Exercises in Classical Real Analysis

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Contents

CHAPTER 1

Numbers

EXERCISE 1.1. *Let a, b, c, d be rational numbers and x an irrational number such that $cx + d \neq 0$. Prove that $(ax + b)/(cx + d)$ is irrational if and only if $ad \neq bc$.*

SOLUTION. Suppose that $(ax + b)/(cx + d) = p/q$, where $p, q \in \mathbb{Z}$. Then $(aq - cp)x = dp - bq$, and so we must have $dp - bq = aq - cp = 0$, since x is irrational. It follows that $ad = bc$. Conversely, if $ad = bc$ then $(ax + b)/(cx + d) = b/d \in \mathbb{Q}$. □

EXERCISE 1.2. *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. Prove that*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right) \leq n \sum_{k=1}^{n} a_k b_k$$

and that equality obtains if and only if either $a_1 = a_n$ or $b_1 = b_n$.

SOLUTION. Since $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are both increasing, we have

$$0 \leq \sum_{1 \leq i,j \leq n} (a_i - a_j)(b_i - b_j) = 2n \sum_{k=1}^{n} a_k b_k - 2\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{j=1}^{n} b_j\right).$$

If we have equality then the above implies $(a_i - a_j)(b_i - b_j) = 0$ for all i, j. In particular $(a_1 - a_n)(b_1 - b_n) = 0$, and so either $a_1 = a_n$ or $b_1 = b_n$. □

EXERCISE 1.3. *(a) If $a_1, a_2, \ldots, a_n$ are all positive, then*

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \geq n^2$$

and equality obtains if and only if $a_1 = a_2 = \cdots = a_n$.

(b) If a, b, c are positive and $a + b + c = 1$, then

$$(1/a - 1)(1/b - 1)(1/c - 1) \geq 8$$

and equality obtains if and only if $a = b = c = 1/3$.

SOLUTION. (a) By the Cauchy-Schwarz inequality we have

$$n = \sum_{i=1}^{n} a_i^{1/2} \left(\frac{1}{a_i}\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i\right)^{1/2} \left(\sum_{i=1}^{n} \frac{1}{a_i}\right)^{1/2}.$$

(b) Since $a + b + c = 1$, (a) implies $1/a + 1/b + 1/c \geq 9$ and therefore

$$(1/a - 1)(1/b - 1)(1/c - 1) = 1/a + 1/b + 1/c - 1 \geq 8.$$

□

EXERCISE 1.4. *Prove that for all $n \in \mathbb{N}$ we have*

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n+1}}$$

end equality obtains if and only if $n = 1$.

SOLUTION. Note that

$$\frac{2k-1}{2k} \leq \frac{\sqrt{3k-2}}{\sqrt{3k+1}}$$

and therefore the product telescopes. □

EXERCISE 1.5. *(a) For all $n \in \mathbb{N}$ we have*

$$\sqrt{n+1} - \sqrt{n} < \frac{1}{\sqrt{n}} < \sqrt{n} - \sqrt{n-1}.$$

(b) If $n \in \mathbb{N}$ ad $n > 1$ then

$$2\sqrt{n+1} - 2 < \sum_{k=1}^{n} \frac{1}{\sqrt{k}} < 2\sqrt{n} - 1.$$

SOLUTION. (a) We have

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}},$$

$$\sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n} + \sqrt{n-1}} > \frac{1}{2\sqrt{n}}.$$

(b) Sum inequalities (a) for $k = 2, 3, \ldots, n$. □

EXERCISE 1.6. *Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then*
(a) $-1 < x < 0$ implies $(1+x)^n \leq 1 + nx + (n(n-1)/2)x^2$.
(b) $x > 0$ implies $(1+x)^n \geq 1 + nx + (n(n-1)/2)x^2$.

SOLUTION. Induction on n. □

EXERCISE 1.7. *If $n \in \mathbb{N}$, then $n! \leq ((n+1)/2)^n$.*

SOLUTION. In the Geometric-Arithmetic Means Inequality, take $a_k = k$. □

EXERCISE 1.8. *If $b_1, b_2, \ldots, b_n$ are positive real numbers, then*

$$\frac{n}{\frac{1}{b_1} + \frac{1}{b_2} + \cdots \frac{1}{b_n}} \leq (b_1 b_2 \cdots b_n)^{1/n}.$$

SOLUTION. In the Geometric-Arithmetic Means Inequality, take $a_k = 1/b_k$. □

EXERCISE 1.9. *If $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then*
(a) $[x+y] \geq [x] + [y]$,
(b) $[[x]/n] = [x/n]$,
(c) $\sum_{k=0}^{n-1} [x + k/n] = [nx]$.

SOLUTION. (a) $[x]+[y]$ is an integer and satisfies $[x]+[y] \leq x+y$, therefore $[x]+[y] \leq [x+y]$.

(b) We claim that $[x/n] \leq [x]/n$. Indeed, if this were not the case we would have $[x]/n < [x/n] \leq ([x]+\epsilon)/n$, for some $0 \leq \epsilon < 1$. Therefore $[x] < n[x/n] \leq [x]+\epsilon$, a contradiction since $n[x/n]$ is an integer. It follows that $[x/n] \leq [[x]/n]$. The converse inequality is obvious.

(c) Let

$$f(x) = \sum_{k=0}^{n-1} [x + k/n] - [nx].$$

Then f is periodic with period $1/n$ and vanishes on the interval $[0, 1/n]$. So, $f = 0$ identically. □

EXERCISE 1.10. *(a) If a, b, c are positive real numbers then*

$$\left(\frac{1}{2}a + \frac{1}{3}b + \frac{1}{6}c\right)^2 \leq \frac{1}{2}a^2 + \frac{1}{3}b^2 + \frac{1}{6}c^2$$

with equality if and only if $a = b = c$.

(b) If $a_1, \ldots, a_n$ and $w_1, \ldots, w_n$ are positive real numbers with $\sum_{i=1}^{n} w_i = 1$ then

$$\left(\sum_{i=1}^{n} a_i w_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 w_i$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

SOLUTION. (a),(b) Cauchy-Schwarz inequality. □

EXERCISE 1.11. *If $n \in \mathbb{N}$, then*

(a) $\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(b) $\sum_{k=1}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}$.

SOLUTION. (a) By the Binomial Theorem we have

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k.$$

But

$$(1+x)^{2n} = (1+x)^n (1+x)^n = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)$$

$$= \sum_{i,j} \binom{n}{i}\binom{n}{j} x^{i+j} = \sum_{k=0}^{2n} x^k \sum_{i+j=k} \binom{n}{i}\binom{n}{j}.$$

Equating the coefficients of x^n we get

$$\binom{2n}{n} = \sum_{i+j=n} \binom{n}{i}\binom{n}{j} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

(b) As in (a) we have

$$(1-x^2)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k x^{2k}$$

and

$$(1-x^2)^{2n} = (1-x)^{2n}(1+x)^{2n} = \left(\sum_{i=0}^{2n} \binom{2n}{i} (-1)^i x^i \right) \left(\sum_{j=0}^{2n} \binom{2n}{j} x^j \right)$$

$$= \sum_{i,j} \binom{2n}{i}\binom{2n}{j} (-1)^i x^{i+j} = \sum_{k=0}^{4n} x^k \sum_{i+j=k} (-1)^i \binom{2n}{i}\binom{2n}{j}.$$

Equating the coefficients of x^{2n} we get

$$(-1)^n \binom{2n}{n} = \sum_{i+j=2n} (-1)^i \binom{2n}{i}\binom{2n}{j} = \sum_{i=0}^{2n} (-1)^i \binom{2n}{i}^2.$$

□

Exercise 1.12. *If $m, n \in \mathbb{N}$, then* $1 + \sum_{k=1}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$.

