THE AVERAGE DEGREE OF AN IRREDUCIBLE CHARACTER OF A FINITE GROUP

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Key words: finite group, character degree, solvable, nilpotent, supersolvable

MSC (2010): 20C15, 20D10, 20D15

The second and third authors were supported by research grant No. 2870 from the University of Crete. The third author was also supported by the Spanish Ministerio de Educación y Ciencia MTM2010-15296 and PROMETEO/2011/030.

1. Introduction. Since a finite group G is abelian if all of its irreducible characters have degree 1, it is reasonable to suppose that if the average of the degrees of the irreducible characters of G is small, then in some sense, G should be "almost" abelian. To make this precise, we write

$$\mathrm{acd}(G) = \frac{1}{|\mathrm{Irr}(G)|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\,,$$

so that $\mathrm{acd}(G)$ is the average (irreducible) character degree of G. Our principal results are the following:

THEOREM A. *If* $\mathrm{acd}(G) \leq 3$, *then* G *is solvable.*

THEOREM B. *If* $\mathrm{acd}(G) < 3/2$, *then* G *is supersolvable.*

THEOREM C. *If* $\mathrm{acd}(G) < 4/3$, *then* G *is nilpotent.*

K. Magaard and H. Tong-Viet proved in Theorem 1.4 of [**7**] that if $\mathrm{acd}(G) \leq 2$, then G is solvable, and they conjectured that our Theorem A might be true. In fact, it seems reasonable to conjecture something even stronger: that G is solvable whenever $\mathrm{acd}(G) < 16/5$. (Note that $\mathrm{acd}(A_5) = 16/5$.) On the other hand, the inequalities in Theorems B and C cannot be improved. To see this, observe that $\mathrm{acd}(A_4) = 3/2$ and $\mathrm{acd}(S_3) = 4/3$, and these groups are respectively not supersolvable and not nilpotent. Nevertheless, for groups of odd order, it is possible to prove stronger results.

THEOREM D. *Suppose that* $|G|$ *is odd.*

(a) *If* $\mathrm{acd}(G) < 27/11$, *then* G *is supersolvable.*

(b) *If* $\mathrm{acd}(G) < 3p/(p+2)$, *where* p *is the smallest prime divisor of* $|G|$, *then* G *is nilpotent.*

Going in the other direction, one might ask how large $\mathrm{acd}(G)$ can be if G is solvable. The answer is that there is no upper bound, and in fact, even for p-groups, $\mathrm{acd}(G)$ is unbounded. Consider, for example, the group $G = C_p \wr C_p$ of order p^{p+1}, where C_p is the cyclic group of order p. Then $|G : G'| = p^2$ and G has an abelian subgroup of index p. It follows that G has exactly p^2 linear characters, and all other irreducible characters have degree p. If the number of these is n, then $p^{p+1} = |G| = p^2 + np^2$, and thus $n = p^{p-1} - 1$. It follows that $\mathrm{acd}(G) = (p^2 + np)/(p^2 + n)$, and it is easy to check that $\mathrm{acd}(G) > p - 1$ if $p > 3$. It follows therefore, that if we let p vary, then as claimed, there is no universal upper bound for $\mathrm{acd}(G)$ as G runs over p-groups. In fact, we shall see that more is true: for a fixed prime p, there is no upper bound for $\mathrm{acd}(G)$ as G runs over p-groups.

Next, we recall that there is often an analogy between theorems about character degrees and theorems about class sizes. In particular, writing $\mathrm{acs}(G)$ to denote the average class size of a group G, it is known that if $\mathrm{acs}(G) < 12$, then G is solvable. (This result appears as part of Theorem 11 in the paper [**3**] of R. Guralnick and G. Robinson.) Although our proof of Theorem A relies on the classification of simple groups, this fact about class sizes is more elementary; it can be proved without an appeal to the classification. (It seems that this result was first proved by P. Lescot. See [**4**] for further bibliographic information.)

