Hyperbolic modules and cyclic subgroups

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Abstract

Let G be a finite group of odd order, $\mathcal{F}$ a finite field of odd characteristic p and $\mathcal{B}$ a finite–dimensional symplectic $\mathcal{F}G$-module. We show that $\mathcal{B}$ is $\mathcal{F}G$-hyperbolic, i.e., it contains a self–perpendicular $\mathcal{F}G$-submodule, iff it is $\mathcal{F}N$-hyperbolic for every cyclic subgroup N of G.

1 Introduction

Let $\mathcal{F}$ be a finite field of odd characteristic p, G a finite group and $\mathcal{B}$ a finite–dimensional $\mathcal{F}G$-module. If $\mathcal{B}$ carries a non-singular alternating bilinear form $< \cdot, \cdot >$ (i.e., a symplectic form) that is invariant by G, then we call $\mathcal{B}$ a *symplectic* $\mathcal{F}G$-module. Following the notation in [3], for any $\mathcal{F}G$-submodule $\mathcal{S}$ of $\mathcal{B}$, we write $\mathcal{S}^{\perp}$ for the *perpendicular* subspace of $\mathcal{S}$, i.e., $\mathcal{S}^{\perp} := \{t \in \mathcal{B} | < \mathcal{S}, t >= 0\}$. We say that $\mathcal{S}$ is *isotropic* if $\mathcal{S} \leq \mathcal{S}^{\perp}$, and $\mathcal{B}$ is *anisotropic* if it contains no non–trivial isotropic $\mathcal{F}G$-submodules. Furthermore, we say that $\mathcal{B}$ is *hyperbolic* if it contains some self–perpendicular $\mathcal{F}G$-submodule $\mathcal{S}$, i.e., $\mathcal{S}$ is an $\mathcal{F}G$-submodule satisfying $\mathcal{S} = \mathcal{S}^{\perp}$.

Symplectic modules play an essential role in studying monomial characters. (An irreducible character χ of a finite group G is monomial if it is induced from a linear character of a subgroup of G.) One of the most representative links between symplectic modules and monomial characters can be found in [3]. (For other examples one could look at [2, 7, 8, 9, 10, 11, 12], and [13].) There E. C. Dade proved the following theorem (Theorem 3.2 in [3]):

Theorem 1.1 (Dade). *Suppose that $\mathcal{F}$ is a finite field of odd characteristic p, that G is a finite p-solvable group, that H is a subgroup of p-power index in G, and that $\mathcal{B}$ is a symplectic $\mathcal{F}G$-module whose restriction $\mathcal{B}_H$ to a symplectic $\mathcal{F}H$-module is hyperbolic. Then $\mathcal{B}$ is hyperbolic.*

Using the above theorem, E. C. Dade was able to prove (Theorem 0 in [3]) that, given a p-solvable odd group G, an irreducible monomial character χ of G, and a subnormal subgroup N of G, every irreducible constituent of the restricted character χ_N is monomial, provided that $\chi(1)$ is a power of p.

In this paper we prove

Theorem A. *Suppose that $\mathcal{F}$ is a finite field of odd characteristic p, that G is a finite group of odd order, and that $\mathcal{B}$ is a symplectic $\mathcal{F}G$–module whose restriction $\mathcal{B}_N$ to a symplectic $\mathcal{F}N$–module is hyperbolic for every cyclic subgroup N of G. Then $\mathcal{B}$ is hyperbolic.*

All groups considered here are of finite order, and all modules have finite dimension over $\mathcal{F}$.

Acknowledgments I am indebted to Professor E. C. Dade for many helpful ideas and suggestions. Also, I would like to thank Professor M. Isaacs for useful conversations that helped me improve this paper.

2 Symplectic modules

We first give some elementary results about symplectic modules.

Assume that $\mathcal{B}$ is a symplectic $\mathcal{F}G$-module, while $\mathcal{S}$ is an isotropic $\mathcal{F}G$-submodule of $\mathcal{B}$. Then the factor $\mathcal{F}G$-module $\bar{\mathcal{S}} = \mathcal{S}^{\perp}/\mathcal{S}$ is again a symplectic $\mathcal{F}G$-module with the symplectic form defined as (see 1.4 in [3]),

$$< s_1 + \mathcal{S}, s_2 + \mathcal{S} >=< s_1, s_2 >, \text{ for all } s_1, s_2 \in \mathcal{S}^{\perp}. \tag{1}$$

Furthermore, if $\mathcal{S}$ is an isotropic $\mathcal{F}G$-submodule of $\mathcal{B}$, then its $\mathcal{F}$-dimension $\dim_{\mathcal{F}} \mathcal{S}$ is at most $(1/2)\dim_{\mathcal{F}} \mathcal{B}$, (see 19.3 in [1]).

We say that an isotropic $\mathcal{F}G$-submodule $\mathcal{S}$ of $\mathcal{B}$ is *maximal isotropic* if $\mathcal{S}$ is not properly contained in any larger isotropic $\mathcal{F}G$-submodule of $\mathcal{B}$. Clearly any self–perpendicular $\mathcal{F}G$-submodule $\mathcal{S}$ of $\mathcal{B}$ is maximal isotropic. The converse is also correct under the extra assumption that $\mathcal{B}$ is G-hyperbolic (see Lemma 3.1 in [3]). Another way to get a self–perpendicular module from a maximal isotropic one is to control its dimension, as the following lemma shows.

Lemma 2.1. *Assume that $\mathcal{B}$ is a symplectic $\mathcal{F}G$-module, and that $\mathcal{S}$ is a maximal isotropic $\mathcal{F}G$-submodule of $\mathcal{B}$. If $\dim_{\mathcal{F}} \mathcal{S} = (1/2)\dim_{\mathcal{F}} \mathcal{B}$ then $\mathcal{S}$ is self–perpendicular and $\mathcal{B}$ is G-hyperbolic.*

Proof. Let $\hat{\mathcal{S}}$ denote the dual of $\mathcal{S}$. Then $\mathcal{B}/\mathcal{S}^{\perp} \cong \hat{\mathcal{S}}$. But $\dim_{\mathcal{F}} \hat{\mathcal{S}} = \dim_{\mathcal{F}} \mathcal{S} = (1/2)\dim_{\mathcal{F}} \mathcal{B}$. Hence $\dim_{\mathcal{F}} \mathcal{S}^{\perp} = (1/2)\dim_{\mathcal{F}} \mathcal{B}$. Since $\mathcal{S} \leq \mathcal{S}^{\perp}$ we conclude that $\mathcal{S} = \mathcal{S}^{\perp}$. Thus the lemma holds. □

The following is Proposition 2.1 in [3].

Proposition 2.2. *Let G be a finite group and $\mathcal{B}$ be an anisotropic symplectic $\mathcal{F}G$–module. Then $\mathcal{B}$ is an orthogonal direct sum:*

$$\mathcal{B} = \mathcal{U}_1 \dot{\perp} \mathcal{U}_2 \dot{\perp} \ldots \dot{\perp} \mathcal{U}_k, \tag{2}$$

where $k \geq 0$ and each $\mathcal{U}_i$ is a simple $\mathcal{F}G$–submodule of $\mathcal{B}$ that is also symplectic.

Remark 1. If G has odd order then according to Proposition (1.10) and Corollary 2.10 in [3] all the $\mathcal{U}_i$ that appear in (2) are distinct.

