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DOCTORAL DISSERTATION:

Closed Range Integral and Composition Operators on Spaces of Analytic Functions

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Preface

If X is a space of analytic functions f in $\mathbb{D}$, the open unit disk, then the integral S_g and the composition C_φ operators on X are defined as $S_g f(z) = \int_0^z f'(w)g(w)dw$ and $C_\varphi(f) = f \circ \varphi$ respectively. This thesis is about finding necessary and sufficient conditions for S_g and C_φ to have closed range or equivalently to be bounded below on some spaces of analytic functions.

Four conditions for the integral operator S_g to have closed range on Hardy $H^p(1 \leq p < \infty)$, BMOA, $Q_p(0 < p < \infty)$, and Besov $B^p(1 < p < \infty)$ spaces, respectively, are proved. All these conditions are based upon the behaviour of function g in the disk $\mathbb{D}$.

We also prove that, two already known conditions for C_φ to have closed range on Hardy space H^2, can be extended to all Hardy spaces $H^p, 0 < p < \infty$. The first condition concerns the behaviour of φ at the boundary of the disk $\mathbb{D}$. The second one is based upon the behaviour of φ in the disk $\mathbb{D}$ and we prove this by using Hardy-Stein identities for one of the directions, and reverse Carleson measures and pull-back measures for the converse.

Moreover, two necessary conditions and one sufficient condition, for C_φ to have the property of being bounded below on $BMOA$ space, are presented.

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To Katerina

Contents

List of Symbols

Symbol	Definition
$\mathbb{D}$	The open unit disk in the complex plane
S_g	The integral operator induced by the analytic function (symbol) g
T_g	The integral operator, companion to integral operator S_g
M_z	The multiplication operator
C_φ	The composition operator induced by the analytic function φ, $C_\varphi(f) = f \circ \varphi$
$\mathbb{T}$	The unit circle in the complex plane
A	The normalized area Lebesgue measure in the open unit disk $\mathbb{D}$
m	The normalized length Lebesgue measure in the unit circle $\mathbb{T}$
$\rho(z,w)$	The pseudohyberbolic distance between $z, w \in \mathbb{D}$
$D_\eta(a)$	The pseudohyberbolic disk of center $a \in \mathbb{D}$ and radius $\eta < 1$
C	A positive and finite constant which may change from one occurrence to another
H^p	The Hardy space H^p of analytic functions in $\mathbb{D}$
H^∞	The space of bounded analytic functions in $\mathbb{D}$
$\Gamma_\beta(\zeta)$	The Stolz angle at $\zeta \in \mathbb{T}$, the conelike region with aperture $\beta \in (0,1)$
A^p	The Bergman space of analytic functions in $\mathbb{D}$
A^p_γ	The weighted Bergman space of analytic functions in $\mathbb{D}(\gamma > -1)$
$BMOA$	Space of analytic functions of bounded mean oscillation
B^p	The Besov space of analytic functions in $\mathbb{D}$
$B_{p,\alpha}$	The Besov type space of analytic functions in $\mathbb{D}$
$\mathcal{D}$	The Dirichlet space of analytic functions in $\mathbb{D}$
$\mathcal{B}$	The Bloch space of analytic functions in $\mathbb{D}$
Q_p	The Q_p space of analytic functions in $\mathbb{D}$
ψ_α	The Möbius transformations, the conformal mappings of the open unit disk to itself
N_φ	The Nevanlinna counting function
n_φ	A counting function, $n_\varphi(w)$ is the number of pre-images of w
$\asymp$	Comparability
$E(z_0; r)$	The euclidean disk of center $z_0 \in \mathbb{D}$ and radius $r < 1, r < 1 - \lvert z_0\rvert$
$\Re(z)$	The real part of complex number z
μ^s_α	The singular part of measure μ_α
$\frac{d\nu}{dm}$	The Radon-Nikodym derivative of measure ν with respect to measure m
S_{h,θ_0}	Carleson square, defined as $S_{h,\theta_0} = \{re^{i\theta} \in \mathbb{D} : 1 - h \leq r < 1, \lvert\theta - \theta_0\rvert \leq h\}$
X/Y	The quotient space of X and Y spaces
$\Delta_\eta(\alpha)$	The euclidean disk of center $\alpha \in \mathbb{D}$ and radius $\eta(1 - \lvert\alpha\rvert), \eta < 1$
χ_B	The characteristic function of set B
$Arg(z)$	The principal value of the argument of complex number z, i.e. $Arg(z) \in (-\pi, \pi]$
$AB \perp CD$	Line segment AB is perpendicular to line segment CD
$\angle AOE$	Angle with arms OA and OE and vertex the point O
(OA)	The length of the line segment with end points O and A
$\triangle ABC$	The triangle with vertices at points A, B and C

Chapter 1

Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $g : \mathbb{D} \to \mathbb{C}$ be an analytic function. If X is a space of analytic functions f in $\mathbb{D}$ then, the integral operator $S_g : X \to X$, induced by g, is defined as

$$S_g f(z) = \int_0^z f'(w)g(w)dw, \;\; z \in \mathbb{D},$$

for every $f \in X$. S_g is companion to the operator $T_g : X \to X$ which is defined as

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \;\; z \in \mathbb{D},$$

for every $f \in X$. If $g(z) = z$ or $g(z) = \log \frac{1}{(1-z)}$, then T_g is the integration operator and the Cesáro operator respectively. Interest in T_g arose originally from studying semigroups of analytic composition operators because, for certain g, T_g are related to the resolvents of such semigroups (see [46]). Results, concerning the boundedness and compactness of T_g on certain spaces of analytic functions, can be found in [5, 6, 17, 47]. It can be easily seen (using integration by parts) that T_g and its companion operator S_g are related to the multiplication operator

$$M_g f(z) = g(z)f(z)$$

by

$$M_g f(z) = f(0)g(0) + T_g f + S_g f.$$

If any two of M_g, S_g and T_g are bounded, then so is the third, but in some situations one operator is bounded while the other two are unbounded.

Moreover, if $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic function then, the composition operator $C_\varphi : X \to X$ induced by φ, is defined as

$$C_\varphi(f) = f \circ \varphi,$$

for every $f \in X$.

In this thesis, we studied the problem of finding conditions, depending only on g in case of the operator S_g and depending only on φ in case of the operator C_φ, so that the operators S_g and C_φ to have closed ranges in space X, which means, the set $S_g(X)$ or $C_\varphi(X)$ to be a closed subset of X.

In general, if X and Y are normed spaces then the operator $T : X \to Y$ is bounded below if there exists $C > 0$ such that

$$\|Tx\|_Y \geq C\|x\|_X$$

for all $x \in X$.

From the basic operator theory (see [1], theorem 2.5) we know the following result.

Theorem 1.0.1. *Let X and Y be Banach spaces and $T : X \to Y$ bounded operator. Then, T is bounded below if and only if T is 1-1 and has closed range.*

It's easy to see that $S_g f(z) \equiv 0$ on $\mathbb{D}$ if and only if $f(z) \equiv C$ on $\mathbb{D}$ (for some constant $C \in \mathbb{C}$) or $g(z) \equiv 0$ on $\mathbb{D}$ and, clearly, S_g cannot be 1-1. In case where $g(z) \equiv 0$ on $\mathbb{D}$, we have that $S_g(X) = \{\mathbf{0}\}$ (where by $\mathbf{0}$ we denote the function which is identically equal to zero on $\mathbb{D}$) and it's obvious that S_g has closed range since $\{\mathbf{0}\}$ is a closed subset of X.

Moreover, in case where $g(z) \not\equiv 0$ on $\mathbb{D}$, if we restrict the study of S_g to space X_0 of analytic functions modulo the constants (the quotient space $X_0 :=$

$X/\mathbb{C}$) or, equivalently, the space of analytic functions f such that $f(0) = 0$, then S_g, obviously, is 1-1. This restriction can be made without loss of generality since it's easy to check that the following equivalence holds: $S_g : X \to X$ has closed range if and only if $S_g : X_0 \to X_0$ has closed range.

According to theorem 1.0.1, for the operators $S_g : X_0 \to X_0$ and $C_\varphi : X \to X$ to have closed range is equivalent to be bounded below, i.e. there is $C_1 > 0$ such that $\|S_g f\|_{X_0} > C_1 \|f\|_{X_0}$ for all $f \in X_0$ and, respectively, there is $C_2 > 0$ such that $\|C_\varphi f\|_X > C_2 \|f\|_X$ for all $f \in X$.

This thesis is organized in three parts. In part I, all the appropriate preliminaries and background results are presented, as well as, a short history of the research and the already published results, concerning conditions for the integral and composition operators to have closed range. In parts II and III, all the new results which came up in this thesis are presented.

In part II, four conditions for the integral operator to have closed range on Hardy, BMOA, Q_p and Besov spaces, respectively, are presented. More specifically, in chapters 4, 5, 6, 7, results (theorems 4.1.2, 5.2.1, 6.2.1 and 7.2.1) concerning necessary and sufficient conditions for the integral operator S_g to have closed range on Hardy ($H^p, 1 \leq p < \infty$), BMOA, Q_p, ($0 < p < \infty$) and Besov ($B^p, 1 < p < \infty$) spaces, respectively, are proved.

Part III is dedicated to the study of property of having closed range for composition operators. In chapter 9 we prove that two already known conditions for C_φ to have closed range on the Hardy space H^2, can be extended to all Hardy spaces $H^p, 0 < p < \infty$. The first condition (theorem 9.1.2 (part (ii)) concerns the behaviour of φ at the boundary of the disk $\mathbb{D}$, while the second one (theorem 9.1.2 (part (iii)) is based upon the behaviour of φ in the disk $\mathbb{D}$.

In chapter 10 two necessary conditions (theorems 10.1.2 and 10.1.3) and one sufficient (theorem 10.2.2) are proved for C_φ to have closed range on the space $BMOA$.

The chapter 11 doesn't contain any new result. We just show that an already known proof of Akeroyd and Fulmer (2008) for C_φ to have closed range on Bergman space A^2 works, with few modifications, for the weighted Bergman spaces A^p $(1 \leq p < \infty,\ \gamma > -1)$.

Part I

Background and Preliminaries

Chapter 2

Background

The basic notation, some definitions, as well as many background results which will widely be used throughout this thesis, are included in this chapter.

Let $\mathbb{T}$ denote the unit circle, A the normalized area Lebesgue measure in the open unit disk $\mathbb{D}$ and m the normalized length Lebesgue measure in $\mathbb{T}$. Let $\rho(z,w)$ denote the pseudohyberbolic distance between $z,w \in \mathbb{D}$,

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right| \tag{2.1}$$

and $D_\eta(a)$ denote the pseudohyberbolic disk of center $a \in \mathbb{D}$ and radius $\eta < 1$:

$$D_\eta(a) = \{z \in \mathbb{D} : \rho(a,z) < \eta\}.$$

Also, in the following, C denotes a positive and finite constant which may change from one occurrence to another.

2.1 Spaces of analytic functions

Next, we present the definitions and the norms of all analytic function spaces which are mentioned in this work.

2.1.1 The Hardy spaces

For $0 < p < \infty$ the Hardy space H^p is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\sup_{0\le r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < +\infty$$

and the corresponding norm in H^p is defined by

$$\|f\|_{H^p}^p = \sup_{0\le r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta). \tag{2.2}$$

When $p = \infty$, we define H^∞ to be the space of bounded analytic functions f in $\mathbb{D}$ and

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}.$$

Let $\zeta \in \mathbb{T}$. Every $f \in H^p$, $0 < p < \infty$, can be extended to the boundary of the unit disk $\mathbb{D}$ by taking the limit

$$f(\zeta) = \lim_{r\to 1-} f(r\zeta), \tag{2.3}$$

since it is well known that this limit exists for $m - a.e.\ \zeta \in \mathbb{T}$. In addition, we have that $\|f\|_{L^p(\mathbb{T})} = \|f\|_{H^p}$ holds.

Another norm in H^p, equivalent to the norm of relation (2.2), is

$$\|f\|_{H^p}^p = \int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta). \tag{2.4}$$

In this work we will mainly make use of two other equivalent norms in H^p. The first one is (see Calderon's theorem in [40], page 213):

$$\|f\|_{H^p}^p = |f(0)|^p + \int_{\mathbb{T}} \Big(\iint_{\Gamma_\beta(\zeta)} |f'(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta), \tag{2.5}$$

where $\Gamma_\beta(\zeta)$ is the Stolz angle at $\zeta \in \mathbb{T}$, the conelike region with aperture $\beta \in (0,1)$, which is defined as

$$\Gamma_\beta(\zeta) = \{z \in \mathbb{D} : |z| < \beta\} \cup \bigcup_{|z|<\beta} [z, \zeta)$$

and the second one (see Hardy-Stein identities in [40], pages 58-59) is:

$$\|f\|_{H^p}^p = |f(0)|^p + \iint_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log \frac{1}{|z|} dA(z). \tag{2.6}$$

In order to gain deeper knowledge about the Hardy spaces see [19], [24], [29] and [43].

2.1.2 The Bergman spaces and the weighted Bergman spaces

For $1 \leq p < \infty$ the Bergman space A^p is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\iint_{\mathbb{D}} |f(z)|^p dA(z) < +\infty \tag{2.7}$$

and the corresponding norm in A^p is defined by

$$\|f\|_{A^p}^p = \iint_{\mathbb{D}} |f(z)|^p dA(z).$$

In this work we will mainly use the following equivalent norm

$$\|f\|_{A^p}^p = |f(0)|^p + \iint_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p dA(z). \tag{2.8}$$

The weighted Bergman space $A^p_\gamma, \gamma > -1$, is defined as the set of all analytic functions f in $\mathbb{D}$ such that

$$\iint_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\gamma dA(z) < \infty.$$

Observe that Bergman spaces as defined in (2.7) are the special case of weighted Bergman spaces A^p_γ, when $\gamma = 0$.

The following theorem, regarding A^p_γ spaces, is proved in [51] (Theorem 4.28).

Theorem 2.1.1. *Suppose $p > 0$, $n \geq 1$, and f analytic function in $\mathbb{D}$. Then, $f \in A^p_\gamma$, $\gamma > -1$ if and only if the function*

$$g(z) = (1 - |z|^2)^n f^{(n)}(z)$$

satisfies the condition

$$\iint_{\mathbb{D}} |g(z)|^p (1 - |z|^2)^\gamma dA(z) < \infty.$$

According to the proof of theorem 2.1.1, we can define in the weighted Bergman spaces A^p_γ the following equivalent norm

$$\|f\|^p_{A^p_\gamma} = |f(0)|^p + \iint_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p+\gamma} dA(z). \tag{2.9}$$

For more information about the Bergman spaces and the weighted Bergman spaces see [18] and [51].

2.1.3 The BMOA space

Let's suppose that $f \in L^1(\mathbb{T})$, I is an interval contained in $\mathbb{T}$ and by $|I|$ we denote the length of I. Then, the mean of f over I is defined as

$$f_I = \frac{1}{|I|} \int_I f(\zeta) dm(\zeta)$$

and the function f is said to have bounded mean oscillation if

$$\sup_I \frac{1}{|I|} \int_I |f(\zeta) - f_I| dm(\zeta) < \infty.$$

The space of all functions $f \in L^1(\mathbb{T})$ that have bounded mean oscillation is called $BMO(\mathbb{T})$ and the corresponding norm is given by

$$\|f\|_* = \sup_I \frac{1}{|I|} \int_I |f(\zeta) - f_I| dm(\zeta). \tag{2.10}$$

Let $z \in \mathbb{D}, \zeta \in \mathbb{T}$. The Poisson kernel is defined as

$$P_z(\zeta) = \frac{1 - |z|^2}{|1 - \bar{z}\zeta|^2} \tag{2.11}$$

and, if $f \in L^1(\mathbb{T})$, the Poisson integral of f,

$$P[f](z) = \int_{\mathbb{T}} f(\zeta) P_z(\zeta) dm(\zeta). \tag{2.12}$$

The BMOA space, the space of analytic functions of bounded mean oscillation, is the set of all analytic functions f in $\mathbb{D}$ which are Poisson integrals of functions that belongs to BMO. An equivalent definition is that BMOA is the space of analytic functions f in $\mathbb{D}$ for which

$$\sup_{\beta \in \mathbb{D}} \iint_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \bar{\beta} z|^2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) < \infty$$

and we may consider the corresponding norm in BMOA given by

$$\|f\|_*^2 = |f(0)|^2 + \sup_{\beta \in \mathbb{D}} \iint_{\mathbb{D}} \frac{1 - |\beta|^2}{|1 - \bar{\beta} z|^2} |f'(z)|^2 \log \frac{1}{|z|} dA(z),$$

which is equivalent to the norm in (2.10).

It's true that $BMOA \subset \bigcap_{p>0} H^p$ and, considering the extension of $f \in BMOA$ to the boundary of the disk, we can also get the following equivalent norms:

$$\|f\|_*^2 = \sup_{\beta \in \mathbb{D}} \int_{\mathbb{T}} |f \circ \psi_\beta(\zeta) - f(\beta)|^2 dm(\zeta) \tag{2.13}$$

and

$$\|f\|_*^2 = \sup_{\beta\in\mathbb{D}} \int_{\mathbb{T}} |f(\zeta) - f(\beta)|^2 P_\beta(\zeta)dm(\zeta), \tag{2.14}$$

where

$$\psi_\beta(\zeta) = \frac{\beta - \zeta}{1-\bar{\beta}\zeta}, \ \zeta \in \mathbb{T}, \beta \in \mathbb{D}.$$

For more details about the $BMOA$ space see [24], [25], [43] and [51].

2.1.4 The Besov spaces and the Besov type spaces

For $1 < p < \infty$ the Besov space B^p is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < +\infty$$

and the corresponding norm in B^p is defined by

$$\|f\|_{B^p}^p = |f(0)|^p + \iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z).$$

If $p > 1$ and $\alpha > -1$ the Besov type space $B_{p,\alpha}$ is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{\alpha} dA(z) < +\infty.$$

In fact, if $p > 1$ and $\alpha = p - 2$ then, the Besov type space $B_{p,p-2}$ is just the Besov space B^p. For more information about the Besov spaces see [30], [48], [51] and [52].

