

**A NONTRIVIAL VARIANT OF HILBERT'S INEQUALITY, AND
AN APPLICATION TO THE NORM OF THE HILBERT MATRIX
ON THE HARDY-LITTLEWOOD SPACES**

V. DASKALOGIANNIS, P. GALANOPOULOS, AND M. PAPADIMITRAKIS

Abstract: Hilbert's inequality for non-negative sequences states that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$

where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. This implies that the norm of the Hilbert matrix as an operator on the sequence space ℓ^p equals $\frac{\pi}{\sin \frac{\pi}{p}}$.

In this article we prove the nontrivial variant

$$\sum_{m,n=1}^{\infty} \binom{n}{m}^{\frac{1}{q} - \frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}$$

of Hilbert's inequality, and we use it to prove that the norm of the Hilbert matrix as an operator on the Hardy-Littlewood space K^p equals $\frac{\pi}{\sin \frac{\pi}{p}}$, where K^p consists of all functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ analytic in the unit disc with $\|f\|_{K^p}^p = \sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p < \infty$. We also see that $\frac{\pi}{\sin \frac{\pi}{p}}$ is the norm of the Hilbert matrix

on the space ℓ_{p-2}^p of sequences (a_m) with $\|(a_m)\|_{\ell_{p-2}^p}^p = \sum_{m=1}^{\infty} m^{p-2} |a_m|^p < \infty$.

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1. PRELIMINARIES

The Hilbert matrix is the infinite matrix, whose entries are

$$\frac{1}{m+n-1}, \quad n, m = 1, 2, \dots$$

The well known Hilbert's inequality [8, Th. 323] (see also [8, Th. 315] for a weaker inequality) states that if $(a_m), (b_n)$ are sequences of non-negative terms such that $(a_m) \in \ell^p, (b_n) \in \ell^q$, then

$$(1.1) \quad \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}},$$

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where $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is the smallest possible for this inequality. This implies that the Hilbert matrix induces a bounded operator \mathcal{H} ,

$$\mathcal{H} : (a_m) \mapsto \mathcal{H}(a_m) = \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n-1} \right)$$

on the spaces ℓ^p , $1 < p < \infty$, with norm

$$\|\mathcal{H}\|_{\ell^p \rightarrow \ell^p} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

The operator \mathcal{H} can also be considered as an operator on spaces of analytic functions by its action on the sequence of Taylor coefficients of any such function.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} .

The Hardy space H^p , $0 < p < \infty$, consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where $M_p^p(r, f)$ are the integral means

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

If $p \geq 1$, then H^p is a Banach space under the norm $\|\cdot\|_{H^p}$. If $0 < p < 1$, then H^p is a complete metric space.

For $f(z) = \sum_{m=0}^{\infty} a_m z^m \in H^1$, Hardy's inequality [6, p.48]

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} \leq \pi \|f\|_{H^1}$$

implies that the power series

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{a_m}{m+n+1} \right) z^n$$

has bounded coefficients. Therefore $\mathcal{H}(f)$ is an analytic function of the unit disk for any $f \in H^1$ and hence for any $f \in H^p$, $p \geq 1$.

The Bergman space A^p , $0 < p < \infty$, consists of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty,$$

where $dA(z)$ is the normalized Lebesgue area measure on \mathbb{D} . If $p \geq 1$, then A^p is a Banach space under the norm $\|\cdot\|_{A^p}$.

If $f(z) = \sum_{m=0}^{\infty} a_m z^m \in A^p$ and $p > 2$, then by [10, Lemma 4.1] we have

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} < \infty.$$

Thus $\mathcal{H}(f)$ is an analytic function in \mathbb{D} for each function $f \in A^p$, $p > 2$.

E. Diamantopoulos and A. G. Siskakis initiated the study of the Hilbert matrix as an operator on Hardy and Bergman spaces in [3, 4] and showed that $\mathcal{H}(f)$ has the following integral representation

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz} dt, \quad z \in \mathbb{D}.$$

Then, considering \mathcal{H} as an average of weighted composition operators, they showed that it is a bounded operator on H^p , $p > 1$, and on A^p , $p > 2$, and they estimated its norm. Their study was further extended by M. Dostanić, M. Jevtić and D. Vukotić in [5] and by V. Božin and B. Karapetrović in [1] (see also [9]). Summarizing their results, we now know that

$$\|\mathcal{H}\|_{H^p \rightarrow H^p} = \|\mathcal{H}\|_{A^{2p} \rightarrow A^{2p}} = \frac{\pi}{\sin \frac{\pi}{p}}, \quad 1 < p < \infty.$$

The Hardy-Littlewood space K^p , $0 < p < \infty$, is defined as the space of all $f(z) = \sum_{m=0}^{\infty} a_m z^m \in H(\mathbb{D})$ such that

$$\|f\|_{K^p}^p = \sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p < \infty.$$

If $p \geq 1$, then K^p is a Banach space under the norm $\|\cdot\|_{K^p}$.