Lemma 2.3. *Let $\mathcal{U}$ be an $\mathcal{F}G$-module that affords a symplectic G-invariant form $< \cdot, \cdot >$. Then $\mathcal{U}$ is self–dual.*

Proof. We write $\widehat{\mathcal{U}}$ for the dual $\mathcal{F}G$-module of $\mathcal{U}$. For every $x \in \mathcal{U}$ the map $\alpha_x : \mathcal{U} \to \mathcal{F}$ defined as:

$$\alpha_x(u) =< u\,,\, x > \text{ for all } u \in \mathcal{U}$$

is an element of $\mathrm{Hom}_{\mathcal{F}}(\mathcal{U}, \mathcal{F}) \cong \widehat{\mathcal{U}}$. Since $< \cdot\,, \cdot >$ is G-invariant the map $\alpha : x \to \alpha_x$ is an $\mathcal{F}G$-homomorphism from $\mathcal{U}$ to $\widehat{\mathcal{U}}$. Furthermore the kernel of α is trivial, as $\mathcal{U}$ is symplectic. Hence $\mathcal{U} \cong \widehat{\mathcal{U}}$. □

Corollary 2.4. *Let $\mathcal{B}$ be an anisotropic symplectic $\mathcal{F}G$–module. Then each of the simple $\mathcal{F}G$–modules $\mathcal{U}_i$ that appears in (2) is self–dual.*

Proof. It follows easily from Proposition 2.2 and Lemma 2.3. □

Proposition 2.5. *Assume that $\mathcal{U}$ is a simple symplectic $\mathcal{F}G$–module. Let N be a normal subgroup of G such that $|G : N|$ is odd. Then any simple $\mathcal{F}N$–submodule of $\mathcal{U}_N$ is self–dual. Hence any $\mathcal{F}N$-submodule of $\mathcal{U}_N$ is self–dual.*

Proof. As N is a normal subgroup of G, Clifford's theorem implies that

$$\mathcal{U}_N \cong e(\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_n) \tag{3}$$

where $\mathcal{V} = \mathcal{V}_1$ is a simple $\mathcal{F}N$–submodule of $\mathcal{U}$ and $\mathcal{V}_1, \ldots, \mathcal{V}_n$ are the distinct G-conjugates of $\mathcal{V}$. So $n\big||G:N|$ and therefore n is odd.

According to Lemma 2.3 the module $\mathcal{U}$ is self–dual. Hence the dual, $\widehat{\mathcal{V}_i}$ of any $\mathcal{V}_i$ should appear in (3). Therefore we can form pairs among the $\mathcal{V}_i$, consisting of a simple $\mathcal{F}N$–module $\mathcal{V}_k$ and its dual for $k \in \{1, \ldots, n\}$, where we take as the second part of the pair the module itself if it is self–dual. Since G acts transitively on the $\mathcal{V}_i$ for $i = 1, \ldots, n$, either all the $\mathcal{V}_i$ are self–dual or none of them is. In the latter case we get that any of the above pairs consists of two distinct modules. This implies that $2|n$. As n is odd, this case can never occur. Hence any one of the $\mathcal{V}_i$ is self–dual and the proposition is proved. □

Proposition 2.6. *Assume that the symplectic $\mathcal{F}G$–module $\mathcal{B}$ is hyperbolic. Assume further that $\mathcal{B}$ is a semi–simple $\mathcal{F}G$-module. Then every self–dual simple $\mathcal{F}G$–submodule of $\mathcal{B}$ appears with even multiplicity in any decomposition of $\mathcal{B}$ as a direct sum of simple $\mathcal{F}G$–submodules.*

Proof. Because $\mathcal{B}$ is hyperbolic it contains a self–perpendicular $\mathcal{F}G$-submodule $\mathcal{S}$. For every $\mathcal{F}G$-submodule $\mathcal{V}$ of $\mathcal{B}$ we have $\mathcal{B}/\mathcal{V}^{\perp} \cong \widehat{\mathcal{V}}$. So

$$\mathcal{B}/\mathcal{S} \cong \widehat{\mathcal{S}} \tag{4}$$

Now the proposition follows from (4) and the fact that $\mathcal{B}$ is semi–simple. □

Corollary 2.7. *Let $\mathcal{B}$ be an anisotropic symplectic $\mathcal{F}G$–module. Let N be a normal subgroup of G such that $|G:N|$ is odd. Assume further that $\mathcal{B}_N$ is a hyperbolic $\mathcal{F}N$-module. Then any simple $\mathcal{F}N$–submodule of $\mathcal{B}_N$ appears with even multiplicity in any decomposition of $\mathcal{B}_N$ as a direct sum of simple $\mathcal{F}N$–submodules.*

Proof. This is a straightforward application of Propositions 2.2, 2.5 and 2.6. □

We close this section with a well known fact that we prove here for completeness.

Lemma 2.8. *Assume that $\mathcal{U}$ is a self–dual absolutely irreducible $\mathcal{F}G$-module, where G has odd order and $\mathcal{F}$ is a finite field whose characteristic does not divide $|G|$. Then $\mathcal{U}$ is trivial.*

Proof. Let χ denote the $\mathcal{F}$-absolutely irreducible character that $\mathcal{U}$ affords, while ϕ denotes a Brauer character that $\mathcal{U}$ affords. Then ϕ is defined for every element of G, since the characteristic of $\mathcal{F}$ is coprime to $|G|$. Because $\mathcal{U}$ is self–dual, the character ϕ is real valued. Let $\nu_2(\phi) = |G|^{-1}\sum_{g\in G}\phi(g^2)$ be the Frobenius–Schur indicator (see Chapter 4 in [6]) of ϕ. Then Theorem 4.5 in [6] implies that $\nu_2(\phi) \neq 0$, since ϕ is real valued. But

$$\nu_2(\phi) = |G|^{-1}\sum_{g\in G}\phi(g^2) = |G|^{-1}\sum_{g\in G}\phi(g),$$

because G has odd order. Hence $\nu_2(\phi)$ is the inner product $\nu_2(\phi) = [\phi, 1_G]$, where 1_G is the trivial character of G. We conclude that $[\phi, 1_G] \neq 0$. Hence $\phi = 1_G$. Therefore $\chi = 1_G$, and the lemma follows. □

3 Proof of Theorem A

We can now prove our main result. The proof will follow from a series of lemmas, based on the hypothesis that $\mathcal{F}, \mathcal{B}, G$ form a minimal counter–example. All the groups considered in this section have odd order. We also fix the odd prime p that is the characteristic of $\mathcal{F}$, and we assume that

Inductive Hypothesis. *$\mathcal{F}, \mathcal{B}, G$ have been chosen among all triplets satisfying the hypothesis but not the conclusion of Theorem A so as to minimize first the order $|G|$ of G and then the $\mathcal{F}$–dimension $\dim_{\mathcal{F}} \mathcal{B}$ of $\mathcal{B}$.*

Remark 2. For any proper subgroup H of G the minimality of $|G|$ easily implies that the restriction $\mathcal{B}_H$ is a hyperbolic $\mathcal{F}H$-module.

Lemma 3.1. *$\mathcal{B}$ is non–zero and anisotropic.*

Proof. If $\mathcal{B}$ were zero it would be hyperbolic contradicting the Inductive Hypothesis. So $\mathcal{B}$ is non–zero. If $\mathcal{B}$ is not anisotropic then it contains a non–zero isotropic $\mathcal{F}G$–module $\mathcal{U}$. Let N be an arbitrary cyclic subgroup of G. Then the isotropic $\mathcal{F}N$-submodule $\mathcal{U}_N$ of $\mathcal{B}_N$ is contained in some maximal isotropic $\mathcal{F}N$–submodule $\mathcal{V}$ of $\mathcal{B}_N$. Since $\mathcal{B}_N$ is hyperbolic this maximal isotropic submodule is self–perpendicular, i.e., $\mathcal{V} = \mathcal{V}^{\perp}$. Hence

$$\mathcal{U} \subseteq \mathcal{V} = \mathcal{V}^{\perp} \subseteq \mathcal{U}^{\perp}.$$

Therefore the factor module $\bar{\mathcal{V}} = \mathcal{V}/\mathcal{U}$ is a self–perpendicular $\mathcal{F}N$–submodule of the symplectic $\mathcal{F}G$–module $\bar{\mathcal{U}} = \mathcal{U}^{\perp}/\mathcal{U}$. Hence $\mathcal{F}, G, \bar{\mathcal{U}}$ satisfy the hypothesis of the Main Theorem. As $\dim(\bar{\mathcal{U}}) < \dim(\mathcal{B})$, the minimality of $\dim(\mathcal{B})$ implies that $\bar{\mathcal{U}}$ is a hyperbolic $\mathcal{F}G$–module. So there is a self–perpendicular $\mathcal{F}G$–submodule $\bar{\mathcal{J}}$ in $\bar{\mathcal{U}}$. From the definition of the symplectic form on $\bar{\mathcal{U}}$ (see (1)) it follows that the inverse image $\mathcal{J}$ of $\bar{\mathcal{J}}$ in $\mathcal{U}^{\perp}$ is a self–perpendicular $\mathcal{F}G$–submodule of $\mathcal{B}$ containing $\mathcal{U}$. Therefore $\mathcal{B}$ is hyperbolic, contradicting the Inductive Hypothesis. So the lemma holds. □