2.1.5 The Dirichlet space

The Dirichlet space is, in fact, the special case of the Besov spaces when $p = 2$. So, the Dirichlet space $\mathcal{D}$ is defined as the set of all analytic functions f in $\mathbb{D}$

for which

$$\iint_{\mathbb{D}} |f'(z)|^2 dA(z) < +\infty$$

and the corresponding norm in $\mathcal{D}$ is defined by

$$\|f\|^2_{\mathcal{D}} = |f(0)|^2 + \iint_{\mathbb{D}} |f'(z)|^2 dA(z).$$

In order to gain deeper knowledge about the Dirichlet space see [9] and [20].

2.1.6 The Bloch space

The Bloch space $\mathcal{B}$ is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\sup_{|z|<1} (1 - |z|^2)|f'(z)| < +\infty$$

and the corresponding norm in $\mathcal{B}$ is defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{|z|<1} (1 - |z|^2)|f'(z)|.$$

For more information on Bloch space see [30], [48] and [51].

2.1.7 The Q_p spaces

For $0 \leq p < \infty$ the Q_p space is defined as the set of all analytic functions f in $\mathbb{D}$ for which

$$\sup_{\beta \in \mathbb{D}} \iint_{\mathbb{D}} \frac{(1 - |\beta|^2)^p}{|1 - \overline{\beta} z|^{2p}} |f'(z)|^2 (1 - |z|^2)^p dA(z) < \infty$$

and we may define the corresponding norm in Q_p space by

$$\|f\|^2_{Q_p} = |f(0)|^2 + \sup_{\beta \in \mathbb{D}} \iint_{\mathbb{D}} \frac{(1 - |\beta|^2)^p}{|1 - \overline{\beta} z|^{2p}} |f'(z)|^2 (1 - |z|^2)^p dA(z).$$

It's clear that $Q_0 = \mathcal{D}$, the Dirichlet space, and also, $Q_1 = BMOA$. If $p \in (1, +\infty)$ then $Q_p = \mathcal{B}$, the Bloch space with an equivalent norm, see [50] (Corollary 1.2.1). For more details and results about the Q_p spaces see [50].

2.2 The Schwarz's and the Pick-Schwarz's lemmas

The Schwarz's and Pick-Schwarz's lemmas are included in every classical book of Complex Analysis and, in addition, are two powerful and very useful tools in the study of many subjects in Analysis and, in particular, in the study of composition operators.

Lemma 2.2.1 (Schwarz)**.** *Let φ analytic on $\mathbb{D}$, $\varphi(0) = 0$ and $|\varphi(z)| \leq 1$ for all $z \in \mathbb{D}$. Then we have that*

- $|\varphi(z)| \leq |z|$, *for all* $z \in \mathbb{D}$ *and*
- $|\varphi'(0)| \leq 1$.

The so-called Pick-Schwarz's lemma is a generalization of Schwarz's lemma.

Lemma 2.2.2 (Pick-Schwarz)**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then for all $\alpha \in \mathbb{D}$ and for all $z \in \mathbb{D}$ we have that*

- $\left|\frac{\varphi(z)-\varphi(\alpha)}{1-\overline{\varphi(\alpha)}\varphi(z)}\right| \leq \left|\frac{z-\alpha}{1-\overline{\alpha}z}\right|$ *and*
- $|\varphi'(\alpha)| \leq \frac{1-|\varphi(\alpha)|^2}{1-|\alpha|^2}$.

2.3 Inner functions

Definition 2.3.1. Inner function is called an analytic function $g \in H^\infty$ with $\|g\|_\infty = 1$ and $|g(z)| = 1$ $m - a.e.$ $z \in \mathbb{T}$.

In this definition, we have considered that g is extended to the boundary of the unit disk according to the relation (2.3).

Special cases of inner functions are the Blaschke products. The Möbius transformations, a special category of Blaschke products, are the conformal mappings of the open unit disk to itself and have the form

$$\psi_\alpha(z) = \lambda \frac{\alpha - z}{1 - \bar{\alpha} z}, \tag{2.15}$$

where $|\lambda| = 1$ and $\alpha \in \mathbb{D}$. Some basic properties of Möbius transformations, which can easily be proved, are $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$ and $\psi_\alpha^{-1} = \psi_\alpha$. Moreover, we have $\psi_\alpha(\mathbb{D}) = \mathbb{D}$, $\psi_\alpha(\mathbb{T}) = \mathbb{T}$ and the following very useful identity.

$$1 - |\psi_\alpha(z)|^2 = \frac{(1 - |z|^2)(1 - |\alpha|^2)}{|1 - \bar{\alpha} z|^2}. \tag{2.16}$$

Also ψ_α is an isometry with respect to pseudohyperbolic metric ρ (defined in (2.1))

$$\rho(\psi_\alpha(z), \psi_\alpha(w)) = \rho(z, w).$$

A Blaschke product is defined as the infinite product

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z}, \tag{2.17}$$

where $m \geq 0$ and $\{a_n\}$ is a non-vanishing sequence satisfying

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty.$$

Some worth mentioning properties of Blaschke products are:

- $B(z) \in H^\infty$
- If $m = 0$ then, the zeros of $B(z)$, are exactly the terms of sequence $\{a_n\}$, while if $m > 0$, the zeros of $B(z)$ are the elements of the set $\{0\} \cup \{a_n\}$.
- $\|B\|_\infty = 1$ and $|B(\zeta)| = 1$ $m - a.e.$ $\zeta \in \mathbb{T}$.

2.4 The Nevanlinna counting function

A powerful tool in the study of composition operators is the Nevanlinna counting function. If $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic function, then, the Nevanlinna counting function N_φ is defined as

$$N_\varphi(w) = \begin{cases} \sum\limits_{\varphi(z)=w} \log \frac{1}{|z|}, & \text{if } w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\} \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

Very useful are the estimates for N_φ listed in the following lemma, which is proved in [44] (page 188).

Lemma 2.4.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then*

- $N_\varphi(w) = \mathcal{O}\Big(\log \frac{1}{|w|}\Big)$, *when* $|w| \to 1$.
- *If* $\varphi(0) = 0$ *then* $N_\varphi(w) \leq \log \frac{1}{|w|}$ *for all* $w \in \mathbb{D}$.

Another counting function, associated to φ, is n_φ, which is defined as

$$n_\varphi(w) = card\{z \in \mathbb{D} : \varphi(z) = w\}.$$

Actually, the value of $n_\varphi(w)$ is exactly the number of pre-images $\{\varphi^{-1}(w)\}$ of w.

2.5 Change of variable

We mention below two change of variable formulas $w = \varphi(z)$ in integrals' calculation, when φ is not 1-1 (non-univalent change of variable). A proof for the first proposition can be found in [51] (page 307).

Proposition 2.5.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. If g is a measurable positive function on $\mathbb{D}$ then:*

$$\iint_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) = \iint_{\mathbb{D}} g(w) N_\varphi(w) dA(w).$$

The second proposition, which is proved in [16] (page 36), is a generalization of the previous one.

Proposition 2.5.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. If g and W are measurable non-negative functions on $\mathbb{D}$ then:*

$$\iint_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 W(z) dA(z) = \iint_{\varphi(\mathbb{D})} g(w) \Big(\sum_{z:\varphi(z)=w} W(z) \Big) dA(w).$$

2.6 Comparable quantities

By writing $K(z) \asymp L(z)$ for the non-negative quantities $K(z)$ and $L(z)$ we mean that $K(z)$ is comparable to $L(z)$ if z belongs to a specific set: there are positive constants C_1 and C_2 independent of z such that

$$C_1 K(z) \leq L(z) \leq C_2 K(z).$$

Next, we list some relations concerning comparable quantities, which will be widely used in the rest of this work.

Let $\alpha \in \mathbb{D}$ and $\eta \in (0,1)$. If $z \in D_\eta(\alpha)$ then, the relations

$$1 - |z|^2 \asymp 1 - |\alpha|^2 \asymp |1 - \overline{\alpha} z|, \tag{2.19}$$

and

$$A(D_\eta(\alpha)) \asymp (1 - |\alpha|^2)^2, \tag{2.20}$$

hold and all the underlying constants depend only on η. For a proof of (2.19) and (2.20), see [51] (proposition 4.5).

Another widely used relation is the following. It holds that

$$\log \frac{1}{|z|} \asymp 1 - |z|^2, \tag{2.21}$$

when $0 < \delta \le |z| < 1$, where δ is fixed but arbitrary.

Let $E(z_0; r)$ denote the euclidean disk of center $z_0 \in \mathbb{D}$ and radius $r < 1 - |z_0|$:

$$E(z_0; r) = \{z \in \mathbb{D} : |z - z_0| < r\}. \tag{2.22}$$

Using (2.21) we can prove the following proposition.

Proposition 2.6.1. *For every analytic function f on $\mathbb{D}$,*

$$\iint_{\mathbb{D}} |f(z)| \log \frac{1}{|z|} dA(z) \asymp \iint_{\mathbb{D}} |f(z)|(1 - |z|^2) dA(z). \tag{2.23}$$

Proof. It is well known that,

$$1 - |z|^2 \le \log \frac{1}{|z|},$$

for $z \in \mathbb{D} \setminus \{0\}$. So, clearly, we have

$$\iint_{\mathbb{D}} |f(z)|(1 - |z|^2) dA(z) \le \iint_{\mathbb{D}} |f(z)| \log \frac{1}{|z|} dA(z). \tag{2.24}$$

Suppose $|z| \le \frac{1}{4}$. Using (2.21) with fixed $\delta = \frac{1}{4}$ and, because of the subharmonicity of $|f|$ and the fact that $1 - |w|^2 \ge \frac{1}{2}$ when $w \in E(z; \frac{1}{4})$, we get

$$\begin{aligned} |f(z)| &\le \frac{2}{A(E(z; \frac{1}{4}))} \iint_{E(z; \frac{1}{4})} |f(w)|(1 - |w|^2) dA(w) \\ &= C \iint_{E(z; \frac{1}{4})} |f(w)|(1 - |w|^2) dA(w) \\ &\le C \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w). \end{aligned}$$

Hence, for $z \in E(0; \frac{1}{4})$, we have

$$|f(z)| \log \frac{1}{|z|} \leq C \log \frac{1}{|z|} \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w)$$

and

$$\iint_{E(0;\frac{1}{4})} |f(z)| \log \frac{1}{|z|} dA(z) \leq C \iint_{E(0;\frac{1}{4})} \log \frac{1}{|z|} \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w) dA(z). \tag{2.25}$$

Using polar coordinates we can easily see that

$$\iint_{E(0;\frac{1}{4})} \log \frac{1}{|z|} dA(z) = C < \infty,$$

so, from (2.25), we get

$$\iint_{E(0;\frac{1}{4})} |f(z)| \log \frac{1}{|z|} dA(z) \leq C \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w). \tag{2.26}$$

From (2.21) we have that there exists $C_0 > 0$ such that

$$\log \frac{1}{|w|} \leq C_0 (1 - |w|^2) \tag{2.27}$$

when $w \in \mathbb{D} \setminus E(0; \frac{1}{4})$. Therefore

$$\iint_{\mathbb{D} \setminus E(0;\frac{1}{4})} |f(z)| \log \frac{1}{|z|} dA(z) \leq C \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w). \tag{2.28}$$

From (2.26) and (2.28) we get

$$\iint_{\mathbb{D}} |f(z)| \log \frac{1}{|z|} dA(z) \leq C \iint_{\mathbb{D}} |f(w)|(1 - |w|^2) dA(w). \tag{2.29}$$

Finally, from (2.24) and (2.29), we have the desired result. $\square$

Let $\beta \in \mathbb{D}$ and f analytic on $\mathbb{D}$. Using (2.23), with the analytic on $\mathbb{D}$ function $h_\beta(z) = \frac{1-|\beta|^2}{(1-\bar{\beta}z)^2} f'(z)^2$, we have the following:

Proposition 2.6.2. *For $\beta \in \mathbb{D}$ and for every analytic function f on $\mathbb{D}$,*

$$\iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\bar{\beta}z|^2}|f'(z)|^2 \log\frac{1}{|z|} dA(z) \asymp \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\bar{\beta}z|^2}|f'(z)|^2(1-|z|^2)dA(z).$$

2.7 The Alexandrov - Clark measures

Alexandrov - Clark measures provide us important information regarding the behaviour of an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ on the boundary of the unit disk and they proved to be useful tools for the study of composition operators. More results and details for Alexandrov - Clark measures can be found in [14] (chapter 9). The definition of Alexandrov - Clark measures is based on a well-known theorem of Herglotz (see [19], pages 3-4).

Theorem 2.7.1 (Herglotz)**.** *If u is a non-negative harmonic function on $\mathbb{D}$, then there exists a unique positive Borel measure μ such that*

$$u(z) = \int\limits_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta).$$

The integral in the last relation is called Poisson integral of measure μ.

Now, if $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function and $\alpha \in \mathbb{T}$, then the function

$$\Re\Big(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\Big) = \frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2}$$

is positive and harmonic in $\mathbb{D}$ and from Herglotz's theorem we have

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \int\limits_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu_\alpha(\zeta)$$

for a unique Borel measure μ_α in $\mathbb{T}$. The measures μ_α, $\alpha \in \mathbb{T}$, are called Alexandrov-Clark measures. Now we have the Lebesgue decomposition

$$d\mu_\alpha = h_\alpha dm + d\mu_\alpha^s, \;\; h_\alpha \in L^1(\mathbb{T}), \;\; \mu_\alpha^s \perp m.$$

It is well known (see [14], pages 204-208) that the total variation of μ_α is given by

$$\|\mu_\alpha\| = \frac{1-|\varphi(0)|^2}{|\alpha-\varphi(0)|^2}, \tag{2.30}$$

that the absolutely continuous part $h_\alpha dm$ is carried by the set $\{\zeta \in \mathbb{T} : |\varphi(\zeta)| < 1\}$,

$$h_\alpha(\zeta) = \frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2} \tag{2.31}$$

and that the singular part $d\mu_\alpha^s$ is carried by the set $\{\zeta \in \mathbb{T} : \varphi(\zeta) = \alpha\}$. From (2.30) and (2.31), we see that

$$\begin{aligned} \|\mu_\alpha^s\| &= \|\mu_\alpha\| - \int_{\mathbb{T}} h_\alpha(\zeta) dm(\zeta) \\ &= \frac{1-|\varphi(0)|^2}{|\alpha-\varphi(0)|^2} - \int_{\mathbb{T}} \frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2} dm(\zeta). \end{aligned} \tag{2.32}$$

A useful result for the study of closed rangeness of composition operators is that of proposition 2.7.2 which is proved in [13].

Proposition 2.7.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $\alpha \in \mathbb{T}$. Then*

$$\|\mu_\alpha^s\| = \lim_{r \to 1^-} \|C_\varphi K_{r\alpha}\|_{H^2}^2$$

where $\{K_\lambda\}_{\lambda \in \mathbb{D}}$ are the functions of reproducing kernel in H^2 defined in (3.6).

The result which connects the Alexandrov-Clark measures with C_φ having closed range is a proposition due to K. Luery in [35] (page 56). If we extend φ on $\mathbb{T}$ as $\varphi(\zeta) = \lim_{r \to 1} \varphi(r\zeta)$ (this limit exists for $m-a.e.$ $\zeta \in \mathbb{T}$) and define the measure ν_φ on Borel $E \subset \mathbb{T}$ as

$$\nu_\varphi(E) = m(\varphi^{-1}(E)), \tag{2.33}$$

then, for the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ of measure ν_φ with respect to Lebesgue measure m, we have the following result.

Proposition 2.7.3 (Luery). *For m-a.e $\alpha \in \mathbb{T}$,*

$$\frac{d\nu_\varphi}{dm}(\alpha) = \|\mu_\alpha^s\|. \tag{2.34}$$

More information regarding Alexandrov-Clark measures, as well as their applications to the study of composition operators, can be found in [13], [14], [36], [37], [41] and [45].

2.8 Boundedness criteria for integral operators

In [7] (theorem 2.2), a result concerning the boundedness of the integral operator S_g is proved. We restate it here with its proof.

Theorem 2.8.1. *Let g be analytic, X, Y be Banach spaces of analytic functions and let Λ_{z_0} be the linear functional on X and Y defined by $\Lambda_{z_0} f = f'(z_0)$ (point evaluation of the derivative functional). Let's suppose that Λ_{z_0} is bounded. Then, if S_g maps X boundedly into Y then*

$$|g(z)| \leq \|S_g\| \frac{\|\Lambda_{z_0}\|_Y}{\|\Lambda_{z_0}\|_X} \tag{2.35}$$

Proof. For $f \in X$ and $z_0 \in \mathbb{D}$ we have

$$|f'(z_0)||g(z_0)| = |\Lambda_{z_0} S_g(f)| \leq \|\Lambda_{z_0}\|_Y \|S_g\| \|f\|_X. \tag{2.36}$$

Since $\sup_{\|f\|_X=1} |f'(z_0)| = \|\Lambda_{z_0}\|_X$, taking the supremum over $\|f\|_X = 1$ of both sides of (2.36) gives us

$$\|\Lambda_{z_0}\|_X |g(z_0)| \leq \|S_g\| \|\Lambda_{z_0}\|_Y,$$

which is what we had to prove. □

A consequence of theorem 2.8.1 is the following corollary (see [7] (corollary 2.3)).

Corollary 2.8.2. *If X is a Banach space of analytic functions on which point evaluation of the derivative Λ_{z_0} is a bounded linear functional, and S_g is bounded on X, then g is bounded.*

Chapter 3

Brief history of the research

In this chapter, we mention most of the already published results, concerning conditions for the integral operators S_g and the composition operators C_φ, respectively, to have closed range on some spaces of analytic functions.

3.1 History of the research for integral operators S_g

Anderson, in [7] (Corollary 3.6), formulated the following result, providing a condition for the operator S_g to have closed range on the quotient space $H^2/\mathbb{C}$ (H^2 modulo the constants).

Corollary 3.1.1 (Anderson)**.** *Let $g \in H^\infty$ and $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$. S_g has closed range on $H^2/\mathbb{C}$ if and only if there exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that*

$$A(G_c \cap D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)) \tag{3.1}$$

for all $\alpha \in \mathbb{D}$.

If $h \in [0,1]$ and $\theta_0 \in [0,2\pi]$ then, we denote S_{h,θ_0} the so-called Carleson square, defined as

$$S_{h,\theta_0} = \{re^{i\theta} \in \mathbb{D} : 1-h \leq r < 1, |\theta - \theta_0| \leq h\}. \tag{3.2}$$

Actually, Anderson formulated corollary 3.1.1 using, in (3.1), Carleson squares $S_{h,\theta}$ instead of pseudohyperbolic disks. It is well known that both these conditions are equivalent.