According to the classical Hardy-Littlewood inequalities, [7, Th. 5 & 6], [6, Th. 6.2 & 6.3], if $f(z) = \sum_{m=0}^{\infty} a_m z^m \in H^p$, $0 < p \leq 2$, then

$$\sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p \leq c_p \|f\|_{H^p}^p$$

and hence $f \in K^p$. Also, if $2 \leq p < \infty$ and $f(z) = \sum_{m=0}^{\infty} a_m z^m \in K^p$, then

$$\|f\|_{H^p}^p \leq c_p \sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p$$

and hence $f \in H^p$. In both cases c_p is a constant independent of f .

If $p \geq 1$, and in the special case where the sequence (a_m) is real and decreasing to zero, then for $f(z) = \sum_{m=0}^{\infty} a_m z^m$ we have that $f \in H^p$ if and only if $f \in K^p$ [11, Th. A & 1.1].

Now it is clear that the proper domain of definition of the operator \mathcal{H} acting on analytic functions in the unit disc is the space K^1 . Indeed, if $f(z) = \sum_{m=0}^{\infty} a_m z^m \in K^1$ then

$$\sum_{m=0}^{\infty} \frac{|a_m|}{m+1} < \infty$$

and hence $\mathcal{H}(f) \in H(\mathbb{D})$.

Moreover, when $1 < p < \infty$ and $f \in K^p$, we consider q so that $\frac{1}{p} + \frac{1}{q} = 1$ and we apply Hölder's inequality to find

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{|a_m|}{m+1} &= \sum_{m=0}^{\infty} (m+1)^{\frac{2}{p}-2} (m+1)^{1-\frac{2}{p}} |a_m| \\ &\leq \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^2} \right)^{\frac{1}{q}} \left(\sum_{m=0}^{\infty} (m+1)^{p-2} |a_m|^p \right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Hence $K^p \subseteq K^1$ and so, if $f \in K^p$, then $\mathcal{H}(f)$ defines an analytic function in \mathbb{D} .

Recently, in [12, Theorem 1] (see also [2]), the authors associated the boundedness of the generalized Volterra operators

$$T_g(f)(z) = \int_0^z f(w)g'(w)dw, \quad z \in \mathbb{D},$$

induced by symbols $g \in H(\mathbb{D})$ with non-negative Taylor coefficients and acting from a space X to H^∞ , to the K^p -norm of the function $\mathcal{H}(g')$. In this result X can be H^p or K^p or the Dirichlet-type space D_{p-1}^p .

2. A VARIANT OF HILBERT'S INEQUALITY

Our first result is a nontrivial variant of the classical Hilbert's inequality.

Before we state our first main result we shall mention two more variants of Hilbert's inequality. The first, in [13], is

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m} \right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}$$

and the second, in [14], is

$$\sum_{m,n=1}^{\infty} \left(\frac{n-\frac{1}{2}}{m-\frac{1}{2}} \right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$

In fact Yang proves a whole family of such inequalities depending on a parameter. In all these variants, as well as in the original Hilbert's inequality, the kernel involved in the double sum is of the form

$$\left(\frac{k(n)}{k(m)} \right)^{c_p} \frac{1}{(Ak(m) + Bk(n))^\lambda}$$

which is homogeneous of degree $-\lambda$. As a consequence, in order to prove these variants one needs to apply the standard arguments used in the proof of the original Hilbert's inequality. The kernel

$$\left(\frac{n}{m} \right)^{\frac{1}{q}-\frac{1}{p}} \frac{1}{m+n-1}$$

in our variant of Hilbert's inequality, which appears in the following Theorem 1, lacks any homogeneity and the standard arguments do not apply.

Theorem 1. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $(a_m) \in \ell^p$, $(b_n) \in \ell^q$ are sequences of non-negative terms, then*

$$\sum_{m,n=1}^{\infty} \left(\frac{n}{m} \right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left(\sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$

The constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is the smallest possible for this inequality.