Solution.

$$1 + \sum_{k=1}^{m} \binom{n+k}{k} = 1 + \sum_{k=1}^{m} \left(\binom{n+k+1}{k} - \binom{n+k}{k-1} \right) = \binom{n+m+1}{m}.$$

□

Exercise 1.13. *Prove Lagrange's inequality for real numbers*

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2.$$

Solution. We have

$$\sum_{1 \le k < j \le n} (a_k b_j - a_j b_k)^2 = \sum_{1 \le k < j \le n} (a_k^2 b_j^2 + a_j^2 b_k^2 - 2 a_k b_j a_j b_k).$$

But

$$\sum_{1 \le k < j \le n} a_k^2 b_j^2 + \sum_{1 \le k < j \le n} a_j^2 b_k^2 = \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) - \sum_{k=1}^{n} a_k^2 b_k^2$$

and

$$\sum_{1 \le k < j \le n} 2 a_k b_j a_j b_k = \left(\sum_{k=1}^{n} a_k b_k \right)^2 - \sum_{k=1}^{n} a_k^2 b_k^2.$$

The result follows. □

Exercise 1.14. *Given a real x and an integer $N > 1$, prove that there exist integers p and q with $0 < q \le N$ such that $|qx - p| < 1/N$.*

Solution. For $k = 0, 1, \ldots, N$ let $a_k = kx - [kx]$. Then $\{a_k\}_{k=0}^N \subset [0, 1)$, and therefore there exist $0 \le k_1, k_2 \le N$ such that $|a_{k_1} - a_{k_2}| < 1/N$. □

Exercise 1.15. *If x is irrational prove that there are infinitely many rational numbers p/q with $q > 0$ and such that $|x - p/q| < 1/q^2$.*

SOLUTION. Assume there are finitely many, say, $p_1/q_1, \ldots, p_n/q_n$. Then, by the preceding exercise, there exists p/q such that $|x - p/q| < 1/(qN)$ with $q \leq N$ and $1/N < \min\{|x - p_i/q_1| : 1 \leq i \leq n\}$. (The minimum is positive because x is irrational.) □

CHAPTER 2

Sequences, Series and Limits

EXERCISE 2.1. *Evaluate* $\lim_{n\to\infty}\prod_{k=0}^{n}(1+a^{2^k})$ *where* $a\in\mathbb{C}$.

SOLUTION. If $a\neq 1$, then for all $n\in\mathbb{N}$ we have

$$\prod_{k=0}^{n}(1+a^{2^k})=\frac{1-a^{2^{n+1}}}{1-a}.$$

Therefore the sequence converges to $1/(1-a)$ for $|a|<1$. It diverges for $|a|>1$ or $a=1$. The limit does not exist if $|a|=1$ and $a\neq 1$. □

EXERCISE 2.2. *Evaluate* $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$.

SOLUTION. Note that

$$\frac{n}{\sqrt{n^2+n}}\leq\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\leq 1.$$

Therefore the sum converges to 1. □

EXERCISE 2.3. *Let* $x=2+\sqrt{2}$ *and* $y=2-\sqrt{2}$. *Then* $n\in\mathbb{N}$ *implies*
(a) $x^n+y^n\in\mathbb{N}$ *and* $x^n+y^n=[x^n]+1$.
(b) $\lim_{n\to\infty}(x^n-[x^n])=1$.

SOLUTION. (a) By the Binomial Theorem, we have

$$x^n+y^n=\sum_{k=0}^{n}\binom{n}{k}2^{k+\frac{n-k}{2}}(1+(-1)^{n-k})=\sum_{\substack{0\leq k\leq n\\ n-k\text{ even}}}\binom{n}{k}2^{k+1+\frac{n-k}{2}}\in\mathbb{N}.$$

Since $x^n+y^n-1<x^n<x^n+y^n$, we conclude that $[x^n]=x^n+y^n-1$.
(b) By (a), $x^n-[x^n]=1-y^n\to 0$ as $n\to\infty$. □

EXERCISE 2.4. *If* $\{x_n\}_{n=1}^{\infty}\subset\mathbb{R}$, $\{y_n\}_{n=1}^{\infty}\subset(0,\infty)$ *and* $\{x_n/y_n\}_{n=1}^{\infty}$ *is monotone, then the sequence* $\{z_n\}_{n=1}^{\infty}$ *defined by*

$$z_n=\frac{x_1+\cdots+x_n}{y_1+\cdots+y_n}$$

is also monotone.

SOLUTION. Assume that $\{x_n/y_n\}_{n=1}^{\infty}$ is increasing and prove inductively that $z_n\leq z_{n+1}\leq x_{n+1}/y_{n+1}$ using the fact

$$\frac{a}{b}\leq\frac{c}{d}\Rightarrow\frac{a}{b}\leq\frac{a+c}{b+d}\leq\frac{c}{d}.$$

□

EXERCISE 2.5. *Let* $0 < a < b < \infty$. *Define*

$$x_1 = a, \quad x_2 = b, \quad x_{2n+1} = \sqrt{x_{2n}x_{2n-1}}, \quad x_{2n+2} = \frac{x_{2n} + x_{2n-1}}{2}.$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges.*

SOLUTION. Note that $[x_{2n+1}, x_{2n+2}] \subset [x_{2n-1}, x_{2n}]$ and

$$x_{2n+2} - x_{2n+1} \leq \frac{x_{2n} - x_{2n-1}}{2} \leq \cdots \leq \frac{x_2 - x_1}{2^{n-1}} \to 0.$$

Therefore the sequence converges and

$$\lim_{n\to\infty} x_n = \bigcap_{n=1}^{\infty} [x_{2n-1}, x_{2n}].$$

□

EXERCISE 2.6. *Let* $0 < a < b < \infty$. *Define*

$$x_1 = a, \quad x_2 = b, \quad x_{n+2} = \frac{x_n + x_{n+1}}{2}.$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ *converges and determine its limit.*

SOLUTION. Note that $x_{n+1} - x_n = (-1/2)^{n-1}(x_2 - x_1)$. Therefore

$$x_n = x_1 + (x_2 - x_1)\sum_{k=0}^{n-2}(-\frac{1}{2})^k \to a + (b-a)\frac{2}{3} = \frac{a+2b}{3}.$$

□

EXERCISE 2.7. *Let* $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *satisfy* $0 < x_n < 1$ *and* $4x_{n+1}(1 - x_n) \geq 1$ *for all* $n \in \mathbb{N}$. *Show that* $\lim_{n\to\infty} x_n = 1/2$.

SOLUTION. Note that

$$x_{n+1} \geq \frac{1}{4(1 - x_n)} \geq x_n.$$

Therefore the sequence is increasing. Since it is bounded, it converges to a limit l which must satisfy $4l(1 - l) \geq 1$. We conclude that $l = 1/2$. □

EXERCISE 2.8. *Let* $1 < a < \infty$, $x = 1$, *and* $x_{n+1} = a(1 + x_n)/(a + x_n)$. *Show that* $x_n \to \sqrt{a}$.

SOLUTION. Prove inductively that the sequence is decreasing and bounded from below by $\sqrt{a}$. □

EXERCISE 2.9. *Define* $x_0 = 0$, $x_1 = 1$, *and*

$$x_{n+1} = \frac{1}{n+1}x_{n-1} + \frac{n}{n+1}x_n.$$

Prove that $\{x_n\}_{n=1}^{\infty}$ *converges and determine its limit.*

SOLUTION. Note that $x_{n+1} - x_n = (-1)^n/(n + 1)!$, and so

$$x_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \to \frac{1}{e}.$$

□

EXERCISE 2.10. *Let* $a \in \mathbb{R}$, $a \notin \{0, 1, 2\}$ *and define* $x_1 = a$, $x_{n+1} = 2 - 2/x_n$ *for* $n \in \mathbb{N}$. *Find the limit points of the sequence* $\{x_n\}_{n=1}^{\infty}$.

Solution. Note that $x_{n+4} = x_n$ for all $n \in \mathbb{N}$. Therefore the sequence takes on the values $\{x_1, x_2, x_3, x_4\}$ only. □

Exercise 2.11. *For* $n \in \mathbb{N}$, *write* $n = 2^{j-1}(2k-1)$ *where* $j, k \in \mathbb{N}$ *and write*

$$S_n = \frac{1}{j} + \frac{1}{k}.$$

Find all limit points of the sequence $\{S_n\}_{n=1}^{\infty}$. *Evaluate* $\underline{\lim}\, S_n$ *and* $\overline{\lim}\, S_n$.

Solution. Let A be the set of limit points of $\{S_n\}_{n=1}^{\infty}$. We claim that $A = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Indeed, let $n_k = 2^{k-1}(2k-1)$ and $m_{p,k} = 2^{p-1}(2k-1)$.Then

$$S_{n_k} = \frac{2}{k} \to 0, \quad S_{m_{p,k}} = \frac{1}{p} + \frac{1}{k} \to \frac{1}{p} \quad \text{as } k \to \infty.$$

Hence $A \supset \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Now take $l \in A$, $l \neq 0$. Then there exists a subsequence $\{S_{n_m}\}_{m=1}^{\infty}$ such that $S_{n_m} \to l$. Write $n_m = 2^{j_m-1}(2k_m - 1)$. Note that at least one of the sets $\{j_m : m \in \mathbb{N}\}$, $\{k_m : m \in \mathbb{N}\}$ is unbounded, and so we may assume, without loss of generality, that there exists $\{j_{m_i}\}_{i=1}^{\infty}$ with $j_{m_i} \to \infty$. Then, since $S_{n_{m_i}} \to l$, we have $k_{m_i} \to 1/l$. Therefore $\{k_{m_i}\}_{i=1}^{\infty}$ is eventually constant and $l \in \{1/k_{m_i} : i \in \mathbb{N}\}$. $\underline{\lim}\, S_n = \inf A = 0$, $\overline{\lim}\, S_n = \sup A = 1$. □

Exercise 2.12. *Prove that* $(n/e)^n < n!$ *for all* $n \in \mathbb{N}$.

