COUNTING CHARACTERS OF SMALL DEGREE IN UPPER UNITRIANGULAR GROUPS

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ABSTRACT. Let U_n denote the group of upper $n \times n$ unitriangular matrices over a fixed finite field $\mathbf{F}$ of order q. That is, U_n consists of upper triangular $n \times n$ matrices having every diagonal entry equal to 1. It is known that the degrees of all irreducible complex characters of U_n are powers of q. It was conjectured by Lehrer that the number of irreducible characters of U_n of degree q^e is an integer polynomial in q depending only on e and n. We show that there exist recursive (for n) formulas that this number satisfies when e is one of $1, 2$ and 3, and thus show that the conjecture is true in those cases.

1. Introduction

We fix a prime p. Let q be a fixed power of p and $\mathbf{F}_q = \mathbf{F}$ the finite field of order q. We write $U_n(q) = U_n$ for the group of upper triangular $n \times n$ matrices over $\mathbf{F}$, whose diagonal elements are all equal to 1. We also write $GL_n(q)$ for the general linear group of all $n \times n$ invertible matrices over $\mathbf{F}$ and note that $U_n(q)$ is a p-Sylow subgroup of $GL_n(q)$. Furthermore, for every finite group G and every integer k we write

$$N_k(G) = |\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = k\}|,$$

for the number of irreducible characters of G of degree k.

In 1974 G. I. Lehrer, see [6], conjectured two results. First, he claimed that the degrees of the irreducible representations of U_n are of type q^e for some $e \in \{0, 1, \cdots, \mu(n)\}$, where

$$\mu(n) = \begin{cases} m(m-1) \text{ if } n = 2m \text{ and} \\ m^2 \text{ if } n = 2m+1. \end{cases}$$

Next he conjectured that for any fixed n, the number of irreducible characters of U_n whose degree is q^e, i.e., $N_{q^e}(U_n)$ in our notation, is an integer polynomial in q depending only on e.

As far as the first of his conjecture is concerned, it was shown by M. Isaacs [5], that every irreducible character of U_n has degree a power of q. In addition, B. Huppert, [2], proved that the degrees of the irreducible characters of U_n is exactly the set $\{q^e \mid 0 \leq e \leq \mu(n)\}$.

As for the second part of his conjecture, it still remains open apart for some specific values of e.

The case $e = 0$ is well known and easy to compute, that is, $N_1(U_n(q)) = N_1(U_n) = q^{n-1}$. For greater values of e, M. Marjoram [7] provided some first formulas. In particular, he proved that there exist formulas for the number of irreducible characters having one of the next two lowest degrees, that is $N_q(U_n)$ and $N_{q^2}(U_n)$. Also in his unpublished thesis [8], M. Marjoram established formulas for the three highest degrees when $n = 2m$ is even, that is $N_{q^{\mu(n)}}(U_{2m})$, $N_{q^{\mu(n)-1}}(U_{2m})$ and $N_{q^{\mu(n)-2}}(U_{2m})$, as well as a formula for the number of irreducible characters of highest degree when n is odd, that is $N_{q^{\mu(n)}}(U_{2m+1})$.

In addition, I. M. Isaacs, in his paper [4], using a different method, constructed specific polynomials for the number of irreducible characters of $U_n(q)$ of degree $q, q^{\mu(n)}$ and $q^{\mu(n)-1}$.

In this paper we use an elementary method to prove the conjecture for the functions $N_q(U_n)$, $N_{q^2}(U_n)$ and $N_{q^3}(U_n)$. In particular we provide recursive formulas that the number of irreducible characters of degree q, q^2 and q^3 satisfy. In a forthcoming paper we prove analogous recursive formulas for the degrees $q^{\mu(n)}, q^{\mu(n)-1}, q^{\mu(n)-2}$ and consequently we show that the corresponding functions $N_{q^{\mu(n)}}(U_n), N_{q^{\mu(n)-1}}(U_n), N_{q^{\mu(n)-2}}(U_n)$ are integer polynomials in q.

We follow the notation used in [3]. In addition, for any matrix $X = (x_{i,j}) \in GL_n(q)$ we write $R_i(X)$ for its i-row written as an $1 \times n$ matrix. We also write $C_j(X)$ for its j-column written as an $n \times 1$ matrix. Also if $A = (a_{i,j}) \in U_n$ then we say that its i-row is trivial if the only nonzero element in that row is the diagonal element $a_{i,i} = 1$. Similarly, we say that the j-column of A is trivial if every entry in the j-column of A is 0 except $a_{j,j} = 1$. We will often consider the additive abelian group of the st-dimensional vector space $\mathbf{F}^{st}$ (of order q^{st}) as the additive group of all $s \times t$ matrices over $\mathbf{F}$. When viewed as such we write it as $\mathbf{F}^{s\times t}$.

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2. Orbits of unitriangular actions on $\mathbf{F}^{s\times t}$

The aim of this section is to compute the orbits of a specific action of $H = U_s \times U_t$ on $\mathbf{F}^{s\times t}$.

