ . Also  $M' \trianglelefteq N' \trianglelefteq G'$ . Furthermore, the group  $Z' = Z'_1 \times \dots \times Z'_k$  is a normal subgroup of  $G'$  contained in  $M'$ . The character  $\phi' = \phi'_1 \times \dots \times \phi'_k$  is an irreducible character of  $M'$  that lies above the  $G'$ -invariant linear character  $\zeta' = \zeta'_1 \times \dots \times \zeta'_k$  of  $Z'$ . In view of Remark 3.8 the quintuple  $T' = (G', M', \phi')$  is a linear limit of  $T = (G, M, \phi)$ , while  $Z' = Z(T')$ ,  $\zeta'$  is the central character  $\zeta^{(T')}$  of  $T'$ , and  $K' = K'_1 \times \dots \times K'_k$  is the kernel  $\text{Ker}(T') = \text{Ker}((\phi')^{G'})$  of  $T'$ .

In order to complete the proof of the lemma, it suffices to show that  $N'/K'$  is a nilpotent group. According to Lemma 4.1, it is enough to prove that every  $p$ -Sylow subgroup of  $N'/K'$  centralizes every  $q$ -Sylow subgroup of  $M'/K'$ , whenever  $q$  is a prime divisor of  $|M'/K'|$  and  $p \neq q$  is a prime divisor of  $|N'/K'|$ . We fix a prime  $p = p_r$  that divides  $|N'/K'|$ , for some  $r = 1, \dots, k$ . Let  $S'$  be a  $p$ -Sylow subgroup of  $N'$ . Clearly  $N' \leq N'_i$  for all  $i = 1, \dots, k$ . Hence for all  $i \neq r$ , there exists a  $p'_i$ -Hall subgroup  $H'_i$  of  $N'_i$  so that  $S' \leq H'_i$ . Therefore (4.3) implies

$$[S', M'_i] \leq K'_i, \text{ for all } i \neq r, i = 1, \dots, k.$$

So  $[(S'K')/K', (M'_i K')/K'] = 1$  for all  $i \neq r$ . Hence  $N'/K'$  is nilpotent. This proves Lemma 4.2.  $\square$

Now we can prove Theorem C that we restate here.

**Theorem C.** *Assume that  $G$  is a finite monomial group. Assume further that  $G$  has normal subgroups  $M \leq N$  such that  $M$  is nilpotent with odd order and  $N/M$  is nilpotent. Let  $\psi$  be*

an irreducible character of  $N$ . Then there exists a linear limit  $T' = (G', N', \psi')$  of the triple  $T = (G, N, \psi)$  so that the factor group  $N'/\text{Ker}(S')$  is nilpotent. Hence  $N$  is a monomial group.

*Proof.* We fix the character  $\psi \in \text{Irr}(N)$  and an irreducible character  $\phi$  of  $M$  lying under  $\psi$ . This way we can form two triples  $S = (G, M, \phi)$  and  $T = (G, N, \psi)$ . According to Lemma 4.2, there exists a linear limit  $S'' = (G'', M'', \phi'')$  of  $S = (G, M, \phi)$  so that the factor group  $N''/\text{Ker}(S'')$  is nilpotent, where  $N'' = G'' \cap N$ . If  $\psi'' \in \text{Irr}(N'')$  is the  $S''$ -reduction of  $\psi$ , then the triple  $T'' = (G'', N'', \psi'')$  is a multilinear reduction of  $T = (G, N, \psi)$ , see Remark 3.6. Because  $M'' \trianglelefteq N''$  while  $\psi''$  lies above  $\phi''$  the same remark implies that

$$Z(S'') \leq Z(T'') \text{ and } \text{Ker}(S'') \leq \text{Ker}(T'').$$

Of course, the linear reduction  $T''$  of  $T$  does not need to be a linear limit of the latter, but certainly any linear limit of  $T''$  is also a linear limit of  $T$ . Let  $T' = (G', N', \psi')$  be a linear limit of  $T''$  and  $T$ . Then Remark 3.2 implies that  $\text{Ker}(T'') \leq \text{Ker}(T')$ , while  $N' \leq N''$ . Hence  $\text{Ker}(S'') \leq \text{Ker}(T') \leq N' \leq N''$ . This, along with the fact that  $N''/\text{Ker}(S'')$  is nilpotent, implies that  $N'/\text{Ker}(T')$  is also a nilpotent group. Therefore the first part of Theorem C is proved.

For the rest of the theorem, observe that  $\psi'$  is a monomial character of  $N'$ , because  $N'/\text{Ker}(T')$  is nilpotent and  $\text{Ker}(T') \leq \text{Ker}\psi'$ . Because  $\psi'$  induces  $\psi$  in  $N$ , Theorem C follows.  $\square$

## 5. PROOF OF THEOREM D

The proof of Theorem D is heavily based on

**Theorem 5.1.** *Let  $Q$  be a  $q$ -group acting on a  $p$ -group  $P$ , with  $p \neq q$  odd primes. We identify both  $P$  and  $Q$  with their images in the semidirect product  $QP = Q \rtimes P$ . Let  $\mathcal{T}$  be a finite-dimensional right  $\mathbb{Z}_q QP$ -module such that the action of  $P$  on  $\mathcal{T}$  is faithful. Then there exists an element  $\tau \in \mathcal{T}$  such that its stabilizer  $(QP)(\tau)$  in  $Q \rtimes P$  equals  $Q$ .*

*Proof of Theorem 5.1.* We will prove a series of claims under the

**Inductive Assumption.**  *$Q, P, \mathcal{T}$  are chosen among all the triplets satisfying the hypothesis, but not the conclusion, of Theorem 5.1, so as to minimize first the order  $|QP|$  of the semidirect product  $Q \rtimes P$ , and then the  $\mathbb{Z}_q$ -dimension  $\dim_{\mathbb{Z}_q} \mathcal{T}$  of  $\mathcal{T}$ .*

These claims will lead to a contradiction, thus proving the theorem. First note that  $\mathcal{T} \neq 0$ .

**Claim 1.**  *$\mathcal{T}$  is an indecomposable  $\mathbb{Z}_q QP$ -module.*

*Proof.* Suppose not. Let  $\mathcal{T} = \mathcal{T}_1 \dot{+} \mathcal{T}_2$  be a direct decomposition of  $\mathcal{T}$ , where  $\mathcal{T}_1, \mathcal{T}_2$  are nontrivial  $\mathbb{Z}_q QP$ -submodules of  $\mathcal{T}$ . For  $i = 1, 2$  let  $K_i$  be the kernel of the action of  $P$  on  $\mathcal{T}_i$ . Hence  $\mathcal{T}_i$  is a  $\mathbb{Z}_q Q \rtimes (P/K_i)$ -module such that  $P/K_i$  acts faithfully on it. As  $\dim_{\mathbb{Z}_q} \mathcal{T}_i$  is strictly smaller than  $\dim_{\mathbb{Z}_q} \mathcal{T}$ , the minimality in Inductive Assumption provides an element  $\tau_i \in \mathcal{T}_i$  such that  $(Q \rtimes (P/K_i))(\tau_i) = Q$ . (Here we have identifying  $Q$  with its image in the semidirect product  $Q \rtimes (P/K_i)$ .) If we take as  $\tau$  the sum,  $\tau = \tau_1 + \tau_2$ , then  $\tau$  is an element of  $\mathcal{T}$  fixed by  $Q$ , as  $Q$  fixes each one of the  $\tau_i$  for  $i = 1, 2$ . Furthermore for the stabilizer of  $\tau$  in  $P$  we have

$$P(\tau) = \bigcap_{i=1}^2 P(\tau_i) = \bigcap_{i=1}^2 K_i.$$

Since  $P$  acts faithfully on  $\mathcal{T}$  the last intersection is trivial. Therefore  $(QP)(\tau) = Q$ , which contradicts the Inductive Assumption. Hence  $\mathcal{T}$  is an indecomposable  $\mathbb{Z}_qQP$ -module.  $\square$

**Claim 2.** *The restriction  $\mathcal{T}_P$  of  $\mathcal{T}$  to  $P$  is a multiple of an irreducible  $Q$ -invariant  $\mathbb{Z}_qP$ -module.*

*Proof.* Because  $q$  does not divide  $|P|$  we can write  $\mathcal{T}_P$  as a direct sum of its  $\mathbb{Z}_qP$ -homogeneous components, i.e.,

$$\mathcal{T}_P = \mathcal{U}_1 \dot{+} \mathcal{U}_2 \dot{+} \cdots \dot{+} \mathcal{U}_r.$$

So  $r \geq 1$  and there exist distinct simple  $\mathbb{Z}_qP$ -modules  $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_r$  and positive integers  $m_1, m_2, \dots, m_r$  such that  $\mathcal{U}_i \cong m_i \mathcal{B}_i$  as  $\mathbb{Z}_qP$ -modules, for all  $i = 1, \dots, r$ . Observe that right multiplication by any element in  $QP$  permutes among themselves the  $\mathcal{U}_i$ . So each  $QP$ -orbit  $\Omega$  of the  $\mathcal{U}_i$  leads to a  $\mathbb{Z}_qQP$ -direct summand  $\sum_{\mathcal{U}_i \in \Omega} \mathcal{U}_i$  of  $\mathcal{T}_P$ . According to Claim 1, the group  $QP$  acts transitively on the  $\mathcal{U}_i$ . Hence  $m_1 = m_2 = \cdots = m_r = m$ . Furthermore, if  $\mathcal{B}_1 = \mathcal{B}$  and  $(QP)(\mathcal{B})$  is the stabilizer of the isomorphism class of  $\mathcal{B}$  in  $QP$ , then  $V_i = \mathcal{B}^{\sigma_i}$ , where  $1 = \sigma_1, \dots, \sigma_r$  are representatives for the cosets in  $Q \cdot P$  of  $(QP)(\mathcal{B})$ . Thus,  $\mathcal{U} := \mathcal{U}_1 \cong m\mathcal{B} = m\mathcal{B}^{\sigma_1}, \mathcal{U}_2 \cong m\mathcal{B}^{\sigma_2}, \dots, \mathcal{U}_r \cong m\mathcal{B}^{\sigma_r}$ . We may pick  $\sigma_1, \dots, \sigma_r$  to be representatives of the cosets in  $Q$  of the stabilizer,  $Q(\mathcal{B})$ , of the isomorphism class of  $\mathcal{B}$  in  $Q$ . Note that  $Q(\mathcal{B}) = Q(\mathcal{U})$  as  $\mathcal{U} \cong m\mathcal{B}$ , where  $Q(\mathcal{U})$  is the stabilizer in  $Q$  of  $\mathcal{U}$  under multiplication in  $\mathcal{T}$ . If  $\mathcal{T}_P$  is not homogeneous, then  $r > 1$  and  $Q(\mathcal{U}) = Q(\mathcal{B}) < Q$ . For  $i = 1, \dots, r$  let  $K_i$  be the kernel of the action of  $P$  on  $\mathcal{U}_i$ . Then for every  $i = 1, \dots, r$  the stabilizer  $Q(\mathcal{U}_i)$  of  $\mathcal{U}_i$  in  $Q$  equals the  $\sigma_i$ -conjugate,  $Q(\mathcal{U})^{\sigma_i}$ , of  $Q(\mathcal{U}) = Q(\mathcal{B})$ . For the corresponding kernels we similarly have  $K_i = K_1^{\sigma_i}$ .

As  $\mathcal{U}$  is a faithful  $\mathbb{Z}_qP/K_1$ -module and  $Q(\mathcal{U}) < Q$ , the minimality of  $|QP|$  in the Inductive Assumption implies that there exists an element  $\mu \in \mathcal{U}$  such that

$$(Q(\mathcal{U}) \times (P/K_1))(\mu) = Q(\mathcal{U}).$$

For every  $i = 1, \dots, r$  we can define an element  $\mu_i = \mu \sigma_i$  of  $\mathcal{U}_i$ . Then  $Q(\mathcal{U}_i) = Q(\mathcal{U}_i)^{\sigma_i}$  fixes  $\mu_i$  as  $Q(\mathcal{U})$  fixes  $\mu$ . Furthermore if  $x$  is any element of  $P$  fixing  $\mu_i$  then  $x^{\sigma_i^{-1}}$  is an element of  $P$  fixing  $\mu$ . Therefore  $x^{\sigma_i^{-1}} \in K_1$ , which implies that  $x \in K_i$ . Thus

$$(Q(\mathcal{U}_i) \times (P/K_i))(\mu_i) = Q(\mathcal{U}_i)$$

for every  $i = 1, \dots, r$ .

Let  $\tau$  be the sum of the  $\mu_i$  for  $i = 1, \dots, r$ . Then  $\tau$  is an element of  $\mathcal{T}$  fixed by  $Q$ , since multiplication by any element in  $Q$  permutes the  $\mathcal{U}_i$  and the  $\mu_i$  among themselves. The stabilizer  $P(\tau)$  of  $\tau$  in  $P$  equals the intersection of the stabilizers of  $\mu_i$  in  $P$  for  $i = 1, \dots, r$ . Since  $(Q(\mathcal{U}_i) \times (P/K_i))(\mu_i) = Q(\mathcal{U}_i)$  for every such  $i$ , the latter equals the intersection of  $K_i$  for  $i = 1, \dots, r$ . The faithful action of  $P$  on  $\mathcal{T}$  implies that

$$P(\tau) = \bigcap_{i=1}^r K_i = 1.$$

Hence  $\mathcal{T}$  has an element  $\tau$  with  $(QP)(\tau) = Q$ , contradicting the Inductive Assumption. This contradiction proves Claim 2.  $\square$

**Claim 3.** *There are no  $Q$ -invariant subgroup,  $H < P$ , and  $\mathbb{Z}_qQH$ -submodule,  $\mathcal{S}$ , of  $\mathcal{T}_{QH}$  such that  $\mathcal{T}$  is the  $\mathbb{Z}_qQP$ -module  $\mathcal{S}^{QP}$  induced from  $\mathcal{S}$ , i.e.,*

$$\mathcal{T} = \sum_{1 \leq i \leq n} \mathcal{S}\sigma_i,$$

where the  $\sigma_i$  are representatives for the cosets  $H\sigma_i$  of  $H$  in  $P$ .

*Proof.* Suppose Claim 3 is false. We choose  $H$  to have maximal order among all  $Q$ -invariant subgroups of  $P$  that contradict Claim 3. Hence  $\mathcal{T}_{QH}$  has a  $\mathbb{Z}_qQH$ -submodule,  $\mathcal{S}$ , such that  $\mathcal{S}^{QP} = \mathcal{T}$ . If  $H$  is not normal in  $P$  then its normalizer,  $N_P(H)$ , in  $P$  satisfies  $H \triangleleft N_P(H) < P$ . Since  $H$  is  $Q$ -invariant,  $N_P(H)$  is also  $Q$ -invariant. Hence  $\mathcal{S}^{QN_P(H)}$  is a  $\mathbb{Z}_qQN_P(H)$ -submodule of  $\mathcal{T}_{QN_P(H)}$ . Furthermore  $\mathcal{S}^{QN_P(H)}$  induces  $\mathcal{T}$ . Thus  $N_P(H)$  is among the  $Q$ -invariant subgroups of  $P$  that contradict Claim 3, while  $|N_P(H)| > |H|$ . So the maximality of  $|H|$  implies that  $H$  is normal in  $P$ .

Let  $1 = \sigma_1, \dots, \sigma_k$  be coset representatives of  $H$  in  $P$ , and let  $\bar{\sigma}_m$  denote the image of  $\sigma_m$  in  $P/H$  for  $m = 1, \dots, k$ . Then  $\bar{1} = \bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_k$  are the distinct elements of  $P/H$ . As  $Q$  acts on  $P/H$ , it has to divide the  $\bar{\sigma}_m$ , for  $m = 1, \dots, k$ , into orbits,  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_l$ , for some  $l \in \{1, \dots, k\}$ . We may choose  $\bar{R}_1$  to be equal to  $\{\bar{\sigma}_1\} = \{1\}$ . For every  $i = 1, \dots, l$ , we pick some element  $\bar{\sigma}_{i,1} \in \bar{R}_i$ . Then  $\bar{R}_i = \{\bar{\sigma}_{i,1}^{q_j}\}_{j=1}^{k_i}$  where  $k_i = |\bar{R}_i|$  and  $q_j$  runs over a set  $Q_j$  of coset representatives of the stabilizer,  $C_Q(\bar{\sigma}_{i,1})$ , in  $Q$ . For every  $i = 1, \dots, l$  the stabilizer  $C_Q(\bar{\sigma}_{i,1})$  acts by conjugation on  $H$  and on  $\sigma_{i,1}H$ , where  $\sigma_{i,1} \in P$  has image  $\bar{\sigma}_{i,1} \in P/H$ . Furthermore,  $H$  acts transitively by right multiplication on  $\sigma_{i,1}H$  and  $(xh)^c = x^c h^c$  for all  $x \in \sigma_{i,1}H, h \in H, c \in C_Q(\bar{\sigma}_{i,1})$ . Hence Glauberman's Lemma (13.8 in [7]) provides an element  $t_{i,1} \in \sigma_{i,1}H$  that is fixed by  $C_Q(\bar{\sigma}_{i,1})$ . So  $C_Q(t_{i,1}) \geq C_Q(\bar{\sigma}_{i,1})$ . Furthermore, the opposite inclusion,  $C_Q(t_{i,1}) \leq C_Q(\bar{\sigma}_{i,1})$ , also holds as  $\bar{\sigma}_{i,1} = t_{i,1}H$ . Hence,

$$C_Q(t_{i,1}) = C_Q(\bar{\sigma}_{i,1}).$$

In this way we can pick a  $t_{i,1} \in \sigma_{i,1}H$ , for every  $i = 1, \dots, l$ , such that  $C_Q(t_{i,1}) = C_Q(\bar{\sigma}_{i,1})$ . We can even assume that  $t_{1,1} = 1$ . Let  $t_{i,j}$  denote the  $q_j$ -conjugate,  $t_{i,1}^{q_j}$ , of  $t_{i,1}$  for every  $j = 1, \dots, k_i$ . Hence the set of all  $t_{i,j}$ , for  $i = 1, \dots, l$  and for  $j = 1, \dots, k_i$ , is a complete set of coset representatives of  $H$  in  $P$ . Furthermore the  $Q$ -orbit  $\bar{R}_i$  corresponds to a  $Q$ -orbit  $R_i = \{t_{i,1}, \dots, t_{i,k_i}\}$ , for every  $i = 1, \dots, l$ .

