Hilbert’s inequality is

\[ \left| \sum_{m,n \neq n} \frac{x_m y_n}{m-n} \right| \leq C \left( \sum_m |x_m|^2 \right)^{1/2} \left( \sum_n |y_n|^2 \right)^{1/2} \]

for arbitrary \( x_m, y_n \in \mathbb{C} \).

Hilbert had given the value \( 2\pi \) for the constant \( C \), while the best value \( \pi \) for \( C \) was found by Schur (1911).

There are two generalizations of Hilbert’s inequality:

\[ \left| \sum_{m,n \neq n} \frac{x_m y_n}{\lambda_m - \lambda_n} \right| \leq C \delta \left( \sum_m \frac{|x_m|^2}{\delta_m} \right)^{1/2} \left( \sum_n \frac{|y_n|^2}{\delta_n} \right)^{1/2}, \tag{1} \]

where \( \{\lambda_k\} \) is a strictly increasing real sequence such that \( |\lambda_m - \lambda_n| \geq \delta > 0 \) for \( m \neq n \), and

\[ \left| \sum_{m,n \neq n} \frac{x_m y_n}{\lambda_m - \lambda_n} \right| \leq C \left( \sum_m \frac{|x_m|^2}{\delta_m} \right)^{1/2} \left( \sum_n \frac{|y_n|^2}{\delta_n} \right)^{1/2}, \tag{2} \]

where \( \{\lambda_k\} \) is a strictly increasing real sequence and \( \delta_k = \min_{l,l \neq k} |\lambda_l - \lambda_k| \).

Both inequalities were proven by Montgomery and Vaughan in [2]. For (1) they calculated the best value \( \pi \) of the constant \( C \). For (2) they gave the value \( \frac{3\pi}{2} \) for \( C \), but this is not the best possible. The conjecture is that the best value of \( C \) for (2) is also \( \pi \). If this is true, (1) is a particular case of (2).

For the recent history of these inequalities see [1].

In this note we shall prove the continuous analogue of (2) with the best value \( \pi \) of \( C \). That is

\[ \left| \iint_{\mathbb{R} \times \mathbb{R}} f(x)g(y) K(x-K(y)) - K(y)dy \right| \leq \pi \|F\|_2 \|G\|_2 \]

where \( K : \mathbb{R} \to \mathbb{R} \) has strictly positive continuous derivative and \( f, g \) have compact support in \( \mathbb{R} \).

We define \( F = \frac{f}{\sqrt{K}} \) and \( G = \frac{g}{\sqrt{K}} \) and we get the equivalent

\[ \left| \iint_{\mathbb{R} \times \mathbb{R}} \sqrt{K(x)} \sqrt{K'(y)} F(x)G(y) dx dy \right| \leq \pi \|F\|_2 \|G\|_2. \]

By the Cauchy-Schwarz inequality it suffices to prove

\[ \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\sqrt{K'(y)}}{K(x) - K(y)} G(y) dy \right|^2 K'(x) dx \right)^{1/2} \leq \pi \|G\|_2. \]
We change variables: \( \xi = K(x), \ x = L(\xi), \ \eta = K(y), \ y = L(\eta) \) and \( G^*(\xi) = G(x), \ G^*(\eta) = G(y) \). Therefore,

\[
\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\sqrt{K'(y)}}{K(x) - K(y)} G(y) \, dy \right|^2 K'(x) \, dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\sqrt{L'(\eta)}G^*(\eta)}{\xi - \eta} \, d\eta \right|^2 d\xi
\]

\[
= \pi^2 \int_{\mathbb{R}} \left| \sqrt{L'(\eta)}G^*(\eta) \right|^2 d\eta
\]

\[
= \pi^2 \|G\|^2_2.
\]

The next to last equality is just the isometric property of the Hilbert transform \( Hk(\xi) = P.V. \frac{1}{\pi} \int_{\mathbb{R}} \frac{k(\eta)}{\xi - \eta} \, d\eta \).

**References**


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