Let g an analytic function on $\mathbb{D}$. The multiplication operator, induced by g, is defined as $M_g f(z) = f(z)g(z)$. Apart from corollary 3.1.1, Anderson proved a more general result (see [7] (theorem 3.9)).

Theorem 3.1.2 (Anderson). *The following are equivalent for $g \in H^\infty$:*

(i) $g = BF$ *for a finite product* B *of interpolating Blaschke products and* F *such that* $F, \frac{1}{F} \in H^\infty$.

(ii) S_g *is bounded below on* $\mathcal{B}/\mathbb{C}$ *(the Bloch space* $\mathcal{B}$ *modulo the constants).*

(iii) There exist $r < 1$ *and* $\eta > 0$ *such that for all* $\alpha \in \mathbb{D}$,

$$\sup_{z \in D_r(\alpha)} |g(z)| > \eta.$$

(iv) S_g *is bounded below on* $H^2/\mathbb{C}$.

(v) M_g *is bounded below on* A^p_γ *for* $\gamma > -1$.

(vi) S_g *is bounded below on* $A^p_\gamma/\mathbb{C}$ *for* $\gamma > -1$ *(the weighted Bergman space* A^p_γ *modulo the constants).*

Earlier, in 1987, the equivalence, between (3.1) and part (i) of theorem 3.1.2, had been proved by Bourdon in [11] (theorem 2.3, corollary 2.5).

3.2 History of the research for composition operators C_φ

3.2.1 On Hardy spaces

Cima, Thomson and Wogen in [15](1974), characterized the closed rangeness of composition operator on H^2 by a contidion which depends only on the behaviour of φ in the boundary of the unit disk. They extended the definition of φ in the boundary by

$$\varphi(\zeta) = \lim_{r \to 1^-} \varphi(r\zeta), \ \ \zeta \in \mathbb{T}.$$

It is well known that the above limit exists for $m - a.e.$ $\zeta \in \mathbb{T}$. Then they defined the measure

$$\nu(E) = A(\varphi^{-1}(E)), E \subset \mathbb{T},$$

where E is a Borel subset of $\mathbb{T}$. The measure ν is absolutely continuous with respect to the Lebesgue measure A and let $\frac{d\nu}{dA}$ be the Radon-Nikodym derivative of ν with respect to A. The result was the following.

Theorem 3.2.1 (Cima, Thomson, Wogen)**.** *$C_\varphi : H^2 \to H^2$ has closed range if and only if the Radon-Nikodym derivative $\frac{d\nu}{dA}$ of measure ν with respect to A is bounded below from a positive constant.*

Cima, Thomson and Wogen posed the question if a condition for C_φ to have closed range could be found depending only on range of φ on $\mathbb{D}$ rather than on $\mathbb{T}$. This question was answered by Zorboska in 1994 (for details see [53]).

She defined the function

$$\tau_\varphi(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}}, w \in \mathbb{D}, \tag{3.3}$$

for $\varepsilon > 0$ the set

$$G_\varepsilon = \Big\{ w \in \mathbb{D} : \tau_\varphi(w) > \varepsilon \Big\}, \tag{3.4}$$

and formulated the following result:

Theorem 3.2.2 (Zorboska). *$C_\varphi : H^2 \to H^2$ has closed range if and only if there are $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that, the set G_ε to satisfy the condition*

$$A(G_\varepsilon \cap D_\eta(\alpha)) > \delta A(D_\eta(\alpha))$$

for all $\alpha \in \mathbb{D}$.

In 2010, Lefèvre, Li, Queffélec and Rondriguez-Piazza (see [31]), proved another condition (theorem 3.2.3) for the operator $C_\varphi : H^p \to H^p, 1 \leq p < \infty$ to have closed range, using the averages of the Nevanlinna counting function N_φ on sets of the form $S(\xi, h) = \{z \in \mathbb{D} : |z - \xi| \leq h\}$, where $h \in (0,1)$ and $\xi \in \mathbb{T}$.

Theorem 3.2.3 (Lefèvre, Li, Queffélec, Rondriguez-Piazza). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, non-constant function and $1 \leq p < \infty$. The operator $C_\varphi : H^p \to H^p$ has closed range if and only if there is $C > 0$ such that for $h \in (0,1)$ to have*

$$\frac{1}{A(S(\xi,h))} \iint\limits_{S(\xi,h)} N_\varphi(w) dA(w) \geq ch$$

for all $\xi \in \mathbb{T}$.

If $\lambda, z \in \mathbb{D}$ then, the reproducing kernel in H^2 consists of the functions

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}, \tag{3.5}$$

where

$$K_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^2}} \tag{3.6}$$

is their normalized companion. These functions are also called Szegő kernel functions.

In [35](2013), K. Luery, using the functions of reproducing kernel in H^2 and Alexandrov-Clark measures (see section 2.7) proved the following theorem, which is a result of Reproducing Kernel Thesis type.

Theorem 3.2.4 (Luery). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. The operator $C_\varphi : H^2 \to H^2$ has closed range if and only if C_φ has closed range on the set of normalized Szegő kernels $\{K_\lambda\}_{\lambda \in \mathbb{D}}$, i.e. there exists $C > 0$ such that*

$$\|C_\varphi K_\lambda\|_{H^2} \geq C\|K_\lambda\|_{H^2} = C,$$

since $\|K_\lambda\|_{H^2} \asymp 1$. In fact, this result asserts that the property of having closed range for C_φ on H^2 can be tested just by studying the action of C_φ on the functions $\{K_\lambda\}_{\lambda \in \mathbb{D}}$ of reproducing kernel in H^2.

3.2.2 On Bergman spaces and on weighted Bergman spaces

For $\varepsilon > 0$, we consider the sets

$$\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{(1 - |z|^2)}{1 - |\varphi(z)|^2} > \varepsilon\}$$

and

$$G_\varepsilon = \varphi(\Omega_\varepsilon). \tag{3.7}$$

In 2008, Akeroyd and Ghatage proved in [2] the following theorem for C_φ to have closed range on A^2.

Theorem 3.2.5 (Akeroyd, Ghatage). *The operator $C_\varphi : A^2 \to A^2$ has closed range if and only if there are $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0, 1)$ such that the set G_ε to satisfy the condition*

$$A(G_\varepsilon \cap D_\eta(z)) > \delta A(D_\eta(z))$$

for all $z \in \mathbb{D}$.

In 2012, Akeroyd and Fulmer proved in [3] the following theorem regarding the weighted Bergman spaces A^p_α.

Theorem 3.2.6 (Akeroyd, Fulmer). *Let $\alpha > -1$ and $1 \leq p < \infty$. The following are equivalent:*

- *$C_\varphi : A^p_\alpha \to A^p_\alpha$ has closed range*
- *There exist $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that the set G_ε to satisfy the condition:*

$$A(G_\varepsilon \cap D_\eta(z)) > \delta A(D_\eta(z))$$

for all $z \in \mathbb{D}$.

- *There are $\varepsilon > 0$, $M > 1$ and a set $U \subseteq \Omega_\varepsilon$ such that*
  1. *$\varphi(U)$ contains an external annulus, which means that there exists $r \in (0,1)$ such that $\{w \in \mathbb{D} : r < |w| < 1\} \subseteq \varphi(U)$ and*
  2. *$\frac{1}{M} \leq |\varphi'(z)| < M$, for all $z \in U$.*

3.2.3 On BMOA space

In [21], Erdem and Tjani proved a necessary condition and a sufficient condition for $C_\varphi : BMOA \to BMOA$ to have closed range (Theorem 3.2.7) and also they formulated a characterization for C_φ to have closed range on Möbius transformations in BMOA (Theorem 3.2.8). They defined $H \subseteq \mathbb{D}$ to be a sampling set for BMOA if for all $f \in BMOA$

$$\sup_{\alpha \in \mathbb{D}} \iint_H |f'(z)|^2 (1 - |\psi_\alpha(z)|^2) dA(z) \asymp \|f\|_*^2.$$

If

$$N_{\alpha,\varphi}(w) = \sum_{z:\varphi(z)=w} (1 - |\psi_\alpha(w)|^2)$$

then, for $\varepsilon > 0$ and $\alpha, \alpha' \in \mathbb{D}$, they also defined the sets $G_{\varepsilon,\alpha}$ as

$$G_{\varepsilon,\alpha} = \{w \in \mathbb{D} : N_{\alpha,\varphi}(w) > \varepsilon(1 - |\psi_\alpha(w)|^2)\}$$

and

$$G_{\varepsilon,\alpha',\alpha} = \{w \in \mathbb{D} : N_{\alpha',\varphi}(w) > \varepsilon(1 - |\psi_\alpha(w)|^2)\}.$$

Theorem 3.2.7 (Erdem, Tjani). *Let φ be a non-constant analytic self map of $\mathbb{D}$. If C_φ is closed range on BMOA then there exists $\varepsilon > 0$ such that $\cup_{\alpha\in\mathbb{D}} G_{\varepsilon,\alpha}$ is a sampling set for BMOA. Moreover, if $\cap_{\alpha\in\mathbb{D}} G_{\varepsilon,\alpha}$ is a sampling set for BMOA, then C_φ is closed range on BMOA.*

Having in mind the simple result that $\|\psi_\alpha\|_* \asymp 1$, we may proceed to the following theorem proved in the same paper.

Theorem 3.2.8 (Erdem, Tjani). *For each φ analytic self map of $\mathbb{D}$, the following conditions are equivalent.*

- *There exists $k \in (0, 1]$ such that for every $\alpha \in \mathbb{D}$, $\|\psi_\alpha \circ \varphi\|_* \geq k$.*
- *There exists $k \in (0, 1]$ such that for every $\alpha \in \mathbb{D}$ there exists $\alpha' \in \mathbb{D}$ and*

$$\lim_{n\to\infty} \|\psi_{\varphi(\alpha_n)} \circ \varphi \circ \psi_{\alpha_n}\|_{H^2} \geq k.$$

Moreover, in [21], Erdem and Tjani proved a necessary and sufficient condition for $C_\varphi : \mathcal{B} \to BMOA$ to have closed range, where $\mathcal{B}$ is the Bloch space (see 2.1.6).

Theorem 3.2.9 (Erdem, Tjani). *Let φ be an analytic self-map of D such that $C_\varphi : \mathcal{B} \to BMOA$ is a bounded operator. Then the composition operator $C_\varphi : \mathcal{B} \to BMOA$ is closed range if and only if there exists an $\varepsilon > 0$ and $r \in (0, 1)$ such that for all $\alpha \in \mathbb{D}$, there exists $\alpha' \in \mathbb{D}$ such that*

$$\frac{A(G_{\varepsilon,\alpha',\alpha} \cap D_r(\alpha))}{A(D_r(\alpha))} \asymp 1.$$

3.2.4 On weighted composition operators on Hardy and Bergman spaces

In general, if X is a space of analytic on $\mathbb{D}$ functions and $h \in X$ then, the weighted composition operator on X is defined as $W_{h,\psi} f = h(z) f(\psi(z))$, $z \in$

$\mathbb{D}$. We observe that if $h(z) = 1$ for all $z \in \mathbb{D}$ then, clearly, $W_{h,\psi}$ coincides with C_φ. Next, we formulate two results proved by Chalendar and Partington in [12], which assert, in the spirit of theorem 3.2.4, that checking the property of being bounded below (having closed range) for weighted composition operators on Hardy and Bergman spaces can be tested by their action on a set of simple test functions, including reproducing kernels.

In [12], if $h \in H^p$ $(1 \leq p < \infty)$ and $\psi : \mathbb{D} \to \mathbb{D}$ analytic, the weighted composition operator on Hardy spaces is defined as $W_{h,\psi} f = h(z) f(\psi(z))$, $z \in \mathbb{D}$, and also, the normalized reproducing kernel functions are defined by $\tilde{l}_w(z) = (1 - |w|^2)^{\frac{1}{p}} / (1 - \overline{w}z)^{\frac{2}{p}}$ so that $\|\tilde{l}_w\|_{H^p} \asymp 1$ for all $w \in \mathbb{D}$. In [12], Chalendar and Partington proved the following result.

Theorem 3.2.10 (Chalendar, Partington). *Let $1 \leq p < \infty$, $h \in H^p$ and $\psi : \mathbb{D} \to \mathbb{D}$ analytic such that the weighted composition operator $W_{h,\psi}$ is bounded. Then $W_{h,\psi}$ is bounded below if and only if there is a constant $C > 0$ such that $\|W_{h,\psi}\tilde{l}_w\|_{H^p} \geq C$ for all $w \in \mathbb{D}$.*

In the same spirit as in Hardy spaces, if $h \in A^p$ $(1 \leq p < \infty)$ and $\psi : \mathbb{D} \to \mathbb{D}$ analytic, the weighted composition operator on Bergman spaces is defined as $W_{h,\psi} f = h(z) f(\psi(z))$, $z \in \mathbb{D}$, and also, the normalized reproducing kernel functions are defined by $\tilde{l}_w(z) = (1 - |w|^2)^{\frac{2}{p}} / (1 - \overline{w}z)^{\frac{4}{p}}$ so that $\|\tilde{l}_w\|_{A^p} \asymp 1$ for all $w \in \mathbb{D}$. The result concerning Bergman spaces is the following.

Theorem 3.2.11 (Chalendar, Partington). *Let $1 \leq p < \infty$, $h \in A^p$ and $\psi : \mathbb{D} \to \mathbb{D}$ analytic such that the weighted composition operator $W_{h,\psi} : A^p \to A^p$ is bounded. Then $W_{h,\psi}$ is bounded below on A^p if and only if there is a constant $C > 0$ such that $\|W_{h,\psi}\tilde{l}_w\|_{A^p} \geq C$ for all $w \in \mathbb{D}$.*

3.2.5 On Dirichlet space

In [28] (1997), Jovovic and MacCluer studied the problem of finding conditions for the operator $C_\varphi : \mathcal{D} \to \mathcal{D}$ to have closed range on Dirichlet space

$\mathcal{D}$. Considering the sets $S(\zeta,h) = \Big\{z \in \mathbb{D} : |z-\zeta| < h\Big\}$, where $\zeta \in \mathbb{T}$ and $0 < h < 1$, they formulated a necessary and sufficient condition in case where there is $M > 0$ such that $n_\varphi(w) < M$ for all $w \in \mathbb{D}$, which means that φ is of bounded valence.

Theorem 3.2.12 (Jovovic, MacCluer). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, $C_\varphi : \mathcal{D} \to \mathcal{D}$ bounded and suppose that there exists $M > 0$ such that $n_\varphi(w) < M$, for all $w \in \mathbb{D}$. Then C_φ has closed range if and only if*

$$A(\varphi(\mathbb{D}) \cap S(\zeta,h)) \geq \varepsilon h^2$$

for all $\zeta \in \mathbb{D}, 0 < h < 1$.

Moreover, in the same paper, the following neseccary condition is proved.

Theorem 3.2.13 (Jovovic, MacCluer). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $C_\varphi : \mathcal{D} \to \mathcal{D}$ bounded. If C_φ has closed range, then there is $\varepsilon > 0$ such that*

$$\iint\limits_{S(\zeta,h)} n_\varphi(w) dA(w) \leq Ch^2$$

for all $\zeta \in \mathbb{D}, 0 < h < 1$.

In [34], D. Luecking proved that condition of theorem 3.2.13 cannot be sufficient.

3.2.6 On Besov spaces and on Besov type spaces

Main results of this section are due to M. Tjani. Let

$$G_{\varepsilon,p,\alpha} = \Big\{w \in \mathbb{D} : \frac{N_{p,\alpha}(w,\varphi)}{(1-|w|^2)^\alpha} > \varepsilon\Big\}$$

and

$$\Omega_{\varepsilon,p,\alpha} = \varphi^{-1}(G_{\varepsilon,p,\alpha})$$

and

$$N_{p,\alpha}(w,\varphi) = \sum_{w=\varphi(z)} |\varphi'(z)|^{p-2}(1-|z|^2)^{\alpha}.$$

M. Tjani, in 2014, (see [49]), proved the following result, which holds under the assumption that φ is of bounded valence, i.e. there is $M > 0$ such that $n_\varphi(w) < M$ for all $w \in \mathbb{D}$.

Theorem 3.2.14 (Tjani). *Let's suppose that there exists $M > 0$ such that $n_\varphi(w) < M$ for all $w \in \mathbb{D}$ and $p > 2$. Then $C_\varphi : B^p \to B^p$ has closed range if and only if there are $\varepsilon > 0$, $C > 0$ and $\eta \in (0,1)$ such that*

$$A(D_\eta(z) \cap G_{\varepsilon,p,p-2}) > CA(D_\eta(z))$$

for all $z \in \mathbb{D}$.

Moreover, in the same paper, the following result, regarding Besov type spaces, is proved.

Theorem 3.2.15 (Tjani). *Let $p > 2$ and $\alpha > p-1$. The operator $C_\varphi : B_{p,\alpha} \to B_{p,\alpha}$ has closed range if and only if there are $\varepsilon > 0$, $C > 0$ and $\eta \in (0,1)$ such that*

$$A(D_\eta(z) \cap G_{\varepsilon,p,\alpha}) > CA(D_\eta(z))$$

for all $z \in \mathbb{D}$.

3.2.7 On Bloch space

In 2005, Akeroyd, Ghatage and Tjani proved in [4] three equivalent conditions for the operator $C_\varphi : \mathcal{B} \to \mathcal{B}$ to have closed range. Define

$$\tau_\varphi(z) = \frac{(1-|z|^2)\varphi'(z)}{1-|\varphi(z)|^2},$$

$$\Lambda_\varepsilon = \{z \in \mathbb{D} : |\tau_\varphi(z)| > \varepsilon\}$$

and

$$F_\varepsilon = \varphi(\Omega_\varepsilon). \tag{3.8}$$

The following theorem is the result they proved.

Theorem 3.2.16 (Akeroyd, Ghatage, Tjani)**.** *The following are equivalent:*

- *$C_\varphi : \mathcal{B} \to \mathcal{B}$ has closed range.*
- *There exist $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that the set F_ε to satisfy the condition:*

$$A(F_\varepsilon \cap D_\eta(z)) > \delta A(D_\eta(z))$$

for all $z \in \mathbb{D}$.

- *There are $\varepsilon > 0$ and $\eta \in (0,1)$ such that*

$$F_\varepsilon \cap D_\eta(z) \neq \emptyset$$

for all $z \in \mathbb{D}$.