Definition 1. Let T be an $s \times t$ matrix over $\mathbf{F}$. We call T **quasimonomial** if it has at most one non-zero entry in every column and row.

We write $\mathbf{E_{i,j}}$ for the matrix that has 1 in it's (i, j)-entry and 0 everywhere else. Clearly $E_{i,j}$ is quasimonomial. Furthermore, every nonzero quasimonomial T can be written as

$$T = f_1 E_{i_1,j_1} + f_2 E_{i_2,j_2} + \cdots + f_k E_{i_k,j_k} \tag{2.1}$$

with $j_1 < j_2 < \cdots < j_k$, all $i_1, \ldots, i_k$ distinct, and $f_1, \ldots, f_k$ non-zero elements in $\mathbf{F}$. We call 2.1 the **standard form** of the non-zero T and we say that k is the **length of T** .

Theorem 1. *Assume that the group $H = U_s \times U_t$ acts on $\mathbf{F}^{s\times t}$ in the following way*

$$X^{(A,B)} = A^{-1} X B,$$

for all $X \in \mathbf{F}^{s\times t}, A \in U_s$ and $B \in U_t$. Then the set of distinct quasimonomial matrices in $\mathbf{F}^{s\times t}$ forms a complete set of orbit representatives of the action of H on $\mathbf{F}^{s\times t}$.

Proof. Let $X \in \mathbf{F}^{s\times t}$. We show that by performing admissible transformations we can get a quasimonomial matrix. By an admissible transformation we mean adding to a row (respectively a column) a multiple of a subsequent row (resp. a previous column). By induction we can suppose that the $(s-1)\times t$ submatrix of X formed by all rows except the first one is quasimonomial. Let $x_{i_1,j_1},\ldots,x_{i_l,j_l}$, $2\leq i_1<\ldots<i_l$, be the non-zero elements in this submatrix. Then we can suppose that $x_{1,j_1}=\ldots=x_{1,j_l}=0$. If the rest of elements in the first row are now zero we are done. Otherwise let $x_{1,j}$ be the first non-zero element in the first row. Then, except for $x_{1,j}$, the jth column is zero and, by performing admissible column transformations, we can have that $x_{1,j}$ is the unique non-zero element in the first row, and thus obtain a quasimonomial matrix.

To prove uniqueness, we argue again by induction on s and t. If X,Y are quasimonomial matrices in the same orbit, we can suppose that the last $s-1$ rows and the first $t-1$ columns of X and Y are the same. We only need to show that $x_{1,t}=y_{1,t}$. If some element in the first row or in the last column different from the $(1,t)$-entry is non-zero, then $x_{1,t}=y_{1,t}=0$ and $X=Y$. Otherwise comparing the $(1,t)$-entry in $XB=AY$ we get $x_{1,t}=y_{1,t}$ and $X=Y$.

□

When a first version of this paper appeared, Vera-López, Arregi and Ormaetxea told me (I thank them for that) about a more general result concerning conjugacy classes in unitriangular groups (see [9], [10], [11]), whose special case is Theorem 1.

3. Irreducible characters in U_n

In this section we will show how Section 2 is connected to Lehrer's conjecture. We follow Marjoram's approach on the problem, and Proposition 1 below follows from his paper [7].

For a fixed but arbitrary integer n we consider the upper unitriangular group U_n over $\mathbf{F}_q$, and its two subgroups $M_{n,t}$ and $H_{n,t}$ defined in the following way. If $1\leq t\leq n$ and $s=n-t$ then

$$M_{n,t}=\{\begin{pmatrix} I_t & C\\ 0 & I_s\end{pmatrix}: C\in\mathbf{F}^{t\times s}\}=\{X\in U_n \text{ with } x_{i,j}=0 \text{ if either } \; i<j\leq t \text{ or } t<i<j\}$$

and

$$H_{n,t}=\{\begin{pmatrix} B & 0\\ 0 & A\end{pmatrix}: A\in U_s \text{ and } B\in U_t\}=\{X\in U_n \text{ with } x_{i,j}=0 \text{ if } i\leq t \text{ and } j>t\}.$$

It is easy to see that for all $t=1,\ldots,n$, the group $M_{n,t}$ is an abelian normal subgroup of U_n, isomorphic to $\mathbf{F}^{t\times s}$. In addition, $H_{n,t}$ complements $M_{n,t}$ in U_n, and is isomorphic to $U_s\times U_t$. We identify $H_{n,t}$ with $U_s\times U_t$ and we write its elements as (A,B) with $A\in U_s$ and $B\in U_t$. We also identify the $M_{n,t}$ with $\mathbf{F}^{t\times s}$, and thus we write the elements of $M_{n,t}$ as $C\in\mathbf{F}^{t\times s}$. Note that with these identifications, the conjugation action of $H_{n,t}$ on $M_{n,t}$ in $U=M_{n,t}\rtimes H_{n,t}$ is given as

$$C^{(A,B)}=B^{-1}CA, \tag{3.1}$$

for all $(A,B)\in H_{n,t}$ and $C\in M_{n,t}$. The product that appears at the right hand side of the equation above is the standard product of matrices.