Lemma 3.2. *p doesn't divide the order $|G|$ of G.*

Proof. Suppose that p divides $|G|$. Because G is solvable, it contains a Hall p'–subgroup H. If G is a p-group we take $H = 1$. Since p divides $|G|$, the subgroup H is strictly smaller than G. Then according to Remark 2, the $\mathcal{F}H$-module $\mathcal{B}_H$ is hyperbolic. It follows (see Theorem 3.2 of [3]) that $\mathcal{B}$ is a hyperbolic $\mathcal{F}G$-module, contradicting the Inductive Hypothesis. Hence $(p, |G|) = 1$. □

Lemma 3.3. *$\mathcal{B}$ is an orthogonal direct sum*

$$\mathcal{B} = \mathcal{U}_1 \dot{\perp} \ldots \dot{\perp} \mathcal{U}_k \tag{5}$$

where $k \geq 1$ and $\{\mathcal{U}_i\}_{i=1,\ldots,k}$ are distinct, simple $\mathcal{F}G$–submodules of $\mathcal{B}$, that are also symplectic. Furthermore each $\mathcal{U}_i$ is quasi–primitive (i.e., its restriction to every normal subgroup of G is homogeneous).

Proof. The first statement follows from Lemma 3.1, Proposition 2.2 and Remark 1. For the rest of the proof we fix $\mathcal{U} = \mathcal{U}_i$ for some $i = 1, \ldots, k$. We also fix a normal subgroup K of G. If the restriction of $\mathcal{U}$ to K is not homogeneous then Clifford's Theorem implies that

$$\mathcal{U}_K \cong e(\mathcal{V}^{\sigma_1} \oplus \ldots \oplus \mathcal{V}^{\sigma_r})$$

where e is some positive integer, $\mathcal{V} = \mathcal{V}^{\sigma_1}$ is a simple $\mathcal{F}K$–submodule of $\mathcal{U}$ and $\mathcal{V}^{\sigma_1}, \dots, \mathcal{V}^{\sigma_r}$ are the distinct conjugates of $\mathcal{V}$ in G, with $\sigma_1, \dots, \sigma_r$ coset representatives of the stabilizer, $G_{\mathcal{V}}$, of $\mathcal{V}$ in G.

Let $\mathcal{W} = \mathcal{U}(\mathcal{V})$ be the $\mathcal{V}$-primary component of $\mathcal{U}_K$. Then Clifford's Theorem implies that $\mathcal{W}$ is the unique simple $\mathcal{F}G_{\mathcal{V}}$–submodule of $\mathcal{U}$ that lies above $\mathcal{V}$ and induces $\mathcal{U}$, i.e., that satisfies $\mathcal{W}^G \cong \mathcal{U}$ and $\mathcal{W}_K \cong e\mathcal{V}$. Furthermore the dual $\widehat{\mathcal{W}}$ of $\mathcal{W}$ induces in G the dual $\widehat{\mathcal{U}}$ of $\mathcal{U}$ since $\widehat{\mathcal{W}}^G \cong \widehat{\mathcal{W}^G}$. Hence $\widehat{\mathcal{W}}^G \cong \mathcal{U}$, because $\mathcal{U}$ is self–dual (see Lemma 2.3). On the other hand, the restriction of $\widehat{\mathcal{W}}$ to K is isomorphic to $e\mathcal{V}$ since $\mathcal{V}$ is self–dual by Proposition 2.5. Hence the unicity of $\mathcal{W}$ implies that $\mathcal{W}$ is self–dual.

According to Proposition 2.6 the self–dual $\mathcal{F}G_{\mathcal{V}}$-module $\mathcal{W}$ appears with even multiplicity as a direct summand of $\mathcal{B}_{G_{\mathcal{V}}}$, because $\mathcal{B}_{G_{\mathcal{V}}}$ is hyperbolic ($G_{\mathcal{V}} < G$). This, along with the fact that $\mathcal{W}$ appears with multiplicity one in $\mathcal{U}_{G_{\mathcal{V}}}$, implies that there is some $j \in \{1, \dots, k\}$ with $j \neq i$ such that the $\mathcal{V}$-primary component $\mathcal{U}_j(\mathcal{V})$ of $\mathcal{U}_j$ is isomorphic to $\mathcal{W}$. So

$$\mathcal{W} = \mathcal{U}(\mathcal{V}) \cong \mathcal{U}_j(\mathcal{V}).$$

We conclude that

$$\mathcal{U}_i = \mathcal{U} \cong \mathcal{W}^G \cong \mathcal{U}_j(\mathcal{V})^G \cong \mathcal{U}_j,$$

as $\mathcal{F}G$-modules. This contradicts the fact that $\{\mathcal{U}_i\}_{i=1}^k$ are all distinct, by the first statement of the lemma. Hence the lemma is proved. ☐

From now on and until the end of the paper, we write $\mathcal{E}$ for a finite algebraic field extension of $\mathcal{F}$, that is a splitting field of G and all its subgroups.

Lemma 3.4. *Assume that $\mathcal{U}_i$, for $i = 1, \dots, k$, is a direct summand of $\mathcal{B}$ appearing in* (5). *Let $N \trianglelefteq G$. Then $\mathcal{U}_i|_N \cong e_i\mathcal{V}_i$, where $\mathcal{V}_i$ is an irreducible $\mathcal{F}N$-submodule of $\mathcal{U}_i$ and e_i is an integer. If $\mathcal{V}_i$ is non-trivial then e_i is odd.*

Proof. We fix $\mathcal{U} = \mathcal{U}_i$, for some $i = 1, \dots, k$. We also fix a normal subgroup N of G. According to Lemma 3.3, the $\mathcal{F}G$-module $\mathcal{U}$ is quasi–primitive. Hence there exists an irreducible $\mathcal{F}N$-submodule $\mathcal{V}$ of $\mathcal{U}$, and an integer e such that $\mathcal{U}_N \cong e\mathcal{V}$. Thus, it remains to show that e is odd in the case that $\mathcal{V}$ is non-trivial. So we assume that $\mathcal{V}$, and thus $\mathcal{U}$, is non-trivial.

We observe that if $\mathcal{U}$ and $\mathcal{V}$ were absolutely irreducible modules then it would be immediate that e is odd (even if $\mathcal{V}$ was trivial), because for absolutely irreducible modules the integer e divides the order of G (see Corollary 11.29 in [6]). So we assume that $\mathcal{F}$ is not a splitting field of G, and we work with the algebraic field extension $\mathcal{E}$ of $\mathcal{F}$. We define $\mathcal{U}^{\mathcal{E}}$ to be the extended $\mathcal{E}G$-module

$$\mathcal{U}^{\mathcal{E}} = \mathcal{U} \otimes_{\mathcal{F}} \mathcal{E}.$$

According to Theorem 9.21 in [6], there exist absolutely irreducible $\mathcal{E}G$-modules $\mathcal{U}^i$, for $i = 1, \dots, n$, such that

$$\mathcal{U}^{\mathcal{E}} \cong \bigoplus_{i=1}^{n} \mathcal{U}^i.$$

Furthermore the $\mathcal{U}^i$, for all $i = 1, \dots, n$, constitute a Galois conjugacy class over $\mathcal{F}$, and thus they are all distinct. In particular, if $\mathcal{E}_{\mathcal{U}}$ is the subfield of $\mathcal{E}$ that is generated by all the values of the irreducible $\mathcal{E}$-character afforded by $\mathcal{U}^i$ (the same field for all $i = 1, \dots, n$), then $n = [\mathcal{E}_{\mathcal{U}} : \mathcal{F}] = \dim_{\mathcal{F}}(\mathcal{E}_{\mathcal{U}})$. (Note that $\mathcal{E}_{\mathcal{U}}$ is the unique subfield of $\mathcal{E}$ isomorphic to the center of the endomorphism algebra $\mathrm{End}_{\mathcal{F}G}(\mathcal{U})$.) Clearly $n \cdot \dim_{\mathcal{E}} \mathcal{U}^1 = \dim_{\mathcal{F}} \mathcal{U}$. Hence n is even, because $\dim_{\mathcal{F}} \mathcal{U}$ is even (as $\mathcal{U}$ is symplectic) and $\dim_{\mathcal{E}} \mathcal{U}^1$ is odd (as G is odd and $\mathcal{U}^1$ is an absolutely irreducible $\mathcal{E}G$-module). In addition, each $\mathcal{E}G$-module $\mathcal{U}^i$, for $i = 1, \dots, n$, when consider as an $\mathcal{F}G$-module, is isomorphic to a direct sum of $[\mathcal{E} : \mathcal{E}_{\mathcal{U}}]$ copies of $\mathcal{U}$ (see Theorem 1.16 in Chapter 1 of [5]). Hence if we denote by $\mathcal{U}^i_{\mathcal{F}}$ the $\mathcal{E}G$-module $\mathcal{U}^i$ regarded as an $\mathcal{F}G$-module, we get

$$\mathcal{U}^i_{\mathcal{F}} \cong [\mathcal{E} : \mathcal{E}_{\mathcal{U}}]\mathcal{U}^i, \tag{6}$$

for all $i = 1, \dots, n$.