Let  $K_{\mathcal{S}}$  be the kernel of the action of  $H$  on  $\mathcal{S}$ . As  $|H/K_{\mathcal{S}}| < |P|$ , the minimality of  $|QP|$  in the Inductive Assumption implies that there exists  $\mu \in \mathcal{S}$  such that its stabilizer,  $(Q \times H/K_{\mathcal{S}})(\mu)$ , in  $Q \times H/K_{\mathcal{S}}$  equals  $Q$ , or equivalently  $(QH)(\mu) = QK_{\mathcal{S}}$ . We note here that  $K_{\mathcal{S}} < H$ . Indeed, if  $H$  acts trivially on  $\mathcal{S}$ , then  $\mathcal{T}$  is induced from a trivial module and thus contains both trivial and non-trivial irreducible  $\mathbb{Z}_qP$ -submodules, contradicting Claim 1. We also have that  $\mu \neq 0$  since  $Q = (Q \times H/K_{\mathcal{S}})(\mu) < QH/K_{\mathcal{S}}$ . We denote by  $\mu t_{i,j}$  the  $t_{i,j}$ -translation of  $\mu$ , for every  $i = 1, \dots, l$  and for every  $j = 1, \dots, k_i$ . Then  $\mu t_{i,j}$  is an element of  $\mathcal{S}t_{i,j}$  such that

$$(Q^{t_{i,j}}H)(\mu t_{i,j}) = Q^{t_{i,j}}K_{\mathcal{S}}^{t_{i,j}}.$$

Since  $\mathcal{S}^{QP} = \mathcal{T}$  we get that

$$(5.2a) \quad \mathcal{T} = \mathcal{S}^{QP} = \sum_{1 \leq i \leq l} \sum_{1 \leq j \leq k_i} \mathcal{S}t_{i,j} = \mathcal{S} + \sum_{2 \leq i \leq l} \sum_{1 \leq j \leq k_i} \mathcal{S}t_{i,j}.$$



Let  $\tau$  be the element of  $\mathcal{T}$  defined by

$$(5.2b) \quad \tau = -\mu + \sum_{i=2}^l \sum_{j=1}^{k_i} \mu t_{i,j} = -\mu + \sum_{i=2}^l \sum_{t_{i,j} \in R_i} \mu t_{i,j}.$$

We claim that  $\tau$  satisfies the condition in Theorem 5.1, i.e., that  $(QP)(\tau) = Q$ . This will contradict the Inductive Assumption, and thus prove Claim 3. Indeed,  $R_i = \{t_{i,1}, \dots, t_{i,k_i}\}$  is a  $Q$ -orbit for every  $i = 2, \dots, l$ . Also  $\mu$  and  $-\mu$  are  $Q$ -invariant as  $(QH)(-\mu) = (QH)(\mu) = QK_{\mathcal{S}}$ . Hence  $\sum_{t_{i,j} \in R_i} \mu t_{i,j}$  is  $Q$ -invariant. Thus  $\tau$  is a  $Q$ -invariant element of  $\mathcal{T}$ , which implies that  $(QP)(\tau) = Q \cdot P(\tau)$ .

If  $x \in H(\tau)$  then, since  $H \triangleleft P$ , we get that  $(\mu t_{i,j})x = \mu x^{(t_{i,j})^{-1}} t_{i,j}$  is an element of  $\mathcal{S}t_{i,j}$ , for all  $i = 2, \dots, l$  and  $j = 1, \dots, k_i$ , while  $(-\mu)x$  is an element of  $\mathcal{S}$ . Since  $\tau x = \tau$ , it follows from (5.2) that  $(-\mu)x = -\mu$  and  $(\mu t_{i,j})x = \mu t_{i,j}$  for every  $i = 2, \dots, l$  and for every  $j = 1, \dots, k_i$ . Hence  $x$  is an element of:

$$H(\mu) \cap \bigcap_{i=2}^l \bigcap_{j=1}^{k_i} (P(\mu t_{i,j}) \cap H) = \bigcap_{i=1}^l \bigcap_{j=1}^{k_i} H(\mu t_{i,j}) = \bigcap_{i=1}^l \bigcap_{j=1}^{k_i} K_{\mathcal{S}}^{t_{i,j}}.$$

As  $H$  acts faithfully on  $\mathcal{T}$ , we get that  $\bigcap_{i=1}^l \bigcap_{j=1}^{k_i} K_{\mathcal{S}}^{t_{i,j}} = 1$ . Hence  $H(\tau) = 1$ .

Now let  $x \in P \setminus H$ . We claim that  $\tau x \neq \tau$ . Indeed any  $x \in P$  permutes the  $\mathcal{S}t_{i,j}$  among themselves. If  $x$  fixes  $\tau$ , then it also permutes among themselves the summands  $-\mu$  and  $\mu t_{i,j}$ , for  $i \neq 1$ , of  $\tau$ . Since  $\mathcal{S}x \neq \mathcal{S}$  we have  $(-\mu)x = \mu t_{i,j}$  for some  $i = 2, \dots, l$  and some  $j = 1, \dots, k_i$ . But as  $x \in P \setminus H$  we have that  $x = ht$  for some coset representative  $t = t_{i_0, j_0}$  of  $H$  in  $P$  with  $i_0 = 2, \dots, l$  and some element  $h \in H$ . Hence  $\mu t_{i,j} = (-\mu)x = (-\mu)ht \in \mathcal{S}t$ , which implies that  $t_{i,j} = t$  and  $(-\mu)h = \mu$ . This last equation leads to a contradiction as  $h$  has odd order ( $|P|$  is odd) and  $\mu \neq -\mu$  (as  $\mathcal{S} \leq \mathcal{T}$  has odd order, while  $\mu \neq 0$ ). Therefore  $\tau x \neq \tau$  whenever  $x \in P \setminus H$ . Hence  $P(\tau) = H(\tau) = 1$  and  $(QP)(\tau) = Q$ , contradicting the Inductive Assumption. This contradiction proves Claim 3.  $\square$

**Claim 4.** *The restriction  $\mathcal{T}_A$  of  $\mathcal{T}$  to any normal subgroup  $A \leq P$  of  $QP$  is a multiple  $e\mathcal{B}$  of a single faithful  $QP$ -invariant  $\mathbb{Z}_p A$ -module  $\mathcal{B}$ . Hence every normal abelian subgroup  $A$  of  $QP$  contained in  $P$  is cyclic.*

*Proof.* Let  $A$  be a normal subgroup of  $QP$  contained in  $P$ , and let  $\mathcal{T}_A$  be the restriction of  $\mathcal{T}$  to  $A$ . According to Claim 2, and Clifford's Theorem,  $\mathcal{T}_A$  can be written as a direct sum of its  $\mathbb{Z}_q A$ -homogeneous components, i.e.,

$$\mathcal{T}_A = \mathcal{W}_1 \dot{+} \mathcal{W}_2 \dot{+} \dots \dot{+} \mathcal{W}_s.$$

Furthermore,  $P$  acts transitively on the  $\mathcal{W}_i$  for all  $i = 1, \dots, s$ , while  $Q$  permutes the  $\mathcal{W}_i$  among themselves (as  $\mathcal{T}$  is a  $\mathbb{Z}_q QP$ -module). Hence Glauberman's lemma implies that  $Q$  fixes some  $\mathbb{Z}_q A$ -homogeneous component,  $\mathcal{W}$ , of  $\mathcal{T}_A$ . Note that Clifford's theorem implies that the homogeneous component  $\mathcal{W}$  of  $\mathcal{T}_A$  is a  $\mathbb{Z}_q QP(\mathcal{W})$ -submodule of  $\mathcal{T}$ , where  $P(\mathcal{W})$  is the stabilizer of  $\mathcal{W}$  under multiplication of elements of  $\mathcal{T}$  by elements of  $P$ . Furthermore, the  $\mathbb{Z}_q QP(\mathcal{W})$ -submodule  $\mathcal{W}$  induces the  $\mathbb{Z}_q QP$ -module  $\mathcal{T} = \mathcal{W}^{QP}$ . In view of Claim 3, we must have  $\mathcal{W} = \mathcal{T}$ . Hence  $\mathcal{T}_A \cong e\mathcal{B}$

where  $\mathcal{B}$  is an irreducible  $QP$ -invariant  $\mathbb{Z}_q A$ -submodule of  $\mathcal{T}$ . As  $P$  acts faithfully on  $\mathcal{T}$ , the  $\mathbb{Z}_q A$ -module  $\mathcal{B}$  is also faithful. If  $A$  is abelian, the existence of a faithful irreducible  $\mathbb{Z}_q A$ -module implies that  $A$  is cyclic. Therefore, the claim is proved.  $\square$

The  $q$ -group  $Q$  acts on the non-trivial  $q$ -group  $\mathcal{T}$ , fixing the trivial element 0 of  $\mathcal{T}$ . Hence the group  $Q$  fixes at least  $q$  elements of  $\mathcal{T}$ . So  $Q$  fixes some  $\tau$  with

$$(5.3) \quad \tau \in \mathcal{T} \text{ and } \tau \neq 0.$$

Hence, to complete the proof of Theorem 5.1, by contradicting the Inductive Assumption, it is enough to show that  $P(\tau) = 1$

By Claim 4 every characteristic abelian subgroup of  $P$  is cyclic. Since  $p$  is odd, Theorem 4.9 of [6] implies that either  $P$  is cyclic or  $P$  is the central product  $E \odot C$ , of the extra-special  $p$ -group  $E = \Omega_1(P)$  of exponent  $p$ , and the cyclic group  $C = Z(P)$ .

According to Claim 4, the  $\mathbb{Z}_q Z(P)$ -module  $\mathcal{T}_{Z(P)}$  is a multiple of a faithful irreducible  $QP$ -invariant  $\mathbb{Z}_q Z(P)$ -module  $\mathcal{B}$ , i.e.,  $\mathcal{T}_{Z(P)} = m\mathcal{B}$ . Hence  $Z(P)$  acts fix point freely on  $\mathcal{T}$ , as it acts fix point freely on  $\mathcal{B}$  (or else  $\mathcal{B}$  wouldn't be simple and faithful).

If  $P$  is cyclic, then  $P = Z(P)$ . Thus  $P$  acts fix point freely on  $\mathcal{T}$ . Hence no element of  $Z(P) - \{1\}$  could fix  $\tau$ . Hence  $P(\tau) = 1$ . So  $(QP)(\tau) = Q$ , contradicting the Inductive Assumption. Therefore,  $P$  can't be cyclic.

Hence,

$$(5.4) \quad P = E \odot C = \Omega_1(P) \odot Z(P),$$

where  $E = \Omega_1(P)$  is an extra special  $p$ -group of exponent  $p$  and  $C = Z(P)$  is cyclic. Therefore the quotient group  $\overline{P} = P/Z(P)$  is an elementary abelian  $p$ -group. Furthermore  $\overline{P}$  affords a bilinear form  $c : \overline{P} \times \overline{P} \rightarrow Z(E)$  defined, for every  $\bar{x}, \bar{y} \in \overline{P}$ , as  $c(\bar{x}, \bar{y}) = [x, y]$ , where  $x, y$  are elements of  $P$  whose images in  $\overline{P}$  are  $\bar{x}$  and  $\bar{y}$  respectively. With respect to that form  $\overline{P}$  is a symplectic  $\mathbb{Z}_p(Q)$ -module.

**Claim 5.** *The symplectic  $\mathbb{Z}_p(Q)$ -module  $\overline{P}$  is anisotropic.*

*Proof.* Assume not. Then there is an isotropic non-zero  $\mathbb{Z}_p(Q)$ -submodule  $\overline{A}$  of  $\overline{P}$ . Hence  $c(\bar{a}, \bar{b}) = 0$  for every  $\bar{a}, \bar{b} \in \overline{A}$ , because  $\overline{A} \subseteq \overline{A}^\perp$ . Therefore, the definition of the symplectic form  $c$  implies that the inverse image  $A$  of  $\overline{A}$  in  $P$  is an abelian subgroup of  $P$  containing  $Z(P)$ . Since  $\overline{A}$  is a  $\mathbb{Z}_p(Q)$ -submodule of  $\overline{P}$ , the abelian group  $A$  is a normal subgroup of  $QP$  contained in  $P$ . Hence by Claim 4,  $A$  is cyclic and properly contains  $Z(P)$ . Therefore there exists an element  $a \in A \setminus Z(P)$  such that  $a^p$  is a generator of  $Z(P)$ . On the other hand, equation (5.4) implies that  $a = \omega \cdot c$  where  $\omega \in \Omega_1(P)$  and  $c \in C = Z(P)$ . Hence  $a^p = \omega^p \cdot c^p = c^p$ . Since  $a^p$  is a generator of the cyclic non-trivial  $p$ -group  $Z(P)$  and  $c \in Z(P)$ , this last equation leads to a contradiction. This proves the claim.  $\square$

Now we can complete the proof of Theorem 5.1. If  $(QP)(\tau) \neq Q$  then there exists a  $Q$ -invariant subgroup  $D = P(\tau) \neq 1$  of  $P$  such that  $(QP)(\tau) = QD$ . Hence the center  $Z(D)$  of  $D$  is a non-trivial  $Q$ -invariant abelian subgroup of  $P$ . Therefore its image  $\overline{Z(D)} = Z(D)Z(P)/Z(P)$  in  $\overline{P}$  is

an isotropic  $\mathbb{Z}_p(Q)$ -submodule of  $\overline{P}$ . Since  $\overline{P}$  is anisotropic,  $\overline{Z(D)} = \overline{1}$ , i.e.,  $Z(D)$  is contained in  $Z(P)$ .

As we saw,  $Z(P)$  acts fix point freely on  $\mathcal{T}$ . This implies that no element of  $Z(P) - \{1\}$  could fix  $\tau$ . Hence  $Z(D) = 1$ , contradicting the fact that  $Z(D) \neq 1$ . So  $(QP)(\tau) = Q$ , contradicting the Inductive Assumption. This final contradiction completes the proof of Theorem 5.1.  $\square$

In terms of characters, Theorem 5.1 implies

**Corollary 5.5.** *Let  $Q$  be a  $q$ -group acting on a  $p$ -group  $P$  with  $p \neq q$  odd primes. Suppose that the semi-direct product  $Q \rtimes P$  acts on a  $q$ -group  $S$  such that the action of  $P$  on  $S$  is faithful. Then there exists a linear character  $\lambda$  of  $S$  whose kernel  $\text{Ker}(\lambda)$  contains the Frattini subgroup  $\Phi(S)$  and whose stabilizer  $(QP)(\lambda)$  in  $Q \rtimes P$  is  $Q$ .*

*Proof.* Let  $\mathcal{T}$  be the quotient group  $\mathcal{T} := S/\Phi(S)$ . Then  $\mathcal{T}$  is a  $\mathbb{Z}_qQP$ -module. We write  $\mathcal{T}^*$  for its dual  $\mathbb{Z}_qQP$ -module, i.e.,  $\mathcal{T}^* = \text{Hom}_{\mathbb{Z}_q}(\mathcal{T}, \mathbb{Z}_q)$ . Then  $P$  acts faithfully on both  $\mathcal{T}$  and  $\mathcal{T}^*$ . Furthermore, according to Theorem 5.1 there is an element  $\tau \in \mathcal{T}^*$  whose stabilizer in  $QP$  equals  $Q$ . Since the linear characters of  $\mathcal{T}$  can be considered as the elements of  $\mathcal{T}^*$  composed with some faithful linear character of  $\mathbb{Z}_q$ , we conclude that there is a linear character  $\lambda^* \in \text{Lin}(\mathcal{T})$  whose stabilizer in  $QP$  is  $Q$ . Let  $\lambda$  be the linear character of  $S$  to which  $\lambda^*$  inflates. Then  $\Phi(S) \leq \text{Ker}(\lambda)$ . Furthermore,  $(QP)(\lambda) = (QP)(\lambda^*) = Q$ , and the corollary follows.  $\square$

The following is a straightforward lemma.

**Lemma 5.6.** *Let  $P$  be a  $p$ -subgroup of a finite group  $G$  and let  $Q_1 \leq Q$  be  $q$ -subgroups of  $G$ , for some distinct odd primes  $p$  and  $q$ . If  $P$  normalizes  $Q_1$ , and  $Q$  normalizes their product  $Q_1P$ , then  $QP$  is also a subgroup of  $G$  with  $Q \in \text{Syl}_q(QP)$ ,  $P \in \text{Syl}_p(QP)$ , while  $Q_1P \trianglelefteq QP$  and  $Q_1 \trianglelefteq QP$ . Furthermore,  $Q$  is the product  $Q = [Q_1, P]N_Q(P)$ , where  $[Q_1, P] \trianglelefteq QP$  and  $[Q_1, P] \cap N_Q(P) = C_{[Q_1, P]}(P) \leq \Phi([Q_1, P])$ .*

*Proof.* Since  $P$  normalizes  $Q_1$ , the latter is a characteristic subgroup of  $Q_1P$ . Therefore, the fact that  $Q$  normalizes  $Q_1P$  implies that  $Q$  normalizes  $Q_1$ . So  $Q_1 \trianglelefteq Q$ .

The product,  $QP = Q(Q_1P)$ , is a subgroup of  $G$ , since  $Q$  normalizes the semidirect product  $Q_1 \rtimes P$ . That same product  $Q_1P$  is a normal subgroup of  $QP = Q(Q_1P)$ . We obviously have that  $Q \in \text{Syl}_q(QP)$  and  $P \in \text{Syl}_p(QP)$ .