Marjoram has given a nice characterization for the abelian group $\mathrm{Irr}(M_{n,t})$, and the way $H_{n,t}$ acts on that group. We collect his results (Lemma 2 and 3 in [7]) in the next proposition.

Proposition 1. *Let $M_{n,t}$ and $H_{n,t}$ defined as above, for fixed but arbitrary n and t. Then $\mathrm{Irr}(M_{n,t})$ is isomorphic to the abelian additive group $\mathbf{F}^{s\times t}$ of all the $s\times t$ matrices over $\mathbf{F}=\mathbf{F}_q$. The isomorphism is given as $D\in\mathbf{F}^{s\times t}\to\lambda_D\in\mathrm{Irr}(M_{n,t})$, where the map $\lambda_D : M_{n,t}\to\mathbf{C}$ is defined as*

$$\lambda_D(C)=\omega^{T(tr(DC))},\ \textit{for all } C\in M_{n,t},$$

where ω is a primitive p-root of unity, $T:\mathbf{F}_q\to\mathbf{F}_p$ is the usual trace map from the extension field of q elements $\mathbf{F}_q$ to the ground field $\mathbf{F}_p$ of p-elements and $tr(DC)$ denotes the trace of the square $s\times s$ matrix DC. Furthermore identifying $H_{n,t}$ with $U_s\times U_t$, we get that the action of $H_{n,t}$ on $\mathrm{Irr}(M_{n,t})$ is given as

$$\lambda_D^{(A,B)}(C)=\lambda_D(C^{(A^{-1},B^{-1})})=\lambda_D(BCA^{-1})=\omega^{T(tr(DBCA^{-1}))}=\omega^{T(tr(A^{-1}DBC))}=\lambda_{A^{-1}DB}(C),$$

for all $D\in\mathbf{F}^{s\times t}, C\in\mathbf{F}^{t\times s}$ and all $(A,B)\in U_s\times U_t\cong H_{n,t}$. Thus $U_s\times U_t\cong H_{n,t}$ acts on $\mathrm{Irr}(M_{n,t})\cong\mathbf{F}^{s\times t}$ as

$$D^{(A,B)}=A^{-1}DB.$$

What the above proposition says is that, identifying $\mathrm{Irr}(M_{n,t})$ with $\mathbf{F}^{s\times t}$ and $H_{n,t}$ with $U_s\times U_t$, then Theorem 1 provides a complete set of orbit representatives of the action of $H_{n,t}$ on $\mathrm{Irr}(M_{n,t})$. In particular,

$$\Omega_{n,t}=\{T\in\mathbf{F}^{s\times t}\mid T \text{ quasimonomial}\}. \tag{3.2}$$

is such a set of representatives.

Now, let G be any finite group N an abelian normal subgroup of G and H a complement of N in G, then it is easy to characterize the irreducible characters of G. In particular, if $\lambda\in\mathrm{Irr}(N)$, and G_λ is the stabilizer of λ in G, then Gallagher's theorem and Clifford Theory implies that λ extends to G_λ and a canonical extension λ^e is given as $\lambda^e(hn)=\lambda(n)$, for all $h\in H_\lambda=G_\lambda\cap H$ and $n\in N$. Every character $\Psi\in\mathrm{Irr}(H_\lambda)$ defines a unique irreducible character $\Psi\cdot\lambda^e$ of G_λ lying above λ and inducing irreducibly on G. Distinct irreducible characters $\Psi\in\mathrm{Irr}(H_\lambda)$ define distinct irreducible characters $(\Psi\cdot\lambda^e)^G$ of G. In addition, every $\chi\in\mathrm{Irr}(G)$ lies above some $\lambda\in\mathrm{Irr}(N)$ and thus $\chi=(\Psi\cdot\lambda^e)^G$, for some $\Psi\in\mathrm{Irr}(H_\lambda)$. Note that $\chi(1)=\Psi(1)\cdot[|H|:|H_\lambda|]$.

The group G acts on $\mathrm{Irr}(N)$ and divides its members into conjugacy classes. (Observe that the G-classes of $\mathrm{Irr}(N)$ are also the H-conjugacy classes of $\mathrm{Irr}(N)$.) Let $\Omega\subseteq\mathrm{Irr}(N)$ consisting of one representative from every G-conjugacy class of irreducible characters of N. Then

$$\mathrm{Irr}(G)=\cup_{\lambda\in\Omega}\{(\Psi\lambda^e)^G\mid\Psi\in\mathrm{Irr}(H_\lambda)\}.$$

Hence if $N_k(G)=|\{\chi\in\mathrm{Irr}(G)\mid\chi(1)=k\}|$, for any finite group G, and any $k=1,2\cdots$, then

$$N_k(G)=\sum_{\lambda\in\Omega}N_{\frac{k}{|H:H_\lambda|}}(H_\lambda)=\sum_{\lambda\in\Omega}N_{\frac{k}{|O_\lambda|}}(H_\lambda)$$

where O_λ is the H-orbit of λ in $\mathrm{Irr}(N)$.