We also write $\mathcal{V}^{\mathcal{E}}$ for the extended $\mathcal{E}N$-module $\mathcal{V}^{\mathcal{E}} = \mathcal{V} \otimes_{\mathcal{F}} \mathcal{E}$. Then according to Theorem 9.21 in [6] there exist absolutely irreducible $\mathcal{E}N$-modules $\mathcal{V}^j$ for $j = 1, \dots, d$, such that

$$\mathcal{V}^{\mathcal{E}} \cong \bigoplus_{j=1}^{d} \mathcal{V}^j. \tag{7}$$

In addition, the absolutely irreducible modules $\mathcal{V}^j$, for all $j = 1, \dots, d$, form a Galois conjugacy class, and thus they are all distinct. Furthermore, $d = [\mathcal{E}_{\mathcal{V}} : \mathcal{F}] = \dim_{\mathcal{F}} \mathcal{E}_{\mathcal{V}}$, where $\mathcal{E}_{\mathcal{V}}$ is the subfield of $\mathcal{E}$ generated by all the values of the irreducible $\mathcal{E}$-character afforded by $\mathcal{V}^j$ (the same field for all $j = 1, \dots, d$). The field $\mathcal{E}_{\mathcal{V}}$ is the unique subfield of $\mathcal{E}$ isomorphic to the center of the endomorphism algebra $\mathrm{End}_{\mathcal{F}N}(\mathcal{V})$. Note that, according to Proposition 2.5, the $\mathcal{F}N$-submodule $\mathcal{V}$ of $\mathcal{U}$ is self–dual. Hence $\mathcal{V}^{\mathcal{E}}$ is also a self–dual $\mathcal{E}N$-module. Because $\mathcal{V}$ is non-trivial, $\mathcal{V}^j$ is also non-trivial, for all $j = 1, \dots, d$. Therefore the absolutely irreducible $\mathcal{E}N$-module $\mathcal{V}^j$ can't be self–dual, because N has odd order and $\mathcal{V}^j$ is non–trivial (see Lemma 2.8), for all such j. The fact that none of the $\mathcal{V}^j$ is self–dual, for all $j = 1, \dots, d$, while they all appear in (7) in dual pairs, implies that d is even. Even more, if $\mathcal{V}^j_{\mathcal{F}}$ denotes the module $\mathcal{V}^j$ regarded as an $\mathcal{F}N$-module, then Theorem 1.16 of Chapter 1 in [5] implies that

$$\mathcal{V}^j_{\mathcal{F}} \cong [\mathcal{E} : \mathcal{E}_{\mathcal{V}}]\mathcal{V}^j, \tag{8}$$

for all $j = 1, \dots, d$.

Without loss we may assume that $\mathcal{V}^1, \dots, \mathcal{V}^c$ are exactly those among the $\mathcal{V}^j$, for $j = 1, \dots, d$, that lie under $\mathcal{U}^1$, for some $c = 1, \dots, d$. Thus Clifford's theorem implies that

$$\mathcal{U}^1_N \cong e'(\mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^c), \tag{9}$$

where $\mathcal{V}^1, \dots, \mathcal{V}^c$ are the distinct G-conjugates of $\mathcal{V}^1$, and e', c are integers that divide $|G|$ and thus are odd. (Note that here we are dealing with absolutely irreducible modules so e' does divide $|G|$.) If we regard the modules of (9) as modules over the field $\mathcal{F}$ then we clearly have $\mathcal{U}^1_{\mathcal{F}}|_N \cong e'(\mathcal{V}^1_{\mathcal{F}} \oplus \cdots \oplus \mathcal{V}^c_{\mathcal{F}})$. This, along with (6) and (8), implies

$$[\mathcal{E} : \mathcal{E}_{\mathcal{U}}]\mathcal{U}_N \cong e'c[\mathcal{E} : \mathcal{E}_{\mathcal{V}}]\mathcal{V}.$$

Since $\mathcal{U}_N \cong e\mathcal{V}$, we have

$$[\mathcal{E} : \mathcal{E}_{\mathcal{U}}]e = e'c[\mathcal{E} : \mathcal{E}_{\mathcal{V}}]. \tag{10}$$

If $\mathcal{D}$ is the subfield of $\mathcal{E}$ generated by $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_{\mathcal{U}}$, then dividing both sides of (10) by $[\mathcal{E} : \mathcal{D}]$ we obtain

$$e[\mathcal{D} : \mathcal{E}_{\mathcal{U}}] = e'c[\mathcal{D} : \mathcal{E}_{\mathcal{V}}]. \tag{11}$$

Assume that e is even. Then (11) implies that $[\mathcal{D} : \mathcal{E}_{\mathcal{V}}]$ is even, as e' and c are known to be odd. Let Γ be the Galois group $\Gamma = \mathrm{Gal}(\mathcal{D}/\mathcal{F})$ of $\mathcal{D}$ over $\mathcal{F}$. Because Γ is cyclic, it contains a unique involution ι. Let $\mathcal{E}^*_{\mathcal{V}}$ and $\mathcal{E}^*_{\mathcal{U}}$ be the subgroups of Γ consisting of those elements of Γ that fix pointwise $\mathcal{E}_{\mathcal{V}}$ and $\mathcal{E}_{\mathcal{U}}$, respectively. Then Galois theory implies that $\mathcal{E}_{\mathcal{V}}^* = [\mathcal{E}_{\mathcal{V}}^* : 1] = [\mathcal{D} : \mathcal{E}_{\mathcal{V}}]$ is even. We conclude that the unique involution ι of Γ is an element of $\mathcal{E}_{\mathcal{V}}^*$. Therefore, ι fixes the field $\mathcal{E}_{\mathcal{V}}$ pointwise. So ι fixes, to within isomorphisms, each of the $\mathcal{E}N$-modules $\mathcal{V}^j$. Because ι acts non–trivially on $\mathcal{D}$ and fixes $\mathcal{E}_{\mathcal{V}}$, it must act non–trivially on $\mathcal{E}_{\mathcal{U}}$. We conclude that ι viewed as an $\mathcal{F}$-automorphism of $\mathcal{E}_{\mathcal{U}}$ must coincide with the unique involution in the Galois group $\mathrm{Gal}(\mathcal{E}_{\mathcal{U}}/\mathcal{F})$ of $\mathcal{E}_{\mathcal{U}}$ above $\mathcal{F}$. Furthermore, this unique involution must send $\mathcal{U}^i$ to its dual $\widehat{\mathcal{U}^i}$, for every $i = 1, \dots, n$. (Of course $\mathcal{U}^i$ is not self–dual, because it is a non–trivial absolutely irreducible module of the odd order group G (see Lemma 2.8).) Hence, applying ι to both sides of (9) we get

$$\widehat{\mathcal{U}^1}_N \cong e'(\widehat{\mathcal{V}^1} \oplus \cdots \oplus \widehat{\mathcal{V}^c}) \cong e'(\mathcal{V}^1 \oplus \cdots \oplus \mathcal{V}^c).$$

Hence the dual $\widehat{\mathcal{V}^1}$ of $\mathcal{V}^1$ should be among the G-conjugates $\mathcal{V}^1, \dots, \mathcal{V}^c$ of $\mathcal{V}^1$. Because $\mathcal{V}^1$ is not self–dual the G-conjugates of $\mathcal{V}^1$ should appear in dual pairs. Hence c is even. But c is also odd as a divisor of $|G|$. This contradiction implies that e is odd. So the lemma holds. □