By Frattini's argument for the Sylow  $p$ -subgroup  $P$  of  $Q_1P \trianglelefteq QP$  we get

$$(5.7a) \quad QP = Q_1PN_{QP}(P).$$

The normalizer,  $N_{QP}(P)$ , of  $P$  in  $QP$  contains  $P$ . So it is equal to  $PN_Q(P)$ . Hence (5.7a) can be written as  $QP = Q_1N_Q(P)P$ . Since  $Q_1N_Q(P) \leq Q$  and  $Q \cap P = 1$ , we conclude that

$$(5.7b) \quad Q = Q_1N_Q(P).$$

Because  $(|Q_1|, |P|) = 1$ , and  $P$  acts on  $Q_1$ , we can write  $Q_1$  as the product  $Q_1 = [Q_1, P]N_{Q_1}(P)$ . The commutator subgroup  $[Q_1, P]$  is a characteristic subgroup of  $Q_1P$  and thus is also a normal subgroup of  $Q$ , as  $Q$  normalizes  $Q_1P$ . Therefore, (5.7b) implies

$$Q = [Q_1, P]N_Q(P).$$

That  $[Q_1, P] \cap N_Q(P) = C_{[Q_1, P]}(P)$  is obvious as  $(|Q_1|, |P|) = 1$ . Also the quotient group  $K := [Q_1, P]/\Phi([Q_1, P])$  is abelian and thus  $K = [K, P] \times C_K(P)$ . As  $[Q_1, P, P] = [Q_1, P]$  (by Theorem 3.6 in [6]), we get that  $K = [K, P]$  and  $C_K(P) = 1$ . This implies that  $C_{[Q_1, P]}(P) \leq \Phi([Q_1, P])$ .  $\square$

As an easy consequence of Corollary 5.5 and Lemma 5.6 we have:

**Proposition 5.8.** *Let  $Q$  be a  $q$ -group acting on a  $p$ -group  $P$  with  $p \neq q$  odd primes. Suppose that the semi-direct product  $Q \ltimes P$  acts on a  $q$ -group  $S$  such that the action of  $P$  on  $S$  is faithful. Then there exists a linear character  $\lambda$  of  $S$  such that  $C_S(P) \leq \text{Ker}(\lambda)$  and  $(QP)(\lambda) = Q$ .*

*Proof.* As  $P$  acts on  $S$  we can write  $S$  as the product  $S = [S, P] \cdot C_S(P)$ . It is clear that the product  $QC_S(P)$  forms a group. Furthermore,  $QC_S(P)$  normalizes  $P$  and the semidirect product  $(QC_S(P)) \ltimes P$  acts on  $[S, P]$ , while the action of  $P$  on  $[S, P]$  is faithful. Then according to Corollary 5.5 there exists a linear character  $\lambda_1$  of  $[S, P]$  such that  $(QC_S(P)P)(\lambda_1) = QC_S(P)$ , while  $\Phi([S, P]) \leq \text{Ker}(\lambda_1)$ .

As we have seen in Lemma 5.6

$$[S, P] \cap C_S(P) = C_{[S, P]}(P) \leq \Phi([S, P]).$$

Since  $\lambda_1$  is a linear character of  $[S, P]$  that is trivial on  $\Phi([S, P])$  and  $C_S(P)$ -invariant, the above inclusion implies that  $\lambda_1$  has a unique extension to a linear character  $\lambda$  of  $S$  trivial on  $C_S(P)$ . Furthermore,  $(QP)(\lambda) = (QP)(\lambda_1) = Q$ , and the proposition follows.  $\square$

We can now prove a special case of Theorem D where  $P(\beta)$  is trivial. In this special case the new character we get is linear. In particular we have

**Lemma 5.9.** *Let  $P$  be a  $p$ -subgroup of a finite group  $G$ , where  $p$  is an odd prime. Let  $Q_1, Q$  be  $q$ -subgroups of  $G$  for some odd prime  $q \neq p$ , with  $Q_1 \leq Q$ . Assume that  $P$  normalizes  $Q_1$  while  $Q$  normalizes the product  $Q_1P$ . Assume further that  $\beta$  is an irreducible character of  $Q_1$  such that  $P(\beta) = 1$ . Then there exists a linear character  $\lambda$  of  $Q_1$  such that  $1 = P(\beta) = P(\lambda)$ , while  $Q(\beta) \leq Q(\lambda) = Q$  and  $\lambda$  extends to  $Q$ .*

*Proof.* Because  $Q$  normalizes the product  $Q_1 \times P$ , it normalizes its characteristic subgroup  $Q_1$ . Let  $C = N_Q(P)$  be the normalizer of  $P$  in  $Q$ . Frattini's argument implies that

$$Q = N_Q(P)Q_1 = CQ_1.$$

In addition,  $C$  normalizes  $P$  and the semidirect product  $CP$  acts on  $Q_1$ . The fact that  $P(\beta) = 1$  implies that  $P$  acts faithfully on  $Q_1$ . Therefore we can apply Proposition 5.8 to the groups  $C, P$  and  $Q_1$  here in the place of  $Q, P$  and  $S$  there respectively. We conclude that there exists a linear character  $\lambda \in \text{Lin}(Q_1)$  such that  $C_{Q_1}(P) \leq \text{Ker}(\lambda)$  and  $(CP)(\lambda) = C$ .

This last equation implies that  $P(\lambda) = P(\beta) = 1$ . Since  $Q = CQ_1$  and  $C$  fixes  $\lambda$  we conclude that  $Q$  also fixes  $\lambda$ . Furthermore we have

$$C \cap Q_1 = N_Q(P) \cap Q_1 = C_{Q_1}(P) \leq \text{Ker}(\lambda).$$

Therefore  $\lambda$  can be extended to  $Q$ .

As  $Q(\beta) \leq Q = Q(\lambda)$ , the proof of Lemma 5.9 is complete.  $\square$

Finally we can now prove Theorem D, that we restate here.

**Theorem D.** *Let  $P$  be a  $p$ -subgroup, for some odd prime  $p$ , of a finite group  $G$ . Let  $Q_1, Q$  be  $q$ -subgroups of  $G$ , for some odd prime  $q \neq p$ , with  $Q_1 \leq Q$ . Assume that  $P$  normalizes  $Q_1$ , while  $Q$  normalizes the product  $Q_1 \rtimes P$ . Assume further that  $\beta$  is an irreducible character of  $Q_1$ . Then there exists an irreducible character  $\beta^\nu$  of  $Q_1$  such that*

$$\begin{aligned} P(\beta) &= P(\beta^\nu), \\ Q(\beta) &\leq Q(\beta^\nu) \text{ and } N_Q(P(\beta)) \leq Q(\beta^\nu), \\ \beta^\nu &\text{ extends to } Q(\beta^\nu). \end{aligned}$$

*Proof of Theorem D.* Let  $P(\beta)$  be the stabilizer of  $\beta$  in  $P$  and  $P_1$  be the normalizer of  $P(\beta)$  in  $P$ . Let  $\bar{P}_1$  denote the quotient group  $P_1/P(\beta)$ . We write  $C_1$  for the centralizer,  $C_1 = C(P(\beta) \text{ in } Q_1)$ , of  $P(\beta)$  in  $Q_1$ . Then it is clear that  $\bar{P}_1$  acts on  $C_1$ .

The Glauberman correspondence (Theorem 13.1 in [7]), applied to the groups  $P(\beta)$  and  $Q_1$ , provides an irreducible character  $\beta^*$  of  $C_1$  corresponding to the irreducible character  $\beta$  of  $Q_1$ . Because  $P_1$  normalizes both  $P(\beta)$  and  $Q_1$  we get that  $P_1(\beta^*) = P_1(\beta) = P(\beta)$ . If  $P_1(\beta^*) < P(\beta^*)$  then  $P_1(\beta^*) < N(P_1(\beta^*) \text{ in } P(\beta^*)) = N(P(\beta) \text{ in } P(\beta^*)) = P_1(\beta^*)$ , which is impossible. Therefore

$$P(\beta^*) = P_1(\beta^*) = P(\beta) = P_1(\beta).$$

We remark here that because  $P(\beta)$  centralizes  $C_1 = C(P(\beta) \text{ in } Q_1)$ , we have  $P(\beta) \leq C(C_1 \text{ in } P_1) \leq P_1(\beta^*) = P(\beta)$ . Hence  $C(C_1 \text{ in } P_1) = P(\beta)$  and  $\bar{P}_1$  acts faithfully on  $C_1$ .

Let  $C := N(P(\beta) \text{ in } Q)$  be the normalizer of  $P(\beta)$  in  $Q$ . Then  $C_1$  is a normal subgroup of  $C$  as  $Q_1 \trianglelefteq Q$ . Furthermore,  $C$  normalizes  $N(P(\beta) \text{ in } PQ_1)$  because  $Q$  normalizes the product  $PQ_1$ . As  $P_1C_1 = N(P(\beta) \text{ in } PQ_1)$  we conclude that  $C$  normalizes the product  $P_1C_1$ . Hence Frattini's argument implies that

$$C = N(P_1 \text{ in } C)C_1.$$

Now we can apply Lemma 5.9 to the groups  $C, C_1$  and  $\bar{P}_1$  and the character  $\beta^* \in \text{Irr}(C_1)$ , in the place of  $Q, Q_1, P$  and  $\beta$  respectively. We conclude that there exists a linear character  $\lambda \in \text{Lin}(C_1)$  such that

$$(5.10a) \quad \bar{P}_1(\beta^*) = 1 = \bar{P}_1(\lambda),$$

$$(5.10b) \quad C(\beta^*) \leq C(\lambda) = N_C(C_1) = C$$

$$(5.10c) \quad \lambda \text{ extends to } C(\lambda) = C.$$

Equation (5.10a) above implies that  $P_1(\lambda) = P(\beta)$ . Thus  $P(\beta) = P_1(\lambda) \leq P(\lambda)$ . We actually have that  $P(\lambda) = P(\beta)$ . Indeed, if  $P(\beta) < P(\lambda)$ , then  $P(\beta)$  would be a proper subgroup of  $N(P(\beta) \text{ in } P(\lambda))$ . Thus  $P(\beta) < N(P(\beta) \text{ in } P(\lambda)) = N(P(\beta) \text{ in } P)(\lambda) = P_1(\lambda) = P(\beta)$ . So

$$P_1(\lambda) = P(\beta) = P(\lambda).$$

Let  $\beta^\nu \in \text{Irr}(Q_1)$  be the Glauberman  $P(\beta)$ -correspondent to  $\lambda$ . Because  $CP_1$  normalizes both  $P(\beta)$  and  $Q_1$  we get  $(CP_1)(\beta^\nu) = (CP_1)(\lambda) = CP(\beta)$ . Hence  $P(\beta^\nu) \geq P_1(\beta^\nu) = P_1(\lambda) = P(\beta)$ . If

$P(\beta^\nu) > P(\beta)$  then  $P(\beta) < N(P(\beta) \text{ in } P(\beta^\nu)) = P_1(\beta^\nu) = P(\beta)$ . Thus

$$P(\beta^\nu) = P(\lambda) = P(\beta)$$

and  $(C(PQ_1))(\beta^\nu) = CP(\beta)Q_1$ . Since  $C$  fixes  $\beta^\nu$  and normalizes  $P(\beta)$  we have  $C \leq N(P(\beta) \text{ in } Q(\beta^\nu)) \leq N(P(\beta) \text{ in } Q) = C$ . Hence

$$N(P(\beta) \text{ in } Q) = C = N(P(\beta) \text{ in } Q(\beta^\nu)) \leq Q(\beta^\nu).$$

In order to show that  $\beta^\nu$  extends to  $Q(\beta^\nu)$ , we first observe that  $Q_1P(\beta) = (Q_1P)(\beta^\nu) = (Q_1P_1)(\beta^\nu)$  as  $P(\beta^\nu) = P(\beta) = P_1(\beta^\nu)$ . So the group  $Q_1P(\beta) = (Q_1P)(\beta^\nu)$  is a normal subgroup of  $Q(\beta^\nu)P(\beta)$  as  $Q$  normalizes the product  $Q_1P$ . Because  $\lambda$  extends to  $C$  while  $P(\beta)$  fixes  $\lambda$  and has coprime order to that of  $C$ , we conclude that  $\lambda$  can be extended to  $CP(\beta)$ , (see Theorem 6.26 in [7]). So we can apply the Main Theorem in [12] to the groups  $P(\beta)Q(\beta^\nu)$ ,  $P(\beta)Q_1$  and  $Q_1$ . We conclude that  $\beta^\nu$  extends to  $P(\beta)Q(\beta^\nu)$  as its  $P(\beta)$ -Glauberman correspondent  $\lambda$  can be extended to  $P(\beta)C = P(\beta)N(P(\beta) \text{ in } Q(\beta^\nu))$ . We write  $\beta^{\nu,e}$  for an extension of  $\beta^\nu$  to  $Q(\beta^\nu)$ .

To complete the proof of the theorem it remains to show that  $Q(\beta) \leq Q(\beta^\nu)$ . The group  $(Q_1P)(\beta) = Q_1P(\beta)$  is a normal subgroup of  $Q(\beta)P(\beta)$ , as  $Q$  normalizes  $Q_1P$ . Hence Frattini's argument implies that

$$Q(\beta) = Q_1N(P(\beta) \text{ in } Q(\beta)).$$

Therefore  $Q(\beta) \leq Q_1N(P(\beta) \text{ in } Q) = Q_1C$ . But we have already seen that  $Q_1P(\beta)$  is a normal subgroup of  $Q(\beta^\nu)P(\beta)$ . Hence the Frattini argument implies that

$$Q(\beta^\nu) = Q_1N(P(\beta) \text{ in } Q(\beta^\nu)) = Q_1C.$$

Thus  $Q(\beta) \leq Q(\beta^\nu)$  and the theorem follows.  $\square$

## 6. PROOF OF THEOREM E

We first need some lemmas.

**Lemma 6.1.** *Assume that  $G$  is a finite group and that  $S \leq H$  are subgroups of  $G$  with  $S$  normal in  $G$ . Assume further that  $\theta \in \text{Irr}(H)$  lies above  $\lambda \in \text{Irr}(S)$ . Let  $\theta_\lambda \in \text{Irr}(H(\lambda))$  denote the unique  $\lambda$ -Clifford correspondent of  $\theta$ . If  $\theta$  extends to its stabilizer  $G(\theta)$  in  $G$ , then  $\theta_\lambda$  also extends to  $G(\theta, \lambda)$ .*

*Proof.* A straight forward application of Clifford Theory implies that  $G(\theta, \lambda) \leq G(\theta_\lambda)$ . Furthermore, as  $G(\theta)$  fixes  $\theta$  it permutes among themselves the members of the  $H$ -conjugacy class of characters in  $\text{Irr}(S)$  lying under  $\theta$ . Since  $\lambda \in \text{Irr}(S)$  lies under  $\theta$  we get

$$(6.2a) \quad G(\theta) = H \cdot G(\theta, \lambda) \leq H \cdot G(\theta_\lambda).$$

In addition,

$$(6.2b) \quad G(\theta, \lambda) \cap H = H(\lambda).$$

Let  $\theta^e \in \text{Irr}(G(\theta))$  be an extension of  $\theta$  to  $G(\theta)$ . Then  $\theta^e$  lies above  $\lambda$ . Let  $\Psi \in \text{Irr}(G(\theta, \lambda))$  denote the unique  $\lambda$ -Clifford correspondent of  $\theta^e \in \text{Irr}(G(\theta))$ . So  $\Psi$  lies above  $\lambda$  and induces  $\theta^e$ . Therefore,

$$(\Psi^{G(\theta)})|_H = \theta^e|_H = \theta.$$

Mackey's Theorem, along with (6.2), implies that

$$(\Psi^{G(\theta)})|_H = (\Psi|_{H(\lambda)})^H.$$

Hence  $(\Psi|_{H(\lambda)})^H = \theta$  is an irreducible character of  $H$ . So the restriction  $\Psi|_{H(\lambda)}$  is an irreducible character of  $H(\lambda)$  that induces  $\theta$  and lies above  $\lambda$  (as  $\Psi$  lies above  $\lambda$ ). We conclude that  $\Psi|_{H(\lambda)}$  is the  $\lambda$ -Clifford correspondent of  $\theta$ . Hence  $\Psi|_{H(\lambda)} = \theta_\lambda$ . Thus  $\Psi$  is an extension of  $\theta_\lambda$  to  $G(\theta, \lambda)$ , and the lemma follows.  $\square$

Applying this lemma to linear reductions we can prove

**Lemma 6.3.** *Let  $T = (G, N, \psi)$  be a triple, and  $T' = (G', N', \psi')$  be a multilinear reduction of  $T$ . Let  $H$  be any subgroup of  $G$  with  $N \leq H$ . If  $\zeta'$  is the central character of  $T'$ , then  $H' = H \cap G' = H(\zeta')$ . Also if  $\psi$  extends to  $H(\psi)$  then  $\psi'$  extends to  $H'(\psi) = H'(\psi')$ .*

*Proof.* Assume first that  $T'$  is a linear reduction of  $T$ . So  $T' = T(\lambda)$  where  $\lambda$  is a linear character of a normal subgroup  $L$  of  $G$  with  $Z(T) \leq L \leq N$ . Furthermore,  $\lambda$  lies above the central character  $\zeta \in \text{Irr}(Z(T))$  of  $T$ . In addition, if  $Z(T')$  is the center of  $T'$ , and  $\zeta' \in \text{Lin}(Z(T'))$  its central character, then  $Z(T) \leq L \leq Z(T') \leq N$  while  $\zeta'$  lies above  $\lambda$ .