Solution. Induction on n. It is clearly true for $n = 1$. Assuming $(n/e)^n < n!$ we have

$$\left(\frac{n+1}{e}\right)^{n+1} = \frac{n+1}{e}\left(1 + \frac{1}{n}\right)^n \left(\frac{n}{e}\right)^n < \frac{n+1}{e}\, e\, n! = (n+1)!.$$

□

Exercise 2.13. *Evaluate*
(a) $\lim_{n\to\infty} ((2n)!/(n!)^2)^{1/n}$,
(b) $\lim_{n\to\infty} (1/n)[(n+1)(n+2)\cdots(n+n)]^{1/n}$,
(c) $\lim_{n\to\infty} [(2/1)(3/2)^2(4/3)^3 \cdots ((n+1)/n)^n]^{1/n}$.

Solution. Let

$$a_n = \frac{(2n)!}{(n!)^2}, \quad b_n = \frac{(n+1)(n+2)\cdots(n+n)}{n^n},$$

$$c_n = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n.$$

Then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+2)}{(n+1)^2} \to 4, \quad \frac{b_{n+1}}{b_n} = \left(\frac{n}{n+1}\right)^n \frac{(2n+1)(2n+2)}{(n+1)^2} \to \frac{4}{e},$$

$$\frac{c_{n+1}}{c_n} = \left(1 + \frac{1}{n+1}\right)^{n+1} \to e.$$

Therefore

$$\sqrt[n]{a_n} \to 4, \quad \sqrt[n]{b_n} \to \frac{4}{e}, \quad \sqrt[n]{c_n} \to e.$$

□

Exercise 2.14. *Evaluate* $\lim n \to \infty(\sqrt[n]{n} - 1)^n$.

SOLUTION. Since $\sqrt[n]{n} \to 1$, there exists $n_0 \in \mathbb{N}$ such that $0 < \sqrt[n]{n} - 1 < 1/2$ for all $n \geq n_0$, and so $0 < (\sqrt[n]{n} - 1)^n < (1/2)^n$. Therefore $0 \leq \underline{\lim}(\sqrt[n]{n} - 1)^n \leq \overline{\lim}(\sqrt[n]{n})^n \leq 0$. We conclude that $\lim_{n\to\infty}(\sqrt[n]{n} - 1)^n = 0$. □

EXERCISE 2.15. *If* $\{x_n\}_{n=1}^{\infty} \subset (0, \infty)$ *and* $x_n \to x$, *then* $(x_1 \cdots x_n)^{1/n} \to x$.

SOLUTION. By the Harmonic-Geometric-Arithmetic Means Inequality we have

$$\frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} \leq (x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

Therefore $(x_1 \cdots x_n)^{1/n} \to x$. □

EXERCISE 2.16. *(a) Let* $S_n = \sum_{k=1}^{n} 1/k$ *for* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty} |S_{n+p} - S_n| = 0$ *for all* $p \in \mathbb{N}$, *but* $\{S_n\}_{n=1}^{\infty}$ *diverges to* ∞.
(b) Find a divergent sequence $\{x_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *such that* $\lim_{n\to\infty} |x_{n^2} - x_n| = 0$.

SOLUTION. (a) $|S_{n+p} - S_n| = 1/(n+1) + \cdots + 1/(n+p) \leq p/(n+1) \to 0$

(b) For $n \geq 4$ let $k(n)$ be the unique integer such that $2^{2^{k(n)}} \leq n < 2^{2^{k(n)+1}}$ and define $x_n = \sum_{j=1}^{k(n)} 1/j$. Note that $k(n) \to \infty$ and $k(n^2) = k(n) + 1$. Therefore $x_n \to \infty$ and $|x_{n^2} - x_n| = 1/(k(n) + 1) \to 0$. □

EXERCISE 2.17. *There exist two divergent series* $\sum a_n$ *and* $\sum b_n$ *of positive terms with* $a_1 \geq a_2 \geq \cdots$ *and* $b_1 \geq b_2 \geq \cdots$ *such that if* $c_n = \min\{a_n, b_n\}$, *then* $\sum c_n$ *converges.*

SOLUTION. Let

$$a_k = 1/2^k, \quad b_k = 1/2^n \quad \text{if } 2^n \leq k < 2^{n+1}, n \text{ even}$$

and

$$a_k = 1/2^n, \quad b_k = 1/2^k \quad \text{if } 2^n \leq k < 2^{n+1}, n \text{ odd}.$$

□

EXERCISE 2.18. *Evaluate the sums*
(a) $\sum_{n=1}^{\infty} 1/(n(n+1)(n+2))$,
(b) $\sum_{n=1}^{\infty} (n-1)!/(n+p)!$, *where* $p \in \mathbb{N}$ *is fixed.*

SOLUTION. (a) Note that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right].$$

Consequently

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right] \to \frac{1}{4}.$$

(b) We have

$$\frac{(n-1)!}{(n+p)!} = \frac{1}{n\cdots(n+p)} = \frac{1}{p}\left[\frac{1}{n\cdots(n+p-1)} - \frac{1}{(n+1)\cdots(n+p)}\right].$$

Therefore

$$\sum_{k=1}^{n} \frac{(k-1)!}{(k+p)!} = \frac{1}{p}\left[\frac{1}{p!} - \frac{1}{(n+1)\cdots(n+p)}\right] \to \frac{1}{p\,p!}.$$

□

EXERCISE 2.19. *Let $\sum a_n$ be a convergent series of nonnegative terms. Then*
(a) $\underline{\lim}\, na_n = 0$,
(b) possibly $\overline{\lim}\, na_n > 0$,
(c) if $a_n \geq a_{n+1}$ for all $n > n_0$, then $\lim na_n = 0$.

SOLUTION. (a) Suppose that $\underline{\lim}\, na_n > c > 0$ for some c. Then there exists $n_0 \in \mathbb{N}$ such that $na_n > c$ for $n \geq n_0$. Consequently,

$$\sum_{n=n_0}^{N} a_n > c \sum_{n=n_0}^{N} \frac{1}{n} \to \infty \quad \text{as} \quad n \to \infty,$$

a contradiction.

(b) Let $a_k = 1/2^k$ if $k \neq 2^n$ and $a_k = 1/2^n$ if $k = 2^n$. Then

$$\sum_{k=1}^{N} a_k = \sum_{k \neq 2^n} a_k + \sum_{k=2^n} a_k \leq \sum_{k=1}^{N} \frac{1}{2^k} + \sum_{n:2^n \leq N} \frac{1}{2^n} \leq 2 \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$$

and $\lim\limits_{n\to\infty} 2^n a_{2^n} = 1$.

(c) Note that

$$na_{2n} \leq \sum_{k=n+1}^{2n} a_k \to 0 \quad \text{and} \quad na_{2n+1} \leq \sum_{k=n+2}^{2n+1} a_k \to 0.$$

Therefore $\lim\limits_{n\to\infty} 2na_{2n} = \lim\limits_{n\to\infty}(2n+1)a_{2n+1} = 0$. We conclude that $\lim na_n = 0$. □

EXERCISE 2.20. *If $\{c_m\}_{m=1}^{\infty} \subset [0, \infty]$ and*

$$b_n = \frac{1}{n(n+1)} \sum_{m=1}^{n} mc_m,$$

then

$$\sum_{n=1}^{\infty} b_n = \sum_{m=1}^{\infty} c_m.$$

SOLUTION. Define

$$a_{m,n} = \begin{cases} \frac{mc_m}{n(n+1)} & \text{if } 1 \leq m \leq n, \\ 0 & \text{if } m > n. \end{cases}$$

Then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} mc_m = \sum_{n=1}^{\infty} b_n.$$

On the other hand

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} mc_m \sum_{n=m}^{\infty} \frac{1}{n(n+1)} = \sum_{m=1}^{\infty} c_m.$$

□

EXERCISE 2.21. *(a) Prove that $\sum\limits_{n=1}^{\infty} 1/n^2 < 2$.*

(b) Prove that

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{(m+n)^2} \right) = \infty.$$

Solution. (a) We have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1 + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 2.$$

(b) We have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^2} = \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \frac{1}{n^2} \geq \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$
$$= \sum_{m=1}^{\infty} \frac{1}{m+1} = \infty.$$

□

Exercise 2.22. *Let b be an integer > 1 and let d be a digit $(0 \leq d < b)$. Let A denote the set of all $k \in \mathbb{N}$ such that the b-adic expansion of k fails to contain the digit d.*

(a) If $a_k = 1/k$ for $k \in A$ and $a_k = 0$ otherwise, then $\sum_{k=1}^{\infty} a_k < \infty$.