- *There are $C, r, s \in (0,1)$ such that for every $w \in \mathbb{D}$, there exists $z_w \in \mathbb{D}$ with the property: the function φ is 1-1 on the disk $D_s(z_w)$ and $\varphi(D_s(z_w)) \subseteq D_r(w)$ and $A(\varphi(D_s(z_w))) \geq C(1 - |w|^2)^2$.*

Part II

Closed Range Integral Operators

Chapter 4

Closed range integral operators on Hardy spaces

Hardy spaces H^p and some equivalent norms in these spaces were defined in section 2.1.1. Let denote $H^p/\mathbb{C}$ as H_0^p.

We recall that the integral operator $S_g : H_0^p \to H_0^p$, induced by the analytic function $g : \mathbb{D} \to \mathbb{C}$, is defined as

$$S_g f(z) = \int_0^z f'(w)g(w)dw, \;\; z \in \mathbb{D},$$

for every $f \in H_0^p$.

In this chapter, we prove a necessary and sufficient condition (theorem 4.1.2) for the integral operator S_g to have closed range on Hardy spaces $H_0^p, 1 \leq p < \infty$.

4.1 The main result

Let $g : \mathbb{D} \to \mathbb{C}$ be an analytic function and, for $c > 0$, let $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$. It is well known (see [7], theorem 2.2, corollary 2.3 and the notes after corollary 2.3) that the integral operator $S_g : H^p \to H^p$ $(1 \leq p < \infty)$ is bounded if and only if $g \in H^\infty$.

We say that S_g, on H^p, is bounded below, if there is $C > 0$ such that $\|S_g f\|_{H^p} > C\|f\|_{H^p}$ for every $f \in H^p$. As already mentioned (see chapter 1), in order S_g operator to be 1-1, we are obliged to consider it on the quotient space $H^p/\mathbb{C}$, the space H^p modulo the constants or, equivalently, the space of analytic functions $f \in H^p$ such that $f(0) = 0$. Theorem 3.2 in [7] states that S_g is bounded below on $H^p/\mathbb{C}$ if and only if it has closed range on $H^p/\mathbb{C}$.

Corollary 3.6 in [7] states that $S_g : H_0^2 \to H_0^2$ has closed range if and only if there exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))$$

for all $a \in \mathbb{D}$.

In the end of [7] A. Anderson posed the question, if the above condition for H_0^2 holds also for all H_0^p. In this paper, theorem 4.1.2 gives an affirmative answer to this question, for the case $1 \leq p < \infty$. Although the answer in case $p = 2$ is an immediate consequence of D. Luecking's theorem (see [7], Proposition 3.5), the answer in case $1 \leq p < \infty$ requires much more effort.

In the following by $\Delta_\eta(\alpha)$ is denoted the euclidean disk of center $\alpha \in \mathbb{D}$ and radius $\eta(1-|\alpha|)$, $\eta < 1$:

$$\Delta_\eta(\alpha) = \{z \in \mathbb{D} : |z-\alpha| < \eta(1-|\alpha|)\}.$$

For $\lambda \in (0,1)$ and $f \in H^p$ we set

$$E_\lambda(\alpha) = \{z \in \Delta_\eta(\alpha) : |f'(z)|^2 > \lambda |f'(\alpha)|^2\}$$

and

$$B_\lambda f(\alpha) = \frac{1}{A(E_\lambda(\alpha))} \iint_{E_\lambda(\alpha)} |f'(z)|^2 dA(z).$$

Lemma 4.1.1 is due to D. Luecking (see [32], lemma 1).

Lemma 4.1.1. *Let f analytic in $\mathbb{D}$, $a \in \mathbb{D}$ and $\lambda \in (0,1)$. Then*

$$\frac{A(E_\lambda(\alpha))}{A(\Delta_\eta(\alpha))} \geq \frac{\log \frac{1}{\lambda}}{\log \frac{B_\lambda f(\alpha)}{|f'(\alpha)|^2} + \log \frac{1}{\lambda}}.$$

Moreover in [32], the following sentence is proved: If $\alpha \in \mathbb{D}$ and $\frac{2\eta}{1+\eta^2} \leq r < 1$ then

$$\Delta_\eta(\alpha) \subseteq D_r(\alpha). \tag{4.1}$$

We proceed with the main result of this section, which appears in [39] (Theorem 2.2).

Theorem 4.1.2. *Let $1 \leq p < \infty$, $g \in H^\infty$ and g not be identically equal to zero. Then the following are equivalent:*

(i) $S_g : H_0^p \to H_0^p$ *has closed range*

(ii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a)) \tag{4.2}$$

for all $a \in \mathbb{D}$.

(iii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that

$$A(G_c \cap \Delta_\eta(a)) \geq \delta A(\Delta_\eta(a)) \tag{4.3}$$

for all $a \in \mathbb{D}$.

We first prove two lemmas which will play an important role in the proof of theorem 4.1.2.

For $\zeta \in \mathbb{T}$ and $0 < \beta < \beta' < 1$ we consider the Stolz angles $\Gamma_\beta(\zeta)$ and $\Gamma_{\beta'}(\zeta)$, where β' has been chosen so that $\Delta_\eta(\alpha) \subset \Gamma_{\beta'}(\zeta)$ for every $\alpha \in \Gamma_\beta(\zeta)$.

Lemma 4.1.3. *Let $\varepsilon > 0$, f analytic in $\mathbb{D}$ and*

$$A = \Big\{\alpha \in \mathbb{D} : |f'(\alpha)|^2 < \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(z)|^2 dA(z)\Big\}.$$

There is $C > 0$ depending only on η such that

$$\iint\limits_{A\cap\Gamma_\beta(\zeta)} |f'(z)|^2 dA(z) \le \varepsilon C \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)$$

Proof. Integrating

$$|f'(\alpha)|^2 < \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(z)|^2 dA(z)$$

over $\alpha \in A \cap \Gamma_\beta(\zeta)$ and using Fubini's theorem on the right side, we get

$$\iint\limits_{A\cap\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha) < \varepsilon \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(z)|^2 \Big[\iint\limits_{A\cap\Gamma_\beta(\zeta)} \frac{\chi_{\Delta_\eta(\alpha)}(z)}{A(\Delta_\eta(\alpha))} dA(\alpha)\Big] dA(z)$$

Using (4.1) with $r = \frac{2\eta}{1+\eta^2}$, we have $\chi_{\Delta_\eta(\alpha)}(z) \le \chi_{D_r(\alpha)}(z) = \chi_{D_r(z)}(\alpha)$. From (2.19) and (2.20) we have that $A(D_r(z)) \asymp (1-|z|)^2$ and, for $\alpha \in D_\eta(z)$, we have $(1-|z|) \asymp (1-|\alpha|)$, where the underlying constants in these relations depend only on η. In addition, $A(\Delta_\eta(\alpha)) = \eta^2(1-|\alpha|)^2$. So,

$$\begin{aligned}\iint\limits_{A\cap\Gamma_\beta(\zeta)} \frac{\chi_{\Delta_\eta(\alpha)}(z)}{A(\Delta_\eta(\alpha))} dA(\alpha) &\le \iint\limits_{A\cap\Gamma_\beta(\zeta)} \frac{\chi_{D_r(z)}(\alpha)}{\eta^2(1-|\alpha|)^2} dA(\alpha) \\ &\le C \iint\limits_{D_r(z)} \frac{1}{\eta^2(1-|z|)^2} dA(\alpha) = C\frac{A(D_r(z))}{\eta^2(1-|z|)^2} \le C,\end{aligned} \tag{4.4}$$

where $C > 0$ depends only on η. □

Lemma 4.1.4. *Let* $0 < \varepsilon < 1$, f *analytic in* $\mathbb{D}$, $0 < \lambda < \frac{1}{2}$ *and*

$$B = \left\{ \alpha \in \mathbb{D} : |f'(\alpha)|^2 < \varepsilon^3 B_\lambda f(\alpha) \right\}.$$

There is $C > 0$ *depending only on* η *such that*

$$\iint\limits_{B \cap \Gamma_\beta(\zeta)} |f'(z)|^2 dA(z) \leq \varepsilon C \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)$$

Proof. We write

$$\iint\limits_{B \cap \Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha) = \iint\limits_{B \cap \Gamma_\beta(\zeta) \cap A} |f'(\alpha)|^2 dA(\alpha) + \iint\limits_{(B \cap \Gamma_\beta(\zeta)) \setminus A} |f'(\alpha)|^2 dA(\alpha),$$

where A is as in lemma 4.1.3. The first integral is estimated by lemma 4.1.3, so it remains to show the desired result for the second integral. Integrating the relation

$$|f'(\alpha)|^2 < \varepsilon^3 B_\lambda f(\alpha) = \varepsilon^3 \frac{1}{A(E_\lambda(\alpha))} \iint\limits_{E_\lambda(\alpha)} |f'(z)|^2 dA(z)$$

over the set $(B \cap \Gamma_\beta(\zeta)) \setminus A$ and using Fubini's theorem on the right side, we get

$$\begin{aligned} \iint\limits_{(B \cap \Gamma_\beta(\zeta)) \setminus A} |f'(\alpha)|^2 dA(\alpha) &\leq \varepsilon^3 \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(z)|^2 \Big[\iint\limits_{(B \cap \Gamma_\beta(\zeta)) \setminus A} \frac{1}{A(E_\lambda(\alpha))} \chi_{E_\lambda(\alpha)}(z) dA(\alpha) \Big] dA(z) \\ &\leq \varepsilon^3 \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(z)|^2 \Big[\iint\limits_{(B \cap \Gamma_\beta(\zeta)) \setminus A} \frac{1}{A(E_\lambda(\alpha))} \chi_{\Delta_\eta(\alpha)}(z) dA(\alpha) \Big] dA(z) \end{aligned} \tag{4.5}$$

where the last inequality is justified by $E_\lambda(\alpha) \subseteq \Delta_\eta(\alpha)$. Let $\alpha \notin A$, i.e.

$$|f'(\alpha)|^2 \geq \frac{\varepsilon}{A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(z)|^2 dA(z). \tag{4.6}$$

Set $r = \eta(1-|\alpha|)$ and suppose $\lambda < \frac{1}{2}$ and $|z-\alpha| < \frac{r}{4}$. We have that

$$\begin{aligned} |f'(z)^2 - f'(\alpha)^2| &= \frac{1}{2\pi}\left| \int\limits_{|w-\alpha|=\frac{r}{2}} f'(w)^2 \left(\frac{1}{w-z} - \frac{1}{w-\alpha} \right) dw \right| \\ &= \frac{1}{2\pi}\left| \int\limits_{|w-\alpha|=\frac{r}{2}} f'(w)^2 \frac{z-\alpha}{(w-z)(w-\alpha)} dw \right|. \end{aligned} \tag{4.7}$$

For $|w-\alpha| = \frac{r}{2}$, by the subharmonicity of $|f'|^2$ we have

$$|f'(w)|^2 < \frac{1}{\frac{r^2}{4}} \iint\limits_{|u-w|\le\frac{r}{2}} |f'(u)|^2 dA(u) \le \frac{C}{A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(u)|^2 dA(u).$$

Since $|w-z| > \frac{r}{4}$ when $|w-\alpha| = \frac{r}{2}$, from (4.7) we get

$$|f'(z)^2 - f'(\alpha)^2| \le \frac{C|z-\alpha|}{r} \frac{1}{A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(u)|^2 dA(u).$$

Since we may assume that $C > 2$, taking $|z-\alpha| < \frac{\varepsilon r}{2C}$, then we have $|z-\alpha| < \frac{r}{4}$ and we get

$$|f'(z)^2 - f'(\alpha)^2| \le \frac{\varepsilon}{2A(\Delta_\eta(\alpha))} \iint\limits_{\Delta_\eta(\alpha)} |f'(u)|^2 dA(u). \tag{4.8}$$

Combining (4.6) and (4.8), we get

$$|f'(z)|^2 > \frac{1}{2}|f'(\alpha)|^2 > \lambda |f'(\alpha)|^2.$$

This means that if $\Delta' = \{z \in \mathbb{D} : |z-\alpha| < \frac{\varepsilon r}{2C}\}$ then $\Delta' \subset E_\lambda(\alpha)$ and

$$A(E_\lambda(\alpha)) \ge A(\Delta') = \frac{\varepsilon^2}{4C^2} r^2 = \frac{\varepsilon^2}{4C^2} A(\Delta_\eta(\alpha)).$$

We finally use this last inequality in (4.5) and we complete the proof. $\square$

Proof of theorem 4.1.2. $(ii) \Leftrightarrow (iii)$ This is easy and it is proved in [32].

$(iii) \Rightarrow (i)$ Let $\alpha \in \mathbb{D} \setminus B$, where B is as in lemma 4.1.4, where $0 < \varepsilon < 1$,

$0 < \lambda < \frac{1}{2}$. Then $\frac{B_\lambda f(\alpha)}{|f'(\alpha)|^2} \leq \frac{1}{\varepsilon^3}$ and, if we choose $\lambda < \varepsilon^{\frac{6}{\delta}}$, then, from lemma 4.1.1, we get that

$$\frac{A(E_\lambda(\alpha))}{A(\Delta_\eta(\alpha))} > \frac{\frac{2}{\delta}\log\frac{1}{\varepsilon^3}}{\log\frac{1}{\varepsilon^3} + \frac{2}{\delta}\log\frac{1}{\varepsilon^3}} > 1 - \frac{\delta}{2}. \tag{4.9}$$

Combining (4.3) and (4.9), we get

$$\begin{aligned} A(G_c \cap E_\lambda(\alpha)) &= A(G_c \cap \Delta_\eta(\alpha)) - A(G_c \cap (\Delta_\eta(\alpha) \setminus E_\lambda(\alpha))) \\ &\geq \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha) \setminus E_\lambda(\alpha)) \\ &= \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha)) + A(E_\lambda(\alpha)) \\ &\geq \delta A(\Delta_\eta(\alpha)) - A(\Delta_\eta(\alpha)) + A(\Delta_\eta(\alpha)) - \frac{\delta}{2} A(\Delta_\eta(\alpha)) \\ &= \frac{\delta}{2} A(\Delta_\eta(\alpha)) \end{aligned}$$

Now let $f \in H_0^p, \zeta \in \mathbb{T}$ and $\alpha \in \Gamma_\beta(\zeta) \setminus B$. Then, using the last relation and $E_\lambda(\alpha) \subset \Delta_\eta(\alpha) \subset \Gamma_{\beta'}(\zeta)$, we get

$$\begin{aligned} &\frac{1}{A(\Delta_\eta(\alpha))} \iint\limits_{G_c \cap \Gamma_{\beta'}(\zeta)} \chi_{\Delta_\eta(\alpha)}(z)|f'(z)|^2 dA(z) \\ &\quad \geq \frac{\delta}{2A(G_c \cap E_\lambda(\alpha))} \iint\limits_{G_c \cap E_\lambda(\alpha)} \chi_{\Delta_\eta(\alpha)}(z)|f'(z)|^2 dA(z) \\ &\quad = \frac{\delta}{2A(G_c \cap E_\lambda(\alpha))} \iint\limits_{G_c \cap E_\lambda(\alpha)} |f'(z)|^2 dA(z) \geq \frac{\delta\lambda}{2}|f'(\alpha)|^2. \end{aligned}$$

Integrating the last relation over the set $\Gamma_\beta(\zeta) \setminus B$ and using Fubini's theorem on the left side, we have

$$\iint\limits_{G_c \cap \Gamma_{\beta'}(\zeta)} |f'(z)|^2 \Big[\iint\limits_{\Gamma_\beta(\zeta) \setminus B} \frac{\chi_{\Delta_\eta(\alpha)}(z)}{A(\Delta_\eta(\alpha))} dA(\alpha) \Big] dA(z) \geq \frac{\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta) \setminus B} |f'(\alpha)|^2 dA(\alpha).$$

With similar arguments as in relation (4.4), we can show that the integral in the brackets is bounded above from a constant $C > 0$ depending only on η.