According to Proposition 3.14, we have  $G(\lambda) = G' = G(\zeta')$ . Hence if  $H' = H \cap G'$  then  $H(\lambda) = H' = H(\zeta')$ . This, along with Lemma 6.1, implies that  $\psi'$  extends to  $H(\psi, \lambda) = H'(\psi)$  whenever  $\psi$  extends to  $H(\psi)$ . But  $H(\psi, \lambda) = H(\psi', \lambda) = H'(\psi')$ , since  $\psi'$  is the  $\lambda$ -Clifford correspondent of  $\psi$ . This implies the lemma in the case that  $T'$  is a linear reduction. If  $T'$  is a multilinear reduction, then repeated applications of the above argument completes the proof of the lemma.  $\square$

**Lemma 6.4.** *Let  $Q$  be a normal  $q$ -subgroup of a finite group  $G$ , for some prime  $q$ . Assume further that  $E \trianglelefteq G$  with  $E \leq Q$ , while  $\lambda \in \text{Irr}(E)$  and  $\chi \in \text{Irr}(Q|\lambda)$ . If  $A$  is a  $q'$ -subgroup of  $G$  then there exists a  $Q$ -conjugate  $\lambda_1 \in \text{Irr}(E)$  of  $\lambda$  that is  $A(\chi)$ -invariant and lies under  $\chi$ .*

*Proof.* Clifford's Theorem implies that  $\chi$  lies above the  $Q$ -conjugacy class of  $\lambda$ . The  $\pi$ -group  $A(\chi)$  fixes  $\chi$ , and normalizes  $E$ , as the latter is normal in  $G$ . Hence  $A(\chi)$  permutes among themselves the  $Q$ -conjugates of  $\lambda$ . As  $(|A(\chi)|, |Q|) = 1$ , Glauberman's Lemma (Lemma 13.8 in [7]) implies that  $A(\chi)$  fixes at least one character  $\lambda_1$  of the  $Q$ -conjugates of  $\lambda$ .  $\square$

The above lemma implies

**Proposition 6.5.** *Let  $Q$  be a normal  $q$ -subgroup of  $G$ , while  $A$  is any  $q'$ -subgroup of  $G$ . If  $\chi \in \text{Irr}(Q)$  we write  $T$  for the triple  $T = (G, Q, \chi)$ . Then there exists a chain of linear subtriples  $T_0, T_1, \dots, T_n$  of  $T$  starting with  $T_0 = T$  and ending with a linear limit  $T_n = T'$  of  $T$ , so that for all  $i = 0, 1, \dots, n$ , the central character  $\zeta^{(T_i)}$  of  $T_i$  is  $A(\chi)$ -invariant. We call such a  $T'$ , an  $A(\chi)$ -invariant linear limit of  $T$ .*

*Proof.* Let  $T_1 = (G_1, Q_1, \chi_1)$  be a linear reduction of  $T = (G, Q, \chi)$ . Then  $T_1 = T(\lambda_1)$ , where  $\lambda_1$  is a linear character of a normal subgroup  $E_1$  with  $E_1 \leq Q$ . According to Lemma 6.4 there is a  $Q$ -conjugate of  $\lambda_1$  that is  $A(\chi)$ -invariant. Thus, without loss, we may assume that  $\lambda_1$  is  $A(\chi)$ -invariant. So  $A(\chi) \leq G_1 = G(\lambda_1)$ . Because  $A(\chi)$  fixes  $\lambda_1$ , while  $\chi_1 \in \text{Irr}(Q_1)$  is the  $\lambda_1$ -Clifford correspondent of  $\chi$  in  $Q_1 = Q(\lambda_1)$ , we get  $A(\chi)(\chi_1) = A(\chi)$ . Let  $Z_1$  be the central subgroup of

$T_1$ , so  $E_1 \leq Z_1 \leq Q_1$ . If  $\zeta_1 \in \text{Lin}(Z_1)$  is the central character of  $T_1$ , then  $\zeta_1$  lies above  $\lambda_1$  while  $\chi_1|_{Z(T_1)} = \chi_1(1) \cdot \zeta_1$ . Hence  $A(\chi) = A(\chi, \chi_1)$  also fixes  $\zeta_1$ .

Let  $T_2 = (G_2, Q_2, \chi_2)$  be a linear reduction of  $T_1$ . So  $T_2 = T_1(\lambda_2)$ , for some linear character  $\lambda_2 \in \text{Irr}(E_2)$ , where  $E_2 \trianglelefteq G_1$ . Since  $A(\chi) = A(\chi, \chi_1)$ , Lemma 6.4 implies that we can pick  $\lambda_2$  to be  $A(\chi)$ -invariant. Hence the  $\lambda_2$ -Clifford correspondent  $\chi_2$  of  $\chi_1$  is also  $A(\chi)$ -invariant. Therefore, the central character  $\zeta_2$  of  $T_2$  is also  $A(\chi)$ -invariant, while  $A(\chi) = A(\chi, \chi_2)$ . We proceed similarly at every linear reduction until we reach  $T'$ .  $\square$

**Proposition 6.6.** *Let  $G$  be finite group of odd order such that  $G = NK$ , where  $N$  is a normal subgroup of  $G$  and  $(|G/N|, |N|) = 1$ . Let  $H = N \cap K$  and let  $\theta$  be any irreducible  $K$ -invariant character of  $H$  that induces an irreducible character  $\theta^N$  of  $N$ . Then  $\theta$  has a unique canonical extension,  $\theta^e$ , to  $K$  such that  $(|K/H|, o(\theta^e)) = 1$  (where  $o(\theta^e)$  is the determinantal order of  $\theta^e$ , see p. 88 in [7]). Also  $\theta^N$  has a unique canonical extension,  $(\theta^N)^e$ , to  $G$  such that  $(|G/N|, o((\theta^N)^e)) = 1$ . Furthermore,  $\theta^e$  induces*

$$(\theta^e)^G = (\theta^N)^e.$$

*Proof.* Let  $\pi$  be the set of primes that divide  $|N|$ . Then  $|K/H| = |G/N|$  is a  $\pi'$ -number, and thus is coprime to  $|H|$ . Because  $\theta \in \text{Irr}(H)$  is  $K$ -invariant, there exists a unique extension  $\theta^e$  to  $K$  such that

$$(6.7) \quad o(\theta) = o(\theta^e),$$

by Corollary 6.28 in [7].

According to Corollary 4.3 in [8], induction defines a bijection  $\text{Irr}(K|\theta) \rightarrow \text{Irr}(G|\theta^N)$ . Therefore,

$$(6.8) \quad \chi := (\theta^e)^G \in \text{Irr}(G|\theta^N).$$

But  $\theta^N$  is  $G$ -invariant since  $\theta$  is  $K$ -invariant and  $G = NK$ . This, and the fact that  $(|N|, |G/N|) = 1$ , implies that  $\theta^N$  extends to  $G$ . Let  $\Psi = (\theta^N)^e \in \text{Irr}(G)$  be the unique extension of  $\theta^N$  such that  $o(\Psi) = o(\theta^N)$  is a  $\pi$ -number. Since  $\chi$  lies above  $\theta^N$ , Gallagher's theorem (see Corollary 6.17 in [7]) implies that

$$\chi = \mu \cdot \Psi,$$

for some  $\mu \in \text{Irr}(G/N)$ . We compute the degree  $\deg(\chi)$  in two ways. First

$$\deg(\chi) = \deg(\mu) \cdot \deg(\Psi) = \deg(\mu) \cdot \deg(\theta^N) = \deg(\mu) \cdot |N : H| \cdot \deg(\theta).$$

As  $\chi = (\theta^e)^G$  we also have that

$$\deg(\chi) = |G : K| \cdot \deg(\theta^e) = |G : K| \cdot \deg(\theta) = |N : H| \cdot \deg(\theta).$$

We conclude that  $\deg(\mu) = 1$ . Thus  $\mu \in \text{Lin}(G/N)$ . Therefore

$$(6.9) \quad \det(\chi) = \mu^{\Psi(1)} \det(\Psi).$$

We can now compute  $o(\chi)$  in two ways. First,  $o(\Psi) = o(\theta^N)$  and  $\Psi(1) = \theta^N(1)$  are  $\pi$ -numbers. Since  $\mu \in \text{Irr}(G/N)$ , we get that  $o(\mu)$  is a  $\pi'$ -number. Therefore, (6.9) implies that the  $\pi'$ -number  $o(\mu)$  divides  $o(\chi)$ .

On the other hand, (6.8) and Lemma 2.2 in [9] imply that

$$o(\chi) = o((\theta^e)^G) \text{ divides } 2 \cdot o(\theta^e).$$



As  $G$  has odd order, we get that  $o(\chi)$  divides  $o(\theta^e)$ . In view of (6.7), we have  $o(\theta^e) = o(\theta)$ , while  $o(\theta) \mid |H|$ . We conclude that  $o(\chi)$  is a  $\pi$ -number.

Hence the only way the  $\pi'$ -number  $o(\mu)$  can divide  $o(\chi)$ , is if  $o(\mu) = 1$ . So  $\mu = 1$ , and

$$(\theta^e)^G = \chi = \Psi = (\theta^N)^e,$$

as desired.  $\square$

**Lemma 6.10.** *Let  $E \trianglelefteq Q$  be subgroups of a finite group  $G$ , and  $\zeta, \beta$  be irreducible characters of  $E$  and  $Q$ , respectively, with  $\beta$  lying above  $\zeta$ . Let  $T$  be any subgroup of  $G$  that normalizes both  $E$  and  $Q$ . If  $Q \leq T \leq G(\beta)$  then  $T = T(\zeta) \cdot Q$ . If in addition  $T$  is a Sylow subgroup of  $G(\beta)$  then  $T(\zeta)$  is a Sylow subgroup of  $G(\beta, \zeta)$ .*

*Proof.* Since  $T$  fixes  $\beta$ , it permutes among themselves the  $Q$ -conjugacy class of characters in  $\text{Irr}(E)$  lying under  $\beta$ . So  $T \leq T(\zeta) \cdot Q$ . The other inclusion is trivial.

Now assume that  $T$  is a  $q$ -Sylow subgroup of  $G(\beta)$ , for some prime  $q$ . Since  $G(\beta) = G(\beta, \zeta)Q$  with  $G(\beta, \zeta) \cap Q = Q(\zeta)$ , the index  $[G(\beta) : G(\beta, \zeta)]$  equals the index  $[Q : Q(\zeta)]$ . Similarly,  $[Q : Q(\zeta)] = [T : T(\zeta)]$ . Hence the index of  $T(\zeta) = G(\beta, \zeta) \cap T$  in  $G(\beta, \zeta)$  is a  $q'$ -number, which implies that  $T(\zeta)$  is a Sylow  $q$ -subgroup of  $G(\beta, \zeta)$ .  $\square$

The following is a well known result that we will use repeatedly below.

**Lemma 6.11.** *Let  $M = A \times B$  be a finite solvable group where  $(|A|, |B|) = 1$ . Let  $\beta \in \text{Irr}(B)$  be  $M$ -invariant. Then there is a unique canonical extension  $\beta^e$  of  $\beta$  to  $M$ . Furthermore, for any irreducible character  $\chi$  of  $M$  lying above  $\beta$  there exists a unique irreducible character  $\alpha$  of  $A$  such that  $\chi = \alpha \cdot \beta^e$ , where*

$$\chi(x \cdot y) = \alpha(x) \cdot \beta^e(xy),$$

for all  $x \in A$  and  $y \in B$ . We will write  $\chi = \alpha \times \beta$ , to denote the above product.

*Proof.* The existence of a unique canonical extension  $\beta^e$  of  $\beta$  follows from Corollary 6.28 in [7]. The rest of the lemma follows from Corollary 6.17 in [7].  $\square$

**Definition 6.12.** If  $M = A \times B$  where  $(|A|, |B|) = 1$ , and  $\alpha \in \text{Irr}(A)$  while  $1_B$  is the trivial character of  $B$ , we write  $\alpha \times 1_B$  (or simply  $\alpha \times 1$  if  $B$  is clear), for the unique irreducible character of  $M$  defined as  $\alpha \times 1(x \cdot y) = \alpha(x)$ , for all  $x \in A$  and  $y \in B$ . Notice that if  $M \leq G$  then  $G(\alpha) \leq G(\alpha \times 1)$ , while  $G(\alpha) = G(\alpha \times 1)$  if  $G$  normalizes  $A$ . Furthermore, if  $M \trianglelefteq G$  then Frattini's Argument implies that  $G(\alpha \times 1) = G(\alpha) \cdot B$ .

In view of the above definition we slightly generalize Proposition 6.5 to

**Proposition 6.13.** *Let  $M = P \times B$  be a normal subgroup of  $G$ , where  $P$  is a  $p$ -group and  $B$  is a normal  $p'$ -subgroup of  $G$ . Let  $\chi \in \text{Irr}(M)$  where  $\chi = \alpha \times 1$  and  $\alpha \in \text{Irr}(P)$ . Let  $T$  be the triple  $T = (G, M, \chi)$ . If  $A$  is any  $p'$ -subgroup of  $G$  then there exists an  $A(\chi)$ -invariant linear limit  $T'$  of  $T$ , i.e., there exists a chain of linear subtriples  $T_i = (G_i, M_i, \chi_i)$ , for  $i = 0, \dots, n$  starting with  $T = T_0$  and ending with the linear limit  $T_n = T'$  of  $T$  so that for all  $i = 0, 1, \dots, n$ , the central character  $\zeta_i$  of  $T_i$  is  $A(\chi)$ -invariant. Furthermore,  $T_i = T_{i-1}(\lambda_i)$  for all  $i = 1, \dots, n$  where  $\lambda_i$  is an*

$A(\chi)$ -invariant linear character of  $L_i$  lying under  $\chi_{i-1}$  and above  $\zeta_{i-1}$ , and  $L_i$  is a normal subgroup of  $G_{i-1}$  with  $B \leq Z(T_{i-1}) \leq L_i \leq M_{i-1}$ .

*Proof.* Let  $\bar{T}$  be the triple  $\bar{T} = (\bar{G}, \bar{M}, \bar{\chi})$ , where  $\bar{G} = G/B$ ,  $\bar{M} = M/B \cong P$  and  $\bar{\chi} \in \text{Irr}(\bar{M})$  is the unique irreducible character of  $\bar{M}$  that inflates to  $\chi \in \text{Irr}(M)$ . We also write  $\bar{A}$  for the quotient group  $\bar{A} = (A(\chi)B)/B$ . Clearly  $\bar{A}$  fixes  $\bar{\chi}$ , and its order is coprime to that of  $\bar{M}$ . Hence Proposition 6.5 implies that there exists a chain of  $\bar{A}$ -invariant linear subtriples  $\{\bar{T}_i\}_{i=0}^n$ , so that  $\bar{T}_0 = \bar{T}$ ,  $\bar{T}_i$  is a linear reduction of  $\bar{T}_{i-1}$ , while  $\bar{T}_n$  is a linear limit of  $\bar{T}$  and the central character  $\bar{\zeta}_i$  of each  $\bar{T}_i$  is  $\bar{A}$ -invariant.

Let  $\bar{T}_i = (\bar{G}_i, \bar{M}_i, \bar{\chi}_i)$ , for  $i = 0, 1, \dots, n$ . Then  $\bar{G}_i = G_i/B$ ,  $\bar{M}_i = M_i/B$ , where  $G_i$  and  $M_i$  are subgroups of  $G$  with  $M_i \leq M$ . In addition,  $\bar{\chi}_i$  inflates to a unique irreducible character  $\chi_i$  of  $M_i$ . So  $T_i = (G_i, M_i, \chi_i)$  is a subtriple of  $T$  for all such  $i$ . Furthermore, it is easy to see (with the use of Proposition 3.1) that if  $Z(\bar{T}_i)$  is the center of  $\bar{T}_i$  then  $Z(\bar{T}_i) = Z_i/B$  where  $Z_i$  is the center of  $T_i$ . In addition, the central character  $\zeta^{(\bar{T}_i)} \in \text{Irr}(Z(\bar{T}_i))$  of  $\bar{T}_i$  is inflated from the central character  $\zeta_i \in \text{Irr}(Z_i)$  of  $T_i$ . Hence the fact that  $\bar{\zeta}_i$  is  $\bar{A}$ -invariant implies that  $\zeta_i$  is  $A(\chi)$ -invariant, for all  $i = 1, \dots, n$ .

Even more,  $T_i$  is a linear reduction of  $T_{i-1}$ , for all  $i = 1, \dots, n$ . Indeed, the fact that  $\bar{T}_i$  is a linear reduction of  $\bar{T}_{i-1}$  implies that  $\bar{T}_i = \bar{T}_{i-1}(\bar{\lambda}_i)$ , where  $\bar{\lambda}_i$  is a linear character of the normal subgroup  $\bar{L}_i$  of  $\bar{G}_{i-1}$  that satisfies  $Z(\bar{T}_{i-1}) \leq \bar{L}_i \leq \bar{M}_{i-1}$ , while  $\bar{\lambda}_i$  lies under  $\bar{\chi}_{i-1}$ . Hence  $\bar{L}_i = L_i/B$ , where  $L_i$  is a normal subgroup of  $G_{i-1}$  with  $Z_{i-1} \leq L_i \leq M_{i-1}$  and  $\bar{\lambda}_i$  inflates to a unique linear character  $\lambda_i$  of  $L_i$  that lies under  $\chi_{i-1}$ . Now it is easily to see that  $\bar{G}_i = \bar{G}_{i-1}(\bar{\lambda}_i) = G_{i-1}(\lambda)/B$ . So  $G_i = G_{i-1}(\lambda)$  and similarly,  $M_i = M_{i-1}(\lambda)$ . Because  $\bar{\chi}_i \in \text{Irr}(\bar{M}_i)$  induces  $\bar{\chi}_{i-1}$  to  $\bar{M}_{i-1}$ , we have that  $\chi_i \in \text{Irr}(M_i)$  induces  $\chi_{i-1}$  to  $M_{i-1}$ . Also  $\chi_i$  lies above  $\lambda_i$  and thus  $\chi_i$  is the  $\lambda_i$ -Clifford correspondent of  $\chi_{i-1}$ , for all  $i = 1, \dots, n$ . We conclude that  $T_i = T_{i-1}(\lambda_i)$ , for all  $i = 1, \dots, n$ . In addition,  $\lambda_i$  is  $A(\chi)$ -invariant since  $\bar{\lambda}_i$  is  $\bar{A}$  invariant for all such  $i$ .

To see that  $T_n$  is a linear limit of  $T$  observe that the procedure described above works both ways. So any linear reduction of  $T_n$  determines a linear reduction of  $\bar{T}_n$ . As the latter triple is irreducible, we have that  $T_n$  is a linear limit of  $T$ .  $\square$

We first prove:

**Theorem 6.14.** *Assume that  $G$  is a finite group. Let  $L \leq M \leq N$  be normal subgroups of  $G$ , so that  $N/M, M/L$  and  $L$  are nilpotent group while  $N$  has order  $p^\alpha q^b$  for distinct odd primes  $p, q$ . So  $L = L_p \times Q$  where  $Q$  is the  $q$ -Sylow and  $L_p$  the  $p$ -Sylow group subgroup of  $L$ . Let  $M_p$  be a  $p$ -Sylow subgroup of  $M$  and let  $H = M_p L$ . Assume that  $\phi \in \text{Irr}(L)$  and  $\theta \in \text{Irr}(H)$  lies above  $\phi$ . Let  $\theta_\phi \in \text{Irr}(H(\phi))$  be the  $\phi$ -Clifford correspondent of  $\theta$ . Write  $\phi = \phi_p \times \beta$ , for  $\phi_p \in \text{Irr}(L_p)$  and  $\beta \in \text{Irr}(Q)$ . Assume further that all irreducible characters of  $G$  lying above  $\beta$  are monomial, while  $\beta$  extends to  $G(\theta_\phi)$ . Then there exists a linear limit  $(G', H', \theta')$  of  $(G, H, \theta)$  so that*

$$[N'_q, H'_p] \leq K',$$

where  $K'$  is the kernel of the triple  $(G', H', \theta')$ , while  $H'_p$  is a  $p$ -Sylow subgroup of  $H'$  and  $N'_q$  is a  $q$ -Sylow subgroup of  $N' = G' \cap N$ .