(b) For $n \in \mathbb{N}$ let $A(n)$ denote the number of elements of A that are $\leq n$. Then $\lim_{n\to\infty}(A(n)/n) = 0$.

Solution. Let

$$A_n = \{k : k \text{ is an } n\text{-digit number and does not contain the digit } d\}$$
$$= \{k : b^{n-1} \leq k < b^n\} \cap A.$$

Note that $|A_n| = (b-2)(b-1)^{n-1}$.

(a) We have

$$\sum_{k=1}^{\infty} a_k = \sum_{n=1}^{\infty} \sum_{k\in A_n} a_k \leq \sum_{n=1}^{\infty} \frac{|A_n|}{b^{n-1}} = (b-2) \sum_{n=1}^{\infty} \left(\frac{b-1}{b} \right)^{n-1} < \infty.$$

(b) If $b \neq 2$ then

$$A(n) \leq \sum_{k:b^{k-1}\leq n} |A_k| = (b-2) \sum_{k:b^k \leq n} (b-1)^k \leq n^{1/\log_{b-1} b} - 1.$$

If $b = 2$ then $A(n) = |\{k : 2^k \leq n\}| \leq \log_2 n$. Therefore $\lim_{n\to\infty}(A(n)/n) = 0$. □

Exercise 2.23. *Let $0 < x < 1$. Then x has a terminating decimal expansion if and only if there exist nonnegative integers m and n such that $2^m 5^n x$ is an integer.*

Solution. If x has a terminating decimal expansion, then $x = p/10^k = p/(2^k 5^k)$. Conversely, if $2^m 5^n x = N \in \mathbb{N}$ for some, say, $m \leq n$, then $x = 2^{n-m} N/10^n$. □

Exercise 2.24. *Evaluate $\lim_{n\to\infty}(n!e - [n!e])$.*

Solution. Let $S_n = \sum_{k=0}^{n} 1/k!$. Then, using the error estimate for the "tail", we have $0 < n!e - n!S_n < 1/n$. We conclude that $[n!e] = n!S_n$ and therefore $n!e - [n!e] \to 0$. □

Exercise 2.25. *Show that $\lim_{n\to\infty} n \sin(2\pi e n!) = 2\pi$.*

Solution. Since $\lim_{n\to\infty}(en! - [en!]) = 0$ we have

$$\lim_{n\to\infty}\frac{\sin(2\pi en! - 2\pi[en!])}{2\pi en! - 2\pi[en!]} = 1 \Rightarrow \lim_{n\to\infty}\frac{\sin(2\pi en!)}{en! - [en!]} = 2\pi.$$

Note that the error estimate for the Maclaurin series expansion of e implies $1/(n+1) < en! - [en!] < 1/n$, and so $\lim_{n\to\infty} n(en! - [en!]) = 1$. It follows that

$$n\sin(2\pi en!) = n(en! - [en!])\frac{\sin(2\pi en!)}{en! - [en!]} \to 2\pi.$$

□

Exercise 2.26. *Find the sum of the series*

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}.$$

Solution.

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{n}}{n} - \frac{\sqrt{n+1}}{n+1}\right) = 1.$$

□

Exercise 2.27. *Let $a_n > 0$ for each $n \in \mathbb{N}$. Then*
(a) $\sum_{n=1}^{\infty} a_n < \infty$ implies $\sum_{n=1}^{\infty}\sqrt{a_n a_{n+1}} < \infty$,
(b) the converse of (a) is false,
(c) $\sum_{n=1}^{\infty} a_n < \infty$ implies $\sum_{n=1}^{\infty}(a_n^{-1} + a_{n+1}^{-1})^{-1} < \infty$,
(d) the converse of (c) is false.

Solution. By the Harmonic-Geometric-Arithmetic Means Inequality, we have

$$2(a_n^{-1} + a_{n+1}^{-1})^{-1} \le \sqrt{a_n a_{n+1}} \le \frac{1}{2}(a_n + a_{n+1}),$$

proving (a) and (c). For (b) and (d), let $a_n = 1/n$ if n is even and $a_n = 1/n^3$ if n is odd. □

Exercise 2.28. *Suppose that $d_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} = \infty$. What can be said of the following series?*
(a) $\sum_{n=1}^{\infty} d_n/(1 + d_n)$,
(b) $\sum_{n=1}^{\infty} d_n/(1 + nd_n)$,
(c) $\sum_{n=1}^{\infty} d_n/(1 + d_n^2)$.

Solution. (a) If $\{d_n\}_{n=1}^{\infty}$ is bounded then $1/(1 + d_n)$ is bounded from below, therefore

$$\sum_{n=1}^{\infty}\frac{d_n}{1 + d_n} \ge C\sum_{n=1}^{\infty} d_n = \infty.$$

If $\{d_n\}_{n=1}^{\infty}$ is unbounded then there exists a subsequence $\{d_{k_n}\}_{n=1}^{\infty}$ with $d_{k_n} \to \infty$. Therefore there exists n_0 such that $d_{k_n}/(1+d_{k_n}) > 1/2$ for all $n \ge n_0$. Consequently $\sum_{n=1}^{\infty} d_n/(1+d_n) = \infty$.

(b) Let $d_n = 1$ for all $n \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty} \frac{d_n}{1 + nd_n} = \infty.$$

Let $d_k = 1/2^k$ if $k \neq 2^n$ and $d_k = 2^n$ if $k = 2^n$. Then $\sum\limits_{n=1}^{\infty} d_n = \infty$ and

$$\frac{d_k}{1 + kd_k} = \begin{cases} \frac{1}{k+2^k} & \text{if } k \neq 2^n, \\ \frac{2^n}{1+4^n} & \text{if } k = 2^n. \end{cases}$$

Therefore $\sum\limits_{n=1}^{\infty} d_n/(1 + nd_n) < \infty$.

(c) Let $d_n = 1$ for all n. Then $\sum\limits_{n=1}^{\infty} d_n/(1 + d_n^2) = \infty$. Let $d_n = n^2$. Then $\sum\limits_{n=1}^{\infty} d_n/(1 + d_n^2) < \infty$. □

Exercise 2.29. *Let $0 < a < b < \infty$ and define $x_1 = a$, $x_2 = b$, and $x_{n+2} = \sqrt{x_n x_{n+1}}$ for $n \in \mathbb{N}$. Find $\lim\limits_{n\to\infty} x_n$.*

Solution. Let $y_n = \log x_n$ and use Exercise 2.6. □

Exercise 2.30. *Let $0 < a < b < \infty$ and define $x_1 = a$, $y_1 = b$, $x_{n+1} = 2(x_n^{-1} + y_n^{-1})^{-1}$, and $y_{n+1} = \sqrt{x_n y_n}$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge and have the same limit.*

Solution. Prove inductively, using the Harmonic-Geometric Means Inequality, that

$$a < x_n \leq x_{n+1} \leq y_{n+1} \leq y_n < b \quad \text{and} \quad y_{n+1} - x_{n+1} \leq \frac{1}{2}(y_n - x_n).$$

□

Exercise 2.31. *Show that if $\sum\limits_{k=1}^{\infty} a_k = 1$ and $0 < a_n \leq \sum\limits_{k=n+1}^{\infty}$, $n = 1, 2, \ldots$, then for every $x \in (0, 1)$ there is a subseries $\sum\limits_{k=1}^{\infty} a_{n_k}$ whose sum is x.*

Solution. Note that, since the sum of the series is 1 and $x \in (0, 1)$, there exists $n_1 \in \mathbb{N}$ such that

$$\sum_{k=n_1}^{\infty} a_k > x \quad \text{and} \quad \sum_{k=n_1+1}^{\infty} a_k \leq x$$

implying

$$\sum_{k=n_1+1}^{\infty} a_k > x - a_{n_1} \quad \text{and} \quad a_{n_1} \leq x.$$

Therefore there exists $n_2 > n_1$ such that

$$\sum_{k=n_2}^{\infty} a_k > x - a_{n_1} \quad \text{and} \quad \sum_{k=n_2+1} \leq x - a_{n_1}.$$

Continuing this way, we can find a sequence of integers $n_1 < n_2 < \cdots$ such that

$$0 \leq x - \sum_{k=1}^{m} a_{n_k} < \sum_{k=n_m+1}^{\infty} a_k.$$

Letting $m \to \infty$, we conclude that $\sum\limits_{k=1}^{\infty} a_{n_k} = x$. □