Applying the above argument to the groups $U_n=M_{n,t}\rtimes H_{n,t}$ for any arbitrary but fixed integer n and any $t=1,\ldots,n-1$, we conclude that

$$N_k(U_n)=\sum_{T\in\Omega_{n,t}}N_{\frac{k}{|H_{n,t}:H_{n,t,T}|}}(H_{n,t,T})=\sum_{T\in\Omega_{n,t}}N_{\frac{k}{|O_T|}}(H_{n,t,T}), \tag{3.3}$$

where $\Omega_{n,t}$ is the set of quasimonomial matrices in $\mathbf{F}^{s\times t}$, O_T is the $H_{n,t}$-orbit of $T \in \mathbf{F}^{s\times t} \cong \mathrm{Irr}(M_{n,t})$ and $H_{n,t,T}$ is the stabilizer of T in $H_{n,t} \cong U_s \times U_t$.

Case 1: $t = 1$. So $s = n-1$ and the groups $H_{n,1}$ and $M_{n,1}$ become $U_{n-1} \times U_1 \cong U_{n-1}$ and $\mathbf{F}^{1\times n-1}$ respectively. Furthermore, $\mathrm{Irr}(M_{n,1}) \cong \mathbf{F}^{n-1\times 1}$ and $\Omega_{n,1} = \{T \in \mathbf{F}^{n-1,1} \mid T \text{ quasimonomial}\}$ consists of the matrices $T_i = fE_{i,1}$, for all $i = 1, \dots, n-1$, and $f \neq 0 \in \mathbf{F}$, along with the zero matrix. So we get $q-1$ matrices of type $fE_{i,1}$. For any n and any $i = 1, \dots, n$ we define $P_{n,i}$ as

$$P_{n,i} = \{A \in U_n \mid C_i(A) \text{ is trivial}\}.$$

Then it is easy to check that $H_{n,1,T_i} = P_{n-1,i}$ while $|O_{T_i}| = q^{i-1}$. Thus in view of equation 3.3 we get

$$N_k(U_n) = \sum_{T\in\Omega_{n,1}} N_{\frac{k}{|O_T|}}(H_{n,1,T}) = (q-1)\sum_{i=1}^{n-1} N_{\frac{k}{q^{i-1}}}(P_{n-1,i}) + N_k(U_{n-1}), \tag{3.4}$$

where the last summand is the contribution of the zero matrix whose orbit size is 1 and the stabilizer group is $H_{n,1} \cong U_{n-1}$ itself. For $k = q^e$, $e = 0, 1, \dots, \mu(n)$ the above equation, along with the fact that $P_{n-1,1} = U_{n-1}$, implies

$$N_k(U_n) = qN_k(U_{n-1}) + (q-1)\sum_{i=2}^{n-1} N_{\frac{k}{q^{i-1}}}(P_{n-1,i}). \tag{3.5}$$

Observe that for $k = 1$ equation (3.5) provides the well known formula $N_1(U_n) = qN_1(U_{n-1})$, for all $n \geq 2$.

Case 2: $t = 2$ and thus $s = n-2$. (Assume $n \geq 4$ for the rest of the section.) Now the groups $H_{n,2}$ and $M_{n,2}$ become $U_{n-2} \times U_2 \cong U_{n-2} \times \mathbf{F}$ and $\mathbf{F}^{2\times n-2}$ respectively. Furthermore, $\mathrm{Irr}(M_{n,2}) \cong \mathbf{F}^{n-2\times 2}$ and $\Omega_{n,2} = \{T \in \mathbf{F}^{n-2,2} \mid T \text{ quasimonomial}\}$ consists of matrices whose length is either 1 or 2 along with the zero matrix. In particular, the non-zero matrices in $\Omega_{n,2}$ are of the following two types:
Those of length 1, i.e. $T_{i,j} = fE_{i,j}$, $j = 1, 2$ and $i = 1, \dots, n-2$, while $f \neq 0 \in \mathbf{F}$. For any fixed i and j we get $q-1$ such. If $j = 1$ then $T_{i,1} = fE_{i,1}$, for $i = 1, \dots, n-2$. In this case it is left to the reader to check that $|O_{T_{i,1}}| = q^i$ while $H_{n,2,T_{i,1}} = \{(A, B) \mid A \in U_{n-2}, B \in U_2 \text{ with } C_i(A) \text{ and } R_1(B) \text{ trivial }\}$. Thus $H_{n,2,T_{i,1}} \cong P_{n-2,i}$.
If $j = 2$ then $T_{i,2} = fE_{i,2}$, for some $i = 1, \dots, n-2$. In this case $|O_{T_{i,2}}| = q^{i-1}$ while $H_{n,2,T_{i,2}} = \{(A, B) \mid A \in U_{n-2}, B \in U_2 \text{ with } C_i(A) \text{ and } R_2(B) \text{ being trivial }\} \cong P_{n-2,i} \times \mathbf{F}$.