*Proof.* First note that  $H = M_p L$  is a normal subgroup of  $G$ , because  $H/L$  is the  $p$ -Sylow subgroup of the nilpotent group  $M/L$ . We fix  $\theta \in \text{Irr}(H)$  and  $\phi = \phi_p \times \beta \in \text{Irr}(L)$  satisfying the hypothesis of the theorem. We also pick a  $p$ -Sylow subgroup  $A$  of  $G$  so that  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G(\beta)$ .

Because  $H/Q$  is a  $p$ -group,  $H$  is the semidirect product  $H = (A \cap H) \rtimes Q$ , where  $A \cap H$  is a  $p$ -Sylow subgroup of  $H$ . Let  $P = (A \cap H)(\phi)$ . Then  $H(\phi) = P \rtimes Q$ . Furthermore,  $\beta$  extends to  $H(\phi) \leq H(\beta)$ , since  $(|H/Q|, |Q|) = 1$ . Hence, if  $\theta_\phi \in \text{Irr}(H(\phi))$  is the  $\phi$ -Clifford correspondent of  $\theta$ , then (see Lemma 6.11) there exists a unique irreducible character  $\alpha \in \text{Irr}(P)$  such that

$$\theta_\phi = \alpha \rtimes \beta,$$

Because  $H \trianglelefteq G$ , Frattini's argument implies that  $G(\theta_\phi) = G(\alpha, \beta)Q$ . Note also that  $\alpha$  lies above  $\phi_p \in \text{Irr}(L_p)$  as  $\theta_\phi$  lies above  $\phi$ . Hence  $G(\alpha) \leq G(\phi_p) \leq G$ .

At this point we pick a  $q$ -Sylow subgroup  $B$  of  $G$  so that  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$  while  $B(\alpha, \beta)$  is a  $q$ -Sylow subgroup of  $G(\alpha, \beta)$ .

Let  $S = (G, Q, \beta)$ . Because  $A(\beta)$  has coprime order to that of  $Q$ , Proposition 6.5 implies that we can get an  $A(\beta)$ -invariant linear limit  $S_1 = (G_1, Q_1, \beta_1)$  of  $S$ . Hence the central character  $\zeta_1$  of  $S_1$  is  $A(\beta)$ -invariant. Furthermore,  $G_1 = G(\zeta_1)$  while  $\beta_1$  is the unique character of  $Q_1 = Q(\zeta_1)$  lying above  $\zeta_1$ , by Proposition 3.14. We write  $L_1 = L \cap G_1$ ,  $H_1 = G_1 \cap H = H(\zeta_1)$ ,  $M_1 = M \cap G_1 = M(\zeta_1)$  and  $N_1 = G_1 \cap N = N(\zeta_1)$ . Observe that  $L_1 = L_p \times Q_1$ , while  $N_1/M_1$  is a nilpotent group and  $M_1/H_1$  is a  $q$ -group. Let  $\theta_1 \in \text{Irr}(H_1)$  be the  $S_1$ -reduction of  $\theta$ . So  $\theta_1$  lies above  $\phi_1$  the  $S_1$ -reduction of  $\phi$ . Note that  $\phi_1 = \phi_p \times \beta_1$ . So  $\theta_1$  lies above  $\beta_1$  and thus above  $\zeta_1$ , and induces  $\theta$ . Clearly all the irreducible characters of  $G_1$  that lie above  $\beta_1$  (and equivalently above  $\zeta_1$ ) are still monomial (see Proposition 3.17).

Since  $G_1 = G(\zeta_1)$  we have  $A(\beta) \leq G_1$ . Also  $G_1 = G(\zeta_1) = G(\zeta_1, \beta_1)$  by Proposition 3.12. Thus  $G_1 \leq G(\beta)$ , since  $\beta_1$  induces  $\beta$ . Also  $H_1 \leq H(\beta)$ . The group  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G(\beta)$  contained in  $G_1$ . Hence  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G_1$ . Furthermore, the  $p$ -Sylow subgroup  $P$  of  $H(\beta)$  is contained in  $H_1$ , and thus  $P$  is also a  $p$ -Sylow subgroup of  $H_1$ . So  $H_1 = P \rtimes Q_1$ . Because  $\beta$  extends to  $G(\theta_\phi) = QG(\alpha, \beta)$ , Lemma 6.1 implies that  $\beta_1$  extends to  $(QG(\alpha, \beta))(\zeta_1, \beta_1)$ . Hence  $\beta_1$  extends to  $Q_1 G_1(\alpha)$  (note that  $G_1(\beta) = G_1(\beta_1) = G_1$ ). Let  $\beta_1^e$  be the unique canonical extension of  $\beta_1$  to  $H_1$ . Then there exists a unique irreducible character  $\alpha'$  of  $P$  so that  $\theta_1 = \alpha \rtimes \beta_1 = \alpha' \cdot \beta_1^e$ . It is not hard to see that  $\theta_1^{H(\beta)} = \alpha' \cdot (\beta_1^e)^{H(\beta)}$  (see Exercise 5.3 in [7]). Since  $\beta_1$  induces  $\beta \in \text{Irr}(Q)$ , Lemma 6.6 implies that  $(\beta_1^e)^{H(\beta)} = \beta^e$ . So  $\theta_1^{H(\beta)} = \alpha' \cdot \beta^e$ . But  $\theta_1^{H(\beta)}$  lies above  $\beta$  and induces  $\theta \in \text{Irr}(H)$  (because  $\theta_1$  does). Hence  $\theta_1^{H(\beta)} = \theta_\beta$ , and thus  $\alpha' = \alpha$ . Therefore,

$$(6.15) \quad \theta_1 = \alpha \rtimes \beta_1,$$

which implies that  $G_1(\theta_1) = G_1(\alpha, \beta_1)Q_1 = G_1(\alpha)Q_1$ .

Let  $K_1 = \text{Ker}(\zeta_1)$  and  $\overline{G}_1 = G_1/K_1$ ,  $\overline{H}_1 = H_1/K_1$  and  $\overline{Q}_1 = Q_1/K_1$ . Then  $\overline{Q}_1 \leq \overline{H}_1 \leq \overline{N}_1$  are normal subgroups of  $\overline{G}_1$ . Furthermore, the irreducible characters  $\beta_1, \theta_1$  of  $Q_1$  and  $H_1$  are inflated from unique irreducible characters  $\overline{\beta}_1$  and  $\overline{\theta}_1$  of  $\overline{Q}_1$  and  $\overline{H}_1$ , all respectively. If  $Z_1$  is the center of the triple  $S_1 = (G_1, Q_1, \beta_1)$  then we write  $\overline{Z}_1$  for the quotient group  $Z_1/K_1$ . We also write  $\overline{\zeta}_1$  for the unique irreducible character of  $\overline{Z}_1$  that inflates to the central character  $\zeta_1 \in \text{Irr}(Z_1)$  of  $S_1$ . According to Proposition 3.7, the cyclic group  $\overline{Z}_1$  is the center of  $\overline{Q}_1$ , affords the faithful

$\overline{G}_1$ -invariant linear character  $\overline{\zeta}_1$ , and it is maximal among the abelian normal subgroups of  $\overline{G}_1$  contained in  $\overline{Q}_1$ . So  $\overline{Q}_1$  satisfies Condition X. Since every irreducible character of  $G_1$  lying above  $\zeta_1$  is monomial, we get that every irreducible character of  $\overline{G}_1$  lying above  $\overline{\zeta}_1$ , and equivalently above  $\overline{\beta}_1$ , is also monomial. Because the  $q$ -special character  $\overline{\zeta}_1$  is  $\overline{G}_1$ -invariant, there exists a  $q$ -special character  $\overline{\chi}_1$  of  $\overline{G}_1$  lying above  $\overline{\zeta}_1$ . So  $\overline{\chi}_1$  is monomial. Let  $\overline{P} = (PK_1)/K_1$ . Then  $\overline{P} \cong P$  while  $\overline{P} \cdot \overline{Q}_1 = \overline{H}_1 \trianglelefteq \overline{G}_1$ . Hence we can apply Theorem B to the groups  $\overline{Q}_1, \overline{P}$  and  $\overline{G}_1$  in the place of  $P, S$  and  $G$  respectively. We conclude that  $\overline{P}$  centralizes  $\overline{Q}_1$ , and thus  $\overline{H}_1$  is a nilpotent group. Furthermore, if  $\overline{\alpha}$  is the unique irreducible character of  $\overline{P}$  that inflates to  $\alpha \times 1 \in \text{Irr}(PK_1)$ , then

$$\overline{H}_1 = \overline{P} \times \overline{Q}_1 \text{ and } \overline{\theta}_1 = \overline{\alpha} \times \overline{\beta}_1.$$

So both  $\overline{P}$  and  $\overline{Q}_1$  are normal subgroups of  $\overline{G}_1$ . Also, it is easy to see that  $G_1(\alpha)/K_1 = \overline{G}_1(\overline{\alpha})$ , because  $\overline{G}_1$  normalizes  $P$  and thus  $\overline{G}_1(\overline{\alpha}) \leq G_1(\alpha \times 1)/K_1 \leq G_1(\alpha)/K_1$  (the other inclusion holds trivially). Because  $\beta_1$  extends to  $Q_1G_1(\alpha)$ , we get that  $\overline{\beta}_1$  extends to  $(Q_1G_1(\alpha))/K_1$ . So  $\overline{\beta}_1$  extends to  $\overline{G}_1(\overline{\alpha})$  (note that  $\overline{Q}_1$  centralizes  $\overline{P}$  and thus it is a subgroup of  $\overline{G}_1(\overline{\alpha})$ ).

Take  $\overline{E}_1$  to be the triple  $\overline{E}_1 = (\overline{G}_1, \overline{P}, \overline{\alpha})$ . If  $B_1 = B \cap G_1 = B(\zeta_1)$ , then  $\overline{B}_1 = B_1/K_1$  is a  $q$ -subgroup of  $\overline{G}_1$ . Thus we can form a  $\overline{B}_1(\overline{\alpha})$ -invariant linear limit  $\overline{E}_2$  of  $\overline{E}_1$ . So  $\overline{E}_2 = (\overline{G}_2, \overline{P}_2, \overline{\alpha}_2)$ , where  $\overline{P}_2 \leq \overline{P}$  and  $\overline{\alpha}_2$  induces  $\overline{\alpha}$  to  $\overline{P}$ . Hence  $\overline{P}_2 = P_2/K_1$  with  $P_2 \leq P$ , while  $\overline{\alpha}_2$  inflates to a unique irreducible character  $\alpha_2$  of  $P_2$  that induces  $\alpha$  to  $P$ . In addition,  $\overline{G}_2 = G_2/K_1$  where  $G_2$  is a subgroup of  $G_1$ . The central character  $\overline{\zeta}_2$  of  $\overline{E}_2$  is  $\overline{B}_1(\overline{\alpha})$ -invariant. Also  $\overline{G}_2 = \overline{G}_1(\overline{\zeta}_2)$  and  $\overline{P}_2 = \overline{P}(\overline{\zeta}_2)$ . The center  $\overline{Z}_2$  of  $\overline{E}_2$  clearly contains  $K_1$ , even more the kernel  $\overline{K}_2 = \text{Ker}(\overline{\zeta}_2)$  of  $\overline{\zeta}_2$  contains  $K_1$ . According to Propositions 3.12 and 3.14 the character  $\overline{\alpha}_2$  is  $\overline{G}_2$ -invariant and it is the only character of  $\overline{P}_2$  that lies above  $\overline{\zeta}_2 \in \text{Irr}(\overline{Z}_2)$ . So

$$(6.16) \quad \overline{G}_2 = \overline{G}_2(\overline{\alpha}_2) = \overline{G}_2(\overline{\alpha}) \leq \overline{G}_1(\overline{\alpha}),$$

where the last equality follows from the fact that  $\overline{\alpha}_2$  is the only irreducible character of  $\overline{P}_2$  that lies above  $\overline{\zeta}_2$  and induces  $\overline{\alpha}$  to  $\overline{P}$ .

We next observe that the  $\overline{E}_2$ -reductions leave  $\overline{Q}_1$  unchanged, i.e.,  $\overline{Q}_1(\overline{\zeta}_2) = \overline{Q}_1$ , because  $\overline{Q}_1$  centralizes  $\overline{P}$ . Hence  $\overline{Q}_1 \leq \overline{B}_1(\overline{\alpha}) \leq \overline{G}_2$ . Furthermore, the  $\overline{E}_2$ -reduction of  $\overline{H}_1 = \overline{P} \times \overline{Q}_1$  equals  $\overline{H}_2 = \overline{P}_2 \times \overline{Q}_1$ . In addition, the  $\overline{E}_2$  reduction of  $\overline{\theta}_1 \in \text{Irr}(\overline{H}_1)$  equals  $\overline{\theta}_2 = \overline{\alpha}_2 \times \overline{\beta}_1$ . Note that since  $\overline{\beta}_1$  extends to  $\overline{G}_1(\overline{\alpha})$ , equation 6.16 implies that  $\overline{\beta}_1$  extends to  $\overline{G}_2$ . Because every irreducible character of  $\overline{G}_1$  lying above  $\overline{\beta}_1$  is monomial, Proposition 3.17 implies that every irreducible character of  $\overline{G}_2$  lying above  $\overline{\theta}_2$  is still monomial.

We look at the quotient group  $\overline{G}_2/\overline{K}_2$ . Its subgroup  $\overline{H}_2/\overline{K}_2$  is a nilpotent normal subgroup, and splits as  $\overline{H}_2/\overline{K}_2 = \overline{P}_2/\overline{K}_2 \times (\overline{Q}_1\overline{K}_2)/\overline{K}_2$ , where  $(\overline{Q}_1\overline{K}_2)/\overline{K}_2$ , is naturally isomorphic to  $\overline{Q}_1$ . We identify these two isomorphic groups and we consider  $\overline{\beta}_1$  as an irreducible character of  $(\overline{Q}_1\overline{K}_2)/\overline{K}_2$ . As we have seen  $\overline{\beta}_1$  extends to  $\overline{G}_2$ , hence  $\overline{\beta}_1$  (considered as a character of  $(\overline{Q}_1\overline{K}_2)/\overline{K}_2$ ) extends to  $\overline{G}_2/\overline{K}_2$ . Let  $\hat{\beta}_1 \in \text{Irr}(\overline{G}_2/\overline{K}_2)$  be such an extension. Then  $\hat{\beta}_1$  is a  $q$ -special character of  $\overline{G}_2/\overline{K}_2$ . The irreducible character  $\overline{\alpha}_2$  of  $\overline{P}_2$  is inflated from a unique irreducible character  $\overline{\alpha}'_2$  of the  $p$ -group  $\overline{P}_2/\overline{K}_2$ . Because  $\overline{\alpha}_2$  is  $\overline{G}_2$ -invariant,  $\overline{\alpha}'_2$  is a  $\overline{G}_2/\overline{K}_2$ -invariant  $p$ -special character of  $\overline{P}_2/\overline{K}_2$ . Therefore there exists a  $p$ -special irreducible character  $\hat{\alpha}_2$  of  $\overline{G}_2/\overline{K}_2$  lying above  $\overline{\alpha}'_2$ . Then the product  $\hat{\chi} = \hat{\alpha}_2 \cdot \hat{\beta}_1$  is an irreducible character of  $\overline{G}_2/\overline{K}_2$  that lies above  $\overline{\alpha}'_2 \times \overline{\beta}_1 \in \text{Irr}(\overline{H}_2/\overline{K}_2)$ . In addition  $\hat{\chi}_q = \hat{\beta}_1(1)$ . Observe also that  $\hat{\chi}$  is monomial, because every irreducible character of

$\overline{G}_2$  lying above  $\overline{\theta}_2$  is monomial, while  $\overline{\alpha}'_2 \times \overline{\beta}_1$  inflates to  $\overline{\theta}_2 \in \text{Irr}(\overline{H}_2)$ . Furthermore, Proposition 3.7 implies that  $\overline{Z}_2/\overline{K}_2$  is the center of  $\overline{P}_2/\overline{K}_2$ , it is maximal among the abelian normal subgroups of  $\overline{G}_2/\overline{K}_2$  contained in  $\overline{P}_2/\overline{K}_2$  and it affords a faithful  $\overline{G}_2/\overline{K}_2$ -invariant irreducible character that inflates to  $\overline{\zeta}_2 \in \text{Irr}(\overline{Z}_2)$ . We conclude that all the hypothesis of Theorem B are satisfied for the groups  $\overline{G}_2/\overline{K}_2, \overline{P}_2/\overline{K}_2$  and  $(\overline{Q}_1\overline{K}_2)/\overline{K}_2$  in the place of  $G, P$  and  $S$ , respectively. Hence any  $p'$ -subgroup  $\mathcal{Q}$  of  $\overline{G}_2/\overline{K}_2$  centralizes  $\overline{P}_2/\overline{K}_2$  provided that  $\mathcal{Q} \cdot \overline{P}_2/\overline{K}_2$  is a normal subgroup of  $\overline{G}_2/\overline{K}_2$ .

