

# SIZE OF ORTHOGONAL SETS OF EXPONENTIALS FOR THE DISK

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**ABSTRACT.** Suppose  $\Lambda \subseteq \mathbb{R}^2$  has the property that any two exponentials with frequency from  $\Lambda$  are orthogonal in the space  $L^2(D)$ , where  $D \subseteq \mathbb{R}^2$  is the unit disk. Such sets  $\Lambda$  are known to be finite but it is not known if their size is uniformly bounded. We show that if there are two elements of  $\Lambda$  which are distance  $t$  apart then the size of  $\Lambda$  is  $O(t)$ . As a consequence we improve a result of Iosevich and Jaming and show that  $\Lambda$  has at most  $O(R^{2/3})$  elements in any disk of radius  $R$ .

**Keywords:** Spectral sets; Fuglede's Conjecture.

**AMS Primary Classification:** 42B99

## 1. INTRODUCTION

**1.1. Orthogonal sets of exponentials for domains in Euclidean space.** Let  $\Omega \subseteq \mathbb{R}^d$  be a bounded measurable set and let us assume for simplicity that  $\Omega$  has Lebesgue measure 1. The concept of a spectrum of  $\Omega$  that we deal with in this paper was introduced by Fuglede [6] who was studying a problem of Segal on the extendability of the partial differential operators (on  $C_c(\Omega)$ )

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}$$

to commuting operators on all of  $L^2(\Omega)$ .

**Definition 1.** A set  $\Lambda \subseteq \mathbb{R}^d$  is called a spectrum of  $\Omega$  (and  $\Omega$  is said to be a spectral set) if the set of exponentials

$$E(\Lambda) = \left\{ e_\lambda(x) = e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda \right\}$$

is a complete orthogonal set in  $L^2(\Omega)$ .

(The inner product in  $L^2(\Omega)$  is  $\langle f, g \rangle = \int_{\Omega} f \overline{g}$ .)

It is easy to see (see, for instance, [15]) that the orthogonality of  $E(\Lambda)$  is equivalent to the *packing condition*

$$(1) \quad \sum_{\lambda \in \Lambda} |\widehat{\chi_{\Omega}}|^2(x - \lambda) \leq |\Omega|^2, \text{ a.e. } (x),$$

as well as to the condition

$$(2) \quad \Lambda - \Lambda \subseteq \{0\} \cup \{\widehat{\chi_{\Omega}} = 0\}.$$

Here  $\chi_{\Omega}$  is the indicator function of  $\Omega$ .

The orthogonality and completeness of  $E(\Lambda)$  is in turn equivalent to the *tiling condition*

$$(3) \quad \sum_{\lambda \in \Lambda} |\widehat{\chi_{\Omega}}|^2(x - \lambda) = |\Omega|^2, \text{ a.e. } (x).$$

These equivalent conditions follow from the identity  $\langle e_\lambda, e_\mu \rangle = \int_{\Omega} e_\lambda \overline{e_\mu} = \widehat{\chi_{\Omega}}(\mu - \lambda)$  and from the completeness of all the exponentials in  $L^2(\Omega)$ . Condition (1) is roughly expressing the validity

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of Bessel's inequality for the system of exponentials  $E(\Lambda)$  while condition (3) says that Bessel's inequality holds as equality.

If  $\Lambda$  is a spectrum of  $\Omega$  then so is any translate of  $\Lambda$  but there may be other spectra as well.

*Example:* If  $Q_d = (-1/2, 1/2)^d$  is the cube of unit volume in  $\mathbb{R}^d$  then  $\mathbb{Z}^d$  is a spectrum of  $Q_d$ . Let us remark here that there are spectra of  $Q_d$  which are very different from affine images of the lattice  $\mathbb{Z}^d$  [11, 21, 14].