So, we have that

$$\iint\limits_{G_c\cap\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z) \geq \frac{C\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta)\setminus B} |f'(\alpha)|^2 dA(\alpha)$$
$$= \frac{C\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha) - \frac{C\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta)\cap B} |f'(\alpha)|^2 dA(\alpha).$$

Because of lemma 4.1.4 we have that

$$\iint\limits_{G_c\cap\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z) \geq \frac{C\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha) - \varepsilon\frac{C'\delta\lambda}{2} \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)$$

and so

$$\iint\limits_{G_c\cap\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z) + \varepsilon\frac{C'\delta\lambda}{2} \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha) \geq \frac{C\delta\lambda}{2} \iint\limits_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha).$$

Hence,

$$\Big(\iint\limits_{G_c\cap\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)\Big)^{\frac{1}{2}} + \Big(\frac{C'\varepsilon\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big(\iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{1}{2}}$$
$$\geq \Big(\frac{C\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big(\iint\limits_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{1}{2}}.$$

Then, we raise the last relation to the p power, so

$$\Big[\Big(\iint\limits_{G_c\cap\Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)\Big)^{\frac{1}{2}} + \Big(\frac{C'\varepsilon\delta\lambda}{2} \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{1}{2}}\Big]^p$$
$$\geq \Big(\frac{C\delta\lambda}{2}\Big)^{\frac{p}{2}} \Big(\iint\limits_{\Gamma_\beta(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}}.$$

Integrating both sides of the last relation over $\mathbb{T}$, we obtain

$$\int\limits_{\mathbb{T}} \Big[\Big(\iint\limits_{G_c \cap \Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)\Big)^{\frac{1}{2}} + \Big(\frac{C'\varepsilon\delta\lambda}{2} \iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{1}{2}}\Big]^p dm(\zeta)$$
$$\geq \int\limits_{\mathbb{T}} \Big[\Big(\frac{C\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big(\iint\limits_{\Gamma_{\beta}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{1}{2}}\Big]^p dm(\zeta).$$

Raising last relation to $\frac{1}{p}$ power and then applying Minkowski's inequality, we get

$$\Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{G_c \cap \Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}} + \Big(\frac{C'\varepsilon\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}}$$
$$\geq \Big(\frac{C\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}}$$

and so

$$\Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{G_c \cap \Gamma_{\beta'}(\zeta)} |f'(z)|^2 dA(z)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}} \geq \Big(\frac{C\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}}$$
$$- \Big(\frac{C'\varepsilon\delta\lambda}{2}\Big)^{\frac{1}{2}} \Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}}. \tag{4.10}$$

According to (2.5), both integrals at the right side of (4.10), represent equivalent norms in H^p_0. Due to the relation between β and β' there is $C'' > 0$ which depends only on η, such that

$$\Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta'}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}} \leq C'' \Big[\int\limits_{\mathbb{T}} \Big(\iint\limits_{\Gamma_{\beta}(\zeta)} |f'(\alpha)|^2 dA(\alpha)\Big)^{\frac{p}{2}} dm(\zeta)\Big]^{\frac{1}{p}}. \tag{4.11}$$

Combining relations (4.10) and (4.11), we get

$$\Big[\int_{\mathbb{T}}\Big[\Big(\iint_{G_c\cap\Gamma_{\beta'}(\zeta)}|f'(z)|^2dA(z)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}} \geq \Big(\frac{C\delta\lambda}{2}\Big)^{\frac{1}{2}}\Big[\int_{\mathbb{T}}\Big(\iint_{\Gamma_\beta(\zeta)}|f'(\alpha)|^2dA(\alpha)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}}$$

$$-\Big(\frac{C'\varepsilon\delta\lambda}{2}\Big)^{\frac{1}{2}}C''\Big[\int_{\mathbb{T}}\Big(\iint_{\Gamma_\beta(\zeta)}|f'(\alpha)|^2dA(\alpha)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}}$$

$$=\Big(\frac{\delta\lambda}{2}\Big)^{\frac{1}{2}}[C^{\frac{1}{2}}-\varepsilon^{\frac{1}{2}}C'^{\frac{1}{2}}C'']\|f\|_{H_0^p}.$$

Choosing ε small enough so that $C-\varepsilon^{\frac{1}{2}}C'^{\frac{1}{2}}C''>0$, we have that

$$\Big[\int_{\mathbb{T}}\Big(\iint_{G_c\cap\Gamma_{\beta'}(\zeta)}|f'(z)|^2dA(z)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}} \geq C\|f\|_{H_0^p},$$

and since $G_c=\{z\in\mathbb{D}:|g(z)|>c\}$, we have

$$\|S_gf\|_{H_0^p}\asymp\Big[\int_{\mathbb{T}}\Big(\iint_{\Gamma_{\beta'}(\zeta)}|(S_gf(z))'|^2dA(z)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}}$$

$$=\Big[\int_{\mathbb{T}}\Big(\iint_{\Gamma_{\beta'}(\zeta)}|f'(z)|^2|g(z)|^2dA(z)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}}$$

$$\geq c\Big[\int_{\mathbb{T}}\Big(\iint_{G_c\cap\Gamma_{\beta'}(\zeta)}|f'(z)|^2dA(z)\Big)^{\frac{p}{2}}dm(\zeta)\Big]^{\frac{1}{p}}\geq C\|f\|_{H_0^p}.$$

So the integral operator S_g has closed range.

$(i)\Rightarrow(ii)$ Let $\alpha\in\mathbb{D}$, $\zeta\in\mathbb{T}$, $\eta\in(0,1)$ and the arc $I_\alpha=\{\zeta\in\mathbb{T}:\Gamma_{\frac{1}{2}}(\zeta)\cap D_\eta(\alpha)\neq\emptyset\}$. It's easy to see that $\zeta\in I_\alpha$ is equivalent to $\alpha\in\Gamma_{\eta'}(\zeta)$, where η' depends only on η. In fact, an elementary geometric argument shows that $1-\eta'\asymp1-\eta$, where the underlying constants are absolute.

Set $R_0=\frac{1+\eta'}{2}$. We continue with the proof by considering two cases for α: **(a)** $R_0\leq|\alpha|<1$ and **(b)** $0\leq|\alpha|\leq R_0$.

Case (a) $R_0 \le |\alpha| \le 1$: At first, we consider the case $p > 1$. Another simple geometric argument gives $m(I_\alpha) \asymp \frac{1-|\alpha|}{(1-\eta')^{\frac{1}{2}}}$ and hence:

$$m(I_\alpha) \asymp \frac{1-|\alpha|}{(1-\eta)^{\frac{1}{2}}}. \tag{4.12}$$

More specifically, as we can see in figure 4.1, let's take $\zeta \in \mathbb{T}$ such that

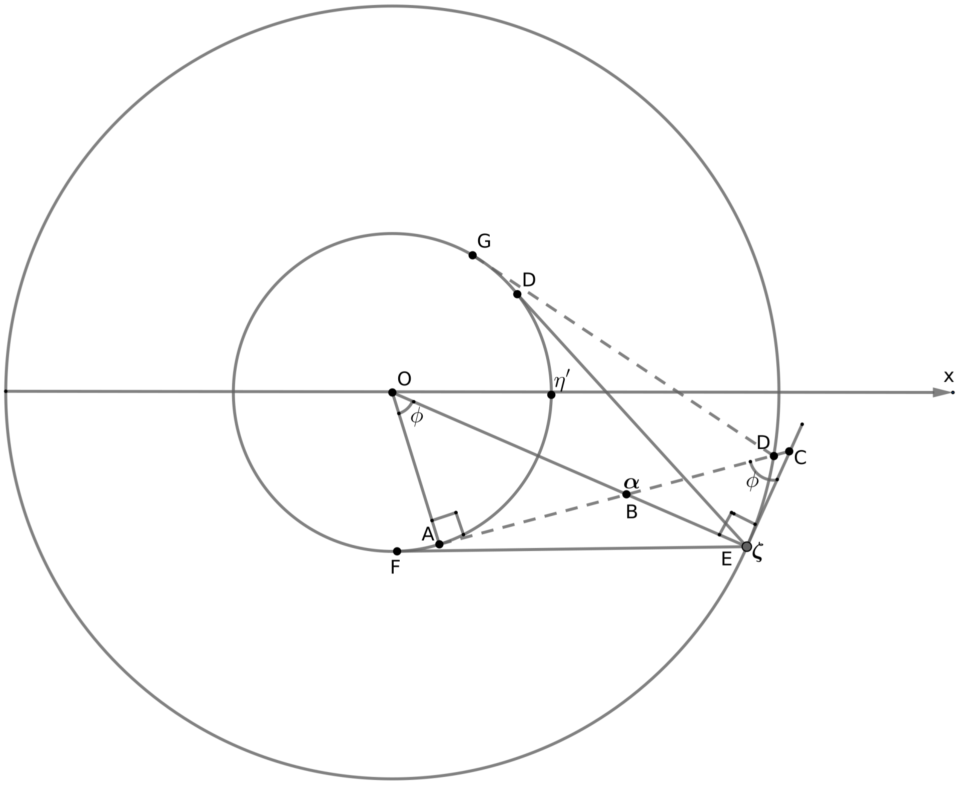

FIGURE 4.1: Geometry for the estimation of arc I_α's length.

$Arg(\zeta) = Arg(\alpha)$ and consider the line tangent to the unit circle at the point, let E, corresponding to ζ. Then, of course, E is the middle point of arc I_α and, having in mind that $\zeta \in I_\alpha$ exactly when $\alpha \in \Gamma_{\eta'}(\zeta)$, we can determine the end point of I_α, let D (corresponding to $\zeta_0 \in \mathbb{T}$), just by requiring that the boundary segment AD of $\Gamma_{\eta'}(\zeta_0)$ (the cone with the dashed boundary segments) to contain the point B which corresponds to α. Since $CE \perp OE$ and $CA \perp AO$ we have that $\angle AOE = \angle ACE = \phi$ and obviously $\cos\phi = \frac{(OA)}{(OB)} = \frac{\eta'}{|\alpha|}$, where by (OA) we denote the length of the line segment with end points

O and A. Since $R_0 \leq |\alpha| < 1$, we have that

$$\cos\phi = \frac{\eta'}{|\alpha|} \geq \eta'$$

and

$$\cos\phi = \frac{\eta'}{|\alpha|} \leq \frac{\eta'}{R_0} = \frac{2\eta'}{1+\eta'} \leq 2\eta'.$$

Hence, $\cos\phi \asymp \eta'$. In addition, we have

$$\sin\phi = \left(1-\cos^2\phi\right)^{\frac{1}{2}} = \left(1-\frac{\eta'^2}{|\alpha|^2}\right)^{\frac{1}{2}} \leq \left(1-\eta'^2\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}\sin\phi &= \left(1-\cos^2\phi\right)^{\frac{1}{2}} = \left(1-\frac{\eta'^2}{|\alpha|^2}\right)^{\frac{1}{2}} \geq \left(1-\frac{\eta'^2}{R_0^2}\right)^{\frac{1}{2}} \\ &= \left(\frac{R_0^2-\eta'^2}{R_0^2}\right)^{\frac{1}{2}} \geq \left(R_0-\eta'\right)^{\frac{1}{2}} = \left(\frac{1-\eta'}{2}\right)^{\frac{1}{2}} \geq \frac{1}{2}\left(1-\eta'^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence, $\sin\phi \asymp \left(1-\eta'^2\right)^{\frac{1}{2}}$ and we have

$$\tan\phi \asymp \frac{(1-\eta'^2)^{\frac{1}{2}}}{\eta'}.$$

But $\frac{1}{2} \leq \eta' < 1$ so, we get

$$\tan\phi \asymp (1-\eta'^2)^{\frac{1}{2}}.$$

Moreover, in the right triangle $\triangle BEC$, we have $\tan\phi = \frac{1-|\alpha|}{(EC)}$. But $(EC) \asymp m(I_\alpha)$ so, we finally have $m(I_\alpha) \asymp \frac{1-|\alpha|}{(1-\eta'^2)^{\frac{1}{2}}}$ and (4.12) is implied.

If S_g has closed range on H_0^p then there exists $C > 0$ such that for every $f \in H_0^p$ we have

$$C\|S_g f\|_{H_0^p}^p \geq \|f\|_{H_0^p}^p. \tag{4.13}$$

Let

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

From lemma A.0.1 in Appendix A we have that $\|\psi_\alpha - \alpha\|_{H^p}^p \asymp (1 - |\alpha|)$.

Setting $f = \psi_\alpha - \alpha$ in (4.13) and using $(x+y)^p \le 2^{p-1}(x^p + y^p)$, we get

$$\begin{aligned}
1 - |\alpha| &\le C\|S_g(\psi_\alpha - \alpha)\|_{H_0^p}^p = C \int_{\mathbb{T}} \Big(\iint_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 |g(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&\le C \int_{I_\alpha} \Big(\iint_{\Gamma_{\frac{1}{2}}(\zeta)\cap G_c \cap D_\eta(\alpha)} |\psi_\alpha'(z)|^2 |g(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&+ C \int_{I_\alpha} \Big(\iint_{\Gamma_{\frac{1}{2}}(\zeta)\cap (D_\eta(\alpha)\setminus G_c)} |\psi_\alpha'(z)|^2 |g(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&+ C \int_{I_\alpha} \Big(\iint_{\Gamma_{\frac{1}{2}}(\zeta)\setminus D_\eta(\alpha)} |\psi_\alpha'(z)|^2 |g(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&+ C \int_{\mathbb{T}\setminus I_\alpha} \Big(\iint_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 |g(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&= C(I_1 + I_2 + I_3 + I_4).
\end{aligned} \tag{4.14}$$

Using $A(D_\eta(\alpha)) = \frac{(1-|\alpha|^2)^2}{(1-\eta^2|\alpha|^2)^2}\eta^2 \le \frac{(1-|\alpha|^2)^2}{(1-\eta^2)^2}$, we get

$$\begin{aligned}
I_1 &\le \|g\|_\infty^p \int_{I_\alpha} \Big(\iint_{G_c \cap D_{\eta(\alpha)}} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}z|^4} dA(z) \Big)^{\frac{p}{2}} dm(\zeta) \\
&\le \|g\|_\infty^p m(I_\alpha) \Big(\frac{A(G_c \cap D_\eta(\alpha))}{(1-|\alpha|^2)^2} \Big)^{\frac{p}{2}} \\
&\le \|g\|_\infty^p m(I_\alpha) \frac{1}{(1-\eta^2)^p} \Big(\frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} \Big)^{\frac{p}{2}}.
\end{aligned}$$

Using $|g(z)| \le c$ in $\mathbb{D} \setminus G_c$ and making the change of variables $w = \psi_\alpha(z)$, we get

$$I_2 \le c^p \int_{I_\alpha} \Big(\iint_{\mathbb{D}} |\psi_\alpha'(z)|^2 dA(z) \Big)^{\frac{p}{2}} dm(\zeta) = c^p \int_{I_\alpha} \Big(\iint_{\mathbb{D}} dA(w) \Big)^{\frac{p}{2}} dm(\zeta) = c^p m(I_\alpha).$$

We increase I_3 by extending it over $\mathbb{D} \setminus D_\eta(\alpha)$ and then we make the change of variables $w = \psi_\alpha(z)$ to get

$$I_3 \leq \|g\|_\infty^p \int_{I_\alpha} \Big(\iint_{\mathbb{D}\setminus D_\eta(0)} dA(w) \Big)^{\frac{p}{2}} dm(\zeta) = \|g\|_\infty^p m(I_\alpha)(1-\eta^2)^{\frac{p}{2}}.$$

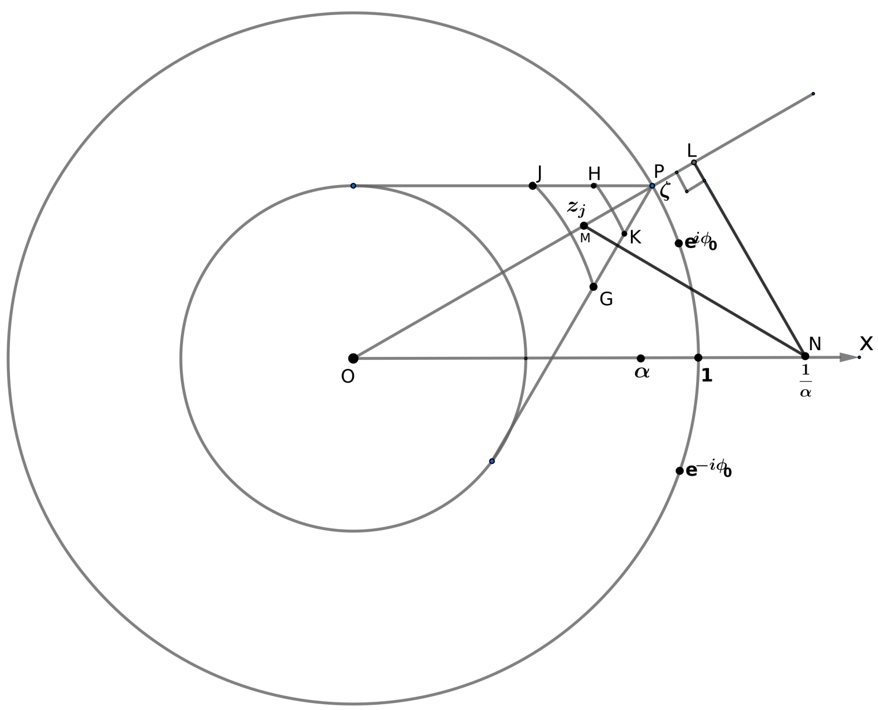

FIGURE 4.2: Geometry for the estimation of integral I_4.

For the estimation of I_4 we have first to estimate $\iint_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 dA(z)$, when $\zeta \in \mathbb{T} \setminus I_\alpha$. In order the process of estimation to be more clearly understood and to have a better geometrical notion for the distances involved, we have to see figure 4.2. Without loss of generality we may assume that $\alpha \in [R_0, 1)$. For $j \in \mathbb{N}, j \geq 2$, we define $r_j = 1 - \frac{1}{2^j}$ and consider the sets $\Omega_1 = E(0; \frac{1}{2})$ and $\Omega_j = (E(0; r_j) \setminus E(0; r_{j-1})) \cap \Gamma_{\frac{1}{2}}(\zeta)$ (disks of the form $E(0;r)$ defined in (2.22)). Then we have that $\Gamma_{\frac{1}{2}}(\zeta) = \bigcup_{j=1}^{+\infty} \Omega_j$ and $A(\Omega_j) \asymp \frac{1}{4^j}$, when $j \geq 1$. We fix $z_j \in \Omega_j$ such that $Arg(z_j) = Arg(\zeta)$. Then, if $z \in \Omega_j$, we have $|1 - \alpha z| \asymp$

$|\frac{1}{\alpha}-z| \asymp |\frac{1}{\alpha}-z_j|$. In figure 4.2, we can consider as Ω_j the region defined by the points $HJGK$ and it's clear that $|\frac{1}{\alpha}-z_j| = (NM)$. In addition, we have $(LP) \asymp |1-\frac{1}{\alpha}| \asymp 1-|\alpha|$, $(PM) \asymp 1-|z_j| \asymp \frac{1}{2^j}$ and $(LN) \asymp |Arg(\zeta)|$. The application of the Pythagorean theorem to the right triangle $\triangle MLN$ gives us $(MN)^2 = (LN)^2 + [(LP)+(PM)]^2$ and, because of the identity $x^2+y^2 \asymp (x+y)^2$, we finally get $(MN) \asymp (LN)+(LP)+(PM)$, which means that $|\frac{1}{\alpha}-z_j| \asymp |\frac{1}{2^j}+1-|\alpha|+|Arg(\zeta)||$. In all these relations, the underlying constants are absolute. If $\zeta \in \mathbb{T}\setminus I_\alpha$, then $a \notin \Gamma_{\frac{1}{2}}(\zeta)$ which means that $1-|\alpha| < |Arg(\zeta)|$, so we have that $|\frac{1}{\alpha}-z_j| \asymp |\frac{1}{2^j}+|Arg(\zeta)||$. There is some j_0 so that $\frac{1}{2^{j_0}} \leq |Arg(\zeta)| \leq \frac{1}{2^{j_0-1}}$. For $j<j_0$ we have $|Arg(\zeta)| < \frac{1}{2^j}$ which implies that $|\frac{1}{\alpha}-z_j| \asymp \frac{1}{2^j}$ and for $j>j_0$ we have $|Arg(\zeta)| > \frac{1}{2^j}$ which implies that $|\frac{1}{\alpha}-z_j| \asymp |Arg(\zeta)|$. Therefore

$$
\begin{aligned}
\iint\limits_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 dA(z) &= \iint\limits_{\Omega_1} \frac{(1-|\alpha|^2)^2}{|1-\alpha z|^4} dA(z) + \sum_{j=2}^{+\infty} \iint\limits_{\Omega_j} \frac{(1-|\alpha|^2)^2}{|1-\alpha z|^4} dA(z) \\
&\asymp (1-|\alpha|^2)^2 + \sum_{j=2}^{j_0} \iint\limits_{\Omega_j} \frac{(1-|\alpha|^2)^2}{|\frac{1}{\alpha}-z_j|^4} dA(z) + \sum_{j=j_0}^{+\infty} \iint\limits_{\Omega_j} \frac{(1-|\alpha|^2)^2}{|\frac{1}{\alpha}-z_j|^4} dA(z) \\
&\asymp (1-|\alpha|^2)^2 + \sum_{j=2}^{j_0} A(\Omega_j)(1-|\alpha|^2)^2(2^j)^4 + \sum_{j=j_0}^{+\infty} A(\Omega_j)\frac{(1-|\alpha|^2)^2}{|Arg(\zeta)|^4} \\
&\asymp (1-|\alpha|^2)^2 + (1-|\alpha|^2)^2 \sum_{j=2}^{j_0} \frac{1}{4^j}16^j + \frac{(1-|\alpha|^2)^2}{|Arg(\zeta)|^4} \sum_{j=j_0}^{+\infty} \frac{1}{4^j}.
\end{aligned}
$$