Proof. It is sufficient to consider the case $1 < q \leq 2 \leq p < \infty$.
We assume

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1, \quad \alpha \geq 0, \beta \geq 0,$$

where α, β will be chosen appropriately later; the choice of α, β will depend on p .
By Hölder's inequality,

$$\begin{aligned} & \sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m b_n}{m+n-1} \\ &= \sum_{m,n=1}^{\infty} \left(\frac{n}{m}\right)^{\left(\frac{1}{pq}-\frac{1}{p}\right)+\left(\frac{1}{q}-\frac{1}{pq}\right)} \frac{a_m b_n}{(m+n)^{\frac{1}{p}}(m+n)^{\frac{1}{q}}} \left(\frac{m+n}{m+n-1}\right)^{\frac{\alpha}{p}} \left(\frac{m+n}{m+n-1}\right)^{\frac{\beta}{q}} \\ &\leq \left(\sum_{m=1}^{\infty} a_m^p \left(\sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}}\right)\right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{n=1}^{\infty} b_n^q \left(\sum_{m=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}} \frac{1}{(m+n)^{1-\beta}(m+n-1)^{\beta}}\right)\right)^{\frac{1}{q}}. \end{aligned}$$

Hence it is enough to prove

$$(2.1) \quad \sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}} \leq \frac{\pi}{\sin \frac{\pi}{p}}, \quad m \geq 1,$$

and

$$(2.2) \quad \sum_{m=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{1}{q}} \frac{1}{(m+n)^{1-\beta}(m+n-1)^{\beta}} \leq \frac{\pi}{\sin \frac{\pi}{q}}, \quad n \geq 1,$$

where, of course, $\sin \frac{\pi}{p} = \sin \frac{\pi}{q}$.

Now we observe that, for all $\alpha \geq 0, p > 0, m \geq 1$, the positive function

$$f(t) = t^{-\frac{1}{p}}(m+t)^{\alpha-1}(m+t-1)^{-\alpha}, \quad t > 0,$$

is convex. Indeed, taking the second derivative of the logarithm of $f(t)$, we get

$$\frac{f(t)f''(t) - f'(t)^2}{f(t)^2} = \frac{t^{-2}}{p} + (m+t)^{-2} + \alpha((m+t-1)^{-2} - (m+t)^{-2}) > 0,$$

which proves that $f''(t) > 0$. In fact, this calculation proves more: that f is logarithmically convex.

The convexity of f implies

$$f(n) \leq \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(t) dt, \quad n \geq 1.$$

Adding these inequalities we get for the left side of (2.1) that

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{m}{n}\right)^{\frac{1}{p}} \frac{1}{(m+n)^{1-\alpha}(m+n-1)^{\alpha}} \\ & \leq \int_{\frac{1}{2}}^{\infty} \left(\frac{m}{t}\right)^{\frac{1}{p}} \frac{1}{(m+t)^{1-\alpha}(m+t-1)^{\alpha}} dt \\ & = \int_{\frac{1}{2m}}^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-\frac{1}{m})^{\alpha}} dt \end{aligned}$$

by the change of variables $t \mapsto mt$.

Therefore, in order to prove (2.1) it is enough to prove

$$(2.3) \quad \int_{\frac{1}{2m}}^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-\frac{1}{m})^{\alpha}} dt \leq \frac{\pi}{\sin \frac{\pi}{p}}, \quad m \geq 1.$$

We consider now the function

$$\begin{aligned} F(y) &= \int_y^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)^{1-\alpha}(t+1-2y)^{\alpha}} dt \\ &= \int_0^{\infty} \frac{1}{(t+y)^{\frac{1}{p}}(t+1+y)^{1-\alpha}(t+1-y)^{\alpha}} dt, \quad 0 \leq y \leq \frac{1}{2}. \end{aligned}$$

Hence in order to prove (2.3) it is enough to prove

$$(2.4) \quad F(y) \leq \frac{\pi}{\sin \frac{\pi}{p}}, \quad 0 \leq y \leq \frac{1}{2}.$$

Now, exactly as before, we observe that, for all $\alpha \geq 0$, $p > 0$, $t > 0$, the positive function

$$g_t(y) = (t+y)^{-\frac{1}{p}}(t+1+y)^{\alpha-1}(t+1-y)^{-\alpha}, \quad 0 \leq y \leq \frac{1}{2},$$

is convex. Indeed, we take the second derivative of the logarithm of $g_t(y)$ and we get

$$\begin{aligned} \frac{g_t(y)g_t''(y) - g_t'(y)^2}{g_t(y)^2} &= \frac{(t+y)^{-2}}{p} + (t+1+y)^{-2} \\ &\quad + \alpha((t+1-y)^{-2} - (t+1+y)^{-2}) > 0, \end{aligned}$$

which proves that $g_t''(y) > 0$.