Lemma 3.5. *The group G is not abelian.*

Proof. Assume that G is abelian. Then any cyclic subgroup $N =< \sigma >$ of G is normal, for every $\sigma \in G$. Because $\mathcal{B}_N$ is hyperbolic, Lemmas 3.1 and 3.3 along with Corollary 2.7 imply that

$$\mathcal{B}_N \cong 2 \cdot \Delta_{(N)},$$

where $\Delta_{(N)}$ is a semi–simple $\mathcal{F}N$-submodule of $\mathcal{B}$. Using the splitting field $\mathcal{E}$ of G, we write $\mathcal{B}^{\mathcal{E}}$ for the extended $\mathcal{E}G$-module $\mathcal{B}^{\mathcal{E}} = \mathcal{B} \otimes_{\mathcal{F}} \mathcal{E}$. Then

$$\mathcal{B}_N^{\mathcal{E}} \cong 2 \cdot \Delta_{(N)}^{\mathcal{E}}, \tag{12}$$

where $\Delta_{(N)}^{\mathcal{E}}$ is the extended $\mathcal{E}N$-module $\Delta_{(N)} \otimes_{\mathcal{F}} \mathcal{E}$. Let ϕ be a Brauer character that the $\mathcal{E}G$-module $\mathcal{B}^{\mathcal{E}}$ affords. Because $(p, |G|) = 1$, ϕ is defined for every element of G. So ϕ coincides with a complex character of G. In view of (12), for every cyclic subgroup $N =< \sigma >$ of G, the restriction ϕ_N of ϕ to N equals $2 \cdot \delta_{(N)}$, where $\delta_{(N)}$ is a complex character of N. Hence, for every element $\sigma \in G$, the integer 2 divides $\phi(\sigma)$ in the ring $\mathbb{Z}[\omega]$, where ω is a $|G|$-primitive root of unity. We conclude that 2 also divides $\sum_{\sigma \in G} \phi(\sigma) \cdot \lambda(\sigma^{-1})$, for any irreducible (linear) complex character λ of G. That is, 2 divides $|G| \cdot < \phi, \lambda >$, for any $\lambda \in \mathrm{Irr}(G)$. The fact that G has odd order, implies that 2 divides $< \phi, \lambda >$ in $\mathbb{Z}[\omega]$, for any $\lambda \in \mathrm{Irr}(G)$. Because $\phi = \sum_{\lambda \in \mathrm{Irr}(G)} < \phi, \lambda > \cdot \lambda$, we get

$$\phi = 2 \cdot \chi, \tag{13}$$

where χ is a complex character of G.

On the other hand, Lemma 3.3 implies that $\mathcal{B} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$, where the $\mathcal{U}_i$ are distinct simple $\mathcal{F}G$-modules, for all $i = 1, \dots, k$. Hence the extended $\mathcal{E}G$-module $\mathcal{B}^{\mathcal{E}}$ will also equal the direct sum of the distinct $\mathcal{E}G$–modules $\mathcal{U}_1^{\mathcal{E}}, \dots, \mathcal{U}_k^{\mathcal{E}}$. By Theorem 9.21 in [6], for each $i = 1, \dots, k$, there exist absolutely irreducible $\mathcal{E}G$–modules $\mathcal{U}_i^j$, for $j = 1, \dots, n_i$ such that

$$\mathcal{U}_i{}^{\mathcal{E}} \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i{}^j.$$

Furthermore, the $\mathcal{U}_i^j$, for $j = 1, \dots, n_i$, constitute a Galois conjugacy class over $\mathcal{F}$, and thus they are all distinct. In addition, the above absolutely irreducible $\mathcal{E}G$-modules $\mathcal{U}_i^j$, for all $i = 1, \dots, k$

and all $j = 1, \dots, n_i$, are distinct. Indeed, for all $i = 1, \dots, k$, the corresponding simple $\mathcal{F}G$-modules $\mathcal{U}_i$ are distinct. We conclude that

$$\mathcal{B}^{\mathcal{E}} \cong \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n_i} \mathcal{U}_i{}^j,$$

where $\mathcal{U}_i^j$ are all distinct absolutely irreducible $\mathcal{E}G$-modules. So the character ϕ that $\mathcal{B}^{\mathcal{E}}$ affords equals

$$\phi = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \chi_i^j,$$

where, for all $i = 1, \dots, k$ and all $j = 1, \dots, n_i$, the character χ_i^j is a Brauer character that $\mathcal{U}_i^j$ affords. So all these characters are distinct. This contradicts (13). Hence the group G is not abelian, and the lemma is proved. $\square$

Lemma 3.6. *G acts faithfully on $\mathcal{B}$.*

Proof. Suppose not. Let K denote the kernel of the action of G on $\mathcal{B}$ and $\bar{G} = G/K$. Thus $|\bar{G}| \lneq |G|$ (as $K \neq 1$).

If $\bar{G}$ is not itself cyclic, then any cyclic subgroup $\bar{N}$ of $\bar{G}$ is the image $\bar{N} = N/G$ of some proper subgroup N of G. Since $\mathcal{B}$ is $\mathcal{F}N$-hyperbolic, it is clearly $\mathcal{F}\bar{N}$-hyperbolic. Hence the triplet $\mathcal{F}, \mathcal{B}, \bar{G}$ satisfies the hypothesis of the Main Theorem. The minimality of $|G|$ implies that $\mathcal{B}$ is a hyperbolic $\mathcal{F}\bar{G}$–module, and therefore a hyperbolic $\mathcal{F}G$–module, because any $\mathcal{F}\bar{G}$–submodule of $\mathcal{B}$ is also an $\mathcal{F}G$–submodule of $\mathcal{B}$. This contradicts the Inductive Hypothesis.

If $\bar{G}$ is cyclic, then $\bar{G} =< \bar{\sigma} >$, where $\bar{\sigma}$ is the image in $\bar{G}$ of some $\sigma \in G$. Let $M =< \sigma >$. Then M is a proper subgroup of G, because G is not cyclic. In addition, the image of M in $\bar{G}$ is $\bar{G}$. So $G = MK$ with $M \lneq G$. Then Remark 2 implies that $\mathcal{B}$ is $\mathcal{F}M$–hyperbolic and thus $\mathcal{F}G$–hyperbolic. This last contradiction implies the lemma. $\square$

Lemma 3.7. *Suppose M is a minimal normal subgroup of G. Then M is cyclic and central.*

Proof. According to Lemmas 3.3 and 3.4 for each $i = 1, \dots, k$ there is a simple $\mathcal{F}M$–submodule $\mathcal{V}_i$ of $\mathcal{U}_i$ and an odd integer e_i, such that $\mathcal{U}_i|_M \cong e_i\mathcal{V}_i$. As G acts faithfully on $\mathcal{B}$, there is some $i \in \{1, \dots, k\}$ such that $\mathcal{V}_i \neq 1$ is non-trivial. Let $K_M(\mathcal{V}_i)$ be the kernel of the action of M on $\mathcal{V}_i$. The fact that $\mathcal{V}_i$ is G-invariant implies that $K_M(\mathcal{V}_i)$ is a normal subgroup of G contained in M. Hence $K_M(\mathcal{V}_i) = 1$. Therefore M admits a faithful simple representation. In addition, M is a q-elementary abelian group, for some prime q that divides $|G|$, because G is solvable. We conclude that $M \cong \mathbb{Z}_q$ is a cyclic group of order q.