Let  $\overline{N}_2$  be the  $\overline{E}_2$ -reduction of  $\overline{N}_1$  (that is  $\overline{N}_2 = \overline{N}_1(\overline{\zeta}_2)$ ), and  $\overline{M}_2$  the  $\overline{E}_2$  reduction of  $\overline{M}_1$ . Then  $\overline{N}_2/\overline{M}_2$  is still a nilpotent group, while  $\overline{M}_2/\overline{H}_2$  is a  $q$ -group. Let  $U$  be a  $q$ -Sylow subgroup of  $\overline{N}_2$ , then  $U\overline{M}_2 = U\overline{H}_2$  is a normal subgroup of  $\overline{G}_2$ , because  $(U\overline{M}_2)/\overline{M}_2$  is the  $q$ -Sylow subgroup of  $\overline{N}_2/\overline{M}_2$ . So  $(U\overline{H}_2)/\overline{K}_2 \trianglelefteq \overline{G}_2$ . Furthermore  $(U\overline{H}_2)/\overline{K}_2 = (U\overline{K}_2)/\overline{K}_2 \times \overline{P}_2/\overline{K}_2$ . Hence the  $q$ -group  $(U\overline{K}_2)/\overline{K}_2$  centralizes  $\overline{P}_2/\overline{K}_2$ , i.e.,  $[U, \overline{P}_2] \leq \overline{K}_2$ . So

$$[U, P_2] \leq K_1K_2.$$

It is easy to see that the triple  $(G_2, P_2 \times Q_1, \alpha_2 \times \beta_1)$  is a multilinear reduction of  $(G, P \times Q, \theta_\phi)$ . This completes the proof of the theorem.  $\square$

For later use we observe that what we actually proved in Theorem 6.14 is

**Remark 6.17.** Assume the hypothesis and the notation of Theorem 6.14. Then any  $A(\beta)$ -invariant linear limit  $S_1 = (G_1, B_1, \beta_1)$  of  $S = (G, B, \beta)$  makes  $(G_1 \cap H)/K_1$  nilpotent, where  $A$  is a  $p$ -Sylow subgroup of  $G$  and  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G(\beta)$  and  $K_1$  is the kernel of  $S_1$ . Furthermore,  $H_1 = P \times Q_1$ , where  $P$  is a  $p$ -Sylow subgroup of  $H(\phi)$ . Let  $\overline{E}_1$  be the factor triple  $\overline{E}_1 = (\overline{G}_1, \overline{P}, \overline{\alpha})$ , where  $\overline{G}_1 = G_1/K_1$  and  $\overline{P} = (PK_1)/K_1 \cong P$ . If  $B$  is a  $p'$ -Hall subgroup of  $G$  so that  $B(\alpha)$  is a  $p'$ -Hall subgroup of  $G(\alpha)$  and  $B(\alpha, \beta)$  is a  $p'$ -Hall subgroup of  $G(\alpha, \beta)$ , let  $\overline{B}_1 = (B \cap G_1)/K_1$ . Then any  $\overline{B}_1(\overline{\alpha})$ -invariant linear limit  $\overline{E}_2 = (\overline{G}_2, \overline{P}_2, \overline{\alpha}_2)$  of  $\overline{E}_1$  satisfies  $[(\overline{N}_1 \cap \overline{G}_2)_q, \overline{P}_2] \leq \overline{K}_2$ , where  $\overline{K}_2$  is the kernel of  $\overline{E}_2$  while  $\overline{N}_1 = (N \cap G_1)/K_1$  and  $\overline{H}_1 = H_1/K_1$ .

**Corollary 6.18.** Assume the hypothesis of Theorem 6.14. Let  $B$  be a  $q$ -Sylow subgroup of  $G$  so that  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$  while  $B(\alpha, \beta)$  is a  $q$ -Sylow subgroup of  $G(\alpha, \beta)$ . Assume further that  $A$  is a  $p$ -Sylow subgroup of  $G$  with  $A(\beta)$  being a  $p$ -Sylow subgroup of  $G(\beta)$ , while  $P = (A \cap H)(\beta)$ . Let  $\zeta_1$  be the central character of the  $A(\beta)$ -invariant linear reduction  $S_1 = (G_1, Q_1, \beta_1)$  of  $S = (G, Q, \beta)$ . Then  $G(\beta) = G(\beta, \zeta_1) \cdot Q$ . If in addition  $C = C_Q(P)$  then  $G(\alpha, \beta) = G(\alpha, \beta, \zeta_1) \cdot C$ . Thus  $B(\alpha, \beta, \zeta_1) \cdot K_1 = B_1(\alpha)K_1$  is a  $p'$ -Hall subgroup of  $G_1(\alpha) \cdot K_1$ .

*Proof.* Since  $S_1$  is an  $A(\beta)$ -invariant linear limit of  $S$ , there exists a chain of linear subtriples  $D_i = (\hat{G}_i, \hat{Q}_i, \hat{\beta}_i)$  of  $S$ , for  $i = 0, 1, \dots, n$ , such that  $S = D_0 \geq D_1 \geq \dots \geq D_n = S_1$ , with  $D_i$  being a linear reduction of  $D_{i-1}$ , for all  $i = 1, \dots, n$ . Hence  $D_i = D_{i-1}(\lambda_i)$  where  $\lambda_i$  is a linear character of a normal subgroup  $L_i$  of  $\hat{G}_{i-1}$ , that lies under  $\hat{\beta}_{i-1}$ . In addition,  $\hat{G}_i = \hat{G}_{i-1}(\lambda_i)$  and  $\hat{Q}_i = \hat{Q}_{i-1}(\lambda_i)$  while  $\hat{\beta}_i \in \text{Irr}(\hat{Q}_i)$  is the  $\lambda_i$ -Clifford correspondent of  $\hat{\beta}_{i-1} \in \text{Irr}(\hat{Q}_{i-1})$ . Thus

$$Z(D_{i-1}) \leq L_i \leq Z(D_i) \leq \hat{Q}_i \leq \hat{Q}_{i-1},$$

while  $\zeta^{(D_i)}$  is an extension of both  $\lambda_i$  and  $\zeta^{(D_{i-1})}$ , and lies under  $\hat{\beta}_i$ . According to Proposition 3.14, we have  $\hat{G}_i = G(\zeta^{(D_i)})$  and  $\hat{Q}_i = Q(\zeta^{(D_i)})$ . The fact that the linear limit  $S_1 = D_n$  is  $A(\beta)$ -invariant, implies that  $\zeta^{(D_i)}$  is  $A(\beta)$ -invariant for all  $i = 0, 1, \dots, n$ .

According to Lemma 6.10, applied to the groups  $L_{i+1} \trianglelefteq \hat{Q}_i \trianglelefteq \hat{G}_i$ , we have  $G(\hat{\beta}_i) = G(\hat{\beta}_i, \lambda_{i+1}) \cdot \hat{Q}_i$ , for all  $i = 0, 1, \dots, n-1$ . Since  $G(\zeta^{(D_i)}) = \hat{G}_i \leq G(\lambda_i)$ , while  $\zeta^{(D_i)}$  extends  $\lambda_i$ , we also have  $G(\hat{\beta}_i, \lambda_{i+1}) = G(\hat{\beta}_i, \zeta^{(D_{i+1})})$ . In addition, the latter group normalizes  $\hat{Q}_{i+1} = Q(\zeta^{(D_{i+1})})$  and fixes its irreducible character  $\hat{\beta}_{i+1}$ . Hence  $G(\hat{\beta}_i, \zeta^{(D_{i+1})}) \leq G(\hat{\beta}_{i+1}, \zeta^{(D_{i+1})})$ . The other inclusion also holds, because  $\hat{\beta}_{i+1}$  induces  $\hat{\beta}_i$  and  $\hat{Q}_i = Q(\zeta^{(D_i)})$ . So for all  $i = 0, 1, \dots, n-1$  we get

$$G(\hat{\beta}_i) = G(\hat{\beta}_{i+1}, \zeta^{(D_{i+1})}) \cdot \hat{Q}_i$$

Therefore,

$$G(\beta) = G(\hat{\beta}_1)(\zeta^{(D_1)}) \cdot Q = G(\hat{\beta}_2)(\zeta^{(D_2)}, \zeta^{(D_1)}) \cdot \hat{Q}_1 \cdot Q = \dots = G(\hat{\beta}_n)(\zeta^{(D_n)}, \dots, \zeta^{(D_1)}) \cdot \hat{Q}_{n-1} \cdots \cdots Q.$$

But  $\zeta^{(D_n)}$  extends all the previous central characters, while  $\hat{Q}_i$  is a subgroup of  $Q$ , for all  $i = 1, \dots, n-1$ . Thus  $G(\beta) = G(\hat{\beta}_n, \zeta^{(D_n)}) \cdot Q$ . As  $\hat{\beta}_n$  induces  $\beta$  we get  $G(\hat{\beta}_n, \zeta^{(D_n)}) \cdot Q \leq G(\beta, \zeta^{(D_n)}) \cdot Q \leq G(\beta)$ . So  $G(\beta) = G(\beta, \zeta_1) \cdot Q$ , since  $\zeta_1 = \zeta^{(D_n)}$ . This completes the proof of the first part of the corollary.

Because  $P \leq A(\beta)$  the characters  $\zeta^{(D_i)}$  are  $P$ -invariant. As  $P$  also fixes  $\beta$  we conclude that  $P$  fixes  $\hat{\beta}_i$ , for all  $i = 1, \dots, n$ . Let  $\beta_i^* \in \text{Irr}(C_{\hat{Q}_i}(P))$  be the  $P$ -Glauberman correspondent of  $\hat{\beta}_i \in \text{Irr}(\hat{Q}_i)$ , for all  $i = 1, \dots, n$ . Similarly we define  $\zeta_i^* \in \text{Irr}(C_{Z(D_i)}(P))$  to be the  $P$ -Glauberman correspondent of  $\zeta^{(D_i)}$ , for all such  $i$ . Then  $\zeta_{i+1}^*$  lies under  $\beta_i^*$ , as  $\zeta^{(D_{i+1})}$  lies under  $\hat{\beta}_i$ . The group  $G(\alpha, \beta, \zeta^{(D_i)})$  fixes  $\zeta^{(D_0)}, \zeta^{(D_1)}, \dots, \zeta^{(D_{i-1})}$  since  $\zeta^{(D_i)}$  is an extension of all these characters. Because  $\hat{\beta}_i$  is the only character of  $\hat{Q}_i$  lying above  $\zeta^{(D_i)}$  (by Proposition 3.14), we conclude that  $G(\alpha, \beta, \zeta^{(D_i)})$  fixes the Glauberman correspondents  $\beta_i^*$ , for all  $i = 0, 1, \dots, n$ . In addition, the group  $G(\alpha, \beta, \zeta^{(D_i)})$  normalizes both  $C_{Z(D_i)}(P)$  and  $C_{\hat{Q}_i}(P)$ . Therefore Lemma 6.10 implies

$$(6.19) \quad G(\alpha, \beta, \zeta^{(D_i)}) = G(\alpha, \beta, \zeta^{(D_i)}, \zeta_{i+1}^*) \cdot C_{Q(\zeta^{(D_i)})}(P),$$

for all  $i = 0, 1, \dots, n-1$ . (For  $i = 0$  the above equation becomes  $G(\alpha, \beta) = G(\alpha, \beta, \zeta_1^*) \cdot C$ .) The group  $L_{i+1}$  was picked to be a normal subgroup of  $G_i = G(\zeta^{(D_i)})$ . Hence  $G(\alpha, \beta, \zeta^{(D_i)})$  normalizes  $L_{i+1}$ , as well as  $P$ . Thus  $G(\alpha, \beta, \zeta^{(D_i)}, \zeta_{i+1}^*) = G(\alpha, \beta, \zeta^{(D_i)}, \zeta^{(D_{i+1})})$ . But  $\zeta^{(D_{i+1})}$  is an extension of  $\zeta^{(D_i)}$  and therefore  $G(\alpha, \beta, \zeta^{(D_i)}, \zeta^{(D_{i+1})}) = G(\alpha, \beta, \zeta^{(D_{i+1})})$  for all  $i = 1, \dots, n-1$ . So (6.19) implies

$$G(\alpha, \beta, \zeta^{(D_i)}) = G(\alpha, \beta, \zeta^{(D_{i+1})}) \cdot C_{Q(\zeta^{(D_i)})}(P),$$

for all  $i = 1, \dots, n-1$ , while for  $i = 0$  we have  $G(\alpha, \beta) = G(\alpha, \beta, \zeta^{(D_1)}) \cdot C$ . Therefore,

$$G(\alpha, \beta) = G(\alpha, \beta, \zeta^{(D_n)}) \cdot C_{Q(\zeta^{(D_1)})}(P) \cdot C_{Q(\zeta^{(D_2)})}(P) \cdots \cdots C_{Q(\zeta^{(D_{n-1})})}(P) \cdot C = G(\alpha, \beta, \zeta^{(D_n)}) \cdot C.$$

Since  $\zeta^{(D_n)} = \zeta_1$ , the second part of the corollary follows.

The group  $B$  of Theorem 6.14 was picked so that  $B(\alpha, \beta)$  is a  $p'$ -Hall subgroup of  $G(\alpha, \beta)$ . Since the latter group equals  $G(\alpha, \beta, \zeta_1) \cdot C$  we conclude (by looking at the indexes) that  $B(\alpha, \beta, \zeta_1)$  is a  $p'$ -Hall subgroup of  $G(\alpha, \beta, \zeta_1)$ . Of course  $B(\alpha, \beta, \zeta_1) = B_1(\alpha)$  and  $G(\alpha, \beta, \zeta_1) = G_1(\alpha)$ , as  $G_1 = G(\zeta_1)$  and it is a subgroup of  $G(\beta)$ . Because  $B_1(\alpha)$  is a  $p'$ -Hall subgroup of  $G_1(\alpha)$  and  $K_1$  is a  $p'$ -subgroup of  $B_1$  we get that  $B_1(\alpha) \cdot K_1$  is a  $p'$ -Hall subgroup of  $G_1(\alpha) \cdot K_1$ . This completes the proof of the corollary.  $\square$

We can remove the additional hypothesis on part b) of Theorem 6.14 that wants  $\beta$  extendible to  $QG(\alpha, \beta)$ , without loosing any of its conclusions, provided firstly that the group  $G$  has order  $p^a q^b$ , for some odd primes  $p$  and  $q$ , and secondly that we have plenty of monomial characters in  $\text{Irr}(G)$ . Thus we can prove the following result.

**Theorem 6.20.** *Assume that  $G$  is a monomial group of order  $p^a q^b$ , where  $p, q$  are two odd primes. Let  $L \leq M \leq N$  be normal subgroups of  $G$ , so that  $N/M, M/L$  and  $L$  are nilpotent groups. So  $L = L_p \times Q$  where  $Q$  is the  $q$ -Sylow subgroup of  $L$  and  $L_p$  its  $p$ -Sylow subgroup. Let  $M_p$  be a  $p$ -Sylow subgroup of  $M$  and let  $H = M_p L$ . Assume that  $\phi \in \text{Irr}(L)$  and  $\theta \in \text{Irr}(H)$  lies above  $\phi$  and let  $\theta_\phi \in \text{Irr}(H(\phi))$  be the  $\phi$ -Clifford correspondent of  $\theta$ . Write  $\phi = \phi_p \times \beta$ , for  $\phi_p \in \text{Irr}(L_p)$  and  $\beta \in \text{Irr}(Q)$ . Let  $A$  be a  $p$ -Sylow subgroup of  $G$  so that  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G(\beta)$ . If  $S_1 = (G_1, Q_1, \beta_1)$  is an  $A(\beta)$ -invariant linear limit of  $S = (G, Q, \beta)$ , then  $(G_1 \cap H)/K_1 = \overline{H}_1 = \overline{P} \times \overline{Q}_1$ , where  $K_1$  is the kernel of  $S_1$ ,  $\overline{Q}_1 = Q_1/K_1$  and  $\overline{P} = (PK_1)/K_1$  where  $P = (A \cap H)(\phi)$  is a  $p$ -Sylow subgroup of  $H(\phi)$ . Furthermore, there exists a multilinear reduction  $\overline{E}^*_1 = (\overline{G}^*_1, \overline{P}^*, \overline{\alpha}^*)$  of  $\overline{E}_1 = (\overline{G}_1, \overline{P}, \overline{\alpha})$  so that  $[(\overline{G}^*_1 \cap \overline{N}_1)_q, (\overline{G}^*_1 \cap \overline{H}_1)_p] \leq \text{Ker}(\overline{E}^*_1)$ .*

*Proof.* The group  $H = M_p L$  is a normal subgroup of  $G$ . We fix  $\theta \in \text{Irr}(H)$  and  $\phi = \phi_p \times \beta \in \text{Irr}(L)$  satisfying the hypothesis of the theorem. We also pick a  $p$ -Sylow subgroup  $A$  of  $G$  so that  $A(\beta)$  is a  $p$ -Sylow subgroup of  $G(\beta)$ .

Because  $H/Q$  is a  $p$ -group,  $H$  is the semidirect product  $H = (A \cap H) \rtimes Q$ , where  $A \cap H$  is a  $p$ -Sylow subgroup of  $H$ . Let  $P = (A \cap H)(\phi)$ . Then  $H(\phi) = P \rtimes Q$ . Furthermore,  $\beta$  extends to  $H(\phi) \leq H(\beta)$ , since  $(|H/Q|, |Q|) = 1$ . Hence, if  $\theta_\phi \in \text{Irr}(H(\phi))$  is the  $\phi$ -Clifford correspondent of  $\theta$ , then (see Lemma 6.11) there exists a unique irreducible character  $\alpha \in \text{Irr}(P)$  such that

$$\theta_\phi = \alpha \rtimes \beta,$$

Because  $H \trianglelefteq G$ , Frattini's argument implies that  $G(\theta_\phi) = G(\alpha, \beta)Q$ . Note also that  $\alpha$  lies above  $\phi_p \in \text{Irr}(L_p)$  as  $\theta_\phi$  lies above  $\phi$ . Hence  $G(\alpha) \leq G(\phi_p) \leq G$ .

We pick a  $q$ -Sylow subgroup  $B$  of  $G$  so that  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$  while  $B(\alpha, \beta)$  is a  $q$ -Sylow subgroup of  $G(\alpha, \beta)$ .

According to Theorem 6.14 and Remark 6.17 there exists an  $A(\beta)$ -invariant linear limit  $S_1 = (G_1, Q_1, \beta_1)$  of  $S = (G, Q, \beta)$  so that the quotient group  $H_1/K_1$  is nilpotent, where  $H_1 = G_1 \cap H$  and  $K_1$  is the kernel of  $S_1$ . Furthermore,  $H_1 = P \times Q_1$  while the  $S_1$ -reduction  $\theta_1$  of  $\theta$  equals  $\theta_1 = \alpha \rtimes \beta_1$ . Let  $B_1 = B \cap G_1$ , and  $N_1 = N \cap G_1$ . We also write  $\overline{G}_1, \overline{N}_1, \overline{Q}_1, \overline{B}_1$  and  $\overline{P}$  for the quotient groups  $G_1/K_1, N_1/K_1, Q_1/K_1, B_1/K_1$  and  $(PK_1)/K_1$ , respectively. If  $\iota$  is the natural epimorphism of  $G_1$  onto  $\overline{G}_1$ , then  $\iota$  sends  $P$  isomorphically onto  $\overline{P}$  and  $\alpha \in \text{Irr}(P)$  to some character  $\overline{\alpha} \in \text{Irr}(\overline{P})$ . Let  $F = N_G(P)$  and  $F_1 = N_{G_1}(P) = F \cap G_1$ , then Frattini's argument implies that  $G_1 = F_1 K_1$ . Furthermore,  $C(Q_1)(P) = F_1 \cap Q_1$  covers  $\overline{Q}_1$ . Hence  $\iota$  sends  $F_1$  onto  $\overline{G}_1$  with kernel  $C_{K_1}(P) = K_1 \cap F_1$ .