EXERCISE 2.32. *Show that if* $a_n, b_n \in \mathbb{R}$, $(a_n+b_n)b_n \neq 0$, $n = 1, 2, \ldots$, *and both* $\sum_{n=1}^{\infty} a_n/b_n$ *and* $\sum_{n=1}^{\infty}(a_n/b_n)^2$ *converge, then* $\sum_{n=1}^{\infty} a_n/(a_n + b_n)$ *converges.*

SOLUTION. Choose $k_0 \in \mathbb{N}$ such that $|1 + a_k/b_k| \geq 1/2$ for all $k \geq k_0$. Then

$$\frac{1}{|a_k b_k + b_k^2|} \leq \frac{2}{|b_k|^2}.$$

Note that

$$\sum_{k=k_0}^{n} \frac{a_k}{a_k + b_k} = \sum_{k=k_0}^{n} \frac{a_k}{b_k} - \sum_{k=k_0}^{n} \frac{a_k^2}{a_k b_k + b_k^2}$$

and

$$\sum_{k=k_0}^{n} \left| \frac{a_k^2}{a_k b_k + b_k^2} \right| \leq 2 \sum_{k=k_0}^{n} \left| \frac{a_k}{b_k} \right|^2.$$

We conclude that $\sum_{n=1}^{\infty} a_n/(a_n + b_n)$ converges. □

EXERCISE 2.33. *Show that if* $b_n \searrow 0$ *and* $\sum_{n=1}^{\infty} b_n = \infty$, *then there is a sequence* $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ *such that* $a_n/b_n \to 1$ *as* $n \to \infty$ *and* $\sum_{n=1}^{\infty}(-1)^n a_n$ *diverges.*

SOLUTION. Let

$$S_n = \sum_{k=1}^{n} b_k, \quad a_n = b_n + (-1)^n \frac{b_n}{S_n}.$$

Note that $a_n > 0$ for large n and

$$\sum_{n=1}^{m}(-1)^n a_n = \sum_{n=1}^{m}(-1)^n b_n + \sum_{n=1}^{m} \frac{b_n}{S_n}.$$

The first series in the above sum converges, being alternating, while the second diverges by Abel's Theorem. Therefore $\sum_{n=1}^{\infty} a_n$ diverges. On the other hand, $a_n/b_n = 1 + (-1)^n/S_n \to 1$ as $n \to \infty$. □

EXERCISE 2.34. *Show that if* $n \geq 2$, *then* $\sum_{k=1}^{\infty}(1 - (1 - 2^{-k})^n) \simeq \log n$.

SOLUTION. Note that

$$\frac{1}{m+1} = \int_0^1 x^m dx = \sum_{k=0}^{\infty} \int_{1-1/2^k}^{1-1/2^{k+1}} x^m dx \leq \sum_{k=1}^{\infty} \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)^m$$

and similarly

$$\sum_{k=1}^{\infty} \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)^m \leq \frac{2}{m+1}.$$

Therefore

$$\sum_{k=1}^{\infty}\left(1 - \left(1 - \frac{1}{2^k}\right)^n\right) = \sum_{k=1}^{\infty}\sum_{m=0}^{n-1} \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)^m = \sum_{m=0}^{n-1}\sum_{k=1}^{\infty} \frac{1}{2^k}\left(1 - \frac{1}{2^k}\right)^m$$

$$\simeq \sum_{m=1}^{n} \frac{1}{m} \simeq \log n.$$

□

EXERCISE 2.35. *Show that if* $r_n \in \mathbb{R}$*, then* $\lim_{n\to\infty} \int_0^\infty e^{-x}(\sin(x+r_n))^n dx = 0$.

SOLUTION.

$$\begin{aligned}\int_0^\infty |e^{-x}(\sin(x+r_n))^n| dx &= \int_0^\infty e^{-x}|\sin(x + r_n \bmod 2\pi)|^n dx \\ &= e^{r_n \bmod 2\pi} \int_{r_n \bmod 2\pi}^\infty e^{-x}|\sin(x)|^n dx \\ &\le e^{2\pi}\int_0^\infty e^{-x}|\sin(x)|^n dx.\end{aligned}$$

Note that $|\sin(x)|^n \to 0$ almost everywhere, and so, by the Dominated Convergence Theorem, $\int_0^\infty e^{-x}|\sin(x)|^n dx \to 0$. □

EXERCISE 2.36. *Let* $f : [0,1] \to \mathbb{R}$ *be defined by*

$$f(x) = \begin{cases} \frac{x\log x}{x-1} & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1. \end{cases}$$

Show that

$$\int_0^1 f(x)dx = 1 - \sum_{n=2}^\infty \frac{1}{n^2(n-1)}.$$

SOLUTION. Note that

$$\frac{x\log x}{x-1} = \sum_{n=0}^\infty \frac{x(1-x)^n}{n+1}$$

and the convergence is uniform on $[0,1]$ by Weierstrass M-test. Therefore

$$\begin{aligned}\int_0^1 f(x)dx &= \sum_{n=0}^\infty \frac{1}{n+1}\int_0^1 x(1-x)^n dx = \sum_{n=0}^\infty \frac{1}{(n+1)^2(n+2)} \\ &= 1 - \sum_{n=2}^\infty \frac{1}{n^2(n-1)}.\end{aligned}$$

□

EXERCISE 2.37. *Show that*

$$\lim_{n\to\infty}\sum_{j=n}^{kn}\frac{1}{j} = \log k.$$

Conclude that

$$\sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} = \log 2.$$

SOLUTION. Note that

$$\int_j^{j+1}\frac{dx}{x} \le \frac{1}{j} \le \int_j^{j+1}\frac{dx}{x-1}.$$

Therefore

$$\int_n^{kn+1}\frac{dx}{x} \le \sum_{j=n}^{kn}\frac{1}{j} \le \int_n^{kn+1}\frac{dx}{x-1},$$

and consequently

$$\log\left(k+\frac{1}{n}\right) \le \sum_{j=n}^{kn} \frac{1}{j} \le \log\left(k+\frac{k}{n-1}\right).$$

Taking the limit as $n \to \infty$, we obtain the first assertion. To prove the second assertion, note that

$$\sum_{j=1}^{2n} \frac{(-1)^{j+1}}{j} = \sum_{j=1}^{2n} \frac{1}{j} - 2\sum_{j=1}^{n} \frac{1}{2j} = \sum_{j=n+1}^{2n} \frac{1}{j} = \sum_{j=n}^{2n} \frac{1}{j} - \frac{1}{n} \to \log 2, \text{ as } n \to \infty.$$

Since $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}$ converges by Leibniz, we conclude that $\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = \log 2$. □

Exercise 2.38. *Show that $e^{x^2/2} \int_x^\infty e^{-t^2/2} dt$ is a decreasing function of x on $[0, \infty)$ and that its limit as $x \to \infty$ is 0.*

Solution. By L'Hospital's Rule we have

$$\lim_{x\to\infty} \frac{\int_x^\infty e^{-t^2/2} dt}{e^{-x^2/2}} = \lim_{x\to\infty} \frac{-e^{-x^2}}{-xe^{-x^2/2}} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Now let

$$g(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt \quad \text{and} \quad h(x) = \frac{e^{-x^2/2}}{x} - \int_x^\infty e^{-t^2/2} dt.$$

Then

$$g'(x) = xe^{x^2/2} \int_x^\infty e^{-t^2/2} dt - 1 \quad \text{and} \quad h'(x) = -\frac{e^{-x^2/2}}{x^2} < 0.$$

Hence h is strictly decreasing. Note that $\lim_{x\to\infty} h(x) = 0$. therefore $h(x) > 0$ and consequently $g'(x) < 0$. □

CHAPTER 3

Topology

EXERCISE 3.1. *Let X be a 2nd countable space. Show that if $\{G_i\}_{i\in I}$ is an arbitrary family of open sets in X then there exists a countable subset $J \subset I$ such that $\bigcup_{i\in I} G_i = \bigcup_{i\in J} G_i$.*

SOLUTION. Suppose $\{U_k\}_{k\in\mathbb{N}}$ is a basis for the topology of X. Let

$$K = \{k \in \mathbb{N} : \exists\, i(k) \in I \text{ such that } U_k \subset G_{i(k)}\}$$

and put $J = \{i(k) : k \in K\}$. □

EXERCISE 3.2. *Let X be a 2nd countable space, and let $A \subset X$ be an uncountable set. Prove that A has at least one condensation point.*

SOLUTION. Suppose that for each $x \in A$ there is an open set $U_x \subset X$ with $x \in U_x$ and $|A \cap U_x| \le \aleph_0$. Since X is 2nd countable there exists $\{x_n\}_{n=1}^{\infty} \subset A$ such that $\bigcup_{x\in A} U_x = \bigcup_{n=1}^{\infty} U_{x_n}$. Hence $A = \bigcup_{n=1}^{\infty}(U_{x_n} \cap A)$ and therefore $U_{x_{n_0}} \cap A$ must be uncountable for some n_0, a contradiction. □

EXERCISE 3.3. *If X is a 2nd countable space and A is a closed subset of X, then there exist a perfect set P and a countable set N, such that $A = P \cup N$. Conclude that any subset of a 2nd countable space can have only countably many isolated points.*

SOLUTION. Let $P = \{x \in X : \text{for each nbd } U_x \text{ of } x,\ U_x \cap A \text{ is uncountable}\}$. Using the preceding exercise, P is perfect and $A \setminus P$ is countable. □

EXERCISE 3.4. *Prove the following assertions.*

(a) *If A is nonempty perfect subset of a complete metric space then A is uncountable.*
(b) *Any countable closed subset of a complete metric space has infinitely many isolated points.*
(c) *There exists a countable closed subset of $\mathbb{R}$ having infinitely many limit.points.*

SOLUTION. Suppose X is a complete metric space.