It remains to show that M is central. If $\mathcal{F}$ is a splitting field of M (that is, it contains a primitive q-root of 1), then the fact that there exists a faithful, simple and thus one–dimensional, G-invariant $\mathcal{F}M$-module $\mathcal{V}_i$ implies that M is central in G. If F is not a splitting field of M, we work with the extension field $\mathcal{E}$ of $\mathcal{F}$. The extended module $\mathcal{B}^{\mathcal{E}} = \mathcal{B} \otimes_{\mathcal{F}} \mathcal{E}$ equals the direct sum of the extended $\mathcal{E}G$–modules $\mathcal{U}_1^{\mathcal{E}}, \dots, \mathcal{U}_k^{\mathcal{E}}$, because $\mathcal{B}$ is the direct sum of $\mathcal{U}_1, \dots, \mathcal{U}_k$. As we have already seen, for each $i = 1, \dots, k$, there exist absolutely irreducible $\mathcal{E}G$–modules $\mathcal{U}_i^j$, for $j = 1, \dots, n_i$, that constitute a Galois conjugacy class over $\mathcal{F}$ and satisfy

$$\mathcal{U}_i{}^{\mathcal{E}} \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i{}^j. \tag{14}$$

Since $\mathcal{U}_i|_M \cong e_i \mathcal{V}_i$ we have $\mathcal{U}_i^{\mathcal{E}}|_M \cong e_i \mathcal{V}_i^{\mathcal{E}}$. In addition,

$$\mathcal{V}_i^{\mathcal{E}} \cong \bigoplus_{r=1}^{s_i} \mathcal{V}_i^r,$$

where the $\mathcal{V}_i^r$, for $r = 1, \ldots, s_i$, are absolutely irreducible $\mathcal{E}M$-modules, and thus of dimension one, that form a Galois conjugacy class over $\mathcal{F}$. Therefore,

$$\mathcal{U}_i^{\mathcal{E}}|_M \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i^j|_M \cong e_i \bigoplus_{r=1}^{s_i} \mathcal{V}_i^r, \tag{15}$$

for all $i = 1, \ldots, k$.

As we have already seen, there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{V}_i$ is a faithful $\mathcal{F}M$-module. Without loss, we may assume that $i = 1$. Then it is clear that the $\mathcal{V}_1^r$ are faithful $\mathcal{E}M$-modules, for all $r \in \{1, \ldots, s_1\}$. If $\mathcal{V}_1^r$ is G-invariant, for some $r \in \{1, \ldots, s_1\}$ (and thus for all such r) we are done.

Thus we may assume that the stabilizer $G_{\mathcal{V}}$ of $\mathcal{V} = \mathcal{V}_1^1$ in G is strictly smaller than G. Then $G_{\mathcal{V}} = C_G(M)$, because $\mathcal{V}$ is $\mathcal{E}M$-faithful. Let $C := G_{\mathcal{V}} = C_G(M)$. Note that C is a normal subgroup of G, since $M \trianglelefteq G$. Furthermore, C is also the stabilizer of $\mathcal{V}_1^r$, for all $r = 1, \ldots, s_1$. According to Lemma 3.4, for all $i = 1, \ldots, k$, we have $\mathcal{U}_i|_C = m_i \cdot \mathcal{Y}_i$, where $\mathcal{Y}_i$ is a simple $\mathcal{F}C$-module, and m_i some positive integer. For the extended $\mathcal{E}C$-modules $\mathcal{Y}_i^{\mathcal{E}}$ we have

$$\mathcal{U}_i^{\mathcal{E}}|_C \cong m_i \mathcal{Y}_i^{\mathcal{E}} \cong m_i \bigoplus_{l=1}^{t_i} \mathcal{Y}_i^l,$$

where the $\mathcal{Y}_i^l$, for $l = 1, 2, \ldots, t_i$, are absolutely irreducible $\mathcal{E}C$-modules that constitute a Galois conjugacy class over $\mathcal{F}$. Hence

$$\mathcal{U}_i^{\mathcal{E}}|_C \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i^j|_C \cong m_i \bigoplus_{l=1}^{t_i} \mathcal{Y}_i^l, \tag{16}$$

for all $i = 1, \ldots, k$. We remark here that, because $\mathcal{U}_i$ is quasi–primitive, all the group conjugates of $\mathcal{Y}_i^1$ are among its Galois conjugates, for every $i = 1, \ldots, k$.

In the case $i = 1$, equations (15) and (16) imply

$$\mathcal{U}_1^{\mathcal{E}}|_M \cong m_1 \bigoplus_{l=1}^{t_1} \mathcal{Y}_1^l|_M \cong e_1 \bigoplus_{r=1}^{s_1} \mathcal{V}_1^r. \tag{17}$$

Without loss we may assume that $\mathcal{U}_1^1$ lies above $\mathcal{Y}_1^1$, and that $\mathcal{Y}_1^1$ lies above $\mathcal{V}_1^1 = \mathcal{V}$. Clearly $\mathcal{Y}_1^1$ is non–trivial as it restricts to a multiple of the non–trivial $\mathcal{F}M$-module $\mathcal{V}_1$. Hence Lemma 3.4 implies that m_1 is an odd integer. Because C is the stabilizer of $\mathcal{V}$ in G, Clifford's theory implies that $\mathcal{Y}_1^1$ is the unique simple $\mathcal{E}C$-module that lies above $\mathcal{V}_1^1$ and induces irreducibly to $\mathcal{U}_1^1$ in G. Note that $\mathcal{Y}_1^1$ appears with odd multiplicity m_1 as a summand of $\mathcal{U}_1^{\mathcal{E}}|_C$, because the $\mathcal{E}C$-modules $\mathcal{Y}_1^l$ are distinct for distinct values of l, as they form a Galois conjugacy class over $\mathcal{F}$. Furthermore, if $\mathcal{Y}_1^1$ lies under some $\mathcal{U}_i^j$, for $i \neq 1$, then it induces $\mathcal{U}_i^j$. So $\mathcal{U}_i^j \cong \mathcal{U}_1^1$. Hence the sum of the Galois conjugates of $\mathcal{U}_i^j$ is isomorphic to the sum of the Galois conjugates of $\mathcal{U}_1^1$. Therefore

$$\mathcal{U}_i^{\mathcal{E}} \cong \bigoplus_{j=1}^{n_i} \mathcal{U}_i^j \cong \bigoplus_{j=1}^{n_1} \mathcal{U}_1^j \cong \mathcal{U}_1^{\mathcal{E}}.$$

The above contradicts the fact that $\mathcal{U}_1$ and $\mathcal{U}_i$ are non-isomorphic simple $\mathcal{F}G$-modules (see Lemma 3.3). We conclude that $\mathcal{Y}_1^1$ appears with odd multiplicity m_1 in the decomposition of

$$\mathcal{B}^{\mathcal{E}}|_C \cong \bigoplus_{i=1}^{k} \mathcal{U}_i^{\mathcal{E}}|_C \cong \bigoplus_{i=1}^{k} m_i \bigoplus_{l=1}^{t_i} \mathcal{Y}_i^l.$$

On the other hand, in view of Corollary 2.7 every simple $\mathcal{F}C$-submodule of $\mathcal{B}$ appears with even multiplicity in any decomposition of $\mathcal{B}_C$, as C is a normal subgroup of G and $\mathcal{B}_C$ is hyperbolic as an $\mathcal{F}C$-module, by Remark 2. Hence every absolutely irreducible $\mathcal{E}C$-submodule of $\mathcal{B}^{\mathcal{E}}$ should also appear with even multiplicity in any decomposition of $\mathcal{B}^{\mathcal{E}}|_C$. This contradicts the conclusion of the preceding paragraph. So we must have $G_{\mathcal{V}} = C = G$. Hence the lemma is proved. □

Clearly Lemma 3.7 implies

Corollary 3.8. *Suppose that M is a minimal normal subgroup of G and $\mathcal{E}$ a splitting field of G and all its subgroups. Then every $\mathcal{E}M$-module is G-invariant.*

We can now show

Lemma 3.9. *Suppose that M is a minimal normal subgroup of G. Then the restriction $\mathcal{B}_M$ is homogeneous. Furthermore $\mathcal{B}_M \cong e\mathcal{V}$, where $\mathcal{V}$ is a simple faithful G-invariant $\mathcal{F}M$-submodule of $\mathcal{B}_M$ and e is a positive integer.*

Proof. As in the previous lemma we write $\mathcal{U}_i|_M = e_i\mathcal{V}_i$, where $i = 1, \ldots, k$, and $\mathcal{V}_i$ is a simple G-invariant $\mathcal{F}M$-submodule of $\mathcal{U}_i$. If $\mathcal{F}M$ is not homogeneous, then there are at least two non-isomorphic simple $\mathcal{F}M$-submodules of $\mathcal{B}_M$, say $\mathcal{V}$ and $\mathcal{W}$. We may suppose that $\mathcal{V}$ is non-trivial. Assume that $\mathcal{V}_i \cong \mathcal{V}$ as $\mathcal{F}M$–modules, for all $i = 1, \ldots, l$ and some l such that $1 \leq l \leq k$, while $\mathcal{V}_i \not\cong \mathcal{V}$ for $i = l+1, \ldots, k$. Let $\mathcal{U}$ be the orthogonal direct sum

$$\mathcal{U} = \mathcal{U}_1 \dot{\perp} \ldots \dot{\perp} \mathcal{U}_l.$$

of the corresponding $\mathcal{F}G$–submodules of $\mathcal{B}$. We also write

$$\mathcal{R} = \mathcal{U}_{l+1} \dot{\perp} \ldots \dot{\perp} \mathcal{U}_k,$$

for the orthogonal direct sum of the remaining simple $\mathcal{F}G$-submodules of $\mathcal{B}$. Clearly

$$\mathcal{B} = \mathcal{U} \dot{\perp} \mathcal{R},$$

while $\mathcal{U}_M$ and $\mathcal{R}_M$ have no simple $\mathcal{F}M$-submodules in common.