The second type are those of length 2, i.e., $T_{i_1,i_2} = f_1E_{i_1,1} + f_2E_{i_2,2}$ for some $i_1 \neq i_2$ and f_1, f_2 non-zero elements in $\mathbf{F}$. We get exactly $(q-1)^2$ such distinct quasimonomial characters.
One can easily check that if $i_1 > i_2$, then $|O_{T_{i_1,i_2}}| = q^{i_1+i_2-1}$, while the stabilizer of T_{i_1,i_2} in $H_{n,2}$ equals

$$H_{n,2,T_{i_1,i_2}} = \{(A, 1) \mid A \in U_{n-2} \text{ with } C_{i_1}(A) \text{ and } C_{i_2}(A) \text{ being trivial}\} \cong P_{n-2,i_1} \cap P_{n-2,i_2}.$$

On the other hand if $i_1 < i_2$, then $|O_{T_{i_1,i_2}}| = q^{i_1+i_2-2}$, while the stabilizer $H_{n,2,T_{i_1,i_2}}$ of T_{i_1,i_2} in $H_{n,2} = U_{n-2} \times U_2$ consists of all matrices $(A, B) \in U_{n-2} \times U_2$ that satisfy $a_{i_1,i_2} = -f_1/f_2 \cdot b_{1,2}$ while $C_{i_1}(A)$ is a trivial column and $a_{x,i_2} = 0$ for all $i_1 \neq x = 1, \dots, i_2 - 1$. For $1 \leq i_1 < i_2 \leq n$ we define

$$Q_{n,i_1,i_2} = \{A \in U_n \mid a_{y,i_1} = 0 = a_{x,i_2}, \text{ for all } i_1 \neq x = 1, \dots, i_2 - 1 \text{ and } y = 1, \dots, i_1 - 1\}. \tag{3.6}$$

Then it is easy to see that $H_{n,2,T_{i_1,i_2}} \cong Q_{n-2,i_1,i_2}$. Finally the zero matrix has orbit length 1 and its stabilizer in $H_{n,2}$ is $H_{n,2} \cong U_{n-2} \times \mathbf{F}$. Collecting all the above and applying equation 3.3 along with equation (3.5) and the fact that $N_k(M \times \mathbf{F}) = |\mathbf{F}| \cdot N_k(M) = qN_k(M)$ for any group M, we get

$$N_k(U_n) = qN_k(U_{n-1}) + N_{\frac{k}{q}}(U_{n-1}) - N_{\frac{k}{q}}(U_{n-2}) + (q-1)^2 \sum_{1\leq i_2<i_1\leq n-2} N_{\frac{k}{q^{i_1+i_2-1}}}(P_{n-2,i_1} \cap P_{n-2,i_2}) + (q-1)^2 \sum_{1\leq i_1<i_2\leq n-2} N_{\frac{k}{q^{i_1+i_2-2}}}(Q_{n-2,i_1,i_2}). \tag{3.7}$$

for all $n \geq 4$ and all k. Some of the summands above are easy to compute. First observe that $P_{n-2,i} \cap P_{n-2,n-2} \cong P_{n-3,i}$ for all $i = 1, \ldots, n-3$, and all $n \geq 5$. Thus (3.5) implies

$$(q-1)^2 \sum_{i=1}^{n-3} N_{\frac{k}{q^{n-3+i}}}(P_{n-2,i} \cap P_{n-2,n-2}) = (q-1)[N_{\frac{k}{q^{n-2}}}(U_{n-2}) - N_{\frac{k}{q^{n-2}}}(U_{n-3})].$$

In addition, $P_{n-2,i} \cap P_{n-2,1} = P_{n-2,i}$, for all $i = 2, \ldots, n-3$ and all $n \geq 5$. Hence

$$(q-1)^2 \sum_{i=2}^{n-3} N_{\frac{k}{q^i}}(P_{n-2,i} \cap P_{n-2,1}) = (q-1)[N_{\frac{k}{q}}(U_{n-1}) - qN_{\frac{k}{q}}(U_{n-2}) - (q-1)N_{\frac{k}{q^{n-2}}}(U_{n-3})].$$

Furthermore, for all $i = 1, \ldots, n-3$ and all $n \geq 5$, the group $Q_{n-2,i,n-2}$ is isomorphic to $P_{n-3,i} \times \mathbf{F}$. This along with (3.5) implies

$$(q-1)^2 \sum_{i=1}^{n-3} N_{\frac{k}{q^{n-4+i}}}(Q_{n-2,i,n-2}) = q(q-1)[N_{\frac{k}{q^{n-3}}}(U_{n-2}) - N_{\frac{k}{q^{n-3}}}(U_{n-3})]. \tag{3.8}$$