Finally, we mention an elementary connection between the quantities $\mathrm{acd}(G)$ and $\mathrm{acs}(G)$. Writing k to denote the number of conjugacy classes of G, we have

$$\mathrm{acs}(G) = \frac{|G|}{k} = \frac{\sum \chi(1)^2}{k} \geq \frac{(1/k)\big(\sum \chi(1)\big)^2}{k} = (\mathrm{acd}(G))^2\,,$$

where the inequality holds by Cauchy-Schwarz. Thus if $\mathrm{acs}(G) \leq 9$, we have $\mathrm{acd}(G) \leq 3$, and so G is solvable by Theorem A. Since Theorem A depends on the classification, this is a hard way to prove a weak version of the result of Lescot to which we referred.

We close this introduction with thanks to Avinoam Mann, who provided helpful comments on an earlier version of this paper.

2. Proof of Theorem A. As we mentioned, our proof of Theorem A relies on the classification of simple groups. The specific consequence of the classification that we need is the following, which appears as Theorem 1.1 of [**7**].

(2.1) LEMMA. *Let N be a nonabelian minimal normal subgroup of a finite group G. Then there exists $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) \geq 5$ and θ is extendible to G.*

Almost all of the work involved in proving Theorem A goes into our next theorem, from which Theorem A follows easily. To state our result, we introduce the following notation. Given a positive integer n and a finite group G, we write $a_n(G)$ to denote the number of irreducible characters of G that have degree n, and we write $b_n(G)$ for the number of irreducible characters of G with degree exceeding n. Also, if λ is an irreducible character of some subgroup of G, we will write $a_n(G|\lambda)$ and $b_n(G|\lambda)$ to denote the numbers of members of $\mathrm{Irr}(G|\lambda)$ that have degrees respectively equal to or exceeding n.

(2.2) THEOREM. *Let G be a finite group, and suppose that*

$$b_4(G) \leq a_1(G) + \frac{a_2(G)}{2}\,.$$

Then either G is solvable, or else equality holds and $a_4(G) > 0$.

Proof. Supposing that G is nonsolvable, we proceed by induction on $|G|$. Let $M \triangleleft G$ be minimal such that M is nonsolvable, and note that $M = M'$. Let S be the solvable radical of M, and observe that by the choice of M, every normal subgroup of G properly contained in M is contained in S. (Of course, we may have $S = 1$.) Finally, let $N \subseteq M$, where N is minimal normal in G, and if S is not central in M, choose N so that $N \subseteq [M, S]$.

Suppose first that N is nonabelian. By Lemma 2.1, there exists $\theta \in \mathrm{Irr}(N)$ such that $\theta(1) \geq 5$ and θ has an extension $\chi \in \mathrm{Irr}(G)$. By Gallagher's theorem (Corollary 6.17 of [**6**]) the map $\beta \mapsto \beta\chi$ is an injection from $\mathrm{Irr}(G/N)$ into the set of irreducible characters of G with degree exceeding 4. In particular, we have $|\mathrm{Irr}(G/N)| \leq b_4(G)$, and if equality holds here, then every irreducible character of G with degree exceeding 4 lies in $\mathrm{Irr}(G|\theta)$.

Since N is minimal normal in G and we are assuming that N is nonabelian, we have $N = N' \subseteq G'$. Then N is contained in the kernel of every linear character of G, and we claim that N is also contained in the kernel of every degree 2 irreducible character of G.

To see this, observe that N is a direct product of simple groups, and thus no irreducible character of N has degree 2. Thus if $\mu \in \mathrm{Irr}(G)$ has degree 2, then μ_N is reducible, and hence it is a sum of linear characters. The principal character is the only linear character of N, however, and thus $N \subseteq \ker(\mu)$, as wanted. We now have

$$\begin{aligned} b_4(G) \geq |\mathrm{Irr}(G/N)| \geq a_1(G/N) + a_2(G/N) &= a_1(G) + a_2(G) \\ &\geq a_1(G) + \frac{a_2(G)}{2} \\ &\geq b_4(G)\,, \end{aligned}$$

where the last inequality holds by hypothesis. Equality thus holds throughout, and to complete the proof in this case, it suffices to show that $a_4(G) > 0$.

