SPHERICAL MEANS AND MEASURES WITH FINITE ENERGY

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Abstract. We prove a restricted weak type inequality for the spherical means operator with respect to measures with finite α-energy, $\alpha \leq 1$. This complements recent results due to D. Oberlin.

Fix a small positive number δ, and for $r > \delta$ let us denote by $S^\delta(\bar{x}, r)$ the δ-neighborhood of the $(n-1)$-dimensional sphere with center $\bar{x} \in \mathbb{R}^n$ and radius r. That is

$$S^\delta(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^n : r - \delta < |\bar{x} - \bar{y}| < r + \delta\}.$$

(Here and for the rest of the paper we assume that $n \geq 3$.) Now, for suitable $f : \mathbb{R}^n \to \mathbb{R}$, consider the spherical means operator

$$T_\delta f : \mathbb{R}^n \times (\delta, \infty) \to \mathbb{R}$$

defined by

$$T_\delta f(\bar{x}, r) = \frac{1}{|S^\delta(\bar{x}, r)|} \int_{S^\delta(\bar{x}, r)} f,$$

where $|\cdot|$ denotes Lebesgue measure. The mapping properties of this operator, its variants, and the corresponding maximal operators have been studied extensively by several authors using Fourier analysis. Recently D. Oberlin [2] proved the following restricted weak type inequality for T_δ with respect to measures more general than the Lebesgue measure.

Theorem 1. *Let $1 < \alpha < n+1$ and suppose μ is a compactly supported non-negative Borel measure in $\mathbb{R}^n \times (0, \infty)$ such that the α-energy $I_\alpha(\mu)$ defined by*

$$I_\alpha(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x - y|^\alpha}$$

is finite. Let

$$r_0 = \inf\{r : \text{there exists } \bar{x} \in \mathbb{R}^n \text{ such that } (\bar{x}, r) \text{ is in the support of } \mu\}.$$

Then for $\lambda > 0$ and $0 < \delta < r_0$ one has the estimate

$$\lambda^2 \mu\left(\{T_\delta \chi_E > \lambda\}\right)^{2/\alpha} \leq C|E|, \tag{1}$$

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for all Borel sets $E \subset \mathbb{R}^n$ (χ_E is the characteristic function). Here C is a positive constant independent of δ and λ (it depends on μ and n).

The case $0 < \alpha \leq 1$ was left open in [2]. The example by the author mentioned in [2] suggests that if $0 < \alpha \leq 1$ then the right-hand side of (1) should be either corrected by a factor which tends to infinity as δ tends to zero, or replaced with a larger norm. In the latter direction, one has the following result due to D. Oberlin, which is a special case of theorem 4_S in [3].

Theorem 2. *Suppose $0 < \alpha \leq 1$, and let $B(x, \rho)$ be the closed ball in $\mathbb{R}^n \times (0, +\infty)$ with center x and radius ρ. If μ satisfies*

$$\mu(B(x, \rho)) \leq \rho^\alpha \tag{2}$$

for all x and ρ, then for every $\varepsilon > 0$ there exists a positive constant C_ε independent of λ and δ such that

$$\lambda^2 \mu(\{T_\delta \chi_E > \lambda\}) \leq C_\varepsilon \|\chi_E\|^2_{W^{2, \frac{1-\alpha}{2}+\varepsilon}}, \tag{3}$$

where the norm on the right-hand side is the Sobolev space norm.

The proof of theorem 2 is Fourier analytic. In this paper we give an elementary proof of the following estimate which may be thought of as the "non δ-free counterpart" of (3) under a weaker energy-finiteness hypothesis ((2) implies that $I_\beta(\mu) < \infty$ for all $\beta < \alpha$).