We finally observe that $Q_{n-2,1,2} = U_{n-2}$. Replacing all the above in equation (3.7) we get

$$\begin{aligned} N_k(U_n) = {} & qN_k(U_{n-1}) + qN_{\frac{k}{q}}(U_{n-1}) - qN_{\frac{k}{q}}(U_{n-2}) + \\ & (q-1)[N_{\frac{k}{q^{n-2}}}(U_{n-2}) - qN_{\frac{k}{q^{n-2}}}(U_{n-3}) + qN_{\frac{k}{q^{n-3}}}(U_{n-2}) - qN_{\frac{k}{q^{n-3}}}(U_{n-3})] + \\ & (q-1)^2 \sum_{2\leq i_2<i_1\leq n-3} N_{\frac{k}{q^{i_1+i_2-1}}}(P_{n-2,i_1} \cap P_{n-2,i_2}) + \\ & (q-1)^2 \sum_{1\leq i_1<i_2\leq n-3 \text{ and } (i_1,i_2)\neq(1,2)} N_{\frac{k}{q^{i_1+i_2-2}}}(Q_{n-2,i_1,i_2}), \end{aligned} \tag{3.9}$$

for all $n \geq 5$ and all k. Note that in the equation above, the sum $i_1 + i_2$ is greater or equal to 5 when $i_2 < i_1$, while $i_1 + i_2 \geq 4$ when $i_1 < i_2$.

4. Linear characters of $P_{n,i}$ and $Q_{n,i,j}$

The aim of this section is to compute the number of linear characters of $P_{n,i}$ and $Q_{n,i,j}$. These groups are examples of *pattern groups* , a term introduced by M. Isaacs. We give here the basic definitions and properties we need, for more details the reader could see [4].

Let $\mathcal{P}$ be a subset of the set of pairs $\{(i,j) \mid 1 \leq i < j \leq n\}$. $\mathcal{P}$ is called a *closed pattern* if it has the property that $(i,k) \in \mathcal{P}$ whenever $(i,j),(j,k) \in \mathcal{P}$, for some $j \in \{i+1,\dots,k-1\}$. The set of unitriangular matrices $X \in U_n$ with $x_{i,j} = 0$ whenever $i < j$ and $(i,j) \neq \mathcal{P}$ is a subgroup of U_n called a *pattern group*. If G is a pattern group corresponding to the closed pettern $\mathcal{P}$ with $|\mathcal{P}| = k$, then G is generated by the matrices $I_n + aE_{i,j}$, $(i,j) \in \mathcal{P}$, $a \in \mathbf{F}^*$ and $|G| = |\mathbf{F}|^n$.

Direct computations show that $[I_n + aE_{i,j}, I_n + bE_{l,k}] = I_n + abE_{i,k}$ if $j = l$ and I_n otherwise. A pair $(i,j) \in \mathcal{P}$ is called *minimal* if it is not possible to find numbers $j_1 < j_2 < \ldots < j_l, l \geq 1$, such that $(i,j_1),(j_1,j_2),\dots,(j_l,k) \in \mathcal{P}$. Then G' is the pattern group associated to $\mathcal{P}_0 = \{(i,k) \in \mathcal{P} \mid (i,k) \text{ is not minimal}\}$. Thus $|G : G'| = q^t$, where t is the number of minimal pairs in $\mathcal{P}$ (see Theorem 2.1 in [4]).

For the group $P_{n,i}$, $n \geq 3, i \leq n-1$ observe that there are $n-1$ minimal pairs: $(k,k+1), k \neq i-1, 1 \leq k \leq n-1$ and $(i-1,i+1)$. Therefore

$$N_1(P_{n,i}) = q^{n-1}. \tag{4.1}$$

For the group $Q_{n,i,i+1}$ with $2 \leq i \leq n-2$ there are $n-1$ minimal pairs: $(k,k+1), k \neq i-1, 1 \leq k \leq n-1$ and $(i-1,i+2)$. For the group $Q_{n,i,j}$ with $1 < i < j-1 \leq n-1$, there are n minimal pairs: $(k,k+1), k \neq i-1, j-1, 1 \leq k \leq n-1$ and $(i-1,i+1),(i,j),(j-1,j+1)$. Therefore

$$N_1(Q_{n,i,j}) = \begin{cases} q^{n-1} \text{ if } i = j-1 \\ q^n \text{ if } i < j-1 \end{cases}. \tag{4.2}$$

5. Computing $N_q(P_{n,2})$

With the aim of $N_1(P_{n,i})$ and $N_1(Q_{n,i,j})$ we give the recursive formulas for $N_k(U_n)$ when $k = q$ and $k = q^2$ and compute $N_q(P_n, 2)$. For $k = q$ and $n \geq 5$, equation (3.5), implies

$$N_q(U_n) = qN_q(U_{n-1}) + (q-1)N_1(P_{n-1,2}) = qN_q(U_{n-1}) + q^{n-2}(q-1). \tag{5.1}$$

It is straight forward to see that $N_q(U_3) = q-1$ while $N_q(U_4) = q(q-1)(q+1)$. So the formula $N_q(U_n) = q^{n-3}(q-1)((n-3)q+1)$ for $N_q(U_n)$ obtained by both Marjoram [7] and Isaacs [4] satisfies (5.1).