Let  $R$  be the triple  $R = (F, P, \alpha)$ , where  $F = N_G(P)$ . We take a  $B(\alpha)$ -invariant linear limit  $R^* = (F^*, P^*, \alpha^*)$  of  $R$ . Then Proposition 3.14 implies that  $F^* = F(\alpha^*) = N_{G(\alpha^*)}(P) \leq G(\alpha)$ . Because  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$  that fixes  $\alpha^*$ , it is also a  $q$ -Sylow subgroup of  $F^*$ . We write  $\zeta^*$  for the central character of  $R^*$ . We also write  $Z^* = Z(R^*)$  and  $K^* = \text{Ker}(R^*)$ .

Let  $\overline{E}_1 = (\overline{G}_1, \overline{P}, \overline{\alpha})$ . Then any linear reduction  $R' = (F', P', \alpha')$  of  $R$  determines a linear reduction  $\overline{E}'_1$  of  $\overline{E}_1$  in the following way. Assume that  $L \trianglelefteq P$  is a normal subgroup  $F$  contained in  $P$ , while  $\lambda \in \text{Lin}(L)$  lies under  $\alpha$  so that  $P' = P(\lambda)$  and  $\alpha'$  is the  $\lambda$ -Clifford correspondent of  $\alpha$ . Then  $LK_1 \trianglelefteq PK_1$ , is a normal subgroup of  $G_1 \leq FK_1$ , while the character  $\lambda \times 1_{K_1}$  lies under  $\alpha \times 1_{K_1}$ . Clearly  $\alpha' \times 1_{K_1}$  is the  $\lambda \times 1_{K_1}$ -Clifford correspondent of  $\alpha \times 1_{K_1}$ . Hence  $\overline{L} = (LK_1)/K_1$  is a normal subgroup of  $\overline{G}_1$  contained in  $\overline{P}$ , while the unique irreducible character  $\overline{\lambda} \in \text{Irr}(\overline{L})$  that inflates to  $\lambda \times 1_{K_1}$  lies under  $\overline{\alpha} \in \text{Irr}(\overline{P})$ . So the triple  $\overline{E}'_1 = (\overline{G}'_1, \overline{P}', \overline{\alpha}')$  is a linear reduction of  $\overline{E}_1$ , where  $\overline{G}'_1 = \overline{G}_1(\overline{\lambda}) = G_1(\lambda)/K_1 \leq (F'K_1)/K_1$ . Assume in addition that  $R'$  is  $B(\alpha)$ -invariant, that is  $\lambda$  is a  $B(\alpha)$ -invariant character. Then the character  $\lambda \times 1_{K_1}$  is  $B_1(\alpha \times 1_{k_1})$ -invariant, since  $B_1(\alpha \times 1_{k_1}) = B_1(\alpha)K_1 \leq B(\alpha)K_1$ . Thus the character  $\overline{\lambda}$  is also  $\overline{B}_1(\overline{\alpha})$ -invariant. Hence  $R'$  determines the multilinear reduction  $\overline{E}^*_1 = (\overline{G}^*_1, \overline{P}^*, \overline{\alpha}^*)$  of  $\overline{E}_1$ , which is  $\overline{B}_1(\overline{\alpha})$ -invariant. Also  $\overline{G}^*_1 = G^*_1/K_1$  where  $G^*_1 \leq (F^*K_1) \cap G_1$ . Furthermore,  $\overline{P}^*$  is isomorphic to  $P^*$ , while  $\overline{\alpha}^* \in \text{Irr}(P^*)$  is mapped under that isomorphism to  $\overline{\alpha}^* \in \text{Irr}(\overline{P}^*)$ . According to Corollary 6.18  $B_1(\alpha) \cdot K_1$  is a  $q$ -Sylow subgroup of  $G_1(\alpha) \cdot K_1$ . Hence  $\overline{B}_1(\overline{\alpha})$  is a  $q$ -Sylow subgroup of  $\overline{G}_1(\overline{\alpha})$ . Because  $\overline{E}^*_1$  is  $\overline{B}_1(\overline{\alpha})$ -invariant the latter group is also a subgroup of  $\overline{G}^*_1$ . But  $\overline{G}^*_1 \leq \overline{G}_1(\overline{\alpha})$ , since  $G^*_1 \leq (F^*K_1) \cap G_1 \leq G_1(\alpha)K_1$ . We conclude that

$$(6.21) \quad \overline{B}_1(\overline{\alpha}) \in \text{Syl}_q(\overline{G}^*_1).$$

Note also that

$$(6.22) \quad Z(\overline{E}^*_1) \geq (Z^* \cdot K_1)/K_1 \text{ and } \text{Ker}(\overline{E}^*_1) \geq (K^* \cdot K_1)/K_1.$$

We apply Theorem D to the  $q$ -groups  $Q \trianglelefteq B$ , the  $p$ -group  $A \cap H$ , and the character  $\beta \in \text{Irr}(Q)$ . (Clearly  $B$  normalizes  $(A \cap H) \cdot Q = H$ ). This way we get an irreducible character  $\beta^\nu$  of  $Q$  that extends to  $B(\beta^\nu)$  and satisfies  $P = (A \cap H)(\beta) = (A \cap H)(\beta^\nu)$ , and  $B(\beta) \leq B(\beta^\nu)$ . In addition, we get that  $N_B(P) \leq B(\beta^\nu)$ . Hence  $B(\alpha) \leq B(\beta^\nu)$ . Note also that  $B(\alpha, \beta^\nu)Q = B(\alpha \times 1_Q, \beta^\nu)$ . Because  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$ , we conclude that  $B(\alpha, \beta^\nu) = B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha, \beta^\nu)$ . So

$$(6.23) \quad B(\alpha, \beta) \leq B(\alpha) = B(\alpha, \beta^\nu) \in \text{Syl}_q(G(\alpha, \beta^\nu)).$$

Note that  $\beta^\nu$  extends not only to  $B(\beta^\nu)$  but to  $QG(\alpha, \beta^\nu)$ . Indeed,  $\beta^\nu$  extends to any  $R$ , where  $R/Q$  is an  $r$ -Sylow subgroup of  $(QG(\alpha, \beta^\nu))/Q$ , for any  $r \neq q$  by Corollary 8.16 in [7]. It also extends to  $QB(\alpha, \beta^\nu)$  where  $(QB(\alpha, \beta^\nu))/Q$  is a  $q$ -Sylow subgroup of  $(QG(\alpha, \beta^\nu))/Q$ . Hence Corollary 11.31 in [7] implies that  $\beta^\nu$  extends to  $QG(\alpha, \beta^\nu)$ .

What is important about this new character is that  $H(\beta) = H(\beta^\nu) = P \times Q$ , (that is, the  $p$ -group  $P$  remains the same for the two characters  $\beta$  and  $\beta^\nu$ ). Furthermore, the product  $\alpha \times \beta^\nu$  is an irreducible character of  $H(\beta^\nu)$  lying above  $\beta^\nu$ . Hence Clifford's Theorem implies that  $\alpha \times \beta^\nu$  induces an irreducible character  $\theta^\nu$  of  $H$ . Also,  $G(\theta^\nu) = QG(\alpha, \beta^\nu)$ . So the groups  $Q \leq H \leq G$  and the characters  $\beta^\nu \in \text{Irr}(Q)$  and  $\theta^\nu \in \text{Irr}(H)$  satisfy all the hypothesis of Theorem 6.14. Then there exists a linear limit  $S_1^\nu = (G_1^\nu, Q_1^\nu, \beta_1^\nu)$  of  $S^\nu = (G, Q, \beta^\nu)$  such that  $H_1^\nu = P \times Q_1^\nu$  while  $H_1^\nu/K_1^\nu$  is a nilpotent group, where  $K_1^\nu$  is the kernel of  $S_1^\nu$ . Let  $B_1^\nu = B \cap G_1^\nu$ . Then

$$(6.24) \quad B_1^\nu = B_1^\nu(\beta_1^\nu) = B_1^\nu(\beta^\nu),$$

by Proposition 3.12. We also write  $\overline{G}^\nu_1, \overline{N}^\nu_1, \overline{Q}^\nu_1, \overline{B}^\nu_1$  and  $\overline{P}$  for the quotient groups  $G_1^\nu/K_1^\nu, N_1^\nu/K_1^\nu, Q_1^\nu/K_1^\nu, B_1^\nu/K_1^\nu$  and  $(PK_1^\nu)/K_1^\nu$ , respectively.



Let  $\overline{E}^{\nu_1} = (\overline{G}^{\nu_1}, \overline{P}, \overline{\alpha})$ . Because  $PK_1^{\nu} \trianglelefteq G_1^{\nu}$ , Frattini's argument also implies that  $G_1^{\nu} = N_{G_1^{\nu}}(P)K_1^{\nu} \leq FK_1^{\nu}$ . So, as with the triple  $\overline{E}_1$  and its reduction  $\overline{E}^*_{1}$ , the linear limit  $R^* = (F^*, P^*, \alpha^*)$  determines a multilinear reduction  $\overline{E}^{\nu^*}_1 = (\overline{G}^{\nu^*}_1, \overline{P}^*, \overline{\alpha}^*)$  of  $\overline{E}^{\nu_1}$ , which is  $\overline{B}^{\nu_1}(\overline{\alpha})$ -invariant. Note that  $\overline{P}^* = (P^*K_1)/K_1 \cong P^*$ . Also if  $\overline{G}^{\nu^*}_1 = G_1^{\nu^*}/K_1^{\nu}$  for some subgroup  $G_1^{\nu^*}$  of  $G_1^{\nu}$ , then  $\overline{G}^{\nu^*}_1 \leq F^*K_1^{\nu}$  and in addition

$$(6.25) \quad \overline{B}^{\nu_1}(\overline{\alpha}) \in \text{Syl}_q(\overline{G}^{\nu^*}_1).$$

Furthermore, similarly to (6.22) we get

$$(6.26) \quad Z(\overline{E}^{\nu^*}_1) \geq (Z^* \cdot K_1^{\nu})/K_1^{\nu} \text{ and } \text{Ker}(\overline{E}^{\nu^*}_1) \geq (K^{\nu^*} \cdot K_1^{\nu})/K_1^{\nu}.$$

Now let  $\overline{E}^{\nu_2} = (\overline{G}^{\nu_2}, \overline{P}^{\nu_2}, \overline{\alpha}^{\nu_2})$  be a  $\overline{B}^{\nu_1}(\overline{\alpha})$ -invariant linear limit of  $\overline{E}^{\nu^*}_1$ . Then  $\overline{E}^{\nu_2}$  is also a  $\overline{B}^{\nu_1}(\overline{\alpha})$ -invariant linear limit of  $\overline{E}^{\nu_1}$ . Hence Remark 6.17 implies that  $[(\overline{N}^{\nu_1} \cap \overline{G}^{\nu_2})_q, \overline{P}^{\nu_2}] \leq \overline{K}^{\nu_2}$  is nilpotent, where  $\overline{K}^{\nu_2}$  is the kernel of  $\overline{E}^{\nu_2}$ . (Note that we have used the fact that the group  $B$  is a  $q$ -Sylow subgroup of  $G$  so that  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $G(\alpha)$  while  $B(\alpha, \beta^{\nu})$  is a  $q$ -Sylow subgroup of  $G(\alpha, \beta^{\nu})$ .) By (6.25) the group  $\overline{B}^{\nu_1}(\overline{\alpha})$  is a  $q$ -Sylow subgroup of  $\overline{G}^{\nu^*}_1$ . Because  $\overline{E}^{\nu_2}$  is a  $\overline{B}^{\nu_1}(\overline{\alpha})$ -invariant linear limit of  $\overline{E}^{\nu^*}_1$ , we get that  $\overline{B}^{\nu_1}(\overline{\alpha})$  is contained in  $\overline{G}^{\nu_2} \leq \overline{G}^{\nu^*}_1$ , and thus it is a  $q$ -Sylow subgroup of  $\overline{G}^{\nu_2}$ . Hence

$$(6.27) \quad [\overline{B}^{\nu_1}(\overline{\alpha}) \cap \overline{N}_1, \overline{P}^{\nu_2}] \leq \overline{K}^{\nu_2},$$

where  $\overline{K}^{\nu_2}$  is the kernel of  $\overline{E}^{\nu_2}$ . In addition, (6.26) along with Remark 3.2 implies

$$(6.28a) \quad Z^* \cong (Z^*K_1^{\nu})/K_1^{\nu} \leq Z(\overline{E}^{\nu^*}_1) \leq Z(\overline{E}^{\nu_2}) \leq \overline{P}^{\nu_2} \leq \overline{P}^* \cong P^*$$

and

$$(6.28b) \quad (K^*K_1^{\nu})/K_1^{\nu} \leq \text{Ker}(\overline{E}^{\nu^*}_1) \leq \overline{K}^{\nu_2}.$$

We identify  $Z^*$  with its isomorphic image  $\overline{Z}^* = (Z^*K_1^{\nu})/K_1^{\nu}$  inside  $Z(\overline{E}^{\nu^*}_1)$ . Under this isomorphism the central character  $\zeta^* \in \text{Lin}(Z^*)$  of  $R^*$  is mapped to a linear character of  $\overline{Z}^*$  that lies under the central character  $\zeta_1^{\nu^*} \in \text{Lin}(Z(\overline{E}^{\nu^*}_1))$  of  $\overline{E}^{\nu^*}_1$ .

Let  $V := P^*/Z^* = P^*/Z(R^*)$ . Then  $V$  is an anisotropic  $F^*/P^*$ -group, by Proposition 3.11. (The  $F^*/P^*$ -invariant bilinear form  $c : V \times V \rightarrow C^{\times}$  is defined (see (3.9)) as  $c(\overline{x}, \overline{y}) = \zeta^*([x, y])$ , for all  $\overline{x}, \overline{y} \in V$ ). Thus  $V$  written additively, is the direct sum

$$V = V_1 \dot{+} V_2,$$

of the perpendicular  $F^*/P^*$ -groups  $V_1 = C_V(N^*)$  and  $V_2 = [V, N^*]$ , where  $N^* = N \cap F^*$ . Because  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $F^*$ , the group  $Q^* = B(\alpha) \cap N^* = B(\alpha) \cap N$  is a  $q$ -Sylow subgroup of  $N^*$ . So

$$N^* = Q^* \rtimes P^*.$$

Therefore the direct summands  $V_1, V_2$  of  $V$  are

$$(6.29) \quad V_1 = C_V(Q^*) \text{ and } V_2 = [V, Q^*].$$

Both  $V_1$  and  $V_2$  are anisotropic  $F^*/P^*$ -groups.

Now let  $U = \overline{P^\nu}_2/Z(\overline{E^\nu}_2)$ . Then  $U$  is isomorphic to a section of  $V$ , by (6.28). Furthermore,  $U$  is isomorphic to the orthogonal direct sum  $U = U_1 \dot{+} U_2$ , where in view of (6.29) we get

$$U_1 = C_U(Q^*) \text{ and } U_2 = [U, Q^*],$$

According to Corollary 6.18 (applied to the  $\nu$ -groups), we have  $G_1^\nu(\alpha, \beta^\nu)C_Q(P) = G(\alpha, \beta^\nu)$ . This, along with (6.24) and (6.23) implies

$$(6.30) \quad B_1^\nu(\alpha) \cdot C_Q(P) = B_1^\nu(\alpha, \beta^\nu) \cdot C_Q(P) = B(\alpha, \beta^\nu) = B(\alpha).$$

Hence the image of  $B(\alpha) \cap N$  in the automorphism group of  $P$  equals that of  $B_1^\nu(\alpha) \cap N$ . So  $U_2 = [U, B_1^\nu(\alpha) \cap N]$ . In view of (6.27) and (6.28b), this latter group is trivial. Hence  $U_2 = 0$  and  $U_1 = C_U(Q^*) = C_U(B_1^\nu(\alpha, \beta^\nu) \cap N)$ .

Because  $V = P^*/Z^*$  is an abelian anisotropic  $F^*/P^*$ -group, the group  $\overline{P^*}/Z(\overline{E^\nu^*}_1)$  is also an abelian group. It also affords a bilinear  $\overline{G^\nu^*}_1/\overline{P^*}$ -invariant form (see (3.9)) defined as  $\hat{c}(\bar{u}, \bar{v}) = \zeta_1^{\nu^*}([u, v])$ , for all  $\bar{u}, \bar{v} \in \overline{P^*}/Z(\overline{E^\nu^*}_1)$ . Identifying  $P^*$  with  $\overline{P^*}$  and  $Z^*$  with  $\overline{Z^*}$  we see that  $[\overline{P^*}, \overline{P^*}] \leq \overline{Z^*} \leq Z(\overline{E^\nu^*}_1)$ . Hence  $\hat{c}(\bar{u}, \bar{v}) = \zeta^*([u, v])$ , for all  $\bar{u}, \bar{v} \in \overline{P^*}/Z(\overline{E^\nu^*}_1)$ . In addition, since  $G_1^{\nu^*} \leq F^*K_1^*$  the factor group  $\overline{G^\nu^*}_1/\overline{P^*}$  is isomorphic to a subgroup of  $F^*/P^*$ . Hence  $\overline{P^*}/Z(\overline{E^\nu^*}_1)$  is an abelian symplectic  $\overline{G^\nu^*}_1/\overline{P^*}$ . Thus we can apply Proposition 3.13 to the linear limit  $\overline{E^\nu}_2$  of  $\overline{E^\nu^*}_1$ . So  $U$  is isomorphic to  $L^\perp/L$ , where  $L$  is maximal among the  $\overline{G^\nu^*}_1/\overline{P^*}$ -invariant isotropic subgroups of  $\overline{P^*}/Z(\overline{E^\nu^*}_1)$ . Furthermore,  $L = Z(\overline{E^\nu}_2)/Z(\overline{E^\nu^*}_1)$  and  $L^\perp = \overline{P^\nu}_2/Z(\overline{E^\nu^*}_1)$ . If  $L = L_1 \dot{+} L_2$ , with  $L_1 = C_L(Q^*)$  and  $L_2 = [L, Q^*]$  then the facts that  $L^\perp/L = U$  while  $U_2 = 0$  implies that  $L_2^\perp = L$ . Hence  $L_2$  is a self perpendicular  $\overline{G^\nu^*}_1/\overline{P^*}$ -invariant subgroup of  $\overline{P^*}/Z(\overline{E^\nu^*}_1)$ .