Since equality holds above, we have $|\mathrm{Irr}(G/N))| = b_4(G)$, and as we have seen, this implies that the restriction to N of every irreducible character of G with degree exceeding 4 is a multiple of θ. Thus if $\varphi \in \mathrm{Irr}(N)$ with $\varphi \neq \theta$, then no member of $\mathrm{Irr}(G|\varphi)$ has degree exceeding 4, and if some character in this set has degree 4 exactly, we are done. We can thus assume that if $\varphi \neq \theta$, then all members of $\mathrm{Irr}(G|\varphi)$ have degree at most 3, and hence $\varphi(1) \leq 3$. Since N has no irreducible character of degree 2, we see that $\mathrm{cd}(N) \subseteq \{1, 3, \theta(1)\}$, and in particular, $|\mathrm{cd}(N)| \leq 3$. By Theorem 12.15 of **[6]**, therefore, N is solvable, and this is a contradiction since N is minimal normal in G, and N was assumed to be nonabelian.

We can now assume that N is abelian. Then G/N is nonsolvable, and the inductive hypothesis applied to G/N yields

$$b_4(G/N) \geq a_1(G/N) + \frac{a_2(G/N)}{2}\,.$$

Since $N \subseteq M = M' \subseteq G'$, we have $a_1(G) = a_1(G/N)$, and thus

$$a_1(G) + \frac{a_2(G)}{2} \geq b_4(G) \geq b_4(G/N) \geq a_1(G/N) + \frac{a_2(G/N)}{2} = a_1(G) + \frac{a_2(G/N)}{2}\,.$$

If $a_2(G) = a_2(G/N)$, then equality holds above, and thus $b_4(G) = a_1(G) + a_2(G)/2$, as required. Also, $b_4(G/N) = a_1(G/N) + a_2(G/N)/2$, and since G/N is not solvable, the inductive hypothesis guarantees that $a_4(G/N) > 0$. Then $a_4(G) > 0$, and there is nothing further to prove. We can assume, therefore, that $a_2(G) > a_2(G/N)$, and hence there exists a character $\mu \in \mathrm{Irr}(G)$ such that $\mu(1) = 2$ and $N \not\subseteq \ker(\mu)$.

Let $L = \ker(\mu)$. Then $N \not\subseteq L$, and in particular $M \not\subseteq L$, and thus ML/L is a nontrivial subgroup of G/L. Also, since M is perfect, we see that ML/L is nonsolvable, and hence G/L is nonsolvable. Now G/L is an in irreducible linear group of degree 2. In fact, G/L is a primitive linear group since it does not have an abelian subgroup of index 2 because it is nonsolvable. By Theorem 14.23 of **[6]**, it follows that $|G : C| \in \{12, 24, 60\}$, where $C/L = \mathbf{Z}(G/L)$. Since G/L is nonsolvable, we deduce that $G/C \cong A_5$, and in particular, G/C is simple. Also, since C/L is abelian and ML/L is nonsolvable, we see that $M \not\subseteq C$, and thus $MC = G$.

Since $M \cap C < M$ and $M \cap C \lhd G$, we have $M \cap C \subseteq S$, the solvable radical of M. Also, $M/(M \cap C) \cong G/C$ is simple, and thus $M \cap C = S$. Then $S \subseteq C$, and since $C/L = \mathbf{Z}(G/L)$, we have $[M, S] \subseteq L$. Recall, however, that $N \not\subseteq L$, and thus $N \not\subseteq [M, S]$. By the choice of N, therefore, we have $S \subseteq \mathbf{Z}(M)$, and in fact, we must have $S = \mathbf{Z}(M)$.

Now $M = M'$, and thus S is isomorphic to a subgroup of the Schur multiplier of $M/S \cong A_5$. Then $|S| \leq 2$, and since N is abelian and $N \subseteq M$, we have $1 < N \subseteq S$, and thus $N = S = C \cap M\mathbf{Z}(M)$ has order 2. In particular, M is a full Schur covering group of A_5, and hence $M \cong SL(2, 5)$.