(a) Note that since A is a closed subset of X, it is complete as a metric space. If A is countable then by the Baire category theorem, at least one of its points must be isolated.

(b) Assume that there exists a countable closed subset of X with finitely many isolated points. Removing these points results in a countable perfect set, contradicting (a).

(c) Take infinite copies of a convergent sequence together with its limit. □

EXERCISE 3.5. *It is impossible to express $[0, 1]$ as a union of disjoint closed nondegenerate intervals of length < 1.*

SOLUTION. Suppose $[0, 1] = \bigcup_{i\in I}[x_i, y_i]$, where $\{[x_i, y_i]\}_{i\in I}$ is disjoint. Note that I must be countable. Then the set of endpoints $(\{x_i : i \in I\} \cup \{y_i : i \in I\}) \setminus \{0, 1\}$ is a countable perfect set, a contradiction by the preceding exercise. □

Exercise 3.6. *It is impossible to express* $[0,1]$ *as a countable union of disjoint closed sets.*

Solution. Suppose $[0,1] = \bigcup_{n=1}^{\infty} F_n$ with the F_n's closed and pairwise disjoint. Since $F_1 \cap F_2 = \emptyset$, we can find a closed interval I_1 such that $I_1 \cap F_1 = \emptyset$, $I_1 \cap F_2 \neq \emptyset$, $I_1 \setminus F_2 \neq \emptyset$. We repeat the same procedure inside I_1 with $I_1 \cap F_2$ playing the role of F_1 and $I_1 \cap F_k$ playing the role of F_2, where F_k is the first set in the sequence $\{F_n\}_{n=3}^{\infty}$ intersecting I_1. We thereby construct a decreasing sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ such that $I_n \cap F_n = \emptyset$, a contradiction. □

Exercise 3.7. *Let A be a bounded subset of* $\mathbb{R}$ *which is not closed. Construct explicitly an open cover of A that has no finite subcover.*

Solution. Let $x \in \mathbb{R} \setminus A$ be a point such that $(x - \epsilon, x + \epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$. For each n choose $x_n \in (x - 1/n, x + 1/n) \cap A$. Without loss of generality we may assume that $\{x_n\}_{n=1}^{\infty}$ is monotone. If $x_1 < \cdots < x_n < \cdots x$, consider the cover $\{(-\infty, x_n)\}_{n=1}^{\infty} \cup \{(x, \infty)\}$. If $x_1 > \cdots > x_n > \cdots x$, then take the covering $\{(x_n, \infty)\}_{n=1}^{\infty} \cup \{(-\infty, x)\}$. □

Exercise 3.8. *Let* (X, ρ) *be a metric space and* $A, B \subset X$ *disjoint closed sets. Show that there exists a continuous function* $f : X \to \mathbb{R}$ *such that* $f \mid A = 0$ *and* $f \mid B = 1$.

Solution. Let

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}.$$

Then f is well-defined and has the required properties. □

Exercise 3.9. *If X is a connected metric space with at least two points, then X is uncountable.*

Solution. Let $x, y \in X$ be two distinct points. By the preceding exercise, there exists a continuous function $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f(y) = 1$. Since X is connected, f has the intermediate value property. Therefore $[0,1] \subset f(X)$. We conclude that X is uncountable. □

Exercise 3.10. *Let S be a nonempty closed subset of* $\mathbb{R}$ *and let* $f : S \to \mathbb{R}$ *be continuous. Then there exists a continuous* $g : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = g(x)$ *for all* $x \in S$ *and* $\sup_{x \in \mathbb{R}} |g(x)| = \sup_{x \in S} |f(x)|$. *This is false for every nonclosed* $S \subset \mathbb{R}$.

Solution. Write $\mathbb{R} \setminus S = \bigcup_{n=1}^{\infty} I_n$, where the I_n's are disjoint open intervals and extend f on each I_n linearly (if $(-\infty, a)$ or (a, ∞) appear among the I_n's take f to be constant on these intervals). If S is not closed we can find a point $x \notin S$ and, say, an increasing sequence $x_1 < \cdots < x_n < \cdots < x$ of points in S such that $\lim_n x_n = x$. Any continuous function f on $\mathbb{R} \setminus \{x\}$, and therefore on S, with $f(x_n) = n$ cannot be extended to the whole line. □

Exercise 3.11. *Let X be a topological space, Y a metric space,* $f : X \to Y$ *an arbitrary function and define* $A_f = \{x \in X : f \text{ is continuous at } x\}$.

(a) *Prove that* A_f *is a* G_δ *set.*
(b) *Assume that there exists a set* $D \subset X$ *such that D and* $X \setminus D$ *are both dense in X. Prove that for any* G_δ *set* $G \subset X$ *there exists a function* $f : X \to \mathbb{R}$ *such that* $A_f = G$.
(c) *Show that there is no function* $f : \mathbb{R} \to \mathbb{R}$ *which is continuous at each rational and discontinuous at each irrational.*

(d) *Construct explicitly a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at each irrational and discontinuous at each rational.*

Proof. For any point $x \in X$ define the oscilation of f at x by

$$w_f(x) = \inf\{\text{diam}(f(U)) : U \text{ is a nbd of } x\}.$$

(a) Note that $x \in A_f$ if and only if $w_f(x) = 0$. Therefore

$$A_f = \bigcap_{n=1}^{\infty} \{x \in X : w_f(x) < 1/n\}.$$

The sets in the intersection are open, hence A_f is G_δ.

(b) Write $G = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open and $X = G_1 \supset G_2 \supset \cdots$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in G, \\ 1/n & \text{if } x \in D \cap (G_n \setminus G_{n+1}), \\ -1/n & \text{if } x \in (X \setminus D) \cap (G_n \setminus G_{n+1}) \end{cases}.$$

(c) If such a function existed, $\mathbb{Q}$ would be G_δ by (a).

(d)

$$f(x) = \begin{cases} 1/n & \text{if } x = m/n, (m, n) = 1, \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

□

Exercise 3.12. *Construct a strictly increasing function that is continuous at each irrational and discontinuous at each rational.*

Solution. Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of the rationals and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{r_n < x} 1/2^n$. Note that $f(r_n-) = f(r_n) = f(r_n+) - 1/2^n$ and $f(x-) = f(x) = f(x+)$ for all $x + in\mathbb{R} \setminus \mathbb{Q}$. □

Exercise 3.13. *Let X be a topological space and (Y, ρ) a metric space. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions from X into Y and that $f : X \to Y$ is some function such that $\lim_n f_n(x) = f(x)$ for all $x \in X$.*

(a) *Show that there exists a set $E \subset X$ that is of 1st category in X such that f is continuous at each point of $X \setminus E$. In particular, if X is a complete metric space, then f is continuous at every point of a dense subset of X.*

(b) *$f^{-1}(V)$ is an F_σ set in X for every open $V \subset Y$.*

(c) *There is no sequence $\{f_n\}_{n=1}^{\infty}$ of continuous real functions on $\mathbb{R}$ such that $f_n(x) \to 1$ for $x \in \mathbb{Q}$ and $f_n(x) \to 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$.*

(d) *Show that $\chi_{\mathbb{Q}}$, the characteristic function of $\mathbb{Q}$, is the pointwise limit of a sequence of functions, so that each of them is the pointwise limit of a sequence of continuous functions.*

Solution. (a) Let $A_{k,m} = \{x \in X : \rho(f_m(x), f_n(x)) \le 1/k \text{ for all } n \ge m\}$. Then each $A_{k,m}$ is closed, and so $A_{k,m} \setminus A^\circ_{k,m}$ is nowhere dense. Now let

$$G = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} A^\circ_{k,m}, \quad E = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} (A_{k,m} \setminus A^\circ_{k,m}).$$

Then E is of 1st category, $X \setminus G \subset E$ (since $X = \bigcup_{m=1}^{\infty} A_{k,m}$ for all k), and each $x \in G$ is a point of continuity of f.