Thus $F(y) = \int_0^{\infty} g_t(y) dt$ is also convex and, as such, it satisfies

$$F(y) \leq \max \left\{ F(0), F\left(\frac{1}{2}\right) \right\}.$$

Since

$$F(0) = \int_0^{\infty} \frac{1}{t^{\frac{1}{p}}(t+1)} dt = \frac{\pi}{\sin \frac{\pi}{p}},$$

in order to prove (2.4) it is enough to prove

$$F\left(\frac{1}{2}\right) \leq \frac{\pi}{\sin \frac{\pi}{p}}.$$

Since

$$F\left(\frac{1}{2}\right) = \int_{1/2}^{\infty} \frac{(t+1)^{\alpha-1}}{t^{\frac{1}{p}+\alpha}} dt = \int_0^2 \frac{(t+1)^{\alpha}}{t^{1-\frac{1}{p}}(t+1)} dt$$

after the change of variables $t \mapsto \frac{1}{t}$, we conclude that in order to prove (2.1) it is enough to prove

$$\int_0^2 \frac{(t+1)^\alpha}{t^{1-\frac{1}{p}}(t+1)} dt \leq \frac{\pi}{\sin \frac{\pi}{p}}.$$

In exactly the same manner, we see that in order to prove (2.2) it is enough to prove

$$\int_0^2 \frac{(t+1)^\beta}{t^{1-\frac{1}{q}}(t+1)} dt \leq \frac{\pi}{\sin \frac{\pi}{q}}.$$

We make the change of notation

$$x = \frac{1}{p}, \quad 1-x = \frac{1}{q},$$

and, after $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we write

$$\beta = \frac{1-\alpha x}{1-x},$$

where $0 \leq \alpha x \leq 1$. Then our last two inequalities become

$$(2.5) \quad \int_0^2 \frac{(t+1)^\alpha}{t^{1-x}(t+1)} dt \leq \frac{\pi}{\sin \pi x} = \int_0^\infty \frac{1}{t^{1-x}(t+1)} dt$$

and

$$(2.6) \quad \int_0^2 \frac{(t+1)^{\frac{1-\alpha x}{1-x}}}{t^x(t+1)} dt \leq \frac{\pi}{\sin \pi x} = \int_0^\infty \frac{1}{t^x(t+1)} dt.$$

Now, inequality (2.5) is equivalent to

$$\int_0^2 \frac{(t+1)^\alpha - 1}{t^{1-x}(t+1)} dt \leq \int_2^\infty \frac{1}{t^{1-x}(t+1)} dt$$

or, after the change of variables $t \mapsto 2t$, to

$$\int_0^1 \frac{(2t+1)^\alpha - 1}{t^{1-x}(2t+1)} dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt,$$

or finally, substituting $t \mapsto \frac{1}{t}$ in the left integral, to the inequality

$$(2.7) \quad \int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^\alpha - 1}{t^x(t+2)} dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt, \quad 0 < x \leq \frac{1}{2}.$$

Similarly, inequality (2.6) is equivalent to

$$(2.8) \quad \int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^{\frac{1-\alpha x}{1-x}} - 1}{t^{1-x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^x(2t+1)} dt, \quad 0 < x \leq \frac{1}{2}.$$

So we have come to the point where, for every x with $0 < x \leq \frac{1}{2}$, we have to prove inequalities (2.7) and (2.8) for a proper choice of α with $0 \leq \alpha \leq \frac{1}{x}$.

A very useful observation for what follows is that for fixed α with $0 \leq \alpha \leq 1$, if (2.7) holds for some x , then it holds for all larger x . The reason is that the left-hand side in (2.7) is a decreasing function of x and the right-hand side in (2.7) is an increasing function of x . Similarly, if (2.8) holds for some x , then it holds for all smaller x . It helps to see that for fixed α with $0 \leq \alpha \leq 1$ the function $\frac{1-\alpha x}{1-x}$ is increasing.

Now we split the interval $0 < x \leq \frac{1}{2}$ in three subintervals in each of which we make the corresponding choices $\alpha = 0$, $\alpha = 1$ and $\alpha = \frac{1}{2}$.

The case $0 < x \leq \frac{1}{3}$.

Let $\alpha = 0$. First of all, it is obvious that (2.7) is true for all $0 < x \leq \frac{1}{2}$. We claim that (2.8) is valid for all $0 < x \leq \frac{1}{3}$ and as we observed it is enough to prove it for $x = \frac{1}{3}$.

Observe now that $0 < x \leq \frac{1}{2}$ implies $0 < \frac{x}{1-x} \leq 1$, so by Bernoulli's inequality we get

$$\begin{aligned} \left(1 + \frac{2}{t}\right)^{\frac{1}{1-x}} &= \left(1 + \frac{2}{t}\right) \left(1 + \frac{2}{t}\right)^{\frac{x}{1-x}} \leq \left(1 + \frac{2}{t}\right) \left(1 + \frac{x}{1-x} \frac{2}{t}\right) \\ &= 1 + \frac{2}{t} + \frac{x}{1-x} \frac{2(t+2)}{t^2}. \end{aligned}$$