Research on spectral sets [22, 13, 20, 19, 3, 2, 16, 10] has been driven for many years by a conjecture of Fuglede [6], sometimes called the Spectral Set Conjecture, which stated that a set  $\Omega$  is spectral if and only if it is a translational tile. A set  $\Omega$  is a translational tile if we can translate copies of  $\Omega$  around and fill space without overlaps. More precisely there exists a set  $S \subseteq \mathbb{R}^d$  such that

$$(4) \quad \sum_{s \in S} \chi_\Omega(x - s) = 1, \text{ a.e. } (x).$$

One can generalize naturally the notion of translational tiling from sets to functions by saying that a nonnegative  $f \in L^1(\mathbb{R}^d)$  tiles when translated at the locations  $S$  if  $\sum_{s \in S} f(x - s) = \ell$  for almost every  $x \in \mathbb{R}^d$  (the constant  $\ell$  is called the *level* of the tiling). Thus the question of spectrality for a set  $\Omega$  is essentially a tiling question for the function  $|\widehat{\chi}_\Omega|^2$  (the *power-spectrum*). Taking into account the equivalent condition (3) one can now, more elegantly, restate the Fuglede Conjecture as the equivalence

$$(5) \quad \chi_\Omega \text{ tiles } \mathbb{R}^d \text{ by translation at level } 1 \iff |\widehat{\chi}_\Omega|^2 \text{ tiles } \mathbb{R}^d \text{ by translation at level } |\Omega|^2.$$

In this form the conjectured equivalence is perhaps more justified. However this conjecture is now known to be false in both directions if  $d \geq 3$  [24, 23, 17, 18, 4, 5], but remains open in dimensions 1 and 2 and it is not out of the question that the conjecture is true in all dimensions if one restricts the domain  $\Omega$  to be convex. (It is known that the direction “tiling  $\Rightarrow$  spectrality” is true in the case of convex domains; see for instance [15].) The equivalence (5) is also known, from the time of Fuglede's paper [6], to be true if one adds the word *lattice* to both sides (that is, lattice tiles are the same as sets with a lattice spectrum).

**1.2. Orthogonal exponentials for the disk.** Already in [6] it was claimed that the disk in the plane (and the Euclidean ball in  $\mathbb{R}^d$ ) is not a spectral set, in agreement with (5). A proof appeared in [9]. Later it was proved in [7, 12] that any orthogonal set of exponentials for the ball must necessarily be finite. It is still unknown however if there is a uniform bound for the size of each orthogonal set. It is still a possibility that there are arbitrarily large orthogonal sets of exponentials for the ball and proving a uniform upper bound is probably very hard as it appears to depend on algebraic relations among the roots of the Bessel function  $J_1$ . In the direction of showing upper bounds for orthogonal sets of exponentials it was proved in [8] that if  $\Lambda$  is a set of orthogonal exponentials for the ball then  $|\Lambda \cap [-R, R]^d| = O(R)$ , with the implicit constant independent of  $\Lambda$ . Completeness would of course require that  $|\Lambda \cap [-R, R]^d| \gtrsim R^d$  (this follows easily from the tiling condition (3)).

The result in this paper, Theorem 1 below, improves the result of [8] mentioned above. We choose to work only in the case of the unit disk in the plane and not in higher dimension or in the larger class of smooth convex bodies in order to present a clear geometric argument, which probably extends to these cases as well.

**Theorem 1.** *There are constants  $C_1, C_2$  such that whenever  $\Lambda \subseteq \mathbb{R}^2$  is an orthogonal set of exponentials for the unit disk in the plane and*

$$t = \inf \{|\lambda - \mu| : \lambda, \mu \in \Lambda, \lambda \neq \mu\}$$

*then  $|\Lambda| \leq C_1 t$ . (It is well known and easy to see from (1) or (2) that  $t > 0$ .)*

*Furthermore,  $|\Lambda \cap [-R, R]^2| \leq C_2 R^{2/3}$  for all  $R \geq 1$ .*

The proof of Theorem 1 is given in the next section.