We will show

Claim 1. $\mathcal{U}$ is $\mathcal{F}N$-hyperbolic for every cyclic subgroup N of G.

We first prove Claim 1 in the case that the product NM is a proper subgroup of G. In this case Remark 2 implies that $\mathcal{B}_{NM}$ is hyperbolic. Hence there exists a self–perpendicular $\mathcal{F}NM$-submodule $\mathcal{S} > 0$ of $\mathcal{B}$. Then $\mathcal{S}$ is a maximal isotropic $\mathcal{F}NM$-submodule of $\mathcal{B}_{NM}$. Furthermore, $\mathcal{B}_{NM} = \mathcal{U}_{NM} \dot{\perp} \mathcal{R}_{NM}$, where $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$ have no simple $\mathcal{F}NM$-submodule in common (otherwise $\mathcal{U}_M$ and $\mathcal{R}_M$ would have some common simple $\mathcal{F}M$-submodule). Hence

$$\mathcal{S} = (\mathcal{S} \cap \mathcal{U}_{NM}) \dot{\perp} (\mathcal{S} \cap \mathcal{R}_{NM}).$$

Because $\mathcal{S}$ is isotropic, both $\mathcal{S} \cap \mathcal{U}_{NM}$ and $\mathcal{S} \cap \mathcal{R}_{NM}$ are also isotropic. Hence their $\mathcal{F}$-dimensions are at most 1/2 the dimensions of $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$, respectively. But $\mathcal{S}$ is self–perpendicular and thus its $\mathcal{F}$-dimension is exactly $(1/2)\dim(\mathcal{B}_{NM})$. We conclude that the $\mathcal{F}$-dimensions of $\mathcal{S}\cap\mathcal{U}_{NM}$ and $\mathcal{S}\cap\mathcal{R}_{NM}$ are exactly $1/2$ the dimensions of $\mathcal{U}_{NM}$ and $\mathcal{R}_{NM}$, respectively. Therefore $\mathcal{S}\cap\mathcal{U}_{NM}$ is a maximal isotropic $\mathcal{F}NM$-submodule of $\mathcal{U}_{NM}$ of dimension $1/2$ the dimension of $\mathcal{U}_{NM}$. So $\mathcal{S}\cap\mathcal{U}_{NM}$ is self–perpendicular, by Lemma 2.1. Thus $\mathcal{U}_{NM}$ is hyperbolic as an $\mathcal{F}NM$-module. Hence it is also hyperbolic as an $\mathcal{F}N$-module. So Claim 1 holds when $NM < G$.

Assume now that N is a cyclic subgroup of G such that $NM = G$. Because M is minimal, Lemma 3.7 implies that $M \cong \mathbb{Z}_q$ is central. Hence $G = MN$ is an abelian group. This contradicts Lemma 3.5. Therefore $NM < G$, for every cyclic subgroup N of G. Thus Claim 1 holds.

Since $\mathcal{U} < \mathcal{B}$, the Inductive Hypothesis, along with Claim 1, implies that $\mathcal{U}$ is $\mathcal{F}G$-hyperbolic. Hence $\mathcal{U}$ contains a self–perpendicular $\mathcal{F}G$-submodule $\mathcal{T}$. Let $\mathcal{T}^{\perp}$ be the submodule of $\mathcal{B}$ that is perpendicular to $\mathcal{T}$. Then $\mathcal{R}$ as well as $\mathcal{T}$ are subsets of $\mathcal{T}^{\perp}$. We conclude that $\mathcal{T}$ is an isotropic $\mathcal{F}G$-submodule of $\mathcal{B}$. Hence $\mathcal{B}$ is not anisotropic. This last contradiction implies that $\mathcal{U} = \mathcal{B}$, and completes the proof of Lemma 3.9. □

Lemma 3.10. *Every abelian normal subgroup of G is cyclic.*

Proof. Let A be an abelian normal subgroup of G. By Lemma 3.4 there is a simple $\mathcal{F}A$–submodule $\mathcal{R}_1$ of $\mathcal{U}_1$ and an integer e_1 such that

$$\mathcal{U}_1|_A \cong e_1\mathcal{R}_1.$$

It follows from Lemma 3.9 that $\mathcal{R}_1$ is non-trivial, since its restriction to any minimal normal subgroup of G is non–trivial. Let K_1 denote the corresponding centralizer of $\mathcal{R}_1$ in A . Then K_1 equals the centralizer $C_A(\mathcal{U}_1)$ of $\mathcal{U}_1$ in A, and therefore is a normal subgroup of G. If K_1 is not trivial then it contains a minimal normal subgroup M of G. In view of Lemma 3.9 the restriction $\mathcal{U}_1|_M$, cannot be trivial, contradicting the definition of K_1. Hence K_1 is trivial. Thus A is cyclic and the lemma is proved. □

Let $F = F(G)$ be the Fitting subgroup of G. Assume further that $\{q_i\}_{i=1}^r$ are the distinct primes dividing $|F|$, and that T_i is the q_i-Sylow subgroup of F, for each $i = 1, \ldots, r$. Then $F = T_1 \times T_2 \times \cdots \times T_r$. Every characteristic abelian subgroup of F is cyclic, according to Lemma 3.10. Hence (see Theorem 4.9 in [4]) either T_i is cyclic or T_i is the central product $T_i = E_i \odot Z(T_i)$ of the extra special q_i–group $E_i = \Omega(T_i)$ of exponent q_i and the cyclic group $Z(T_i)$. We complete the proof of Theorem A exploring the two possible types of T_i.

Assume first that T_i is a cyclic group, for all $i = 1, \ldots, r$. In this case $F = T_1 \times \cdots \times T_r$ is also a cyclic group. Let C/F be a chief factor of G. So $\bar{C} = C/F$ is an elementary abelian q-group, for some prime q, because G is solvable. Then $\bar{C}$ acts coprimely on T_i for all i such that q does not divide $|T_i|$. But T_i is cyclic, and the minimal subgroup of T_i is central in G. Hence $C_{T_i}(\bar{C}) \neq 1$. We conclude that $T_i = [T_i, \bar{C}] \times C_{T_i}(\bar{C}) = C_{T_i}(\bar{C})$. So any q-Sylow subgroup C_q of C centralizes the q'-Hall subgroup R of F that is also a q'-Hall subgroup of C. We conclude that $C = C_q \times R$. But R is nilpotent as a subgroup of F. So C is a nilpotent normal subgroup of G bigger than the Fitting subgroup F of G. Therefore $G = F$ is a cyclic group, contradicting the Inductive Hypothesis. Hence there exists a Sylow subgroup T_i of $F = F(G)$ that is not cyclic.

Let $T = T_i$ be a non-cyclic q-Sylow subgroup of F, where $q = q_i$ for some $i = 1, \ldots, r$. Then $T = E \odot Z(T)$, where $E = \Omega(T)$ is an extra special q-group of exponent q and $Z(T)$ is the center of T. Of course E is a normal subgroup of G, since it is a characteristic subgroup of F. Furthermore, $Z(E)$ is a central subgroup of G because it is a minimal (it has order q) normal subgroup of G. According to Lemma 3.9, there exists a faithful G-invariant $\mathcal{F}Z(E)$-module $\mathcal{V}$ so that the restriction $\mathcal{B}_{Z(E)}$ of $\mathcal{B}$ to $Z(E)$ is a multiple of $\mathcal{V}$.

