A NOTE ON A GENERALIZATION OF HILBERT'S INEQUALITY

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Hilbert's inequality is

$$\Big|\sum_{(m,n),m\neq n}\frac{x_m y_n}{m-n}\Big|\leq C\Big(\sum_m |x_m|^2\Big)^{\frac{1}{2}}\Big(\sum_n |y_n|^2\Big)^{\frac{1}{2}}$$

for arbitrary $x_m, y_n \in \mathbf{C}$.

Hilbert had given the value 2π for the constant C, while the best value π for C was found by Schur (1911).

There are two generalizations of Hilbert's inequality:

$$\Big|\sum_{(m,n),m\neq n}\frac{x_m y_n}{\lambda_m-\lambda_n}\Big| \quad\leq\quad \frac{C}{\delta}\Big(\sum_m |x_m|^2\Big)^{\frac{1}{2}}\Big(\sum_n |y_n|^2\Big)^{\frac{1}{2}}, \tag{1}$$

where $\{\lambda_k\}$ is a strictly increasing real sequence such that $|\lambda_m-\lambda_n|\geq\delta>0$ for $m\neq n$, and

$$\Big|\sum_{(m,n),m\neq n}\frac{x_m y_n}{\lambda_m-\lambda_n}\Big| \quad\leq\quad C\Big(\sum_m \frac{|x_m|^2}{\delta_m}\Big)^{\frac{1}{2}}\Big(\sum_n \frac{|y_n|^2}{\delta_n}\Big)^{\frac{1}{2}}, \tag{2}$$

where $\{\lambda_k\}$ is a strictly increasing real sequence and $\delta_k=\min_{l,l\neq k}|\lambda_l-\lambda_k|$.

Both inequalities were proven by Montgomery and Vaughan in [2]. For (1) they calculated the best value π of the constant C. For (2) they gave the value $\frac{3\pi}{2}$ for C, but this is not the best possible. The conjecture is that the best value of C for (2) is also π. If this is true, (1) is a particular case of (2).

For the recent history of these inequalities see [1].

In this note we shall prove the continuous analogue of (2) with the best value π of C. That is

$$\Big|\iint_{\mathbf{R}\times\mathbf{R}}\frac{f(x)g(y)}{K(x)-K(y)}\,dxdy\Big|\leq\pi\Big(\int_{\mathbf{R}}\frac{|f(x)|^2}{K'(x)}\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbf{R}}\frac{|g(y)|^2}{K'(y)}\,dy\Big)^{\frac{1}{2}},$$

where $K:\mathbf{R}\to\mathbf{R}$ has strictly positive continuous derivative and f, g have compact support in $\mathbf{R}$.

We define $F=\frac{f}{\sqrt{K'}}$ and $G=\frac{g}{\sqrt{K'}}$ and we get the equivalent

$$\Big|\iint_{\mathbf{R}\times\mathbf{R}}\frac{\sqrt{K'(x)}\sqrt{K'(y)}}{K(x)-K(y)}\,F(x)G(y)\,dxdy\Big|\leq\pi\,\|F\|_2\,\|G\|_2\,.$$

By the Cauchy-Schwarz inequality it suffices to prove

$$\int_{\mathbf{R}}\Big|\int_{\mathbf{R}}\frac{\sqrt{K'(y)}}{K(x)-K(y)}\,G(y)\,dy\Big|^2 K'(x)\,dx\leq\pi^2\,\|G\|_2^2\,.$$

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We change variables: $\xi = K(x),\ x = L(\xi),\ \eta = K(y),\ y = L(\eta)$ and $G^*(\xi) = G(x),\ G^*(\eta) = G(y)$. Therefore,

$$\begin{aligned}\int_{\mathbf{R}}\Big|\int_{\mathbf{R}} \frac{\sqrt{K'(y)}}{K(x)-K(y)}\, G(y)\, dy\Big|^2 K'(x)\, dx &= \int_{\mathbf{R}}\Big|\int_{\mathbf{R}} \frac{\sqrt{L'(\eta)}G^*(\eta)}{\xi-\eta}\, d\eta\Big|^2 d\xi \\ &= \pi^2 \int_{\mathbf{R}} \big|\sqrt{L'(\eta)}G^*(\eta)\big|^2\, d\eta \\ &= \pi^2\, \|G\|_2^2\ .\end{aligned}$$

The next to last equality is just the isometric property of the Hilbert transform $Hk(\xi) = P.V.\, \frac{1}{\pi}\int_{\mathbf{R}} \frac{k(\eta)}{\xi-\eta}\, d\eta\,.$

References

[1] H. L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS **84** (1990), Amer. Math. Soc. Publications.

[2] H. L. Montgomery, R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2) **8** (1974), 73–82.

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