## 2. PROOF OF THE MAIN THEOREM

A crucial ingredient of the proof is the asymptotics for the zeros of the Fourier Transform of the indicator function of the unit disk  $D = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ , which is of course a radial function. Since the zeros of  $\widehat{\chi}_D(r)$  are the same as the zeros of the Bessel function  $J_1(2\pi r)$  and since for the zeros of  $J_1$ , written as  $j_{1,n}$ ,  $n = 1, 2, \dots$  we have an asymptotic expansion [1]

$$(6) \quad j_{1,n} = \rho_n + \frac{K_1}{\rho_n} + O\left(\frac{1}{\rho_n^3}\right), \quad \text{where } \rho_n = n\pi + \frac{\pi}{4}, \quad n = 1, 2, \dots,$$

where  $K_1$  is an absolute constant, it follows that the zeros of  $\widehat{\chi}_D(r)$  are at the locations

$$(7) \quad r_n = \frac{1}{2\pi} j_{1,n} = \frac{n}{2} + \frac{1}{8} + \frac{K_1}{2\pi\rho_n} + O(n^{-3}).$$

Moreover, if  $0 \leq m - n \leq K$  and  $m, n \rightarrow \infty$  it follows from (6) that

$$(8) \quad r_m - r_n = \frac{m - n}{2} + O(Kn^{-2}) = \frac{m - n}{2} + O(Kr_n^{-2}) = \frac{m - n}{2} + O((r_m - r_n)r_n^{-2}).$$

**Lemma 1.** *There is a constant  $C > 0$  such that whenever  $a, b, c \in \mathbb{R}^2$  are orthogonal for the unit disk, with  $|a - c|, |b - c|, |a - b| \geq R$  then the two largest angles of the triangle  $abc$  (as well as all its external angles) are*

$$(9) \quad \geq \frac{C}{R^{1/2}}.$$

*Proof.* Assume without loss of generality that  $R = |a - c| \leq |b - c| \leq |a - b|$  (see Fig. 1). Writing  $\theta = \widehat{bac}$  for the second largest angle and  $T = |a - b|$  we have

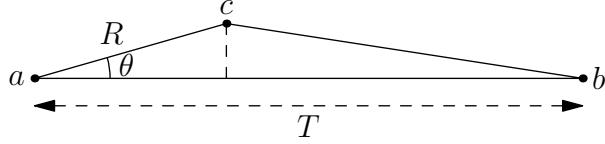


Figure 1: Three points orthogonal for the unit disk

$$|b - c| = \sqrt{(T - R \cos \theta)^2 + R^2 \sin^2 \theta} = \sqrt{(T - R)^2 + 2TR(1 - \cos \theta)}$$

from which we get

$$(10) \quad |b - c| - (T - R) = \frac{2TR(1 - \cos \theta)}{T - R + |b - c|} = \frac{2R(1 - \cos \theta)}{1 - \frac{R}{T} + \frac{|b - c|}{T}} \leq 2R(1 - \cos \theta) \leq R\theta^2.$$

From (7) it follows that as  $R \rightarrow \infty$  the quantities  $|a - b|, |b - c|, |a - c|$  are all of the form

$$\frac{k}{2} + \frac{1}{8} + o(1), \quad \text{for some integer } k.$$

It follows that  $|b - c| - (T - R) = \frac{k}{2} + \frac{1}{8} + o(1)$ , for some integer  $k \geq 0$ . This, together with (10), implies that  $\frac{k}{2} + \frac{1}{8} + o(1) \leq R\theta^2$  which gives us the required inequality with constant  $C$  arbitrarily close to  $\sqrt{1/8}$  when  $R$  is large.  $\square$

**Corollary 1.** *There is a constant  $C' > 0$  such that whenever  $a, b, c \in \mathbb{R}^2$  are orthogonal for the unit disk and their pairwise distances are at least  $L$  then they cannot all belong to a strip of width  $C'L^{1/2}$ .*

*Proof.* Suppose they do belong to such a strip.