We have $[C, M] \subseteq C \cap M = \mathbf{Z}(M)$, so $[C, M, M] = 1$. Also, $[M, C, M] = [C, M, M] = 1$, and thus $[M, M, C] = 1$ by the three-subgroups lemma. Since $M = M'$, we deduce that $[M, C] = 1$, and thus $G = MC$ is a central product.

Let λ be the unique nonprincipal linear character of $N = M \cap C$. Since $G = MC$ is a central product, there is a bijection $\mathrm{Irr}(M|\lambda) \times \mathrm{Irr}(C|\lambda) \to \mathrm{Irr}(G|\lambda)$, where if $(\alpha, \beta) \mapsto \chi$, we have $\chi(1) = \alpha(1)\beta(1)$. Also, since $M \cong SL(2, 5)$, we know that the only possibilities for $\alpha(1)$ are 2, 4 and 6.

Since $N \not\subseteq \ker(\mu)$, we see that $\mu \in \mathrm{Irr}(G|\lambda)$, and thus if $(\alpha, \beta) \mapsto \mu$, we must have $\alpha(1) = 2$ and $\beta(1) = 1$, and hence λ extends to C. By Gallagher's theorem, therefore, there exists a degree-preserving bijection $\mathrm{Irr}(C/N) \to \mathrm{Irr}(C|\lambda)$, and in particular, we have

$$a_1(C|\lambda) = a_1(C/N) = a_1(G/M) = a_1(G)\,,$$

where the second equality holds because $G/M \cong C/N$, and the third follows because $M = M' \subseteq G'$.

Now $a_2(M|\lambda) = 2$ and $a_1(M|\lambda) = 0$, and thus

$$a_2(G|\lambda) = a_2(M|\lambda)a_1(C|\lambda) = 2a_1(C|\lambda) = 2a_1(G)\,.$$

Also, since $G/N = (M/N) \times (C/N)$ and $M/N \cong A_5$, we have $a_1(M/N) = 1$ and $a_2(M/N) = 0$, and thus

$$a_2(G/N) = a_1(M/N)a_2(C/N) = a_2(C/N)\,.$$

We deduce that

$$a_2(G) = a_2(G/N) + a_2(G|\lambda) = a_2(C/N) + 2a_1(G)\,,$$

and thus $a_2(C/N) = a_2(G) - 2a_1(G)$.

Next, we consider characters of larger degree. We have

$$a_5(G) \geq a_5(G/N) \geq a_5(M/N)a_1(C/N) = a_1(C/N) = a_1(G)\,,$$

and

$$\begin{aligned} a_6(G) = a_6(G|\lambda) + a_6(G/N) &\geq a_6(M|\lambda)a_1(C|\lambda) + a_3(M/N)a_2(C/N) \\ &= a_1(C|\lambda) + 2a_2(C/N) \\ &= a_1(G) + 2a_2(C/N)\,. \end{aligned}$$

This yields

$$\begin{aligned}
b_4(G) \geq a_5(G) + a_6(G) &\geq a_1(G) + (a_1(G) + 2a_2(C/N)) \\
&\geq 2a_1(G) + \frac{a_2(C/N)}{2} \\
&= 2a_1(G) + \frac{a_2(G) - 2a_1(G)}{2} \\
&= a_1(G) + \frac{a_2(G)}{2} \\
&\geq b_4(G)\,,
\end{aligned}$$

where the final inequality holds by hypothesis. Equality thus holds throughout, and it suffices to show that $a_4(G) > 0$. But $a_4(G) \geq a_4(G/C) = 1$, since $G/C \cong A_5$. ∎