Hence

$$\int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^{\frac{1}{1-x}} - 1}{t^{1-x}(t+2)} dt \leq \int_1^\infty \frac{2}{t^{2-x}(t+2)} dt + \frac{2x}{1-x} \int_1^\infty \frac{1}{t^{3-x}} dt.$$

Using

$$(2.9) \quad \frac{2}{t(t+2)} = \frac{1}{t} - \frac{1}{t+2}$$

the last inequality becomes

$$\begin{aligned} \int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^{\frac{1}{1-x}} - 1}{t^{1-x}(t+2)} dt &\leq \int_1^\infty \frac{1}{t^{2-x}} dt - \int_1^\infty \frac{1}{t^{1-x}(t+2)} dt + \frac{2x}{(1-x)(2-x)} \\ &= \frac{2+x}{(1-x)(2-x)} - \int_1^\infty \frac{1}{t^{1-x}(t+2)} dt. \end{aligned}$$

Hence in order to prove (2.8) it is enough to have

$$\begin{aligned} \frac{2+x}{(1-x)(2-x)} &\leq \int_1^\infty \frac{1}{t^{1-x}(t+2)} dt + \int_1^\infty \frac{1}{t^x(2t+1)} dt \\ &= \int_0^1 \frac{1}{t^x(2t+1)} dt + \int_1^\infty \frac{1}{t^x(2t+1)} dt = \int_0^\infty \frac{1}{t^x(2t+1)} dt \\ &= 2^{x-1} \int_0^\infty \frac{1}{t^x(t+1)} dt = 2^{x-1} \frac{\pi}{\sin \pi x}. \end{aligned}$$

For $x = \frac{1}{3}$ this becomes $\frac{21}{10} \leq \frac{2^{\frac{1}{3}}\pi}{\sqrt{3}}$ which is true and proves our claim. We proved that when $\alpha = 0$ both (2.7) and (2.8) hold for $0 < x \leq \frac{1}{3}$.

The case $\frac{2}{5} \leq x \leq \frac{1}{2}$.

Let $\alpha = 1$. In this case (2.7) becomes

$$(2.10) \quad \int_1^\infty \frac{2}{t^{1+x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt.$$

We claim that this inequality is true for $\frac{2}{5} \leq x \leq \frac{1}{2}$ and it suffices to prove it for $x = \frac{2}{5}$.

Using (2.9), the left-hand side of (2.10) becomes

$$\begin{aligned} \int_1^\infty \frac{2}{t^{1+x}(t+2)} dt &= \int_1^\infty \frac{1}{t^{1+x}} dt - \int_1^\infty \frac{1}{t^x(t+2)} dt \\ &= \frac{1}{x} - \int_1^\infty \frac{1}{t^x(t+2)} dt. \end{aligned}$$

Therefore, (2.10) amounts to showing the inequality

$$\begin{aligned} \frac{1}{x} &\leq \int_1^\infty \frac{1}{t^x(t+2)} dt + \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt = \int_0^\infty \frac{1}{t^x(t+2)} dt \\ &= 2^{-x} \int_0^\infty \frac{1}{t^x(t+1)} dt = 2^{-x} \frac{\pi}{\sin(\pi x)} \end{aligned}$$

for $x = \frac{2}{5}$. But the inequality $\frac{\sin \pi x}{\pi x} \leq 2^{-x}$ can be easily proved for $x = \frac{2}{5}$ using the first three non-zero terms of the Taylor expansion of $\frac{\sin \pi x}{\pi x}$ and doing a few straightforward calculations. Thus, (2.7) is valid for $\frac{2}{5} \leq x \leq \frac{1}{2}$.

We now turn to (2.8), and we claim that it holds for $0 < x \leq \frac{1}{2}$ and it suffices to prove it for $x = \frac{1}{2}$. When $\alpha = 1$, (2.8) becomes

$$\int_1^\infty \frac{2}{t^{2-x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^x(2t+1)} dt$$

or, by the use of (2.9),

$$\int_1^\infty \frac{1}{t^{2-x}} dt - \int_1^\infty \frac{1}{t^{1-x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^x(2t+1)} dt.$$

This is equivalent to

$$\begin{aligned} \frac{1}{1-x} &\leq \int_1^\infty \frac{1}{t^{1-x}(t+2)} dt + \int_1^\infty \frac{1}{t^x(2t+1)} dt = \int_0^\infty \frac{1}{t^{1-x}(t+2)} dt \\ &= 2^{x-1} \frac{\pi}{\sin \pi x}. \end{aligned}$$

When $x = \frac{1}{2}$ this becomes $2\sqrt{2} \leq \pi$ and it is clearly true. We proved that when $\alpha = 1$ both (2.7) and (2.8) hold for $\frac{2}{5} \leq x \leq \frac{1}{2}$.