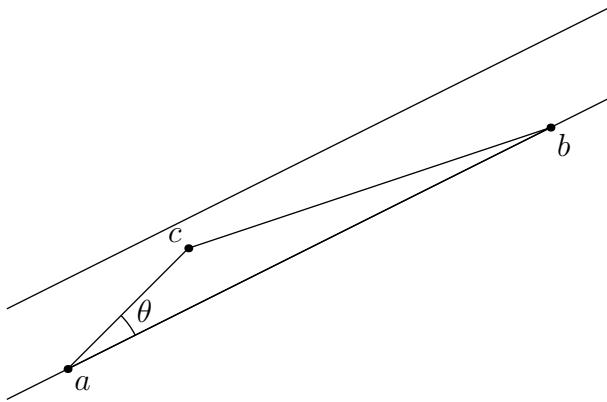


Figure 2: Three points in a strip

Move and turn the strip so that two of the points, those with the largest distance apart, say  $a$  and  $b$  are on one of the strip sides and the other point  $c$  is still in the strip (see Fig. 2). Assume also that  $c$  is closer to  $a$  than to  $b$ . By Lemma 1 it follows that the angle  $\theta = \widehat{bac}$  is at least

$$\frac{C}{|a - c|^{1/2}}$$

from which we obtain that the distance of  $c$  to the line  $ab$  is at least  $C\sqrt{a - c} \geq C\sqrt{L}$ , a contradiction if the constant  $C'$  in the Lemma is sufficiently small.  $\square$

**Corollary 2.** Suppose  $\Lambda \subseteq \mathbb{R}^2$  is a set of orthogonal exponentials for the unit disk,  $R > 0$  and let

$$\Delta = \inf \{|\lambda - \mu| : \lambda, \mu \in \Lambda \cap [-R, R]^2\}.$$

Then

$$(11) \quad |\Lambda \cap [-R, R]^2| \leq C \frac{R}{\Delta^{1/2}},$$

for some constant  $C > 0$ .

*Proof.* Cover  $[-R, R]^2$  by  $O(R/\Delta^{1/2})$  strips of width  $c\Delta^{1/2}$ , for small  $c > 0$ . From Corollary 1 each of these contains at most two points of  $\Lambda$ .  $\square$

We may assume from now on that the points  $V = (\Delta, 0)$  and  $-V = (-\Delta, 0)$  belong to the set  $\Lambda$  and that  $t/2 \leq \Delta \leq t$ . It is also sufficient to bound the size of  $\Lambda$  in the first quadrant only, for reasons of symmetry, so we restrict ourselves to the first quadrant. By Corollary 1 we have that

$$(12) \quad |\Lambda \cap \{(x, y) : x, y \geq 0, \min\{x, y\} \leq \Delta\}| = O(\Delta^{1/2}).$$

So from now on we may assume that the point  $\lambda = (x, y) \in \Lambda$  belongs to the first quadrant and has  $x, y \geq \Delta$ .

To each  $\lambda = (x, y)$  in the open first quadrant we correspond two numbers  $a(\lambda), b(\lambda) \in (0, \Delta)$  such that  $a(\lambda)^2 + b(\lambda)^2 = \Delta^2$  and  $\lambda$  is on the hyperbola

$$H_\lambda : \frac{x^2}{a(\lambda)^2} - \frac{y^2}{b(\lambda)^2} = 1, \quad (x, y \geq 0).$$

This hyperbola  $H_\lambda$  is the locus of all points  $p$  in the first quadrant such that

$$(13) \quad |p + V| - |p - V| = 2a(\lambda).$$

A parametrization of  $H_\lambda$  is

$$(14) \quad x(t) = a(\lambda) \cosh t, \quad y(t) = b(\lambda) \sinh t, \quad (t \geq 0).$$