Proof of Theorem A. Given that $\mathrm{acd}(G) \leq 3$, we want to show that G is solvable, and so by Theorem 2.2, it suffices to show that $b_4(G) \leq a_1(G) + a_2(G)/2$, and that if equality holds, then $a_4(G) = 0$. We have

$$a_1(G) + 2a_2(G) + 3a_3(G) + 4a_4(G) + 5b_4(G) \leq \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)$$

and

$$a_1(G) + a_2(G) + a_3(G) + a_4(G) + b_4(G) = |\mathrm{Irr}(G)|\,.$$

Since

$$3|\mathrm{Irr}(G)| \geq \mathrm{acd}(G)|\mathrm{Irr}(G)| = \sum \chi(1)\,,$$

we deduce that

$$2a_1(G) + a_2(G) - a_4(G) - 2b_4(G) \geq 0\,,$$

and thus

$$a_1(G) + \frac{a_2(G)}{2} \geq b_4(G) + \frac{a_4(G)}{2} \geq b_4(G)\,.$$

Also, we see that if equality holds above, then $a_4(G) = 0$, as required. ∎

3. Supersolvable and nilpotent groups. We begin by recalling the following fact, first proved by J. Ernest [**1**] using characters, with a later character-free proof by P. X. Gallagher [**2**]. We present here a somewhat simplified version of Ernest's short proof.

(3.1) LEMMA. *Let $H \subseteq G$ and write $k(H)$ and $k(G)$ to denote the numbers of conjugacy classes of H and G, respectively. Then $k(G) \leq |G : H|k(H)$.*

Proof. Given a character $\psi \in \mathrm{Irr}(H)$, Frobenius reciprocity guarantees that each irreducible constituent of ψ^G has degree at least $\psi(1)$. Since $\psi^G(1) = |G : H|\psi(1)$, we conclude that the number of distinct irreducible constituents of ψ^G is at most $|G : H|$, and thus as ψ runs over the $k(H)$ characters in $\mathrm{Irr}(H)$, there are at most $|G : H|k(H)$ different irreducible characters of G that appear as constituents of the induced characters ψ^G. Since every one of the $k(G)$ irreducible characters of G appears as a constituent of some ψ^G, we deduce that $k(G) \leq |G : H|k(H)$, as required. ∎

(3.2) THEOREM. *Let $A \triangleleft G$, where A is abelian and G splits over A, and let r be the number of orbits in the action of G on the set of nonprincipal linear characters of A. Then the size t of one of these orbits satisfies $t(r+1)/(t+r) \le \mathrm{acd}(G)$.*

Proof. Let $\{\lambda_0, \lambda_1, \dots, \lambda_r\}$ be a set of representatives for the G-orbits on $\mathrm{Irr}(A)$, where λ_0 is the principal character, and let T_i be the stabilizer of λ_i in G. Write

$$t_i = |G : T_i| \,, \qquad k_i = |\mathrm{Irr}(T_i/A)| \,, \qquad s_i = \sum_{\varphi \in \mathrm{Irr}(T_i/A)} \varphi(1) \,,$$

for $0 \le i \le r$, and note that $s_i \ge k_i$. Since λ_i is linear and T_i splits over A, it follows that λ_i extends to T_i. By Gallagher's theorem and the Clifford correspondence, there exists a bijection $\mathrm{Irr}(T_i/A) \to \mathrm{Irr}(G|\lambda_i)$, and if $\varphi \mapsto \chi$ under this map, then $\chi(1) = t_i\varphi(1)$. Since $\mathrm{Irr}(G)$ is the disjoint union of the sets $\mathrm{Irr}(G|\lambda_i)$ for $0 \le i \le r$, we deduce that

$$|\mathrm{Irr}(G)| = \sum_{i=0}^{r} k_i$$

and

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = \sum_{i=0}^{r} t_i s_i \ge \sum_{i=0}^{r} t_i k_i \,.$$

Writing $a = \mathrm{acd}(G)$, we have

$$\sum_{i=0}^{r} t_i k_i \le \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) = a|\mathrm{Irr}(G)| = a \sum_{i=0}^{r} k_i \,.$$