Theorem 3. *If $0 < \alpha \leq 1$ and $I_\alpha(\mu) < \infty$ then*

$$\lambda^2 \mu\left(\{T_\delta \chi_E > \lambda\}\right)^2 \leq C_\varepsilon |E| \delta^{\alpha - 1 - \varepsilon}. \tag{4}$$

Note that (4) is not entirely satisfactory. A natural conjecture (corresponding to an L^2 bound) would be

$$\lambda^2 \mu\left(\{T_\delta \chi_E > \lambda\}\right) \leq C_\varepsilon |E| \delta^{\alpha - 1 - \varepsilon}.$$

We do not, however, know how to prove (or disprove) this.

Proof of Theorem 3

To simplify the presentation we will be using the standard notation $x \lesssim y$ to denote $x \leq Cy$ for some positive constant C. Similarly, $x \simeq y$ means that x and y are comparable.

Let

$$F = \{T_\delta \chi_E > \lambda\} \subset \mathbb{R}^n \times (0, \infty).$$

We will discretize the problem at scale δ. First we show that F can be decomposed into roughly $|\log\delta|$ sets on which μ behaves as if it were α-dimensional. So, put

$$F_0 = \left\{ x \in F : \sup_{\rho \geq \delta} \frac{\mu(B(x,\rho))}{\rho^\alpha} \leq 1 \right\},$$
$$F_i = \left\{ x \in F : 2^{i-1} < \sup_{\rho \geq \delta} \frac{\mu(B(x,\rho))}{\rho^\alpha} \leq 2^i \right\}, \quad i = 1, 2, \dots,$$
$$I = \{ i \in \mathbb{N} \cup \{0\} : \mu(F_i) \neq 0 \}.$$

Then $\mu(F) = \sum_{i \in I} \mu(F_i)$, and since μ is a finite measure, we have that $|I| \lesssim |\log\delta|$ for δ small enough. Moreover

$$\mu(B(x,\rho)) \leq 2^i \rho^\alpha, \text{ for } x \in F_i, \ \rho \geq \delta. \tag{5}$$

This means that, modulo the factor 2^i, the measure μ is α-dimensional on F_i. To estimate this factor, fix $i \in I$ with $i \geq 1$. Then, by the Besicovitch covering lemma, there exists a countable family of closed balls B_j with radius $\rho_j \geq \delta$ such that

- $\{B_j\}_j$ has bounded overlap.
- $\{B_j\}_j$ covers F_i.
- For all j we have that

$$\mu(B_j) > 2^{i-1} \rho_j^\alpha. \tag{6}$$

Notice that

$$\frac{\mu(B_j)^2}{\rho_j^\alpha} \lesssim \iint_{B_j \times B_j} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}. \tag{7}$$

So, using (6) and (7), we get that

$$2^i \mu(F_i) \leq \sum_j 2^i \mu(B_j) \lesssim \sum_j \rho_j^{-\alpha} \mu(B_j)^2 \lesssim \sum_j \iint_{B_j \times B_j} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \lesssim I_\alpha(\mu), \tag{8}$$

where the last inequality follows from the fact that $\{B_j\}_j$ has bounded overlap. Therefore, (5) and (8) imply that

$$\mu(B(x,\rho)) \lesssim \mu(F_i)^{-1} \rho^\alpha, \text{ for } x \in F_i, \ \rho \geq \delta, \ i \in I, \ i \neq 0. \tag{9}$$

If $i \in I$ and $i = 0$ then (9) follows trivially from (5) because μ is finite.

Now, we use Córdoba's orthogonality argument [1] to estimate the measure of each F_i, $i \in I$. (9) will be important here. We decompose $\mathbb{R}^{n+1}$ into a family $\mathscr{Q}$ of disjoint cubes of side length δ. That is

$$\mathscr{Q} = \left\{ \prod_{l=1}^{n+1} [m_l\delta, (m_l+1)\delta) : m_1, \dots, m_{n+1} \in \mathbb{Z} \right\}.$$

Let $\{Q_j\}_j = \{Q \in \mathscr{Q} : Q \cap F_i \neq \emptyset\}$ and pick $(\bar{x}_j, r_j) \in Q_j$ ($\bar{x}_j \in \mathbb{R}^n$, $r_j > 0$) such that

$$\frac{1}{|S^\delta(\bar{x}_j, r_j)|} \int_{S^\delta(\bar{x}_j, r_j)} \chi_E > \lambda.$$