It follows that in the region of interest  $x, y \geq \Delta$  we have

$$(15) \quad \Delta \leq x \leq a(\lambda)e^t \leq 2|\lambda|, \quad \Delta \leq y \leq b(\lambda)e^t \leq 2|\lambda|.$$

**Lemma 2.** *There is a constant  $K > 0$  such that  $b(\lambda) \geq K\Delta^{1/2}$  with the exception of at most a constant number of points of  $\Lambda$ .*

*Proof.*

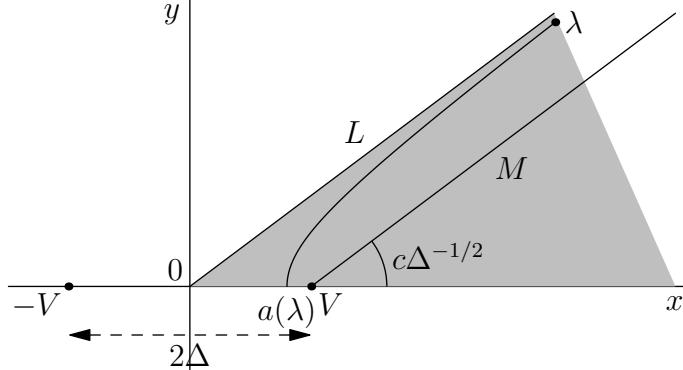


Figure 3: Figure to aid the proof of Lemma 2

It follows from Lemma 1 that  $\lambda$  cannot belong to the sector defined by the  $x$ -axis from  $V$  onward and the straight line  $M$  through  $V$  of angle  $c\Delta^{-1/2}$ , if  $c > 0$  is small enough (refer to Fig. 3). Now draw a parallel line  $L$  to straight line  $M$  through the origin and note that the strip bordered by these two parallel lines,  $L$  and  $M$ , has width  $O(\Delta^{1/2})$ . Therefore, by Corollary 1, there is only a constant number of elements of  $\Lambda$  that can belong to the sector defined by the positive  $x$ -semiaxis and the straight line  $L$  (shaded region in Fig. 3).

Suppose now that  $\lambda \in \Lambda$  is such that  $b(\lambda) \leq K\Delta^{1/2}$ , for an appropriately small constant  $K$ , so that we also have  $a(\lambda) \geq \Delta/2$ . It follows that the asymptote to the hyperbola  $H_\lambda$ , with equation  $y = (b(\lambda)/a(\lambda))x$ , has slope at most  $\frac{K}{2}\Delta^{-1/2}$  which implies that  $\lambda$ , lying below that asymptote, is contained in the (shaded) sector mentioned above.  $\square$

Writing  $H(a, \Delta)$  for the hyperbola  $x^2/a^2 - y^2/b^2 = 1$ , with  $a^2 + b^2 = \Delta^2$ , we consider the finite family of confocal hyperbolas

$$(16) \quad H_k = H\left(\frac{k}{4}, \Delta\right), \quad k = 0, 1, 2, \dots, [4\Delta].$$

The hyperbola  $H_k$  is the locus of all points  $p$  with  $|p + V| - |p - V| = \frac{k}{2}$ . For each  $\lambda$  we define the corresponding  $k$  to be the unique integer such that

$$(17) \quad |\lambda + V| - |\lambda - V| = \frac{k}{2} + 2\epsilon, \quad -\frac{1}{8} \leq \epsilon < \frac{1}{8}.$$

We write  $a = k/4$ ,  $b = \sqrt{\Delta^2 - a^2}$ . It follows from (13) and (17) that

$$a(\lambda) = a + \epsilon, \quad b(\lambda) = b - \epsilon',$$

for some  $\epsilon'$ , of the same sign as  $\epsilon$ . From (8) we have that

$$(18) \quad |\epsilon| \leq C\Delta|\lambda|^{-2},$$

for some absolute finite constant  $C > 0$ .