But $\sum_{j=2}^{j_0} 4^j \asymp 4^{j_0} \asymp \frac{1}{|Arg(\zeta)|^2}$ and $\sum_{j=j_0}^{+\infty} \frac{1}{4^j} \asymp \frac{1}{4^{j_0}} \asymp |Arg(\zeta)|^2$. Therefore

$$
\iint\limits_{\Gamma_{\frac{1}{2}}(\zeta)} |\psi_\alpha'(z)|^2 dA(z) \asymp (1-|\alpha|^2)^2 + \frac{(1-|\alpha|^2)^2}{|Arg(\zeta)|^2}. \tag{4.15}
$$

Since α is positive, there is ϕ_0 such that $\mathbb{T}\setminus I_\alpha = [\phi_0, 2\pi-\phi_0]$ (see figure 4.2) and $\phi_0 \asymp m(I_\alpha)$. Therefore

$$
I_4 \leq C\|g\|_\infty^p \int_{\phi_0}^{\pi} (1-|\alpha|^2)^p d\phi + C\|g\|_\infty^p \int_{\phi_0}^{\pi} \frac{(1-|\alpha|^2)^p}{\phi^p} d\phi
$$

$$\le C\|g\|_\infty^p(1-|\alpha|^2)^p + C\|g\|_\infty^p \frac{(1-|\alpha|^2)^p}{\phi_0^{p-1}}$$
$$\le C\|g\|_\infty^p(1-|\alpha|^2)^p + C\|g\|_\infty^p \frac{(1-|\alpha|^2)^p}{m(I_\alpha)^{p-1}}.$$

Substituting the estimates for I_1, I_2, I_3, I_4 in (4.14), we get

$$1-|\alpha| \le C\Big[\|g\|_\infty^p m(I_\alpha)\frac{1}{(1-\eta^2)^p}\Big(\frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))}\Big)^{\frac{p}{2}} + c^p m(I_\alpha)$$
$$+\|g\|_\infty^p m(I_\alpha)(1-\eta^2)^{\frac{p}{2}} + \|g\|_\infty^p(1-|\alpha|^2)^p + \|g\|_\infty^p\frac{(1-|\alpha|^2)^p}{m(I_\alpha)^{p-1}}\Big].$$

Using (4.12) we get

$$1-|\alpha| \le C\Big[\|g\|_\infty^p \frac{1-|\alpha|}{(1-\eta)^{\frac{1}{2}}}\frac{1}{(1-\eta^2)^p}\Big(\frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))}\Big)^{\frac{p}{2}}$$
$$+c^p\frac{1-|\alpha|}{(1-\eta)^{\frac{1}{2}}} + \|g\|_\infty^p\frac{1-|\alpha|}{(1-\eta)^{\frac{1}{2}}}(1-\eta^2)^{\frac{p}{2}}$$
$$+\|g\|_\infty^p(1-|\alpha|^2)(1-\eta^2)^{p-1} + \|g\|_\infty^p(1-|\alpha|^2)(1-\eta)^{\frac{p-1}{2}}\Big].$$

Thus

$$C \le \|g\|_\infty^p\frac{1}{(1-\eta)^{\frac{2p+1}{2}}}\Big(\frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))}\Big)^{\frac{p}{2}} + \frac{c^p}{(1-\eta)^{\frac{1}{2}}}$$
$$+\|g\|_\infty^p(1-\eta)^{\frac{p-1}{2}} + \|g\|_\infty^p(1-\eta)^{p-1} + \|g\|_\infty^p(1-\eta)^{\frac{p-1}{2}}.$$

Choose η close enough to 1 so that $\|g\|_\infty^p(1-\eta)^{\frac{p-1}{2}} + \|g\|_\infty^p(1-\eta)^{p-1} + \|g\|_\infty^p(1-\eta)^{\frac{p-1}{2}} < \frac{C}{4}$ and then set $C_\eta = (1-\eta)^{\frac{1}{2}}$. We have that

$$\frac{3C}{4} \le \frac{\|g\|_\infty^p}{C_\eta^{2p+1}}\Big(\frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))}\Big)^{\frac{p}{2}} + \frac{c^p}{C_\eta}.$$

Choose c small enough so that $\frac{c^p}{C_\eta} < \frac{C}{4}$. Then

$$\frac{C}{2} \le \frac{\|g\|_\infty^p}{C_\eta^{2p+1}}\Big(\frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))}\Big)^{\frac{p}{2}}$$

and finally

$$\Big(\frac{CC_\eta^{2p+1}}{2\|g\|_\infty^p}\Big)^{\frac{2}{p}} \leq \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))}$$

or

$$A(G_c \cap D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)),$$

for every α with $R_0 \leq |\alpha| < 1$.

Now, we consider the case $p = 1$. Let $\alpha \in \mathbb{D}$ and the functions

$$f_\alpha(z) = \frac{(1-|\alpha|^2)^2}{3\overline{\alpha}(1-\overline{\alpha}z)^3} - \frac{(1-|\alpha|^2)^2}{3\overline{\alpha}}.$$

Obviously $f_\alpha \in H_0^1$. We define the sets $J_\alpha = \{\zeta \in \mathbb{T} : \alpha \in \Gamma_{\frac{1}{2}}(\zeta)\}$ and it's clear that $m(J_{\alpha)} \asymp 1 - |\alpha|$. Then we consider the integral

$$I = \int\limits_{\mathbb{T}\setminus J_\alpha} \Big(\iint\limits_{\Gamma_{\frac{1}{2}}(\zeta)} |f'_\alpha(z)|^2 dA(z)\Big)^{\frac{1}{2}} dm(\zeta).$$

If $\zeta \in \mathbb{T} \setminus J_\alpha$ then $\alpha \notin \Gamma_{\frac{1}{2}}(\zeta)$. Using similar arguments as in the proof of (4.15), we get

$$\iint\limits_{\Gamma_{\frac{1}{2}}(\zeta)} |f'_\alpha(z)|^2 dA(z) = \iint\limits_{\Gamma_{\frac{1}{2}}(\zeta)} \frac{(1-|\alpha|^2)^4}{|1-\overline{\alpha}z|^8} dA(z) \asymp (1-|\alpha|^2)^4 + \frac{(1-|\alpha|^2)^4}{|Arg(\zeta)|^6},$$

hence

$$I \asymp \int_{1-|\alpha|}^{\pi} \frac{(1-|\alpha|^2)^2}{\phi^3} d\phi \asymp 1. \tag{4.16}$$

If $F \in H_0^1$, from lemma B.0.1 in appendix B it follows that there exists $C'' > 0$ such that

$$\|F\|_{H_0^1} \leq C'' \iint\limits_{\mathbb{D}} |F'(z)| dA(z). \tag{4.17}$$

S_g has been supposed to have closed range, so there exists $C' > 0$ such that $\|f_\alpha\|_{H_0^1} \le C'\|S_g f_\alpha\|_{H_0^1}$ and using (4.16) and (4.17) (with $F = S_g f_\alpha$), we get

$$0 < C_0 \le I \le \|f_\alpha\|_{H_0^1} \le C'\|S_g f_\alpha\|_{H_0^1} \le C'C'' \iint_{\mathbb{D}} |(S_g f_\alpha)'(z)| dA(z).$$

Hence, observing that $|f_\alpha'(z)| = |\psi_\alpha'(z)|^2$, we have

$$\begin{aligned}
C_1 &\le \iint_{\mathbb{D}} |(S_g f_\alpha)'(z)| dA(z) = \iint_{\mathbb{D}} |\psi_\alpha'(z)|^2 |g(z)| dA(z) \\
&\le \|g\|_\infty \iint_{G_c \cap D_\eta(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} dA(z) + c \iint_{D_\eta(\alpha)\setminus G_c} |\psi_\alpha'(z)|^2 dA(z) \\
&\qquad\qquad + \|g\|_\infty \iint_{\mathbb{D}\setminus D_\eta(\alpha)} |\psi_\alpha'(z)|^2 dA(z) \\
&\le \|g\|_\infty \iint_{G_c \cap D_\eta(\alpha)} \frac{1}{(1-|\alpha|^2)^2} dA(z) + c \iint_{\mathbb{D}} |\psi_\alpha'(z)|^2 dA(z) + \|g\|_\infty \iint_{\mathbb{D}\setminus D_\eta(0)} dA(w) \\
&\le \frac{\|g\|_\infty}{(1-\eta^2)^2} \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c \iint_{\mathbb{D}} dA(w) + \|g\|_\infty A(\mathbb{D} \setminus D_\eta(0)) \\
&\le C_2 \Big[\frac{\|g\|_\infty}{(1-\eta^2)^2} \frac{A(G_c \cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c + \|g\|_\infty (1-\eta) \Big].
\end{aligned} \tag{4.18}$$

Choosing c close enough to 0 and η close enough to 1 we get

$$A(G_c \cap D_\eta(\alpha)) \ge \frac{C}{\|g\|_\infty} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)).$$

Case (b) $0 \le |\alpha| \le R_0$: There exists η_1, depending only on η, such that $D_\eta(R_0) \subseteq D_{\eta_1}(0)$. Take α' so that $|\alpha'| = R_0$ and $Arg(\alpha') = Arg(\alpha)$. Then $D_\eta(\alpha') \subseteq D_{\eta_1}(\alpha)$. Set $\eta_2 = \max\{\eta, \eta_1\}$. Then from case (a) for α' we have

$$\begin{aligned}
A(G_c \cap D_{\eta_2}(\alpha)) &\ge A(G_c \cap D_{\eta_1}(\alpha)) \ge A(G_c \cap D_\eta(\alpha')) \\
&\ge \delta A(D_\eta(\alpha')) \ge C\delta A(D_{\eta_1}(\alpha)) \ge C\delta A(D_{\eta_2}(\alpha)),
\end{aligned} \tag{4.19}$$

where the constants $C > 0$ depend only on η.

Moreover, when $R_0 \leq |\alpha| < 1$, we have

$$A(G_c \cap D_{\eta_2}(\alpha)) \geq A(G_c \cap D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)) \geq C\delta A(D_{\eta_2}(\alpha)),$$

where the constant $C > 0$ depends only on η. So, we have proved that there are $\eta_2 \in (0,1)$, $c > 0$ and $C > 0$ such that

$$A(G_c \cap D_{\eta_2}(\alpha)) \geq CA(D_{\eta_2}(\alpha)),$$

for every $\alpha \in \mathbb{D}$, which is what we had to prove. $\square$

Chapter 5

Closed range integral operators on BMOA space

$BMOA$ space and the corresponding equivalent norms in this space were defined in section 2.1.3. Let denote as $BMOA_0$ the space $BMOA/\mathbb{C}$, the space $BMOA$ modulo the constants. In this chapter, a necessary and sufficient condition (theorem 5.2.1) for the integral operator S_g to have closed range on $BMOA_0$ space is proved.

5.1 On the boundedness of the integral operator on BMOA

As far as the boundedness of S_g on $BMOA_0$ is concerned, we will prove proposition 5.1.2 by making use of the following result which is proved in [51] (proposition 4.13).

Proposition 5.1.1. *Suppose $p > 0$, γ is real and $r > 0$. Then there exists a positive constant C such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{2+\gamma}} \iint\limits_{D_r(z)} |f(w)|^p (1-|w|^2)^\gamma dA(w), \tag{5.1}$$

for all analytic functions f in $\mathbb{D}$ and for all $z \in \mathbb{D}$.

Proposition 5.1.2. *The operator $S_g : BMOA_0 \to BMOA_0$ is bounded if and only if $g \in H^\infty$.*

Proof. Let $z_0 \in \mathbb{D}$, $f \in BMOA$ and suppose that S_g is bounded. The point evaluation functional of the derivative, on $BMOA$, induced by z_0, is defined as $\Lambda_{z_0} f = f'(z_0), f \in BMOA$. It is easy to check that Λ_{z_0} is bounded on $BMOA$.

If $z \in D_{\frac{1}{2}}(z_0)$ then, from (2.19), we have

$$\frac{(1-|z_0|^2)(1-|z|^2)}{|1-\overline{z_0}z|^2} \asymp 1,$$

so applying (5.1), for $p = 2, \gamma = 0$ and $r = 1/2$, and using proposition 2.6.2, we get

$$\begin{aligned}
|\Lambda_{z_0} f|^2 = |f'(z_0)|^2 &\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} |f'(z)|^2 dA(z) \\
&\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} \frac{(1-|z_0|^2)(1-|z|^2)}{|1-\overline{z_0}z|^2} |f'(z)|^2 dA(z) \\
&\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{\mathbb{D}} \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \leq C\|f\|_*^2,
\end{aligned}$$

where the constant C depends only on z_0 which is fixed. We proved that Λ_{z_0} is bounded and, in addition, we have supposed that S_g is bounded so, corollary 2.8.2 implies that g is a bounded function.

For the converse, let's suppose that g is a bounded function. Then we have

$$\begin{aligned}
\|S_g f\|_*^2 &= \left\| \int_0^z f'(w)g(w)dw \right\|_*^2 \\
&= \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 |g(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\leq \|g\|_\infty^2 \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 \log\frac{1}{|z|} dA(z)
\end{aligned}$$

$$\leq \|g\|_\infty^2 \|f\|_*^2.$$

So S_g is bounded and $\|S_g\| \leq \|g\|_\infty$. □

5.2 The main result

In [7], A. Anderson posed the question of finding a necessary and sufficient condition for the operator S_g to have closed range on $BMOA_0$. Next, we answer this question, proving that conditions (ii) and (iii) of theorem 4.1.2, for H_0^p, are also necessary and sufficient for the integral operator S_g to have closed range on $BMOA_0$. We consider $g \in H^\infty$ and set again $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$.

The following theorem is the main result of this section, which appears in [39] (Theorem 3.1).

Theorem 5.2.1. *Let $g \in H^\infty$ and g not be identically equal to zero. Then the following are equivalent:*

(i) The operator $S_g : BMOA_0 \to BMOA_0$ has closed range

(ii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a)) \tag{5.2}$$

for all $a \in \mathbb{D}$.

In the proof of theorem 5.2.1, we will make use of a theorem (see [32]) due to D. Luecking.

Theorem 5.2.2 (Luecking)**.** *Let G be a measurable subset of $\mathbb{D}$, $0 < p < +\infty$ and $\gamma > -1$. The following are equivalent:*

- *There exists $C > 0$ such that*

$$\iint\limits_{G} |f(w)|^p (1-|w|^2)^\gamma dA(w) \geq C \iint\limits_{\mathbb{D}} |f(w)|^p (1-|w|^2)^\gamma dA(w) \quad (5.3)$$

for all $f \in A^p_\gamma$.

- *There are $\delta > 0$ and $\eta \in (0,1)$, such that*

$$A(G \cap D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)) \quad (5.4)$$

for all $\alpha \in \mathbb{D}$.