Using the extension field $\mathcal{E}$ of $\mathcal{F}$, we write $\mathcal{V}^{\mathcal{E}}$ for the extended $\mathcal{E}Z(E)$-module $\mathcal{V}\otimes_{\mathcal{F}}\mathcal{E}$. Then

$$\mathcal{V}^{\mathcal{E}} \cong \bigoplus_{j=1}^{s} \mathcal{V}^j$$

where $\mathcal{V}^j$ is an absolutely irreducible $\mathcal{E}Z(E)$-module, for all j with $j = 1, \dots, s$. Furthermore, the $\mathcal{V}^j$ constitute a Galois conjugacy class over $\mathcal{F}$, and thus they are all distinct. As we have already seen (see Corollary 3.8 and Lemma 3.9), the module $\mathcal{V}^j$ is a non-trivial G-invariant $\mathcal{E}Z(E)$-module. Because E is extra special, there exists a unique, up to isomorphism, absolutely irreducible $\mathcal{E}E$-module $\mathcal{W}^j$ lying above $\mathcal{V}^j$, for every $j = 1, \dots, s$. Note that for all such j the $\mathcal{E}E$-module $\mathcal{W}^j$ is G-invariant because $\mathcal{V}^j$ is G-invariant. According to Theorem 9.1 in [7] (used for modules) there exists a canonical conjugacy class of subgroups $H \leq G$ such that $HE = G$ and $H \cap E = Z(E)$. Furthermore, for this conjugacy class there exists a one–to–one correspondence between the isomorphism classes of absolutely irreducible $\mathcal{E}G$-modules lying above $\mathcal{W}^j$ and those classes of absolutely irreducible $\mathcal{E}H$-modules lying above $\mathcal{V}^j$. In addition, the fact that G has odd order implies that if Ξ and Ψ are representatives of the above two isomorphism classes, then they correspond iff $\Xi_H \cong \Psi \oplus 2\cdot\Delta$, where Δ is a completely reducible $\mathcal{E}H$-submodule of Ξ_H.

Let $\mathcal{U} = \mathcal{U}_1$ be one of the simple $\mathcal{F}G$-submodules of $\mathcal{B}$ appearing in (5). Then $\mathcal{U}^{\mathcal{E}} \cong \oplus_{j=1}^{n_1}\mathcal{U}^j$, where the $\mathcal{U}^j$ are absolutely irreducible $\mathcal{E}G$-modules that form a Galois conjugacy class. As earlier, we write $\mathcal{E}_{\mathcal{U}}$ for the extension field of $\mathcal{F}$ generated by all the values of the absolutely irreducible character that $\mathcal{U}^1$ affords. Let $\Gamma = \mathrm{Gal}(\mathcal{E}_{\mathcal{U}}/\mathcal{F})$ be the Galois group of that extension. Then (see Theorem 9.21 in [6]),

$$\mathcal{U}^{\mathcal{E}} \cong \bigoplus_{j=1}^{n_1}\mathcal{U}^j \cong \bigoplus_{\tau\in\Gamma}(\mathcal{U}^1)^{\tau}, \tag{18}$$

Clearly $\mathcal{U}^1$ lies above $\mathcal{W}^j$, for some $j = 1, \dots, s$, since $\mathcal{U} = \mathcal{U}_1$ lies above $\mathcal{V}$. Let Ψ be a representative of the isomorphism class of absolutely irreducible $\mathcal{E}H$-modules that corresponds to $\mathcal{U}^1$ and lies above $\mathcal{V}^j$. Then

$$\mathcal{U}^1_H \cong \Psi \oplus 2\cdot\Delta, \tag{19}$$

for some completely reducible $\mathcal{E}H$-module Δ. Let $\mathcal{E}_\Psi$ be the subfield of $\mathcal{E}$ generated by $\mathcal{F}$ and all the values of the absolutely irreducible character that Ψ affords. Then $\mathcal{E}_\Psi$ is a Galois extension of $\mathcal{F}$. Furthermore,

$$\mathcal{E}_\Psi = \mathcal{E}_{\mathcal{U}}. \tag{20}$$

Indeed, for any element σ in the Galois group $\mathrm{Gal}(\mathcal{E}/\mathcal{F})$ of $\mathcal{E}$ above $\mathcal{F}$ we get

$$(\mathcal{U}^1)^\sigma_H \cong \Psi^\sigma \oplus 2\cdot\Delta^\sigma.$$

Hence $(\mathcal{U}^1)^\sigma$ corresponds to Ψ^σ, as Ψ^σ is the only absolutely irreducible $\mathcal{E}H$-module that appears with odd multiplicity in $(\mathcal{U}^1)^\sigma_H$. Therefore, $(\mathcal{U}^1)^\sigma \ncong \mathcal{U}^1$ iff $\Psi^\sigma \ncong \Psi$. This is enough to guarantee that (20) holds. We conclude that the sum $\oplus_{\tau\in\Gamma}\Psi^\tau$ is the extension to $\mathcal{E}$ of an irreducible $\mathcal{F}H$-module, i.e., there exists an irreducible $\mathcal{F}H$-module Π such that

$$\Pi^{\mathcal{E}} \cong \bigoplus_{\tau\in\Gamma}\Psi^\tau,$$

where $\Pi^{\mathcal{E}}$ is the extended $\mathcal{E}H$-module $\Pi\otimes_{\mathcal{F}}\mathcal{E}$. Furthermore, (18) and (19) imply that Π appears with odd multiplicity as a summand of $\mathcal{U}_H = \mathcal{U}_1|_H$.

Next we observe that if Π appears as a summand of $\mathcal{U}_i|_H$, for some $i = 2, \ldots, k$, then it appears with even multiplicity. The reason is that $\mathcal{U}_1 \not\cong \mathcal{U}_i$ for all such i. As in (14) we choose a Galois conjugacy class $\{\mathcal{U}_i^j\}_{j+1}^{n_i}$ of absolutely irreducible $\mathcal{E}G$-modules such that $\mathcal{U}_i^{\mathcal{E}} \cong \oplus_{j=1}^{n_i} \mathcal{U}_i^j$. Then $\mathcal{U}_i \not\cong \mathcal{U} = \mathcal{U}_1$ implies that $\mathcal{U}_i^j \not\cong \mathcal{U}^1$, for all $i = 2, \ldots, k$ and all $j = 1, \ldots, n_i$. So the $\mathcal{E}H$-module Ψ can't correspond to $\mathcal{U}_i^j$, for any such i, j. Therefore if Ψ appears as a summand of the restriction $\mathcal{U}_i^j|_H$ of $\mathcal{U}_i^j$ to H, then it appears only with even multiplicity. Hence the same holds for Π, i.e., Π appears only with even multiplicity as a summand of $\mathcal{U}_i|_H$, whenever $i = 2, \ldots, k$. We conclude that Π appears with odd multiplicity as a summand of $\mathcal{B}_H = \mathcal{U}_1|_H \oplus \cdots \oplus \mathcal{U}_k|_H$.

We complete the proof of Theorem A with one more contradiction, that follows the fact that Π is a self–dual $\mathcal{F}H$-module. That we get a contradiction if Π is self–dual is easy to see, because according to Proposition 2.6, Π should appear with even multiplicity as a summand of the hyperbolic $\mathcal{F}H$-module $\mathcal{B}_H$. Thus it suffices to show that Π is self–dual.

The fact that $\mathcal{U} = \mathcal{U}_1$ is self–dual implies that $\mathcal{U}^{\mathcal{E}}$ is also self–dual. Hence the dual $\widehat{\mathcal{U}^1}$ of $\mathcal{U}^1$ is a Galois conjugate $(\mathcal{U}^1)^\tau$ to $\mathcal{U}^1$, for some $\tau \in \Gamma$. Furthermore, (19) implies that

$$\widehat{\mathcal{U}^1}_H \cong \widehat{\Psi} \oplus 2 \cdot \widehat{\Delta}.$$

Thus the dual $\widehat{\mathcal{U}^1}$ corresponds to the dual $\widehat{\Psi}$ of Ψ. Therefore the dual $\widehat{\Psi}$ of Ψ is a Galois conjugate of Ψ. Hence $\Pi^{\mathcal{E}} \cong \oplus_{\tau \in \Gamma} \Psi^\tau$ is a self–dual $\mathcal{E}H$-module. So Π is also self–dual.

This completes the proof of Theorem A.

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