For $k = q^2$ and $n = 5$ equation (3.9) implies $N_{q^2}(U_5) = q(q-1)(2q^2+q-1)$. In addition, for all $n \geq 6$ we have

$$\begin{aligned} N_{q^2}(U_n) &= qN_{q^2}(U_{n-1}) + qN_q(U_{n-1}) - qN_q(U_{n-2}) + (q-1)^2N_1(Q_{n-2,1,3}) = \\ &\qquad qN_{q^2}(U_{n-1}) + q^{n-4}(q-1)[q^3 + (n-5)q^2 - (n-6)q - 1]. \end{aligned} \tag{5.2}$$

It is straight forward to check that the above recursive formula is satisfied by the equation

$$N_{q^2}(U_n) = q^{n-4}(q-1)\{(n-5)q^3 + (\frac{(n-5)(n-4)}{2} + 2)q^2 + [1 - \frac{(n-6)(n-5)}{2}]q - n + 4\}. \tag{5.3}$$

On the other hand equation (3.5) for $k = q^2$ and $n \geq 5$, implies

$$N_{q^2}(U_n) = qN_{q^2}(U_{n-1}) + (q-1)[q^{n-2} + N_q(P_{n-1,2})]. \tag{5.4}$$

Combining the above with (5.2) we get

$$N_q(P_{4,2}) = \frac{1}{q-1}(N_{q^2}(U_5) - qN_{q^2}(U_4)) - q^3 = q(q^2-1), \tag{5.5}$$

while for $n \geq 6$

$$N_q(P_{n-1,2}) = q^{n-4}(q-1)[q^2 + (n-5)q + 1]. \tag{5.6}$$

6. The group $Q_{n,1,3}$.

The aim in this section is to compute $N_q(Q_{n,1,3})$, for all $n \geq 4$. We point out that we are not able to compute the number of irreducible characters of degree q for every group $Q_{n,i,j}$ where i, and j are arbitrary. But we can do it for the group $Q_{n,1,3}$, and this is enough for the computation of $N_{q^3}(U_n)$.

Assume that $n \geq 4$. Note that, according to its definition, $Q_{n,1,3}$ consists of all $n \times n$ unitriangular matrices whose $(2,3)$-entry is zero. We write $Q_{n,1,3}$ as a semidirect product using the following groups. Let M be the subgroup of $Q_{n,1,3}$ consisting of matrices all of whose non-diagonal elements are zero except for the first row. Assume further that H is the subgroup of $Q_{n,1,3}$ consisting of matrices whose non-diagonal entries in the first row are zero. Then it is clear that M is an abelian normal subgroup of $Q_{n,1,3}$ isomorphic to $\mathbf{F}^{1 \times n-1} \cong \mathbf{F}^{n-1}$. Observe that H is isomorphic to $P_{n-1,2}$. Furthermore, $Q_{n,1,3} = M \rtimes H$ and the conjugation action of H on M is given as

$$\begin{pmatrix} 1 & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} 1 & C \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & CX \\ 0 & I_{n-1} \end{pmatrix},$$

where $X \in P_{n-1,2}$ and $C \in \mathbf{F}^{1 \times n-1}$.

Now we apply Proposition (1) to the group $M = \mathbf{F}^{1 \times n-1}$ (with M in the place of $M_{n,t}$ for $t = 1$). So the group of irreducible characters $\mathrm{Irr}(M)$ of M is isomorphic to the abelian additive group $\mathbf{F}^{n-1 \times 1}$. Thus we regard the irreducible characters of M as column vectors over $\mathbf{F}$, and for every $\chi \in \mathrm{Irr}(M)$ we write $\chi = (\chi_1, \ldots, \chi_{n-1})^t$ with $\chi_i \in \mathbf{F}$. Under the isomorphism between $\mathrm{Irr}(M)$ and $\mathbf{F}^{n-1 \times 1}$ the action of an element $\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}$ becomes multiplication on the left by X^{-1}. It is straightforward to see that the H-invariant irreducible characters of M are those with $\chi_3 = \chi_4 = \ldots = \chi_{n-1} = 0$, and thus they look like $(\chi_1, \chi_2, 0, \ldots, 0)^t$ with $\chi_1, \chi_2 \in \mathbf{F}$. Hence we get q^2 such irreducible characters.

Furthermore, if $\chi = (\chi_1, \ldots, \chi_{n-1})^t \neq (0, \ldots, 0)^t$ is any character in $\mathrm{Irr}(M)$, and k is the biggest index with $\chi_k \neq 0$, then if $k = 1, 2$ the character χ is H-invariant , while if $k \geq 3$ the H-orbit of χ contains all the characters of type $(f_1, \ldots, f_{k-1}, \chi_k, 0, \ldots, 0)^t$, where $f_i \in \mathbf{F}$ are arbitrary. Hence we get orbits of length q^{k-1}. Therefore for any $\chi \in \mathrm{Irr}(M)$ either χ is H-invariant or its stabilizer H_χ in H has index at least q^2. That is, there are no irreducible characters in $\mathrm{Irr}(M)$ whose stabilizer in H has index q in H.