Proof of theorem 5.2.1. $(ii) \Rightarrow (i)$ If (5.2) holds then, because of theorem 5.2.2, (5.3) also holds for $G = G_c$. For $\beta \in \mathbb{D}$, $z \in \mathbb{D}$ and $f \in BMOA_0$, we consider the function $h_\beta(z) = \frac{(1-|\beta|^2)^{\frac{1}{2}}}{1-\overline{\beta}z} f'(z)$. Using proposition 2.6.2 it's easy to see that if $f \in BMOA_0$ then $h_\beta \in A^2_1$. Indeed

$$\begin{aligned}
\|h_\beta\|^2_{A^2_1} &= \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 (1-|z|^2) dA(z) \\
&\leq C \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \leq \|f\|^2_{BMOA_0} < \infty.
\end{aligned}$$

Let $\beta \in \mathbb{D}$. We have that

$$\begin{aligned}
\|S_g f\|^2_{BMOA_0} &= \sup_{z_0 \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} |(S_g f(z))'|^2 \log\frac{1}{|z|} dA(z) \\
&= \sup_{z_0 \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} |f'(z)|^2 |g(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\geq \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 |g(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\geq c^2 \iint\limits_{G_c} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 \log\frac{1}{|z|} dA(z) \\
&= c^2 \iint\limits_{G_c} |h_\beta(z)|^2 \log\frac{1}{|z|} dA(z)
\end{aligned}$$

$$\geq C \iint_{G_c} |h_\beta(z)|^2(1-|z|^2)dA(z)$$
$$\geq C \iint_{\mathbb{D}} |h_\beta(z)|^2(1-|z|^2)dA(z),$$

where the last inequality is justified by theorem 5.2.2. So

$$\|S_g f\|^2_{BMOA_0} \geq C \iint_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2}|f'(z)|^2 \log\frac{1}{|z|}dA(z).$$

Taking the supremum over $\beta \in \mathbb{D}$ in the last relation we get

$$\|S_g f\|^2_{BMOA_0} \geq C\|f\|^2_{BMOA_0}.$$

$(i) \Rightarrow (ii)$ If S_g has closed range then there exist $C_1 > 0$ such that for every $f \in BMOA_0$ we have

$$\|S_g f\|^2_{BMOA_0} \geq C_1\|f\|^2_{BMOA_0}.$$

For $\alpha \in \mathbb{D}$, if we set $f = \psi_\alpha - \alpha$ in the last inequality, just as in the case of Hardy spaces then, lemma A.0.2 in Appendix A implies that $\|\psi_\alpha - \alpha\|_{BMOA} \asymp 1$. In addition $\frac{(1-|\beta|^2)(1-|z|^2)}{|1-\overline{\beta}z|^2} < 1$, for all $z, \beta \in \mathbb{D}$. So we have

$$\begin{aligned}
C_1 &\leq \|S_g(\psi_\alpha - \alpha)\|^2_{BMOA_0} \\
&= \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2}|(S_g(\psi_\alpha-\alpha)(z))'|^2 \log\frac{1}{|z|}dA(z) \\
&\leq C \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2}|\psi'_\alpha(z)|^2|g(z)|^2(1-|z|^2)dA(z) \\
&\leq C \iint_{\mathbb{D}} |\psi'_\alpha(z)|^2|g(z)|^2 dA(z) \qquad (5.5) \\
&\leq C\Big[\|g\|^2_\infty \iint_{G_c\cap D_\eta(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}z|^4}dA(z) + c^2 \iint_{D_\eta(\alpha)\setminus G_c} |\psi'_\alpha(z)|^2 dA(z)
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \|g\|_\infty^2 \iint\limits_{\mathbb{D}\setminus D_\eta(\alpha)} |\psi'_\alpha(z)|^2 dA(z)\Big] \\
&\leq C\Big[\|g\|_\infty^2 \iint\limits_{G_c\cap D_\eta(\alpha)} \frac{1}{(1-|\alpha|^2)^2} dA(z) + c^2 \iint\limits_{\mathbb{D}} |\psi'_\alpha(z)|^2 dA(z) \\
&\qquad\qquad\qquad\qquad + \|g\|_\infty^2 \iint\limits_{\mathbb{D}\setminus D_\eta(\alpha)} |\psi'_\alpha(z)|^2 dA(z)\Big] \\
&= C\Big[\|g\|_\infty^2 \frac{A(G_c\cap D_\eta(\alpha))}{(1-|\alpha|^2)^2} + c^2 \iint\limits_{\mathbb{D}} dA(w) + \|g\|_\infty^2 \iint\limits_{\mathbb{D}\setminus D_\eta(0)} dA(w)\Big] \\
&\leq C\Big[C'\|g\|_\infty^2 \frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c^2 + \|g\|_\infty^2 (1-\eta^2)\Big],
\end{aligned}
$$

where C' depends only on η and C is absolute. Therefore

$$
C_1 \leq C'\|g\|_\infty^2 \frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + c^2 + \|g\|_\infty^2 (1-\eta^2).
$$

First, we choose η close enough to 1 so that $\|g\|_\infty^2(1-\eta^2) < \frac{C_1}{4}$ and c small enough so that $c < \frac{C_1}{4}$. So

$$
A(G_c\cap D_\eta(\alpha)) \geq \frac{C}{2\|g\|_\infty^2} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)),
$$

where C depends only on η. □

Chapter 6

Closed range integral operators on Q_p spaces

Q_p spaces and the corresponding norm in these spaces were defined in section 2.1.7. Let denote as $Q_{p,0}$ the space $Q_p/\mathbb{C}$, the space Q_p modulo the constants. In this chapter, a necessary and sufficient condition (theorem 6.2.1), for the integral operator S_g to have closed range on $Q_{p,0}$, $0 < p < \infty$ spaces, is proved.

The results of this chapter, concerning $Q_{p,0}$ $(0 < p < \infty)$ can come up, by few modifications, from the corresponding results for $BMOA$ proved in the previous chapter.

6.1 On the boundedness of the integral operator on Q_p spaces

The following proposition characterizes the analytic functions g for which $S_g : Q_{p,0} \to Q_{p,0}$ is bounded.

Proposition 6.1.1. *The operator $S_g : Q_{p,0} \to Q_{p,0}$ is bounded if and only if $g \in H^\infty$.*

Proof. Let $z_0 \in \mathbb{D}$, $f \in Q_p$ and suppose that S_g is bounded. The point evaluation functional of the derivative, on Q_p, induced by z_0, is defined as $\Lambda_{z_0} f = f'(z_0), f \in Q_p$. It is easy to check that Λ_{z_0} is bounded on Q_p.

If $z \in D_{\frac{1}{2}}(z_0)$ then, from (2.19), we have

$$\frac{(1-|z_0|^2)^p(1-|z|^2)^p}{|1-\overline{z_0}z|^{2p}} \asymp 1. \tag{6.1}$$

Applying (5.1), for $p = 2$, $\gamma = 0$ and $r = 1/2$, we get

$$|\Lambda_{z_0} f|^2 = |f'(z_0)|^2 \leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} |f'(z)|^2 dA(z). \tag{6.2}$$

Combining (6.1) and (6.2), we get

$$\begin{aligned}
\|\Lambda_{z_0} f\|^2 &\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} |f'(z)|^2 dA(z) \\
&\leq \frac{C}{(1-|z_0|^2)^2} \iint\limits_{D_{\frac{1}{2}}(z_0)} \frac{(1-|z_0|^2)^p}{|1-\overline{z_0}z|^{2p}} |f'(z)|^2 (1-|z|^2)^p dA(z) \leq C\|f\|^2_{Q_p},
\end{aligned}$$

where the constant C depends only on z_0 which is fixed. We proved that Λ_{z_0} is bounded and, in addition, we have supposed that S_g is bounded so, corollary 2.8.2 implies that g is a bounded function.

For the converse, let's suppose that g is a bounded function. Then we have

$$\begin{aligned}
\|S_g f\|^2_{Q_p} &= \left\| \int_0^z f'(w)g(w)dw \right\|^2_{Q_p} \\
&= \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |f'(z)|^2 |g(z)|^2 (1-|z|^2)^p dA(z) \\
&\leq \|g\|^2_\infty \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |f'(z)|^2 (1-|z|^2)^p dA(z) \\
&\leq \|g\|^2_\infty \|f\|^2_{Q_p}.
\end{aligned}$$

So S_g is bounded and $\|S_g\| \leq \|g\|_\infty$. $\square$

6.2 The main result

We require $S_g : Q_{p,0} \to Q_{p,0}$ to be bounded, so consider $g \in H^\infty$ and set again $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$.

The following theorem is the main result of this section, which appears in [39] (Remark 3.1).

Theorem 6.2.1. *Let $g \in H^\infty$ and g not be identically equal to zero. Then the following are equivalent:*

(i) The operator $S_g : Q_{p,0} \to Q_{p,0}$ has closed range

(ii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a)) \tag{6.3}$$

for all $a \in \mathbb{D}$.

In the proof of theorem 6.2.1, we will make use of theorem 5.2.2.

Proof of theorem 5.2.1. $(ii) \Rightarrow (i)$ If (6.3) holds then, because of theorem 5.2.2, (5.3) also holds for $G = G_c$. For $\beta \in \mathbb{D}$, $z \in \mathbb{D}$ and $f \in Q_{p,0}$, we consider the function $h_\beta(z) = \frac{(1-|\beta|^2)^{\frac{p}{2}}}{(1-\overline{\beta}z)^p} f'(z)$. It's easy to see that if $f \in Q_{p,0}$ then $h_\beta \in A^2_p$. Indeed

$$\|h_\beta\|^2_{A^2_p} = \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2} |f'(z)|^2 (1-|z|^2)^p dA(z) \leq \|f\|^2_{Q_{p,0}} < \infty.$$

Let $\beta \in \mathbb{D}$. We have that

$$\begin{aligned}
\|S_g f\|^2_{Q_{p,0}} &= \sup_{z_0 \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|z_0|^2)^p}{|1-\overline{z_0}z|^{2p}} |(S_g f(z))'|^2 (1-|z|^2)^p dA(z) \\
&= \sup_{z_0 \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|z_0|^2)^p}{|1-\overline{z_0}z|^{2p}} |f'(z)|^2 |g(z)|^2 (1-|z|^2)^p dA(z) \\
&\geq \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |f'(z)|^2 |g(z)|^2 (1-|z|^2)^p dA(z)
\end{aligned}$$

$$\begin{aligned}
&\geq c^2 \iint\limits_{G_c} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |f'(z)|^2 (1-|z|^2)^p dA(z) \\
&= c^2 \iint\limits_{G_c} |h_\beta(z)|^2 (1-|z|^2)^p dA(z) \\
&\geq C \iint\limits_{\mathbb{D}} |h_\beta(z)|^2 (1-|z|^2)^p dA(z),
\end{aligned}$$

where the last inequality is justified by theorem 5.2.2. So

$$\|S_g f\|^2_{Q_{p,0}} \geq C \iint\limits_{\mathbb{D}} |h_\beta(z)|^2 (1-|z|^2)^p dA(z).$$

Taking the supremum over $\beta \in \mathbb{D}$ in the last relation we get

$$\|S_g f\|^2_{Q_{p,0}} \geq C \|f\|^2_{Q_{p,0}}.$$

$(i) \Rightarrow (ii)$ If S_g has closed range then there exist $C_1 > 0$ such that for every $f \in Q_{p,0}$ we have

$$\|S_g f\|^2_{Q_{p,0}} \geq C_1 \|f\|^2_{Q_{p,0}}.$$

For $\alpha \in \mathbb{D}$, if we set $f = \psi_\alpha - \alpha$ in the last inequality, just as in the case of $BMOA$ space then, lemma A.0.3 in Appendix A implies that $\|\psi_\alpha - \alpha\|_{Q_p} \asymp 1$. In addition $\frac{(1-|\beta|^2)^p (1-|z|^2)^p}{|1-\overline{\beta}z|^{2p}} < 1$, for all $z, \beta \in \mathbb{D}$ and for all $p \in (0, \infty)$. So we have

$$\begin{aligned}
C_1 &\leq \|S_g(\psi_\alpha - \alpha)\|^2_{Q_{p,0}} \\
&= \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |(S_g(\psi_\alpha - \alpha)(z))'|^2 (1-|z|^2)^p dA(z) \\
&= \sup_{\beta \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |\psi'_\alpha(z)|^2 |g(z)|^2 (1-|z|^2)^p dA(z) \\
&\leq \iint\limits_{\mathbb{D}} |\psi'_\alpha(z)|^2 |g(z)|^2 dA(z).
\end{aligned}$$

So we have

$$C_1 \leq \iint_{\mathbb{D}} |\psi'_\alpha(z)|^2 |g(z)|^2 dA(z). \tag{6.4}$$

But, (6.4) is the same as (5.5). So, following exactly the same steps as these after (5.5), we get the desired result. $\square$

Chapter 7

Closed range integral operators on Besov spaces

Besov spaces B^p and the corresponding norm in these spaces were defined in section 2.1.4. Let denote as B_0^p the space $B^p/\mathbb{C}$. In this chapter a necessary and sufficient condition (theorem 7.2.1) for the integral operator S_g to have closed range on $B_0^p, 1 < p < \infty$ spaces, is proved.

7.1 On the boundedness of the integral operator on Besov spaces

Let consider the operator $S_g : B_0^p \to B_0^p$. As far as the boundedness of S_g on B_0^p is concerned, we prove the following:

Proposition 7.1.1. *The operator $S_g : B_0^p \to B_0^p$ is bounded if and only if $g \in H^\infty$.*

Proof. Let $z_0 \in \mathbb{D}$, $f \in B_0^p$ and suppose that S_g is bounded. The point evaluation functional of the derivative, on B_0^p, induced by z_0, is defined as $\Lambda_{z_0} f = f'(z_0), f \in B_0^p$. It is easy to check that Λ_{z_0} is bounded on B_0^p. Applying (5.1), for $\gamma = p-2$ and $r = 1/2$, we get

$$
\begin{aligned}
|\Lambda_{z_0} f|^p = |f'(z_0)|^p &\leq \frac{C}{(1-|z_0|^2)^p} \iint\limits_{D_{\frac{1}{2}}(z_0)} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\leq \frac{C}{(1-|z_0|^2)^p} \iint\limits_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \leq C \|f\|_{B_0^p}^p,
\end{aligned}
$$

where the constant C depends only on z_0 which is fixed. We proved that Λ_{z_0} is bounded and, in addition, we have supposed that S_g is bounded so, corollary 2.8.2 implies that g is a bounded function.

For the converse, let's suppose that g is a bounded function. Then we have

$$\begin{aligned}
\|S_g f\|^p_{B^p_0} &= \left\| \int_0^z f'(w)g(w)dw \right\|^p_{B^p_0} \\
&= \iint\limits_{\mathbb{D}} |f'(z)|^p |g(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\leq \|g\|^p_\infty \iint\limits_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\leq \|g\|^p_\infty \|f\|^p_{B^p_0}.
\end{aligned}$$

So S_g is bounded and $\|S_g\| \leq \|g\|_\infty$. □

7.2 The main result

We consider $g \in H^\infty$, $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$ and proceed with the proof of the main result of this section (theorem 7.2.1) which appears in [39] (Section 4) and states that condition (5.2) is also necessary and sufficient for the operator S_g to have closed range on B^p_0.

Theorem 7.2.1. *Let $g \in H^\infty$ and g not be identically equal to zero. Then the following are equivalent:*

(i) The operator $S_g : B^p_0 \to B^p_0 (1 < p < \infty)$ has closed range

(ii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0, 1)$ such that

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a)) \tag{7.1}$$

for all $a \in \mathbb{D}$.

Proof. $(ii) \Rightarrow (i)$ For the sufficiency, we observe that, if $f \in B^p$ then $f' \in A^p_{p-2}$, the weighted Bergman space, so we can use theorem 5.2.2. We have

$$
\begin{aligned}
\|S_g f\|^p_{B^p_0} &= \iint_{\mathbb{D}} |(S_g f(z))'|^p (1-|z|^2)^{p-2} dA(z) \\
&\geq \iint_{G_c} |f'(z)|^p |g(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\geq c^p \iint_{G_c} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&\geq C \iint_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&= C\|f\|^p_{B^p_0},
\end{aligned}
$$

where the last inequality is justified by theorem 5.2.2. So S_g has closed range on B^p_0.

$(i) \Rightarrow (ii)$ If S_g has closed range on B^p_0 then there exist $C_1 > 0$ such that for every $f \in B^p_0$ we have

$$
\|S_g f\|^p_{B^p_0} \geq C_1 \|f\|^p_{B^p_0}.
$$

For $\alpha \in \mathbb{D}$, if we set $f = f_\alpha = \frac{(1-|\alpha|^2)^{\frac{2}{p}}}{\frac{2\bar{\alpha}}{p}(1-\bar{\alpha}z)^{\frac{2}{p}}} - \frac{(1-|\alpha|^2)^{\frac{2}{p}}}{\frac{2\bar{\alpha}}{p}}$ in the last inequality, just as in the case of $BMOA$. Lemma A.0.4 in Appendix A implies that $\|f_\alpha\|_{B^p_0} \asymp 1$. Moreover, $|f'_\alpha(z)| = \frac{(1-|\alpha|^2)^{\frac{2}{p}}}{|1-\bar{\alpha}z|^{\frac{2}{p}+1}}$, so we have

$$
\begin{aligned}
C_1 \leq \|S_g f_\alpha\|^p_{B^p_0} &= \iint_{\mathbb{D}} |f'_\alpha(z)|^p |g(z)|^p (1-|z|)^{p-2} dA(z) \\
\leq \|g\|^p_\infty \iint_{G_c \cap D_\eta(\alpha)} &\frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^{2+p}} (1-|z|)^{p-2} dA(z) + c^p \iint_{D_\eta(\alpha)\setminus G_c} |f'_\alpha(z)|^p (1-|z|)^{p-2} dA(z) \\
&+ \|g\|^p_\infty \iint_{\mathbb{D}\setminus D_\eta(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^{2+p}} (1-|z|)^{p-2} dA(z) \\
\leq \|g\|^p_\infty \iint_{G_c \cap D_\eta(\alpha)} &\frac{1}{(1-|\alpha|^2)^2} dA(z) + c^p \iint_{\mathbb{D}} |f'_\alpha(z)|^p (1-|z|)^{p-2} dA(z)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|g\|_\infty^p \iint\limits_{\mathbb{D}\setminus D_\eta(0)} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}\psi_\alpha(w)|^{2+p}}(1-|\psi_\alpha(w)|)^{p-2}|\psi_\alpha'(w)|^2 dA(w)\\
&= \|g\|_\infty^p \iint\limits_{G_c\cap D_\eta(\alpha)} \frac{1}{(1-|\alpha|^2)^2} dA(z) + c^p\|f_\alpha\|_{B_0^p}^p + \|g\|_\infty^p \iint\limits_{\mathbb{D}\setminus D_\eta(0)} \frac{(1-|w|^2)^{p-2}}{|1-\overline{\alpha}w|^{p-2}} dA(w)\\
&\le \|g\|_\infty^p \frac{A(G_c\cap D_\eta(\alpha))}{(1-|\alpha|^2)^2} + c^p\|f_\alpha\|_{B_0^p}^p + \|g\|_\infty^p \iint\limits_{\mathbb{D}\setminus D_\eta(0)} dA(w)\\
&\le C'\|g\|_\infty^p \frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + Cc^p + \|g\|_\infty^p(1-\eta^2),
\end{aligned}
$$

where C' depends only on η and C is absolute. So we have

$$
C_1 \le C'\|g\|_\infty^p \frac{A(G_c\cap D_\eta(\alpha))}{A(D_\eta(\alpha))} + Cc^p + \|g\|_\infty^p(1-\eta^2).
$$

Choosing η close enough to 1 so that $\|g\|_\infty^p(1-\eta^2) < \frac{C_1}{4}$, and c small enough so that $Cc^p < \frac{C_1}{4}$, we get

$$
A(G_c\cap D_\eta(\alpha)) \ge \frac{C_1}{2C'\|g\|_\infty^p} A(D_\eta(\alpha)) = \delta A(D_\eta(\alpha)).
$$

□

Part III

Closed Range Composition Operators

Chapter 8

Three auxiliary lemmas

We recall that if $\varphi : \mathbb{D} \to \mathbb{D}$ is a non-constant analytic function then, the composition operator $C_\varphi : X \to X$ induced by φ, is defined as

$$C_\varphi(f) = f \circ \varphi,$$

for every $f \in X$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and ψ_α, $\alpha \in \mathbb{D}$, the Möbius transformations as defined in (2.15). Next we will prove three very useful lemmas which allow us, many times, to simplify the computations in study of composition operator. If X is one of the spaces Hardy, Bergman and BMOA then, in the study of operator $C_\varphi : X \to X$, these lemmas allow us to suppose, without loss of generality, that $\varphi(0) = 0$ and also to restrict our study in the subspace $X_0 = \{f \in X : f(0) = 0\}$ of X, which means, to consider $f(0) = 0$. The three next lemmas have been formulated for the case of A^2 in [2], without being proved. Here, we have included them with their proofs.