(b) $f^{-1}(V) = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{x \in X : \rho(f_n(x), Y \setminus V) \ge 1/k \text{ for all } n \ge m\}$.

(c) If such a sequence existed then the characteristic function of $\mathbb{Q}$ would be continuous at some point by (a).

(d) Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = |2x - 2k - 1|$ for $x \in [k, k+1]$, $k \in \mathbb{Z}$. Then

$$\lim_{m\to\infty}\left[\lim_{n\to\infty}\phi(m!x)^n\right] = \chi_{\mathbb{Q}}(x) \quad \text{for all } x \in \mathbb{R}.$$

□

Exercise 3.14. *Every compact metric space X is the continuous image of the Cantor space* $\{0,1\}^{\mathbb{N}}$.

Solution. Construct inductively a family of nonempty closed sets $\{B_s\}_{s\in\{0,1\}^{<\omega}}$ such that

$$\lim_{k\to\infty}\operatorname{diam}(B_{\alpha\restriction k}) = 0 \text{ for all } \alpha \in \{0,1\}^{\mathbb{N}},$$

$$\bigcup_{|s|=n} B_s = X \text{ for all } n \in \mathbb{N}, \quad B_s = B_{s^\frown 0} \cup B_{s^\frown 1} \text{ for all } s \in \{0,1\}^{<\omega}.$$

We give the first step. Using compactness, we can find a number N and a covering $\{F_1, \ldots, F_{2^N}\}$ of X by closed sets such that $\operatorname{diam}(F_i) \le 1/2\operatorname{diam}(X)$ for all i. From these sets construct all B_t's with $|t| \le N$. Repeat the same procedure inside each compact space B_s with $|s| = N$. Now define $f : \{0,1\}^{\mathbb{N}} \to X$ by

$$f(\alpha) = \bigcap_{n=1}^{\infty} B_{\alpha\restriction n}.$$

□

Exercise 3.15. *Construct an example of a two-to-one function* $f : [0,1] \to \mathbb{R}$. *Prove that no such* f *can be continuous on* $[0,1]$.

Solution. Let $\{r_n : n \in \mathbb{N}\}$ be an enumeration of the rationals in $[0,1]$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} |2x-1| & \text{if } x \text{ is irrational,} \\ r_{2k-1} & \text{if } x = r_{2k-1}, \\ r_{2k-1} & \text{if } x = r_{2k}. \end{cases}$$

Suppose now that f is a continuous two-to-one function. We can then assume that its, say, minimum is attained at the points $x_1 < x_2$, and x_2 is not an endpoint. Choose disjoint closed intervals $[a_1, b_1]$, $[a_2, b_2]$ with $x_1 \in [a_1, b_1]$, $x_1 \ne b_1$ and $x_2 \in (a_2, b_2)$. Then the intermediate value theorem implies that a value r with $\min\{f(b_1), f(a_2), f(b_2)\} > r > \min f$ is taken on at least three times. □

Exercise 3.16. *Suppose that* $f : [a,b] \to \mathbb{R}$ *satisfies* $f^{-1}(\{y\})$ *is closed for all* $y \in \mathbb{R}$ *and* $f([c,d])$ *is connected for all* $[c,d] \subset [a,b]$. *Prove that* f *is continuous.*

Solution. Let $x \in [a,b]$ and take $\{x_n\}_{n=1}^{\infty} \subset [a,b]$ such that $x_n \uparrow x$. Then $I = \bigcap_{n=1}^{\infty} f([x_n, x])$ is an interval containing $f(x)$. We claim that $I = \{f(x)\}$ and therefore $f(x_n) \to f(x)$. Indeed, take $f(y) \in I$. Then there exist $t_n \in [x_n, x]$ such that $f(t_n) = f(y)$. Hence $t_n \to x$ and $t_n \in f^{-1}(\{f(y)\})$. Since $f^{-1}(\{f(y)\})$ is closed, it follows that $x \in f^{-1}(\{f(y)\})$, and so $f(x) = f(y)$. □

Exercise 3.17. *Let* (X, ρ) *be a metric space. Then there exists a continuous* $f : X \to \mathbb{R}$ *that is not uniformly continuous on X if and only if there exist two nonempty disjoint closed sets A and B such that* $\operatorname{dist}(A, B) = 0$.

Solution. Suppose that A and B are disjoint closed sets with $\operatorname{dist}(A, B) = 0$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)}.$$

Then f is continuous but not uniformly continuous. Now if f is a real continuous function on X which is not uniformly continuous, then we can inductively choose points $x_n, y_n \in X$ such that $\rho(x_n, y_n) < 1/n$, $|f(x_n) - f(y_n)| \geq \epsilon_0$, for a certain ϵ_0, and $\{x_n\}_{n=1}^{\infty} \cap \{y_n\}_{n=1}^{\infty} = \emptyset$. The sets $\{x_n : n \in \mathbb{N}\}$ and $\{y_n : n \in \mathbb{N}\}$ have the required properties. □

Exercise 3.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $|f(x) - f(y)| \geq c|x - y|$ for all $x, y \in \mathbb{R}$, where c does not depend on x and y. Then $f(\mathbb{R}) = \mathbb{R}$.*

Solution. Note that f is one-to-one and that

$$|\lim_{x\to\infty} f(x)| = |\lim_{x\to-\infty} f(x)| = \infty.$$

□

Exercise 3.19. *Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Show that the set E of $x \in \mathbb{R}$ such that f has a simple discontinuity at x is at most countable.*

Solution. Suppose that E is uncountable. Then at least one of the sets $A = \{x : f(x+) \neq f(x-)\}$ and $B = \{x : f(x+) = f(x-),\ f(x) \neq f(x+)\}$ must be uncountable. Without loss of generality, we may assume that A is uncountable, and so there exists a number ϵ_0 such that the set $\{x : |f(x+) - f(x-)| > \epsilon_0\}$ is uncountable and therefore has a point of accumulation a. Then we can find two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that $x_n \uparrow a$, $y_n \uparrow a$ and $|f(x_n) - f(y_n)| \geq \epsilon_0/2$, contradicting the fact that $\lim_{x\uparrow a} f(x)$ exists. □

Exercise 3.20. *If $f : \mathbb{R} \to \mathbb{R}$ has a local maximum at each $x \in \mathbb{R}$, then $f(\mathbb{R})$ is countable.*

Solution. For every $a \in f(\mathbb{R})$ choose $x_a \in \mathbb{R}$ with $f(x_a) = a$ and an open interval I_a with rational endpoints such that $x_a \in I_a$ and for each $x \in I_a$, $f(x) \geq f(x_a) = a$. Then the function

$$f(\mathbb{R}) \ni a \mapsto I_a \in \{(p, q) : p, q \in \mathbb{Q}\}$$

is one-to-one. □

Exercise 3.21. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then the set $A = \{m\alpha + n : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.*

Solution. Note that all the elements of A are distinct since α is irrational. So, the set $\{m\alpha - [m\alpha] : m \in \mathbb{N}\}$ is an infinite subset of $[0, 1]$ and therefore has a limit point. Consequently, there exists $\{r_n\} \subset A$ with $0 < r_n \downarrow 0$. Now let $x > 0$, $\epsilon > 0$. Choose $n \in \mathbb{N}$ with $r_n < \epsilon$ and let m be the smallest integer such that $mr_n > x$. Then $(m - 1)r_n \leq x$ and so, $0 < mr_n - x \leq r_n < \epsilon$. □

CHAPTER 4

Measure and Integration

Exercise 4.1. *Let $\{\phi_n\}_{n=1}^{\infty}$ be an approximate identity in $L^1(\mathbb{R})$ (that is, $\phi_n \geq 0$, $\int \phi_n = 1$, $\lim_{n\to\infty} \int_{|t|\geq\delta} \phi_n(t)dt = 0$ for all $\delta > 0$). Show that $\lim_{n\to\infty} \|\phi_n\|_p = \infty$ for all $p > 1$.*

Solution. Let $M > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\begin{aligned}3/4 \leq \int\limits_{|t|\leq 1/(8M)} \phi_n(t)dt &\leq \int\limits_{\{t:|t|\leq 1/(8M)\}\cap[\phi_n\leq M]} \phi_n(t)dt + \int\limits_{[\phi_n\geq M]} \phi_n(t)dt \\ &\leq 1/4 + \int\limits_{[\phi_n\geq M]} \phi_n(t)dt.\end{aligned}$$

It follows that

$$\int\limits_{[\phi_n\geq M]} \phi_n(t)dt \geq 1/2$$

and therefore

$$\int \phi_n^p(t)dt \geq \int\limits_{[\phi_n\geq M]} \phi_n^{p-1}(t)\phi_n(t)dt \geq M^{p-1} \int\limits_{[\phi_n\geq M]} \phi_n(t)dt \geq 1/2M^{p-1}.$$