This yields

$$\sum_{i=1}^{r} (t_i - a)k_i \le (a - t_0)k_0 = (a-1)k(G/A) \,.$$

By Lemma 3.1 applied in the group G/A with respect to the subgroup T_i/A, we have $k(G/A) \le t_i k_i$ for all i, and since $a - 1 \ge 0$, we obtain

$$\sum_{i=1}^{r} (t_i - a)k_i \le \frac{a-1}{r} \sum_{i=1}^{r} k(G/A) \le \frac{a-1}{r} \sum_{i=1}^{r} t_i k_i \,.$$

It follows that for some subscript i, we have $(t_i - a)k_i \le (a-1)t_i k_i/r$, and thus writing $t = t_i$, we have $t - a \le (a-1)t/r$. Then $rt - ra \le at - t$, and so $t(r+1) \le a(r+t)$, and the desired inequality follows. ∎

We mention that by computing partial derivatives, it is easy to see that for $r \ge 1$ and $t \ge 1$, the function $f(r,t) = t(r+1)/(t+r)$ is monotonically increasing in each of the variables. It follows that if $r \ge r_0 \ge 1$ and $t \ge t_0 \ge 1$, then $f(r_0, t_0) \le f(r,t)$.

Proof of Theorem B. We are assuming that $\mathrm{acd}(G) < 3/2$, and we use induction on $|G|$ to show that G is supersolvable. There is nothing to prove if G is abelian, so we can assume that $G' > 1$, and we choose a minimal normal subgroup A of G with $A \subseteq G'$. Since $\mathrm{acd}(G) < 3/2 < 3$, Theorem A guarantees that G is solvable, and thus A is abelian.

Since $A \subseteq G'$, the irreducible characters of G with kernels not containing A are all nonlinear, and so they all have above-average degree. It follows that the average of the degrees of the remaining irreducible characters of G is less than $\mathrm{acd}(G)$, and so we have $\mathrm{acd}(G/A) < 3/2$. By the inductive hypothesis, therefore, G/A is supersolvable. If A is cyclic, therefore, then G is supersolvable and we are done. We can assume, therefore, that the minimal normal subgroup A is not cyclic, and in particular, $A \not\subseteq \mathbf{Z}(G)$.

If $A \subseteq \Phi(G)$, then since G/A is supersolvable, it follows that G is supersolvable, and we are done. (See Satz VI.8.6(a) of **[5]**, for example.) We can therefore assume that $A \not\subseteq \Phi(G)$, and hence there exists a maximal subgroup M of G such that $A \not\subseteq M$. We have $AM = G$ and $A \cap M < A$, and since A is abelian and $AM = G$, it follows that $A \cap M \triangleleft G$, and thus $A \cap M = 1$, and we see that G splits over A.

Since A is noncentral, we have $1 < [A, G] \subseteq A$, and so by the minimality of A, we deduce that $[A, G] = A$, and thus no nonprincipal linear character of A is G-invariant. Let r be the number of G-orbits of linear characters of A. By Theorem 3.2, therefore, we have

$$\frac{t(r+1)}{t+r} \le \mathrm{acd}(G) < \frac{3}{2}\,,$$

where t is the size of some G-orbit on the set of nonprincipal linear characters of A, and in particular, $t \ge 2$. Since $r \ge 1$ it follows that if $t \ge 3$, then

$$\frac{t(r+1)}{t+r} \ge \frac{3(1+1)}{(3+1)} = \frac{3}{2}\,,$$

which is not the case. We deduce that $t = 2$, and thus there exists a character $\lambda \in \mathrm{Irr}(A)$ in a G-orbit of size 2. If T is the stabilizer of λ in G, then $|G : T| = 2$, and thus $T \triangleleft G$. Then $[A, T] < A$ and $[A, T] \triangleleft G$, and thus $[A, T] = 1$. The group G/T of order 2, therefore, acts irreducibly and without fixed points on $\mathrm{Irr}(A)$, and we deduce that $|A| = |\mathrm{Irr}(A)|$ is prime, and thus A is cyclic. ∎

The proof of Theorem C is very similar, and so we go through the argument giving somewhat less detail.