Since μ is compactly supported, the r_j's are bounded, therefore $|S^\delta(\bar{x}_j, r_j)| \simeq \delta$. Thus

$$\begin{aligned}
\mu(F_i) &= \sum_j \mu(Q_j \cap F_i) = \frac{1}{\lambda\delta} \sum_j \lambda\delta\mu(Q_j \cap F_i) \lesssim \frac{1}{\lambda\delta} \sum_j \mu(Q_j \cap F_i) \int_E \chi_{S^\delta(\bar{x}_j, r_j)} \\
&\leq \frac{|E|^{1/2}}{\lambda\delta} \left[\int_E \left(\sum_j \mu(Q_j \cap F_i) \chi_{S^\delta(\bar{x}_j, r_j)} \right)^2 \right]^{1/2} \\
&\leq \frac{|E|^{1/2}}{\lambda\delta} \left[\int \sum_{j,k} \mu(Q_j \cap F_i)\mu(Q_k \cap F_i) \chi_{S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k)} \right]^{1/2} \\
&= \frac{|E|^{1/2}}{\lambda\delta} \left[\sum_{j,k} \mu(Q_j \cap F_i)\mu(Q_k \cap F_i) \left| S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k) \right| \right]^{1/2}. \qquad (10)
\end{aligned}$$

By Lemma 1 in [2]

$$\left| S^\delta(\bar{x}_j, r_j) \cap S^\delta(\bar{x}_k, r_k) \right| \lesssim \frac{\delta^2}{\delta + |(\bar{x}_j, r_j) - (\bar{x}_k, r_k)|}.$$

Moreover, for all $x \in Q_j$ and $y \in Q_k$ we have that

$$\delta + |x - y| \lesssim \delta + |(\bar{x}_j, r_j) - (\bar{x}_k, r_k)|.$$

Therefore

$$\begin{aligned}
(10) &\lesssim \frac{|E|^{1/2}}{\lambda} \left[\sum_{j,k} \iint_{(Q_j \times Q_k) \cap (F_i \times F_i)} \frac{d\mu(x)\, d\mu(y)}{\delta + |x - y|} \right]^{1/2} \\
&= \frac{|E|^{1/2}}{\lambda} \left[\iint_{F_i \times F_i} \frac{d\mu(x)\, d\mu(y)}{\delta + |x - y|} \right]^{1/2}. \qquad (11)
\end{aligned}$$

To estimate the integral in the square brackets, we use the distribution function. For each $x \in F_i$ we have that

$$\begin{aligned}
\int_{F_i} \frac{d\mu(y)}{\delta + |x - y|} &= \int_0^{1/\delta} \mu\left(\left\{ y \in F_i : \delta + |x - y| < \rho^{-1} \right\} \right) d\rho \\
&\leq \int_0^{1/\delta} \mu\left(B(x, \rho^{-1}) \right) d\rho \qquad (12)
\end{aligned}$$

Since $\rho^{-1} \geq \delta$, (9) implies that

$$(12) \lesssim \frac{1}{\mu(F_i)} \int_0^{1/\delta} \frac{d\rho}{\rho^\alpha} \lesssim \frac{\delta^{\alpha - 1}}{\mu(F_i)}.$$

Consequently, (11) yields

$$\mu(F_i) \lesssim \frac{1}{\lambda}|E|^{1/2}\delta^{(\alpha-1)/2}.$$

Summing up these inequalities in $i \in I$ we obtain

$$\mu(F) \lesssim \frac{1}{\lambda}|E|^{1/2}|\log\delta|\delta^{(\alpha-1)/2} \leq C_\varepsilon \frac{1}{\lambda}|E|^{1/2}\delta^{(\alpha-1)/2-\varepsilon}$$

as claimed.

The same argument shows that if $\alpha = 1$ then

$$\mu(F) \leq C_\varepsilon \frac{1}{\lambda}|E|^{1/2}\delta^{-\varepsilon}.$$

References

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