Now we follow the argument after equation (3.2), for the group $Q_{n,1,3} = M \rtimes H$, to get $N_q(Q_{n,1,3}) = q^2 \cdot N_q(P_{n-1,2})$, for all $n \geq 4$. If $n = 4$ then $P_{3,2} \cong \mathbf{F}^2$ and thus

$$N_q(P_{3,2}) = N_q(Q_{4,1,3}) = 0. \tag{6.1}$$

If $n = 5$ then in view of (5.5) we get

$$N_q(Q_{5,1,3}) = q^3(q^2 - 1) \tag{6.2}$$

In addition, for all $n \geq 6$, we use (5.6) to get

$$N_q(Q_{n,1,3}) = q^{n-2}(q - 1)[q^2 + (n - 5)q + 1]. \tag{6.3}$$

7. Computing $N_{q^3}(U_n)$.

For $n = 5$, equation (3.9) implies that $N_{q^3}(U_5) = q(q - 1)(2q - 1)$. Furthermore, when $k = q^3$ and $n = 6$ equation (3.9) along with (6.1) and (4.2) implies

$$N_{q^3}(U_6) = q^2(q - 1)(4q^2 + q - 3).$$

For the case $n = 7$ we similarly get

$$N_{q^3}(U_7) = q^2(q - 1)[3q^4 + 6q^3 - 2q^2 - 5q + 1]. \tag{7.1}$$

In general, for all $n \geq 8$ equation (3.9) implies

$$N_{q^3}(U_n) = qN_{q^3}(U_{n-1}) + qN_{q^2}(U_{n-1}) - qN_{q^2}(U_{n-2}) + (q - 1)^2[N_q(Q_{n-2,1,3}) + N_1(Q_{n-2,2,3}) + N_1(Q_{n-2,1,4})]. \tag{7.2}$$

According to (5.3), for all $n \geq 8$ we get

$$N_{q^2}(U_{n-1}) - N_{q^2}(U_{n-2}) = q^{n-6}(q - 1)\{(n - 6)q^4 + [\frac{(n - 6)(n - 5)}{2} - (n - 7) + 2]q^3 - [(n - 7)(n - 6) + 1]q^2 + [4 - n + \frac{(n - 8)(n - 7)}{2}]q + n - 6\} \tag{7.3}$$

Furthermore, using (4.2) and (6.3) along with (7.3) in equation (7.2) and we get

$$N_{q^3}(U_n) = qN_{q^3}(U_{n-1}) + q^{n-5}(q - 1)\{q^5 + (2n - 14)q^4 + [25 - 3n + \frac{(n - 6)(n - 5)}{2}]q^3 + [n - 11 - (n - 7)(n - 6)]q^2 + [5 - n + \frac{(n - 8)(n - 7)}{2}]q + n - 6\}, \tag{7.4}$$

for all $n \geq 8$. As we have already computed the formula for $N_{q^3}(U_7)$, we can easily check that the following equation satisfies the recursive formula (7.4) for all $n \geq 8$.

$$N_{q^3}(U_n) = q^{n-5}(q - 1)\{A_nq^5 + B_nq^4 + C_nq^3 + D_nq^2 + E_nq + F_n\}, \tag{7.5}$$

where

- $A_n = n - 7$
- $B_n = 3 + (n - 7)(n - 6)$
- $C_n = 40(n - 7) - \frac{17}{4}(n + 8)(n - 7) + \frac{1}{12}n(n + 1)(2n + 1) - 64$
- $D_n = (n - 7)(7n + 3) - \frac{1}{6}n(n + 1)(2n + 1) + 138$

- $E_n = (n-7)(-\frac{17}{4}n - 1) + \frac{1}{12}n(n+1)(2n+1) - 75$
- $F_n = 1 + \frac{(n-7)(n-4)}{2}$.

It is clear that the polynomials A_n, B_n and F_n in n are integer valued for every n. To show that the same holds for the polynomials C_n, D_n and E_n we make use of the following lemma.

Lemma 1. *Let $P(n)$ be a polynomial in n of degree m with rational coefficients. If $P(n)$ is an integer for $m+1$ consecutive integers, then the polynomial is integer valued.*

Proof. We will use induction on the degree m of $P(n)$. It is clear that for $m = 1$ holds.

Assume it holds for all polynomials of degree less that m, we will show that it also holds for those of degree m. The polynomial $Q(n) := P(n+1) - P(n)$ has degree smaller than m. In addition, if P has integer values for $m+1$ consecutive integers $k, k+1, \ldots, k+m$, then $Q(n)$ is integer valued for the m consecutive integers $k, k+1, \ldots, k+m-1$. Therefore the inductive hypothesis implies that $Q(n)$ is integer valued for every n. This along with the fact that $P(n)$ is integer valued for $n = k+m$, implies that $P(n)$ is an integer for every n. □

Now, it is straight forward to check that C_7, C_8, C_9 and C_{10} are integers. Hence the above lemma implies that C_n is an integer valued polynomial. Similarly we show that D_n and E_n are integer valued. Hence $N_{q^3}(U_n)$ is a polynomial in q with integer coefficients.

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