AN OPTIMAL EXTENSION OF MARSTRAND'S PLANE-PACKING THEOREM

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Abstract. We prove that if F is a subset of the 2-dimensional unit sphere in $\mathbb{R}^3$, with Hausdorff dimension strictly greater than 1, and E is a subset of $\mathbb{R}^3$ such that for each $e \in F$, E contains a plane perpendicular to the vector e, then E must have positive 3-dimensional Lebesgue measure.

1. Introduction

Marstrand [1] proved that a subset of $\mathbb{R}^3$ containing a translate of every plane has positive 3-dimensional Lebesgue measure. The purpose of this paper is to show that Marstrand's result still holds if one restricts the set of the normal directions of the planes to any subset of the unit sphere with Hausdorff dimension strictly greater than one. Namely, we prove the following.

Theorem. *Let $S^2 \subset \mathbb{R}^3$ be the 2-dimensional unit sphere, $F \subset S^2$ be a set of Hausdorff dimension strictly greater than 1, and $E \subset \mathbb{R}^3$ be a Borel set such that for each $e \in F$, E contains a plane perpendicular to the vector e. Then E has positive 3-dimensional Lebesgue measure.*

We also show that this result is optimal, in the sense that the condition on the Hausdorff dimension of F cannot be relaxed.

2. Preliminaries

Ω^2 is the set $\{e \in S^2 : |e - (1,0,0)| \leq 1/10\}$.
$B(x,r)$ is the ball of radius r, centered at the point x.
$\mathcal{L}^k$, $k = 2,3$, denotes k-dimensional Lebesgue measure.
$\dim_H(A)$ is the Hausdorff dimension of the set A.
$l_e(a)$ is the line in the direction $e \in S^2$, passing through the point $a \in \mathbb{R}^3$.
Π_e is the plane through the origin, perpendicular to the vector $e \in S^2$.

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$\Pi_e^\delta(a)$ is the solid cylinder with center $a \in \mathbb{R}^3$, height 2δ, radius 1, and whose axis is in the direction $e \in S^2$, i.e.

$$\Pi_e^\delta(a) = \{x \in R^3 : \mathrm{dist}(x, l_e(a)) \leq 1, \text{ and } |\mathrm{proj}_{l_e(a)}(x) - a| \leq \delta\},$$

where $\mathrm{proj}_{l_e(a)}(x)$ is the orthogonal projection of the point x onto the line $l_e(a)$.

χ_A is the characteristic function of the set A.

A^δ is the δ-neighborhood of the set A, i.e

$$A^\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, A) \leq \delta\}.$$

Throughout this paper, $C, C_1, C_2, \dots$ will denote positive constants.

We finally note the following geometric fact which will allow us to control the measure of the intersection of two cylinders.

Lemma. *There exists a constant C such that for any $e, f \in \Omega^2$, and $a, b \in \mathbb{R}^3$, we have*

$$\mathcal{L}^3(\Pi_e^\delta(a) \cap \Pi_f^\delta(b)) \leq C \frac{\delta^2}{\delta + |e - f|}.$$

Proof. Let θ be the angle between e and f. After a rotation and a translation, $\Pi_e^\delta(a) \cap \Pi_f^\delta(b)$ is contained in the set $(R_1 \cap R_2) \times J$, where R_1 and R_2 are 2-dimensional rectangles of dimensions $2 \times 2\delta$ intersecting at angle θ, and J is an interval of length 2. By trigonometry, the diameter of $R_1 \cap R_2$ is less than

$$C_1 \frac{\delta}{\delta + \sin\theta} \leq C_2 \frac{\delta}{\delta + |e - f|}.$$

So, $R_1 \cap R_2$ is contained in a rectangle of dimensions

$$C_2 \frac{\delta}{\delta + |e - f|} \times 2\delta,$$

and therefore

$$\mathcal{L}^2(R_1 \cap R_2) \leq C_3 \frac{\delta^2}{\delta + |e - f|}.$$

Integrating along J we get that

$$\mathcal{L}^3(\Pi_e^\delta(a) \cap \Pi_f^\delta(b)) \leq C \frac{\delta^2}{\delta + |e - f|}.$$

□

3. Proof of the Theorem

Without loss of generality, we may assume that $F \subset \Omega^2$. Since $\dim_H(F) > 1$, by Frostman's lemma (see, for example, [2]), there exists a positive, finite Borel measure μ supported in F such that the quantity

$$I_1(\mu) := \iint \frac{d\mu(x)d\mu(y)}{|x-y|}$$

is finite.