Next we estimate  $\epsilon'$ :

$$\begin{aligned}
|\epsilon'| &= |b - b(\lambda)| \\
&= \left| \sqrt{\Delta^2 - a^2} - \sqrt{\Delta^2 - a(\lambda)^2} \right| \\
&= \left| \frac{a(\lambda)^2 - a^2}{b + b(\lambda)} \right| \\
&= \left| \epsilon \frac{a + a(\lambda)}{b + b(\lambda)} \right| \\
&\leq |\epsilon| \frac{2\Delta}{b(\lambda)} \\
&= O(|\epsilon|\Delta^{1/2}) && \text{(from Lemma 2, excepting finitely many } \lambda \text{s)} \\
&= O(\Delta^{3/2}|\lambda|^{-2}) && \text{(from (18)).}
\end{aligned}$$

The asymptote  $L(a, \Delta)$  to  $H(a, \Delta) = H_k$  is the line  $y = (b/a)x$  and a unit normal vector to this line is  $u = (b/\Delta, -a/\Delta)$ . We can bound the distance of

$$\lambda = (x, y) = (a(\lambda) \cosh t, b(\lambda) \sinh t)$$

to  $L(a, \Delta)$  as follows:

$$\begin{aligned}
|u \cdot \lambda| &= \left| \frac{bx}{\Delta} - \frac{ay}{\Delta} \right| \\
&= \frac{1}{\Delta} |ba(\lambda) \cosh t - ab(\lambda) \sinh t| \\
&= \frac{1}{\Delta} |(b(\lambda) + \epsilon')a(\lambda) \cosh t - (a(\lambda) - \epsilon)b(\lambda) \sinh t| \\
&= \frac{1}{\Delta} |a(\lambda)b(\lambda)e^{-t} - \epsilon b(\lambda) \sinh t + \epsilon' a(\lambda) \cosh t| \\
&= O(\Delta^2|\lambda|^{-1}) + |\epsilon|O\left(\Delta^{-1}b(\lambda) \sinh t + \Delta^{-1/2}a(\lambda) \cosh t\right) \\
&\quad \text{(since } \epsilon' = O(\Delta^{1/2}\epsilon) \text{ and } a(\lambda)e^t \sim |\lambda| \text{ or } b(\lambda)e^t \sim |\lambda| \text{ from (15))} \\
&= O(\Delta^2|\lambda|^{-1}) + |\epsilon|O\left(\Delta^{-1}b(\lambda)e^t + \Delta^{-1/2}a(\lambda)e^t\right) \\
&= O(\Delta^2|\lambda|^{-1}) + O(|\epsilon||\lambda|\Delta^{-1/2}) && \text{(from (15))} \\
&= O\left(\Delta^2|\lambda|^{-1}\right) && \text{(from (18)).}
\end{aligned}$$

Therefore in the region  $|\lambda| \geq C\Delta^{3/2}$  each point of  $\Lambda$  is at distance  $O(\Delta^{1/2})$  from one of the asymptotes to the hyperbolas  $H_k$ . In each strip of width  $O(\Delta^{1/2})$  around each such asymptote we therefore have at most  $C$  points, a constant. This gives a total of  $O(\Delta)$  points of  $\Lambda$  in that region as there are that many hyperbolas  $H_k$ . In the region  $|\lambda| \leq C\Delta^{3/2}$  we also have  $O(\Delta)$  points because of Corollary 2. This concludes the proof of the first part of Theorem 1.

To prove that  $|\Lambda \cap [-R, R]^2| = O(R^{2/3})$  notice that by Corollary 2 and by the first part of Theorem 1 we have

$$|\Lambda \cap [-R, R]^2| = O\left(\min\left\{\frac{R}{t^{1/2}}, t\right\}\right) = O\left(R^{2/3}\right).$$

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