The case $\frac{1}{3} \leq x \leq \frac{2}{5}$.

Let $\alpha = \frac{1}{2}$. We first deal with inequality (2.7), which we shall prove for $\frac{1}{3} \leq x \leq \frac{2}{5}$. As we know it is enough to prove it for $x = \frac{1}{3}$. When $\alpha = \frac{1}{2}$, (2.7) becomes

$$\int_1^\infty \frac{(1 + \frac{2}{t})^{\frac{1}{2}} - 1}{t^x(t+2)} dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt.$$

Bernoulli's inequality gives

$$\left(1 + \frac{2}{t}\right)^{\frac{1}{2}} \leq 1 + \frac{1}{2} \frac{2}{t} = 1 + \frac{1}{t}$$

and in view of (2.7) it suffices to show that

$$\int_1^\infty \frac{1}{t^{1+x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^{1-x}(2t+1)} dt$$

for $x = \frac{1}{3}$. This is indeed true, since

$$t^{\frac{2}{3}}(2t+1) \leq t^{\frac{4}{3}}(t+2), \quad t \geq 1,$$

as we easily see by raising to the third power.

We now turn to (2.8) which for $\alpha = \frac{1}{2}$ becomes

$$\int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^{\frac{1}{2} \frac{2-x}{1-x}} - 1}{t^{1-x}(t+2)} dt \leq \int_1^\infty \frac{1}{t^x(2t+1)} dt,$$

and we claim it holds for $\frac{1}{3} \leq x \leq \frac{2}{5}$. Again it suffices to prove this inequality for $x = \frac{2}{5}$. Namely, it suffices to show

$$(2.11) \quad \int_1^\infty \frac{\left(1 + \frac{2}{t}\right)^{\frac{4}{3}} - 1}{t^{\frac{3}{5}}(t+2)} dt \leq \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)} dt.$$

Taking into account Bernoulli's inequality, we have

$$\left(1 + \frac{2}{t}\right)^{\frac{4}{3}} = \left(1 + \frac{2}{t}\right) \left(1 + \frac{2}{t}\right)^{\frac{1}{3}} \leq \left(1 + \frac{2}{t}\right) \left(1 + \frac{1}{3} \frac{2}{t}\right) = 1 + \frac{4}{3t^2}(2t+1),$$

so instead of (2.11), it suffices to prove

$$(2.12) \quad \frac{4}{3} \int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)} dt \leq \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)} dt.$$

Observe that the left-hand side of (2.12), in view of (2.9), is equal to

$$\begin{aligned} \frac{4}{3} \int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)} dt &= \frac{2}{3} \int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}} dt - \frac{2}{3} \int_1^\infty \frac{2t+1}{t^{1+\frac{3}{5}}(t+2)} dt \\ &= \frac{4}{3} \int_1^\infty \frac{1}{t^{1+\frac{3}{5}}} dt + \frac{2}{3} \int_1^\infty \frac{1}{t^{2+\frac{3}{5}}} dt - \frac{4}{3} \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)} dt \\ &\quad - \frac{2}{3} \int_1^\infty \frac{1}{t^{1+\frac{3}{5}}(t+2)} dt \\ &= \frac{20}{9} + \frac{5}{12} - \frac{4}{3} \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)} dt - \frac{1}{3} \int_1^\infty \frac{1}{t^{1+\frac{3}{5}}} dt + \frac{1}{3} \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)} dt, \end{aligned}$$

where we used (2.9) for the last equality. Thus, altogether we have

$$\frac{4}{3} \int_1^\infty \frac{2t+1}{t^{2+\frac{3}{5}}(t+2)} dt = \frac{25}{12} - \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)} dt.$$

Therefore, (2.12) is equivalent to the inequality

$$\frac{25}{12} \leq \int_1^\infty \frac{1}{t^{\frac{3}{5}}(t+2)} dt + \int_1^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)} dt = \int_0^\infty \frac{1}{t^{\frac{2}{5}}(2t+1)} dt = \frac{2^{-\frac{3}{5}}\pi}{\sin \frac{3\pi}{5}}.$$

This inequality is an easy consequence of the inequality $\frac{\sin \frac{2\pi}{5}}{\frac{2\pi}{5}} < 2^{-\frac{2}{5}}$, which we proved when we considered the case $\alpha = 1$, and of the equality $\sin \frac{3\pi}{5} = \sin \frac{2\pi}{5}$.

We proved that when $\alpha = \frac{1}{2}$ both (2.7) and (2.8) hold for $\frac{1}{3} \leq x \leq \frac{2}{5}$.

Therefore, we have proved the inequality of our theorem and now we shall show that the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is the best possible in this inequality. The proof follows the lines of Hardy's corresponding proof for the original Hilbert's inequality [8, proof of Theorem 317, p. 232], adapted to our weighted setting. For the sake of completeness, we provide the details.