We conclude that $\|\phi_n\|_p \to \infty$. □

Exercise 4.2. *Let $A \subset \mathbb{R}$ be a measurable set with $|A| > 0$. Then for any $n \in \mathbb{N}$, A contains arithmetic progressions of length n.*

Solution. Let x_0 be a point of density of A. Choose $\epsilon_0 > 0$ such that $n\epsilon_0 < 1/8$. Then there exists $l > 0$ such that $|(x_0 - l', x_0 + l') \cap A| \geq 2(1 - \epsilon_0)l'$ for all $0 < l' \leq l$. Now choose $\epsilon > 0$ such that $n^2\epsilon < 1/8l$. Then for $k = 0, 1, \ldots, n - 1$ we have

$$\begin{aligned}|(x_0 - l, x_0 + l) \cap \epsilon k + A| &= |\epsilon k + (x_0 - l - \epsilon k, x_0 + l - \epsilon k) \cap A| \\ &\geq |\epsilon k + (x_0 - l + \epsilon n, x_0 + l - \epsilon n) \cap A| \\ &= |(x_0 - l + \epsilon n, x_0 + l - \epsilon n) \cap A| \\ &\geq 2(1 - \epsilon_0)(l - \epsilon n).\end{aligned}$$

Hence

$$\begin{aligned}\left|(x_0 - l, x_0 + l) \setminus \bigcap_{k=0}^{n-1} \epsilon k + A\right| &= \left|\bigcup_{k=0}^{n-1}(x_0 - l, x_0 + l) \setminus \epsilon k + A\right| \\ &\leq \sum_{k=0}^{n-1} |(x_0 - l, x_0 + l) \setminus \epsilon k + A| \\ &= \sum_{k=0}^{n-1}(2l - |(x_0 - l, x_0 + l) \cap \epsilon k + A|) \\ &\leq \sum_{k=0}^{n-1}(2l - 2(1 - \epsilon_0)(l - \epsilon n)) \\ &< 2\epsilon n^2 + 2n\epsilon_0 l < l/4 + l/4 = l/2.\end{aligned}$$

Therefore $|\bigcap_{k=0}^{n-1} \epsilon k + A| > 0$. In particular, there exists $x \in \bigcap_{k=0}^{n-1} \epsilon k + A$, and so, $x, x - \epsilon, \ldots, x - (n-1)\epsilon \in A$. ◻

Exercise 4.3. *Let A be a measurable set of reals with arbitrarily small periods (there exist positive numbers p_n with $p_n \to 0$ so that $p_n + A = A$ for all n). Then either A or its complement has measure zero.*

Solution. Suppose that $|A| > 0$ and $|A^\complement| > 0$. Let x_1 be a point of density of A and x_2 a point of density of $A^\complement$ with $x_1 < x_2$. Then there exists $\delta > 0$ such that

$$|(x_1 - \delta, x_1 + \delta) \cap A| \geq 3\delta/2, \quad |(x_2 - \delta, x_2 + \delta) \cap A^\complement| > 3\delta/2.$$

It follows that

$$\begin{aligned}|(x_2 - \delta, x_2 + \delta) \cap x_2 - x_1 + A| &= |(x_2 - x_1) + (x_1 - \delta, x_1 + \delta) \cap A| \\ &= |(x_1 - \delta, x_1 + \delta) \cap A| \geq 3\delta/2.\end{aligned}$$

Consider the function $\phi : [0, \infty) \to \mathbb{R}$ defined by

$$\phi(x) = |(x_2 - \delta, x_2 + \delta) \cap x + A|.$$

Then ϕ is continuous and therefore constant, since it is constant on the dense set $\{mp_n : m, n \in \mathbb{N}\}$. Therefore

$$|(x_2 - \delta, x_2 + \delta) \cap A| = |(x_2 - \delta, x_2 + \delta) \cap x_2 - x_1 + A| \geq 3\delta/2.$$

But this is impossible since $|(x_2 - \delta, x_2 + \delta) \cap A^\complement| \geq 3\delta/2$. ◻

Exercise 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with periods s and t whose quotient is irrational. Prove that f is constant a.e.*

Solution. Note that since s/t is irrational, the set $\{ns + mt : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Therefore the set $f^{-1}([a, b])$ has arbitralily small periods and hence has either full or zero measure for all $a < b$. If it has zero measure for all $a < b$ then $f = +\infty$ or $f = -\infty$ almost everywhere. Suppose that $f^{-1}(I_1)$ has full measure for some interval I_1. Divide I_1 into two subintervals of equal length. Then the inverse image of one of these subintervals must have full measure. Call this interval I_2. Continuing this way we obtain a decreasing sequence $I_1 \supset I_2 \supset \cdots$ of closed intervals whose length tends to zero. Let $\{r\} = \bigcap_{n=1}^{\infty} I_n$. Then the set $f^{-1}(\{r\}) = \bigcap_{n=1}^{\infty} f^{-1}(I_n)$ has full measure and therefore $f = r$ almost everywhere. ◻

EXERCISE 4.5. *Let $A, B \subset \mathbb{R}$ be measurable sets of positive measure. Show that $A - B$ contains an interval.*

SOLUTION. Let x_1 be a point of density of A and x_2 a point of density of B. Then there exists $\delta > 0$ such that

$$|(x_1 - \delta, x_1 + \delta) \cap A| \geq 3\delta/2, \quad |(x_2 - \delta, x_2 + \delta) \cap B| \geq 3\delta/2.$$

It follows that

$$\begin{aligned} |(x_2 - \delta, x_2 + \delta) \cap x_2 - x_1 + A| &= |(x_2 - x_1) + (x_1 - \delta, x_1 + \delta) \cap A| \\ &= |(x_1 - \delta, x_1 + \delta) \cap A| \geq 3\delta/2. \end{aligned}$$

Therefore $|(x_2 - \delta, x_2 + \delta) \cap (x_2 - x_1 + A) \cap B| > 0$. Now consider the function

$$\phi(x) = |(x_2 - \delta, x_2 + \delta) \cap (x + A) \cap B|.$$

Then ϕ is continuous and $\phi(x_2 - x_1) > 0$. Hence there is an interval I such that $\phi(x) > 0$ for all $x \in I$. It follows that $(x + A) \cap B \neq \emptyset$ for all $x \in I$ and so, $I \subset B - A$. □

EXERCISE 4.6. *Suppose (X, μ) is a σ-finite measure space and let $f : X \to \mathbb{C}$ be a measurable function such that $|\int fg| < \infty$ for all $g \in L^p(X)$. Show that $f \in L^q(X)$ where q is the exponent conjugate to p.*

SOLUTION. Write $X = \bigcup_{k=1}^{\infty} A_k$ with A_k disjoint and $\mu(A_k) < \infty$. Suppose that $f \geq 0$, $\int f^q = \infty$ and let $B_n = [2^n \leq f < 2^{n+1}]$, $n \in \mathbb{Z}$. Then

$$\begin{aligned} \infty = \int f^q &= \sum_{n=-\infty}^{\infty} \int_{B_n} f^q = \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} \int_{B_n \cap A_k} f^q \\ &\lesssim \sum_{n=-\infty}^{\infty} \sum_{k=1}^{\infty} 2^{qn} \mu(B_n \cap A_k) = \sum_{i=1}^{\infty} 2^{qN(i)} \mu(B_{N(i)} \cap A_{M(i)}). \end{aligned}$$

Let

$$S_n = \sum_{k=1}^{n} 2^{qN(k)} \mu(B_{N(k)} \cap A_{M(k)})$$

and

$$g = \sum_{i=1}^{\infty} \frac{2^{qN(i)/p}}{S_{N(i)}} \chi_{B_{N(i)} \cap A_{M(i)}}.$$

Then

$$\int g^p = \sum_{i=1}^{\infty} \frac{2^{qN(i)}}{S_{N(i)}^p} \mu(B_{N(i)} \cap A_{M(i)}) < \infty$$

by Abel's Theorem. On the other hand

$$\begin{aligned} \int fg &= \sum_{i=1}^{\infty} \frac{2^{qN(i)/p}}{S_{N(i)}} \int_{B_{N(i)} \cap A_{M(i)}} f \geq \sum_{i=1}^{\infty} \frac{2^{qN(i)/p} 2^{N(i)}}{S_{N(i)}} \mu(B_{N(i)} \cap A_{M(i)}) \\ &= \sum_{i=1}^{\infty} \frac{2^{qN(i)}}{S_{N(i)}} \mu(B_{N(i)} \cap A_{M(i)}) = \infty \end{aligned}$$

By Abel's Theorem again. □