Proof of Theorem C. We have $\mathrm{acd}(G) < 4/3$, and we want to show that G is nilpotent. We can assume that G is nonabelian, and we choose a minimal normal subgroup A of G with $A \subseteq G'$, and we observe that A is abelian since G is solvable by Theorem A. All irreducible characters of G with kernel not containing A have above-average degree, and thus $\mathrm{acd}(G/A) \le \mathrm{acd}(G) < 4/3$. Working by induction on $|G|$, we see that G/A is nilpotent, and so if $A \subseteq \mathbf{Z}(G)$, then G is nilpotent, and we are done. We can thus assume that $[A, G] > 1$, so $[A, G] = A$, and hence all nonprincipal linear characters of A are in G-orbits of size at least 2. Also, if $A \subseteq \Phi(G)$, then G is nilpotent, so we can assume that $A \not\subseteq M$ for some maximal subgroup M of G, and it follows that G splits over A.

In the notation of Theorem 3.2, we have $t(r+1)/(t+r) \leq \mathrm{acd}(G) < 4/3$, and since $r \geq 1$ and $t \geq 2$, we have $t(r+1)/(t+r) \geq 2(1+1)/(2+1) = 4/3$. This contradiction proves the result. ∎

Proof of Theorem D(a). We are assuming that $|G|$ is odd and that $\mathrm{acd}(G) < 27/11$. Our goal is to show that G is supersolvable, so we can assume that G is nonabelian. Choose a minimal normal subgroup $A \subseteq G'$, and observe that A is abelian because G must be solvable. The irreducible characters of G with kernel not containing A are nonlinear, and since $|G|$ is odd, these characters have degrees at least 3, and so their degrees are above average. It follows that $\mathrm{acd}(G/A) \leq \mathrm{acd}(G) < 27/11$, so working by induction on $|G|$, we deduce that G/A is supersolvable. If A is cyclic, then G is supersolvable, and thus we can assume that $|A|$ is a non-prime prime-power, and in particular, $A \not\subseteq \mathbf{Z}(G)$. Also, if $A \subseteq \Phi(G)$, then G is supersolvable, so we can assume that $A \not\subseteq \Phi(G)$, and it follows that G splits over A. By Theorem 3.2, we have

$$\frac{t(r+1)}{(r+t)} \leq \mathrm{acd}(G) < \frac{27}{11}\,,$$

where t and r are as usual. Also, $t > 1$ since $[A, G] = A$, and since t is odd, we have $t \geq 3$.

Complex conjugation permutes the G-orbits on $\mathrm{Irr}(A)$, and we argue that no orbit of nonprincipal linear characters can be fixed by conjugation. To see this, observe that since $|A|$ is odd, a nonprincipal linear character cannot be self-conjugate, and thus an orbit fixed by conjugation would necessarily have even size. The orbits all have odd size, however, and thus conjugation defines a pairing on the set of G-orbits on nonprincipal linear characters of A. In particular, r is even and there are at least two orbits of size t.

If $r = 2$, then each of the two orbits of nonprincipal linear characters of A has size t, and thus $|A| = 2t + 1$. We know that $|A|$ is a non-prime prime-power, and since t is odd, the smallest possibility for t is $t = 13$, corresponding to $|A| = 27$. Thus $t \geq 13$, and so

$$\frac{t(r+1)}{(r+t)} \geq \frac{13(2+1)}{(2+13)} = \frac{13}{5} > \frac{27}{11}\,.$$

This is a contradiction, and we deduce that $r \geq 4$.