We consider any $\epsilon > 0$ and the sequences $(a_m(\epsilon))$ and $(b_n(\epsilon))$ defined by

$$a_m(\epsilon) = m^{-\frac{1+\epsilon}{p}}, \quad b_n(\epsilon) = n^{-\frac{1+\epsilon}{q}}.$$

We then have

$$\|(a_m(\epsilon))\|_{\ell^p}^p = \sum_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}}.$$

Now, since $\frac{1}{x^{1+\epsilon}}$ is decreasing for $x \geq 1$, we have

$$\frac{1}{\epsilon} = \int_1^{\infty} \frac{1}{x^{1+\epsilon}} dx \leq \sum_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}} \leq 1 + \int_1^{\infty} \frac{1}{x^{1+\epsilon}} dx = 1 + \frac{1}{\epsilon}.$$

Setting $\phi(\epsilon) = \sum_{m=1}^{\infty} \frac{1}{m^{1+\epsilon}} - \frac{1}{\epsilon}$, we get

$$(2.13) \quad \|(a_m(\epsilon))\|_{\ell^p}^p = \frac{1}{\epsilon} + \phi(\epsilon), \quad 0 \leq \phi(\epsilon) \leq 1.$$

Respectively, setting $\psi(\epsilon) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} - \frac{1}{\epsilon}$, we have

$$(2.14) \quad \|(b_n(\epsilon))\|_{\ell^q}^q = \frac{1}{\epsilon} + \psi(\epsilon), \quad 0 \leq \psi(\epsilon) \leq 1.$$

In addition, we have that

$$(2.15) \quad \sum_{m,n=1}^{\infty} \binom{n}{m}^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n-1} \geq \sum_{m,n=1}^{\infty} \binom{n}{m}^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n}.$$

Now for (x, y) in the square $[m, m+1) \times [n, n+1)$, $m \geq 1$, $n \geq 1$, we have

$$\begin{aligned} \binom{n}{m}^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n} &= \binom{n}{m}^{\frac{1}{q}-\frac{1}{p}} \frac{m^{-\frac{1+\epsilon}{p}} n^{-\frac{1+\epsilon}{q}}}{m+n} = \frac{m^{-\frac{1}{q}-\frac{\epsilon}{p}} n^{-\frac{1}{p}-\frac{\epsilon}{q}}}{m+n} \\ &\geq \frac{x^{-\frac{1}{q}-\frac{\epsilon}{p}} y^{-\frac{1}{p}-\frac{\epsilon}{q}}}{x+y} = \left(\frac{y}{x}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{x^{-\frac{1+\epsilon}{p}} y^{-\frac{1+\epsilon}{q}}}{x+y}. \end{aligned}$$

Therefore

$$(2.16) \quad \sum_{m,n=1}^{\infty} \binom{n}{m}^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n} \geq I(\epsilon),$$

where $I(\epsilon)$ is defined by

$$I(\epsilon) = \int_1^{\infty} \int_1^{\infty} \left(\frac{y}{x}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{x^{-\frac{1+\epsilon}{p}} y^{-\frac{1+\epsilon}{q}}}{x+y} dx dy = \int_1^{\infty} \int_1^{\infty} \frac{x^{-\frac{1}{q}-\frac{\epsilon}{p}} y^{-\frac{1}{p}-\frac{\epsilon}{q}}}{x+y} dx dy.$$

Applying the change of variables $y \mapsto xy$, we get

$$I(\epsilon) = \int_1^{\infty} \frac{1}{x^{1+\epsilon}} \int_{\frac{1}{x}}^{\infty} \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)} dy dx.$$

Another change of variables $x \mapsto \frac{1}{x}$ gives

$$\begin{aligned} I(\epsilon) &= \int_0^1 x^{\epsilon-1} \int_x^{\infty} \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)} dy dx \\ &= \int_0^1 \frac{1}{\epsilon} (x^{\epsilon})' \int_x^{\infty} \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)} dy dx \\ &= \frac{1}{\epsilon} \left(\int_1^{\infty} \frac{1}{y^{\frac{1}{p}+\frac{\epsilon}{q}}(1+y)} dy + \int_0^1 \frac{1}{x^{\frac{1}{p}-\frac{\epsilon}{q}}(1+x)} dx \right) \end{aligned}$$

by integration by parts. From this we notice that

$$\epsilon I(\epsilon) \rightarrow \int_0^\infty \frac{1}{t^{\frac{1}{p}}(1+t)} dt = \frac{\pi}{\sin \frac{\pi}{p}}$$

when $\epsilon \rightarrow 0^+$. This together with (2.13), (2.14), (2.15) and (2.16) implies