Since  $\overline{P^*}/Z(\overline{E^\nu^*}_1) \cong \frac{P^*/Z^*}{Z(\overline{E^\nu^*}_1)/Z^*}$ , we get that  $L_2$  is isomorphic to a self perpendicular subgroup  $W_2$  of  $V$ . Hence  $W_2 \leq V_2 = [V, Q^*]$ , as  $L_2 = [L, Q^*]$ . Because  $L_2$  is  $\overline{G^\nu^*}_1/\overline{P^*}$ -invariant, it is  $\overline{B^\nu}_1(\bar{\alpha})$ -invariant, by (6.25). According to (6.30) the image of  $B_1^\nu(\alpha)$  in  $\text{Aut}(P)$  equals that of  $B(\alpha)$ . Therefore  $W_2$  is a self perpendicular  $B(\alpha)$ -invariant subgroup of  $V_2$ . Hence  $V_2$  is a hyperbolic  $B(\alpha)$ -group. But  $B(\alpha)$  is a  $q$ -Sylow subgroup of  $F^*$ . Hence Theorem 3.2 in [2] implies that  $V_2$  is a hyperbolic  $F^*$ -group. Because it is also an anisotropic  $F^*/P^*$ -group, we conclude that  $V_2$  is 0. Hence  $Q^*$  centralizes  $V = P^*/Z^*$ . In addition,  $Z^*/K^*$  is a cyclic central  $p$ -section of  $F^*$ . So the  $q$ -subgroup  $Q^*$  of  $F^*$  centralizes both  $p$ -groups  $V = P^*/Z^*$  and  $Z^*/K^*$ . We conclude that  $Q^*$  centralizes  $P^*/K^*$ . Since  $Q^* = B(\alpha) \cap N$  we get  $[P^*, B(\alpha) \cap N] \leq K^*$ . Thus  $[(P^*K_1)/K_1, (B(\alpha)K_1 \cap N)/K_1] \leq (K^*K_1)/K_1$ . This and (6.22) implies that

$$[\overline{P^*}, \overline{B_1(\bar{\alpha})} \cap \overline{N_1}] \leq \text{Ker}(\overline{E^*}_1).$$

But  $\overline{B_1(\bar{\alpha})} \in \text{Syl}_q(\overline{G^*}_1)$ , by (6.21). We conclude that a  $q$ -Sylow subgroup of  $\overline{N^*}_1$  centralizes  $\overline{P^*}$  module  $\text{Ker}(\overline{E^*}_1)$ . Hence the theorem follows.  $\square$

Note that we pick the linear limit  $\overline{E^*}_1$  in the statement of Theorem 6.20 in the following way

**Corollary 6.31.** *Assume the hypothesis and notation of Theorem 6.20. Let  $B$  be a  $q$ -Sylow subgroup of  $G$  so that  $B(\alpha)$  and  $B(\alpha, \beta)$  are  $q$ -Sylow subgroups of  $G(\alpha)$  and  $G(\alpha, \beta)$ , respectively. Let  $F = N_G(P)$ . Then any  $B(\alpha)$ -invariant linear limit  $(F^*, P^*, \alpha^*)$  of  $(F, P, \alpha)$  determines naturally a multilinear reduction  $\overline{E^*}_1 = (\overline{G^*}_1, \overline{P^*}, \overline{\alpha^*})$  of  $\overline{E}_1 = (\overline{G}_1, \overline{P}, \overline{\alpha})$  so that  $[(\overline{G^*}_1 \cap \overline{N^*}_1)_q, \overline{P^*}] \leq \text{Ker}(\overline{E^*}_1)$ , while  $\overline{P^*}$  is a  $p$ -Sylow subgroup of  $\overline{H^*}_1$ .*

We can now prove Theorem E that we restate here.

**Theorem E.** *Assume that  $G$  is a finite odd monomial  $p, q$ -group, while  $N$  is a normal subgroup of  $G$  of nilpotent length 3. So there exists  $L \leq M$  normal subgroups of  $G$  contained in  $N$  so that  $L, M/L$  as well as  $N/M$  are nilpotent groups. Let  $\chi$  be any irreducible character of  $M$ . Then there exists a linear limit  $T' = (G', M', \chi')$  of the triple  $T = (G, M, \chi)$  with  $N'/\text{Ker}(T')$  being nilpotent, where  $N' = N \cap G'$ , and  $\text{Ker}(T')$  is the kernel of the triple  $T'$ . Therefore  $N$  is a monomial group.*

*Proof.* Let  $\phi \in \text{Irr}(L)$  lying under  $\chi \in \text{Irr}(M)$ . Because  $L = L_p \times L_q$  is nilpotent,  $\phi$  splits as  $\phi = \eta \times \beta$ , where  $\eta \in \text{Irr}(L_p)$  and  $\beta \in \text{Irr}(L_q)$ . The groups  $H = M_p \times L_q$  and  $J = M_q \times L_p$  are normal subgroups of  $G$ . Furthermore,  $H(\phi) = P \times L_q$  where  $P = M_p(\phi)$  is a  $p$ -Sylow subgroup of  $H(\phi)$ , and  $J(\phi) = O \times L_p$  where  $O = M_q(\phi)$  is a  $q$ -Sylow subgroup of  $J(\phi)$ . Clearly  $P \geq L_p$  and  $O \geq L_q$ .

Let  $\theta \in \text{Irr}(H)$  be any irreducible character of  $H$  lying above  $\phi$  and under  $\chi$ . Then the  $\phi$ -Clifford correspondent  $\theta_\phi$  of  $\theta$  equals

$$\theta_\phi = \alpha \times \beta,$$

where  $\alpha \in \text{Irr}(P)$  is uniquely determined by  $\theta$  and lies above  $\eta \in \text{Irr}(L_p)$ . Furthermore,  $\alpha$  restricted to  $L_p$  is a multiple of  $\eta$ . Similarly we pick an irreducible character  $\psi \in \text{Irr}(J)$  lying above  $\phi$  and under  $\chi$ . Its  $\phi$ -Clifford correspondent satisfies

$$\psi_\phi = \gamma \times \eta,$$

where  $\gamma \in \text{Irr}(O)$  is uniquely determined by  $\psi$  and lies above  $\beta \in \text{Irr}(L_q)$ . In addition,  $\gamma$  restricted to  $L_q$  is a multiple of  $\eta$ . It is easy to see that

$$(6.32a) \quad G(\alpha, \beta) \leq G(\alpha) \leq G(\eta)$$

and

$$(6.32b) \quad G(\eta, \gamma) \leq G(\gamma) \leq G(\beta).$$

Now we pick a  $p$ -Sylow subgroup  $A$  and a  $q$ -Sylow subgroup  $B$  of  $G$  so that  $A$  intersected with every group in (6.32b) is a  $p$ -Sylow subgroup of that group, and  $B$  intersected with any group in (6.32a) is a  $q$ -Sylow subgroup of that group.

Note that  $A \cap H$  is a  $p$ -Sylow subgroup of  $H$  and thus  $H(\eta, \beta) = H(\phi) = (A \cap H)(\phi) \times L_q$ . Hence there exists  $\sigma \in L_q$  so that  $(A \cap H)(\phi) = P^\sigma$ . Because  $\sigma$  fixes  $\eta, \beta, \gamma$  we may assume (replacing  $A$  with  $A^{\sigma^{-1}}$ ) that  $P = (A \cap H)(\phi)$ . Similarly, we may assume that  $B$  is picked as to satisfy  $(B \cap J)(\phi) = O$ .

Let  $(G', L'_q, \beta')$  be an  $A(\beta)$ -invariant linear limit of  $(G, L_q, \beta)$  and let  $K'$  be the kernel of  $(G', L'_q, \beta')$ . If  $H' = H \cap G'$  then Theorem 6.20 implies that  $H'/K' = [PK']/K' \times L'_q/K'$ , where  $P = (A \cap H)(\phi)$  is a  $p$ -Sylow subgroup of  $H'$  and thus of  $M'$ . Similarly we take  $(G'', L''_p, \eta'')$  to be a  $B(\eta)$ -invariant linear limit of  $(G, L_p, \eta)$ , with kernel  $K''$ . Theorem 6.20 also implies that  $J''/K'' = [OK'']/K'' \times L''_p/K''$ , where  $O = (B \cap J)(\phi)$  is a  $q$ -Sylow subgroup of  $J''$  and thus of  $M''$ .

If  $G_1 = G' \cap G''$ ,  $L_1 = L''_p \times L'_q$  and  $\phi_1 = \eta'' \times \beta'$  then  $(G_1, L_1, \phi_1)$  is a linear limit of  $(G, L, \phi)$ , by Remark 3.8. In addition, (see Lemma 4.2) the factor group  $M_1/K_1$  is a nilpotent group where  $K_1$  is

the kernel of  $(G_1, L_1, \phi_1)$ , and  $M_1 = M \cap G_1$ . Note that  $K_1 = K'' \times K'$ . Let  $\chi_1 \in \text{Irr}(M_1)$  be the  $G_1$ -limit of  $\chi \in \text{Irr}(M)$ . Observe that  $M_1/K_1 = (P_1K_1)/K_1 \times (O_1K_1)/K_1$ , where  $P_1 = P \cap G_1 = P \cap G''$  and  $O_1 = O \cap G_1 = O \cap G'$ . So  $P_1$  is the  $G''$ -reduction of  $P$  while  $O_1$  is the  $G'$ -reduction of  $O$ . Let  $\alpha_1 \in \text{Irr}(P_1)$  and  $\gamma_1 \in \text{Irr}(O_1)$  be the  $G_1$ -reductions of  $\alpha \in \text{Irr}(P)$  and  $\gamma \in \text{Irr}(O)$ , respectively. So  $\alpha_1$  is actually the  $(G'', L_p'', \eta'')$ -reduction of  $\alpha$  while  $\gamma_1$  is the  $(G', L_q', \beta')$ -reduction of  $\gamma$ . Note also that  $(P_1K_1)/K_1 = (P_1K')/K_1 \cong P_1/K''$  while  $(O_1K_1)/K_1 = (O_1K'')/K_1 \cong O_1/K'$ . Let  $\bar{\alpha}_1 \in \text{Irr}((P_1K_1)/K_1)$  be the unique irreducible character of  $(P_1K_1)/K_1$  that inflates  $\alpha \times 1 \in \text{Irr}(P_1K')$ . Similarly we define  $\bar{\gamma}_1 \in \text{Irr}((O_1K_1)/K_1)$ .

Now let  $R = (F, P, \alpha)$  and  $S = (I, O, \gamma)$ , where  $F = N_G(P)$  and  $I = N_G(O)$ . Observe that the  $B(\eta)$ -invariant linear limit  $(G'', L_p'', \eta'')$  of  $(G, L_p, \eta)$  determines naturally a  $B(\eta)$ -invariant multilinear reduction  $R'' = (F \cap G'', P \cap G'', \alpha'')$  of  $R$ , where  $\alpha''$  is the  $(G'', L_p'', \eta'')$ -reduction of  $\alpha$ . But  $P \cap G'' = P_1$  and thus  $R'' = (F'', P_1, \alpha_1)$ , where  $F'' = F \cap G''$ . Note also that in view of (6.32a) the group  $B(\alpha)$  is a subgroup of  $B(\eta)$ . Hence the linear reduction  $R''$  of  $R$  is  $B(\alpha)$ -invariant.

Similarly, the  $A(\beta)$ -invariant linear limit  $(G', L_q', \beta')$  of  $(G, L_q, \beta)$  determines naturally an  $A(\beta)$ -invariant multilinear reduction  $S' = (I', O_1, \gamma_1)$  of  $S$ , where  $I' = G' \cap I$ . In addition, (6.32b) implies that  $A(\gamma) \leq A(\beta)$ . Hence the linear reduction  $S'$  of  $S$  is  $A(\gamma)$ -invariant.

Let  $R^* = (F^*, P^*, \alpha^*)$  be a  $B(\alpha)$ -linear limit of  $R'' = (F'', P_1, \alpha_1)$  and thus of  $R$ . Similarly we pick  $S^* = (I^*, O^*, \gamma^*)$  to be an  $A(\gamma)$ -invariant linear limit of  $S' = (I', O_1, \gamma_1)$  and thus of  $S$ .

Now we can apply Corollary 6.31. Let  $\bar{G}' = G'/K'$  then  $(PK')/K' = \bar{P}$  is isomorphic to  $P$  and under this isomorphism  $\alpha \in \text{Irr}(P)$  gets mapped to  $\bar{\alpha} \in \text{Irr}(\bar{P})$ . If  $\bar{E} = (\bar{G}', \bar{P}, \bar{\alpha})$ , then Corollary 6.31 implies that  $R^*$  determines naturally a multilinear reduction  $\bar{E}^* = (\bar{G}^*, \bar{P}^*, \bar{\alpha}^*)$  of  $\bar{E}$  so that

$$(6.33) \quad [(\bar{G}^* \cap \bar{N}^*)_p, \bar{P}^*] \leq \text{Ker}(\bar{E}^*),$$

where  $\bar{P}^*$  is a  $p$ -Sylow subgroup of  $\bar{H}^* = \bar{G}^* \cap (H'/K')$ . Observe that because  $R^*$  is a linear limit of  $R''$ , the reductions done in  $\bar{E}$  are actually reductions done inside  $(P_1K')/K' \leq \bar{P}$ . Furthermore,  $P_1 \cong (P_1K')/K'$ . This, along with the fact that  $G_1 = G' \cap G''$  and  $K_1 = K' \times K''$ , implies that the multilinear reduction  $\bar{E}^*$  of  $\bar{E}$  determines a multilinear reduction  $U_1$  of  $U = (G_1/K_1, (P_1K_1)/K_1, \bar{\alpha}_1)$ . Note that  $G_1/K_1$  is reduced to  $(G_1 \cap \bar{G}^*K_1)/K_1$  while  $(P_1K_1)/K_1 \cong P_1/K''$  is reduced to  $(P^*K_1)/K_1$ .

Similarly we write  $\hat{G}'' = G''/K''$  then  $(OK'')/K'' = \hat{O}$  is isomorphic to  $O$  and under this isomorphism  $\gamma \in \text{Irr}(O)$  get mapped to  $\hat{\gamma} \in \text{Irr}(\hat{O})$ . If  $\hat{D} = (\hat{G}'', \hat{O}, \hat{\gamma})$ , then Corollary 6.31 implies that  $S^*$  determines naturally a multilinear reduction  $\hat{D}^* = (\hat{G}^*, \hat{O}^*, \hat{\gamma}^*)$  so that

$$(6.34) \quad [(\hat{G}^* \cap \hat{N}^*)_q, \hat{O}^*] \leq \text{Ker}(\hat{D}^*),$$

where  $\hat{Q}^*$  is a  $q$ -Sylow subgroup of  $\hat{J}^* = \hat{G}^* \cap (J''/K'')$ . Furthermore, similarly to  $U_1$  and  $U$  we get that the multilinear reduction  $\hat{D}^*$  of  $\hat{D}$  determines a multilinear reduction  $V_1$  of  $V = (G_1/K_1, (O_1K_1)/K_1, \bar{\gamma}_1)$ . The group  $G_1/K_1$  is being reduced to  $(G_1 \cap \hat{G}^*K_1)/K_1$  while  $(O_1K_1)/K_1 \cong O_1/K'$  is reduced to  $(O^*K_1)/K_1$ .

The fact that  $M_1/K_1 = (P_1K_1)/K_1 \times (O_1K_1)/K_1$  is a nilpotent group, along with Remark 3.8 provides a linear reduction  $\bar{T}_1$  of  $\bar{T} = (G_1/K_1, M_1/K_1, \bar{\alpha}_1 \times \bar{\gamma}_1)$  using the multilinear reductions  $U_1$  and  $V_1$ . The kernel of  $\bar{T}_1$  contains the group  $(\text{Ker}(\bar{E}^*)K_1)/K_1 \times (\text{Ker}(\hat{D}^*)K_1)/K_1$ . So (6.33) and

(6.34) imply that the  $\bar{T}_1$ -reduction of the group  $N_1/K_1$  is a nilpotent group module the kernel of  $\bar{T}_1$ .

Because  $\bar{T}_1$  lifts to a reduction of the triple  $(G_1, M_1, \chi_1)$  and thus of  $(G, M, \chi)$ , the theorem follows. □

## REFERENCES

- [1] T. R. Berger, Hall–Higman Type Theorems V. *Pacific J. Math.* **73** (1977), 1–62
- [2] E. C. Dade, Monomial Characters and Normal Subgroups, *Math. Z.* **178** (1981), 401–420
- [3] E. C. Dade, Normal subgroups of  $M$ -groups need not be  $M$ -groups. *Math. Z.* **133** (1973), 313–317
- [4] E. C. Dade, M. Loukaki, Linear limits of irreducible characters, preprint
- [5] L. Dornhoff,  $M$ -groups and 2-groups, *Math. Z.* **100** (1967) 226–256
- [6] D. Gorenstein, *Finite Groups*. New York: Harper and Row 1968
- [7] I. M. Isaacs, *Character Theory of Finite Groups*. New York–San Francisco–London: Academic Press 197
- [8] I. M. Isaacs, Primitive characters, normal subgroups, and  $M$ -groups, *Math. Z.* **177**, no. 2 (1981), 267–28
- [9] I. M. Isaacs, Induction and restriction of  $\pi$ -special characters. *Can. J. Math.* **38** (1986), 576–604
- [10] I. M. Isaacs, Induction and restriction of  $\pi$ -partial characters and their lifts. *Can. J. Math.* **48**, no. 6 (1996), 1210–1223
- [11] I. M. Isaacs, Characters of Solvable and Symplectic Groups, *Amer. J. Math.* **95** (1973), 594–63
- [12] M. L. Lewis, Characters, coprime actions, and operator groups, *Arch. Math.* **69** (1997), 455–460
- [13] M. L. Lewis, Primitive characters of subgroups of  $M$ -groups, *Proc. Amer. Math. Soc.* **125**, no. 1 (1997), 27–3
- [14] M. Loukaki, Normal subgroups of odd order monomial  $p^a q^b$ -groups, Thesis, University of Illinois at Urbana–Champaign, 2001
- [15] O. Manz, T. R. Wolf, *Representations of Solvable Groups*, Lecture Notes Series **185**
- [16] G. Navarro, Primitive characters of subgroups of  $M$ -groups, *Math. Z.* **218**, no. 3 (1995), 439–445
- [17] A. Parks, Nilpotent by supersolvable  $M$ -groups, *Canad. J. Math.* **37** (1985), no. 5, 934–962
- [18] G. Seitz,  $M$ -groups and the supersolvable residual, *Math. Z.* **110** (1969), 101–122
- [19] R. W. van der Waall, On the embedding of minimal non- $M$ -groups. *Indag. Math.* **36** (1974), 157–16

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