If $t = 3$, then the stabilizer T of some linear character of A has index 3 in G, and in this case, 3 is the smallest prime divisor of $|G|$, and hence $T \triangleleft G$. It follows that $[A, T] \triangleleft G$, and since $[A, T] < A$, we have $[A, T] = 1$, and thus T is the stabilizer of every nonprincipal linear character of A. All G-orbits of nonprincipal linear characters of A, therefore, have size 3, and we have $|A| = 3r + 1$. Since $|A|$ is an odd non-prime prime-power, the smallest possibility is $r = 8$, corresponding to $|A| = 25$. Thus $r \geq 8$, and we have

$$\frac{t(r+1)}{(r+t)} \geq \frac{3(8+1)}{(8+3)} = \frac{27}{11}\,,$$

and this is a contradiction. We now have $r \geq 4$ and $t \geq 5$, and thus

$$\frac{t(r+1)}{(r+t)} \geq \frac{5(4+1)}{(4+5)} = \frac{25}{9} > \frac{27}{11}$$

and this is our final contradiction. ∎

Proof of Theorem D(b). We assume now that $|G|$ is odd and that $\operatorname{acd}(G) < 3p/(p+2)$, where p is the smallest prime divisor of $|G|$. We want to show that G is nilpotent, and by the usual arguments, we can assume that there exists an abelian subgroup $A \triangleleft G$, where A is noncentral and G splits over A. Also, we have

$$\frac{t(r+1)}{(r+t)} \leq \operatorname{acd}(G) < \frac{3p}{(p+2)}\,,$$

where r and t have their customary meanings, and $t > 1$. Then $t \geq p$, and as we saw in the previous proof, r is even, so $r \geq 2$. Then

$$\frac{t(r+1)}{(r+t)} \geq \frac{p(2+1)}{(p+2)}\,,$$

and this contradiction completes the proof. ∎

4. An example As we mentioned in the introduction, $\operatorname{acd}(G)$ is unboundedly large as G runs over p-groups, where p is an arbitrary fixed prime. To see this, let F be the field of order p^e where $e > 1$ is odd, and let G be the set of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & x^p \\ 0 & 0 & 1 \end{bmatrix},$$

where x and y run over F. It is easy to check that G is a group, and clearly, $|G| = p^{2e}$. For notational simplicity, we write $M(x,y)$ to denote the above matrix, and we observe that $M(x,y)M(u,v) = M(x+u, y+v+xu^p)$. It follows that $M(x,y)$ and $M(u,v)$ commute if and only if $xu^p = ux^p$. This commuting condition is always satisfied if $x = 0$, so if we write Z to denote the set of matrices $M(0,y)$ for $y \in F$, we see that $Z \subseteq \mathbf{Z}(G)$ and $|Z| = p^e$.

Now fix $M(x,y)$, where $x \neq 0$. We can compute the centralizer of $M(x,y)$ in G by rewriting the commuting condition $xu^p = ux^p$ as $(u/x)^p = (u/x)$. It follows that $M(u,v)$ commutes with $M(x,y)$ precisely when u/x is in the prime subfield of F, and thus there are p possibilities for u and p^e possibilities for v. This shows that if $g \in G$ and $g \notin Z$, then $|\mathbf{C}_G(g)| = p^{e+1}$, and it follows that $Z = \mathbf{Z}(G)$ and all noncentral classes of G have size p^{e-1}. The total number of conjugacy classes in G, therefore, is

$$k = p^e + \frac{p^{2e} - p^e}{p^{e-1}} = p^e + p^{e+1} - p\,.$$

The map $M(x,y) \mapsto x$ is a homomorphism from G onto the additive group of F, and hence G/Z is abelian group of order p^e. Then G has exactly p^e linear characters with kernel containing Z and exactly $k - p^e = p^{e+1} - p$ other irreducible characters. Since $Z = \mathbf{Z}(G)$ has index p^e and e is odd, it follows that all irreducible characters of G have degree at most $d = p^{(e-1)/2}$, and thus

$$p^{2e} = |G| \leq p^e + (p^{e+1} - p)d^2 = p^e + (p^{e+1} - p)p^{e-1} = p^{2e}\,,$$

and thus equality holds, and we see that in addition to the p^e linear characters, G has exactly $p^{e+1} - p$ irreducible characters of degree d, and this accounts for all of $\operatorname{Irr}(G)$. Since there are more characters of degree d than there are of degree 1, the average degree exceeds $(d+1)/2 > p^{(e-1)/2}/2$, and this is unbounded for large e. ∎

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