$$\frac{\sum_{m,n=1}^\infty \left(\frac{n}{m}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{a_m(\epsilon)b_n(\epsilon)}{m+n-1}}{\|(a_m(\epsilon))\|_{\ell^p} \|(b_n(\epsilon))\|_{\ell^q}} \geq \frac{\epsilon I(\epsilon)}{(1+\epsilon\phi(\epsilon))^{\frac{1}{p}}(1+\epsilon\psi(\epsilon))^{\frac{1}{q}}} \rightarrow \frac{\pi}{\sin \frac{\pi}{p}}$$

when $\epsilon \rightarrow 0^+$. \square

3. THE NORM OF THE HILBERT MATRIX ON THE HARDY-LITTLEWOOD SPACES AND ON WEIGHTED SEQUENCE SPACES

Our second result is the determination of the exact value of the norm $\|\mathcal{H}\|_{K^p \rightarrow K^p}$ for $1 < p < \infty$. To that effect we shall use the variant of Hilbert's inequality in Theorem 1.

Theorem 2. *If $1 < p < \infty$, then the Hilbert matrix operator is bounded on the Hardy-Littlewood space K^p with norm*

$$\|\mathcal{H}\|_{K^p \rightarrow K^p} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

Proof. Let $f(z) = \sum_{m=0}^\infty a_m z^m \in K^p$. Then

$$\mathcal{H}(f)(z) = \sum_{n=0}^\infty \left(\sum_{m=0}^\infty \frac{a_m}{m+n+1} \right) z^n,$$

and

$$\begin{aligned} \|\mathcal{H}(f)\|_{K^p} &\leq \left(\sum_{n=0}^\infty (n+1)^{p-2} \left(\sum_{m=0}^\infty \frac{|a_m|}{m+n+1} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{n=0}^\infty \left(\sum_{m=0}^\infty (n+1)^{\frac{p-2}{p}} \frac{|a_m|}{m+n+1} \right)^p \right)^{\frac{1}{p}}. \end{aligned}$$

Due to the duality of ℓ^p spaces,

$$\|\mathcal{H}(f)\|_{K^p} \leq \sup_{\|(b_n)\|_{\ell^q}=1, b_n \geq 0} \sum_{m,n=0}^\infty (n+1)^{\frac{p-2}{p}} \frac{|a_m|b_n}{m+n+1},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Setting $A_m = |a_m|(m+1)^{\frac{p-2}{p}}$, we have that $\|(A_m)\|_{\ell^p} = \|f\|_{K^p}$ and

$$\sup_{\|f\|_{K^p}=1} \|\mathcal{H}(f)\|_{K^p} \leq \sup_{\substack{\|(A_m)\|_{\ell^p}=1, \\ \|(b_n)\|_{\ell^q}=1, b_n \geq 0}} \sum_{m,n=0}^\infty \left(\frac{n+1}{m+1}\right)^{\frac{1}{q}-\frac{1}{p}} \frac{A_m b_n}{m+n+1} = \frac{\pi}{\sin \frac{\pi}{p}},$$

because of Theorem 1. This proves that $\|\mathcal{H}\|_{K^p \rightarrow K^p} \leq \frac{\pi}{\sin \frac{\pi}{p}}$.

The equality $\|\mathcal{H}\|_{K^p \rightarrow K^p} = \frac{\pi}{\sin \frac{\pi}{p}}$ follows when we consider $f(z) = \sum_{m=0}^\infty a_m z^m \in K^p$ with non-negative coefficients a_m , since then all previous inequalities become equalities. \square

One final remark is that the proof of Theorem 2 applies unchanged and in an obvious way to show that the Hilbert matrix \mathcal{H} induces a bounded operator on the weighted space l_{p-2}^p of sequences (a_m) with norm defined by

$$\|(a_m)\|_{l_{p-2}^p}^p = \sum_{m=1}^{\infty} m^{p-2} |a_m|^p,$$

and that the norm $\|\mathcal{H}\|_{l_{p-2}^p \rightarrow l_{p-2}^p}$ of this operator is again equal to $\frac{\pi}{\sin \frac{\pi}{p}}$.

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Email address: vdaskalo@math.auth.gr

DIVISION OF SCIENCE AND TECHNOLOGY, AMERICAN COLLEGE OF THESSALONIKI, 17 V. SEVENIDI ST., 55535, PYLEA, GREECE.

Email address: `petrosgala@math.auth.gr`

DEPARTMENT OF MATHEMATICS, ARISTOTLE UNIVERSITY OF THESSALONIKI, 54124, THESSALONIKI, GREECE.

Email address: `mpapadim@uoc.gr`

DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CRETE, 70013, HERAKLION, GREECE.