Using measure theory, one can find a subset $F_1 \subset F$ with $\mu(F_1) > 0$ and a compact set $E_1 \subset E$ such that

$$\forall e \in F_1 \ \exists a_e \in \mathbb{R}^3 : \mathcal{L}^2((\Pi_e + a_e) \cap B(a_e, 1) \cap E_1) \geq \frac{\pi}{2},$$

and therefore

$$\mathcal{L}^3(\Pi_e^\delta(a_e) \cap E_1^\delta) \geq \pi\delta, \ \forall e \in F_1. \tag{3.1}$$

Now, decompose F_1 into a finite, disjoint family $\{A_j\}$ of sets with diameter less than δ. Let $c_j = \mu(A_j)$ and pick $e_j \in A_j$. Then

$$\begin{aligned}\mu(F_1) &= \frac{1}{\delta}\sum_j \mu(A_j)\delta \leq \frac{1}{\delta}\sum_j c_j \mathcal{L}^3(\Pi_{e_j}^\delta(a_{e_j}) \cap E_1^\delta) \quad \text{(by (3.1))} \\ &= \frac{1}{\delta}\sum_j c_j \int_{E_1^\delta} \chi_{\Pi_{e_j}^\delta(a_{e_j})} = \frac{1}{\delta}\int_{E_1^\delta} \sum_j c_j \chi_{\Pi_{e_j}^\delta(a_{e_j})}.\end{aligned}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}\mu(F_1) &\leq \frac{1}{\delta}\mathcal{L}^3(E_1^\delta)^{1/2}\Big(\int\Big(\sum_j c_j \chi_{\Pi_{e_j}^\delta(a_{e_j})}\Big)^2\Big)^{1/2} \\ &= \frac{1}{\delta}\mathcal{L}^3(E_1^\delta)^{1/2}\Big(\int \sum_{j,k} c_j c_k \chi_{\Pi_{e_j}^\delta(a_{e_j}) \cap \Pi_{e_k}^\delta(a_{e_k})}\Big)^{1/2} \\ &= \frac{1}{\delta}\mathcal{L}^3(E_1^\delta)^{1/2}\Big(\sum_{j,k} c_j c_k \mathcal{L}^3(\Pi_{e_j}^\delta(a_{e_j}) \cap \Pi_{e_k}^\delta(a_{e_k}))\Big)^{1/2}.\end{aligned}$$

By the Lemma

$$\begin{aligned}\mu(F_1) &\leq C^{1/2}\mathcal{L}^3(E_1^\delta)^{1/2}\Big(\sum_{j,k} \frac{c_j c_k}{\delta + |e_j - e_k|}\Big)^{1/2} \\ &\leq C^{1/2}\mathcal{L}^3(E_1^\delta)^{1/2}\Big(\sum_{j,k} \int_{A_j}\int_{A_k} \frac{2d\mu(x)d\mu(y)}{|x-y|}\Big)^{1/2} \\ &\leq (2C\mathcal{L}^3(E_1^\delta)I_1(\mu))^{1/2}.\end{aligned}$$

We conclude that

$$\mathcal{L}^3(E_1^\delta) \geq \frac{\mu(F_1)^2}{2CI_1(\mu)},$$

for all δ, and therefore $\mathcal{L}^3(E_1) > 0$. Consequently $\mathcal{L}^3(E) > 0$.

To see that the condition $\dim_H(F) > 1$ cannot be relaxed, let $K \subset \mathbb{R}^2$ be a Kakeya set, that is, a set of planar measure zero which contains a line in every direction (see, for example, [2]), and let

$$F = \{(x, y, 0) : x^2 + y^2 = 1\} \subset S^2,$$

$$E = K \times \mathbb{R}.$$

Then

$$\dim_H(F) = 1, \ \mathcal{L}^3(E) = 0, \ \text{and } \forall e \in F \ \exists a_e \in \mathbb{R}^3 : \Pi_e + a_e \subset E.$$

References

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