Lemma 8.0.1. *If one of the operators C_φ, $C_{\varphi\circ\psi_\alpha}$ and $C_{\psi_\alpha\circ\varphi}$ has closed range on X then the same holds for the other ones.*

Proof. Let's suppose that C_φ has closed range. For $\alpha \in \mathbb{D}$, the operator $C_{\psi_\alpha} : X \to X$ is 1-1 and onto, so it has closed range, too. So, there are $C_1 > 0$ and $C_2 > 0$ such that $\|C_{\psi_\alpha} f\|_X \geq C_1 \|f\|_X$ and $\|C_\varphi f\|_X \geq C_2 \|f\|_X$.

If $f \in X$, we have

$$\|C_{\varphi\circ\psi_\alpha}(f)\|_X = \|C_{\psi_\alpha}C_\varphi(f)\|_X \geq C_1\|C_\varphi(f)\|_X \geq C_1C_2\|f\|_X$$

and

$$\|C_{\psi_\alpha\circ\varphi}(f)\|_X = \|C_\varphi C_{\psi_\alpha}(f)\|_X \geq C_2\|C_\varphi(f)\|_X \geq C_1C_2\|f\|_X.$$

So both, $C_{\varphi\circ\psi_\alpha}$ and $C_{\psi_\alpha\circ\varphi}$, have closed range.

The other cases are proved by similar arguments. □

In the proof of lemma 8.0.3, we will make use of the following well known result (a proof of it can be found in [51] (lemma 4.11)).

Lemma 8.0.2. *Suppose $p > 0$ and $0 < r < 1$. Then*

$$|f(0)|^p \leq \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta,$$

for all analytic functions f in $\mathbb{D}$.

We will prove lemma 8.0.3 just for the case of Hardy H^p spaces since, it can be proved for the other spaces by similar arguments.

Lemma 8.0.3. *Let denote $H_0^p = \{f \in H^p : f(0) = 0\}$. Then C_φ has closed range on H^p if and only if C_φ has closed range on H_0^p*

Proof. We will use the norm defined in (2.4)

$$\|f\|_{H^p}^p = \int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta)$$

where $f \in H^p$.

Let's suppose that C_φ has closed range on H_0^p and $f \in H^p$. Lemma 8.0.1 allows us to consider $\varphi(0) = 0$ and, for $z \in \mathbb{D}$, to define the function $g(z) = f(z) - f(0)$. It's clear that $g \in H_0^p$. Since C_φ has closed range on H_0^p we have that there exists $C_0 > 0$ such that

$$\|g \circ \varphi\|_{H_0^p} > C_0 \|g\|_{H_0^p}$$

so

$$\|f(\varphi(z)) - f(0)\|_{H^p} > C_0 \|f(z) - f(0)\|_{H^p}. \tag{8.1}$$

The left side of (8.1) gives

$$\begin{aligned} \|f(\varphi(z)) - f(0)\|_{H^p} &= \Big[\int_{\mathbb{T}} |f(\varphi(\zeta)) - f(0)|^p dm(\zeta) \Big]^{\frac{1}{p}} \\ &\leq \Big[\int_{\mathbb{T}} |f(\varphi(\zeta))|^p dm(\zeta) \Big]^{\frac{1}{p}} + |f(0)|. \end{aligned} \tag{8.2}$$

From the right side of (8.1), we have

$$\begin{aligned} C_0 \|f(z) - f(0)\|_{H^p} &= C_0 \Big[\int_{\mathbb{T}} |f(\zeta) - f(0)|^p dm(\zeta) \Big]^{\frac{1}{p}} \\ &\geq C_0 \Big[\int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta) \Big]^{\frac{1}{p}} - C_0 |f(0)|. \end{aligned} \tag{8.3}$$

Because of (8.2) and (8.3), the relation (8.1) implies

$$\Big[\int_{\mathbb{T}} |f(\varphi(\zeta))|^p dm(\zeta) \Big]^{\frac{1}{p}} + |f(0)| \geq C_0 \Big[\int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta) \Big]^{\frac{1}{p}} - C_0 |f(0)|.$$

Hence

$$\Big[\int_{\mathbb{T}} |f(\varphi(\zeta))|^p dm(\zeta) \Big]^{\frac{1}{p}} + (1 + C_0)|f(0)| \geq C_0 \Big[\int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta) \Big]^{\frac{1}{p}}. \tag{8.4}$$

Let $r \in (0,1)$. Using lemma 8.0.2 and the assumption $\varphi(0) = 0$, we obtain

$$|f(0)|^p = |f(\varphi(0))|^p \leq \int_{\mathbb{T}} |f(\varphi(r\zeta))|^p dm(\zeta) \leq \|f \circ \varphi\|_{H^p}^p.$$

Hence, from (8.4),

$$\Big[\int_{\mathbb{T}} |f(\varphi(\zeta))|^p dm(\zeta)\Big]^{\frac{1}{p}} + (1+C_0)\Big(|f(\varphi(0))|^p\Big)^{\frac{1}{p}} \geq C_0\Big[\int_{\mathbb{T}} |f(\zeta)|^p dm(\zeta)\Big]^{\frac{1}{p}}$$

and

$$(2+C_0)\|f\circ\varphi\|_{H^p} \geq C_0\|f\|_{H^p}.$$

So, finally, we have

$$\|C_\varphi(f)\|_{H^p} > C\|f\|_{H^p}$$

which implies that $C_\varphi : H^p \to H^p$ has closed range.

The inverse is obvious. If C_φ has closed range on H^p then it has closed range on the subspace H^p_0. □

Lemma 8.0.4 will be proved for the case of Hardy spaces, in particular, for the corresponding set G_ε defined in (3.4). All sets of this kind (e.g. sets defined in (3.8) and (3.7)) are associated to function φ and, just for the purposes of the proof of lemma 8.0.4, we will change the notation from G_ε to $G_\varepsilon(\varphi)$. The proofs for the sets corresponding to other spaces can be done by similar arguments as that of lemma 8.0.4. So, in the same manner, we consider the functions $\psi_\alpha\circ\varphi$, $\varphi\circ\psi_\alpha$ and the corresponding sets $G_\varepsilon(\psi_\alpha\circ\varphi)$ and $G_\varepsilon(\varphi\circ\psi_\alpha)$. Lemma 8.0.4 guarantees that the assumption $\varphi(0) = 0$, which is achieved by composing φ with the appropriate Möbius transformation ψ_α (see lemma 8.0.1), does not, in fact, affect the validity of the following condition:

There are $C > 0$ and $\eta \in (0,1)$ such that

$$A(G_\varepsilon \cap D_\eta(w) > CA(D_\eta(w)) \tag{8.5}$$

for all $w \in \mathbb{D}$.

Lemma 8.0.4. *If there is $\varepsilon > 0$ such that one of the sets $G_\varepsilon(\varphi)$, $G_\varepsilon(\varphi \circ \psi_\alpha)$ and $G_\varepsilon(\psi_\alpha \circ \varphi)$ satisfy condition (8.5), then there is $\varepsilon' > 0$ such that the other two sets satisfy (8.5).*

Proof. Let $\alpha \in \mathbb{D}$ be arbitrary but fixed and let's suppose that $G_\varepsilon(\varphi)$, as defined in (3.4), satisfy condition (8.5). We will prove that $G_\varepsilon(\varphi \circ \psi_\alpha)$ and $G_\varepsilon(\psi_\alpha \circ \varphi)$ also satisfy (8.5). Because of (2.21), we may consider $G_\varepsilon(\varphi) = \left\{ w \in \mathbb{D} : \frac{\sum_{\varphi(z)=w} 1-|z|^2}{1-|w|^2} > \varepsilon \right\}$ and then we have

$$
\begin{aligned}
G_\varepsilon(\varphi) &= \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=w} (1-|z|^2)}{1-|w|^2} > \varepsilon \right\} \\
&= \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=w} (1-|z|^2)\frac{(1-|\alpha|^2)}{|1-\bar{\alpha}z|^2}\frac{|1-\bar{\alpha}z|^2}{(1-|\alpha|^2)}}{1-|w|^2} > \varepsilon \right\} \\
&= \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=w} (1-|\psi_\alpha(z)|^2)\frac{|1-\bar{\alpha}z|^2}{(1-|\alpha|^2)}}{1-|w|^2} > \varepsilon \right\} \\
&\subseteq \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=w} (1-|\psi_\alpha(z)|^2)}{1-|w|^2} > \varepsilon \frac{1-|\alpha|^2}{4} = \varepsilon' \right\} \\
&= \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi\circ\psi_\alpha(u)=w} (1-|u|^2)}{1-|w|^2} > \varepsilon' \right\} \\
&= G_{\varepsilon'}(\varphi \circ \psi_\alpha),
\end{aligned}
$$

and from (8.5) we have that

$$
A(G_{\varepsilon'}(\varphi \circ \psi_\alpha) \cap D_\eta(w)) \geq A(G_\varepsilon(\varphi) \cap D_\eta(w)) > CA(D_\eta(w))
$$

for all $w \in \mathbb{D}$. So $G_{\varepsilon'}(\varphi \circ \psi_\alpha)$ satisfies (8.5).

Now, we will show that, if $G_\varepsilon(\varphi)$ satisfies (8.5) then $G_\varepsilon(\psi_\alpha \circ \varphi)$ satisfies (8.5), too. Using (2.16), we have

$$\begin{aligned}
G_\varepsilon(\varphi) &= \left\{ w \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=w} (1-|z|^2)}{1-|w|^2} > \varepsilon \right\} \\
&= \psi_\alpha\left(\left\{ u \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=\psi_\alpha(u)} (1-|z|^2)}{1-|\psi_\alpha(u)|^2} > \varepsilon \right\} \right) \\
&= \psi_\alpha\left(\left\{ u \in \mathbb{D} : \frac{\sum\limits_{z:\varphi(z)=\psi_\alpha(u)} (1-|z|^2)|1-\overline{\alpha}u|^2}{(1-|\alpha|^2)(1-|u|^2)} > \varepsilon \right\} \right) \\
&\subseteq \psi_\alpha\left(\left\{ u \in \mathbb{D} : \frac{\sum\limits_{z:\psi_\alpha\circ\varphi(z)=u} (1-|z|^2)}{1-|u|^2} > \varepsilon \frac{1-|\alpha|^2}{4} = \varepsilon' \right\} \right) \\
&= \psi_\alpha(G_{\varepsilon'}(\psi_\alpha \circ \varphi)).
\end{aligned}$$

Hence

$$G_\varepsilon(\varphi) \subseteq \psi_\alpha(G_{\varepsilon'}(\psi_\alpha \circ \varphi))$$

and, obviously,

$$\psi_\alpha(G_\varepsilon(\varphi)) \subseteq G_{\varepsilon'}(\psi_\alpha \circ \varphi).$$

Thus,

$$\begin{aligned}
A(G_{\varepsilon'}(\psi_\alpha \circ \varphi) \cap D_\eta(w)) &\geq A(\psi_\alpha(G_\varepsilon(\varphi)) \cap D_\eta(w)) \\
&= A(\psi_\alpha(G_\varepsilon(\varphi) \cap \psi_\alpha(D_\eta(w)))) \\
&= A(\psi_\alpha(G_\varepsilon(\varphi) \cap D_\eta(\psi_\alpha(w)))) \\
&\geq C_\alpha A(G_\varepsilon(\varphi) \cap D_\eta(\psi_\alpha(w))) \\
&\geq C_\alpha A(D_\eta(\psi_\alpha(w))) \\
&= C_\alpha A(\psi_\alpha(D_\eta(w))) \\
&\geq C_\alpha^2 A(D_\eta(w)),
\end{aligned}$$

where C_α is a positive constant depending only on α. So, the set $G_{\varepsilon'}(\psi_\alpha \circ \varphi)$ satisfies (8.5). $\square$

Chapter 9

Closed range composition operators on Hardy spaces

Hardy spaces H^p and some equivalent norms in these spaces were defined in section 2.1.1. In this chapter we prove that, two already known conditions for C_φ operator to have closed range on Hardy space H^2, can be extended to all Hardy spaces $H^p, 0 < p < \infty$. The first condition (theorem 9.1.2 (part (ii)) concerns the behaviour of φ at the boundary of the disk $\mathbb{D}$, while the second one (theorem 9.1.2 (part (iii)) is based upon the behaviour of φ in the disk $\mathbb{D}$.

9.1 The main result

$C_\varphi : H^p \to H^p$ is always bounded, for all analytic $\varphi : \mathbb{D} \to \mathbb{D}$. It is implied as an immediate consequence of a theorem due to Littlewood (see [51], theorem 11.12).

Theorem 9.1.1 (Littlewood's subordination theorem). *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ and $p > 0$. Then*

$$\int_0^{2\pi} |f(\varphi(e^{i\theta}))|^p dm(\theta) \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \int_0^{2\pi} |f(e^{i\theta})|^p dm(\theta),$$

for all $f \in H^p$.

In [15] and [53] the case of closed range composition operators in Hardy space H^2 is studied. In [15], Cima, Thomson, and Wogen gave an equivalent condition (see theorem 3.2.1) for $C_\varphi : H^2 \to H^2$ to have closed range that depends only on the behaviour of the function φ on the boundary $\mathbb{T}$ of the open unit disk $\mathbb{D}$. In [53], Zorboska proved a criterion (see theorem 3.2.2) for C_φ to have closed range on H^2 based upon properties of φ on pseudohyberbolic disks. We are going to prove that the results of theorems 3.2.1 and 3.2.2 hold, not only for H^2, but for every H^p, $0 < p < \infty$.

First of all, we define the function

$$\tau_\varphi(w) = \frac{N_\varphi(w)}{\log \frac{1}{|w|}}, w \in \mathbb{D}, \tag{9.1}$$

and, for $\varepsilon > 0$, the set

$$G_\varepsilon = \Big\{ w \in \mathbb{D} : \tau_\varphi(w) > \varepsilon \Big\}, \tag{9.2}$$

just as in (3.3) and (3.4). Here is the main result, which appears in [23].

Theorem 9.1.2. *Let $0 < p < \infty$. Then the following are equivalent:*

(i) $C_\varphi : H^p \to H^p$ has closed range.

(ii) The Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is essentially bounded away from zero.

(iii) There exist $c > 0$, $\delta > 0$ and $\eta \in (0, 1)$ such that the set G_c satisfies

$$A(G_c \cap D_\eta(a)) \geq \delta A(D_\eta(a))$$

for all $a \in \mathbb{D}$.

Before proving the theorem 9.1.2, we provide some definitions and background results.

As in [15] (see section 3.2.1), we extend the definition of φ in the boundary by

$$\varphi(\zeta) = \lim_{r \to 1^-} \varphi(r\zeta), \;\; \zeta \in \mathbb{T}.$$

It is well known that the above limit exists for $m - a.e.\ \zeta \in \mathbb{T}$. Then they defined the measure

$$\nu_\varphi(E) = m(\varphi^{-1}(E) \cap \mathbb{T}), E \subset \mathbb{T}, \tag{9.3}$$

where E is a Borel subset of $\mathbb{T}$. The measure ν_φ is absolutely continuous with respect to the Lebesgue measure m and let $\frac{d\nu_\varphi}{dm}$ be the Radon-Nikodym derivative of ν_φ with respect to m.

The equivalence $(i) \Leftrightarrow (ii)$ is actually theorem 3.2.1. Our proof is the same and we include it for the sake of completeness. The equivalence $(i) \Leftrightarrow (iii)$ is theorem 3.2.2. As it is not clear whether Zorboska's proof for $p = 2$ works for every $p > 0$, we prove this equivalence following a different approach. Namely, we use Hardy-Stein identities (see [40], pages 58-59) for one of the directions, and reverse Carleson measures (see [26]) and pull-back measures (see [22]) for the converse.

We will make use of the following lemma 9.1.3 and theorem 9.1.4, proved in [22], as well as theorem 9.1.5. The case $p > 1$ of theorem 9.1.5 is proved in [26].

Let $\Delta = \frac{4\partial^2}{\partial z \partial \bar{z}}$ be the usual Laplacian and, for $\zeta \in \mathbb{T}$ and $0 < h < 1$, let $W(\zeta, h)$ be the usual Carleson square

$$W(\zeta, h) = \{z \in \overline{\mathbb{D}} : 1 - h < |z| \leq 1, |\arg(z\bar{\zeta})| \leq \pi h\}.$$

We will also make use of the measure m_φ defined on Borel sets $E \subset \overline{\mathbb{D}}$ by

$$m_\varphi(E) = m(\varphi^{-1}(E) \cap \mathbb{T}). \tag{9.4}$$

Actually, ν_φ defined in (9.3) is the restriction of m_φ on $\mathbb{T}$.

Lemma 9.1.3. *For every $g \in C^2(\mathbb{C})$ we have*

$$\iint_{\overline{\mathbb{D}}} g(z)dm_\varphi(z) = g(\varphi(0)) + \frac{1}{2}\iint_{\mathbb{D}} \Delta g(w)N_\varphi(w)dA(w).$$

Theorem 9.1.4. *For $0 < c < \frac{1}{8}$, $\zeta \in \mathbb{T}$ and $0 < h < (1 - |\varphi(0)|)/8$, we have*

$$\sup_{z \in W(\zeta,h)\cap\mathbb{D}} N_\varphi(z) \leq \frac{100}{c^2} m_\varphi(W(\zeta,(1+c)h)).$$

Theorem 9.1.5. *Let $0 < p < \infty$ and let μ be a positive measure in $\overline{\mathbb{D}}$. Then the following assertions are equivalent.*

(i) There exists $C > 0$ such that for every $f \in H^p \cap C(\overline{\mathbb{D}})$,

$$\iint_{\overline{\mathbb{D}}} |f(z)|^p d\mu(z) \geq C\|f\|_{H^p}^p.$$

(ii) There exists $C > 0$ such that for every $\lambda \in \mathbb{D}$

$$\iint_{\overline{\mathbb{D}}} |K_\lambda(z)|^p d\mu(z) \geq C.$$

where, for $p > 1$, $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ is the reproducing kernel in H^p and $K_\lambda = \frac{k_\lambda}{\|k_\lambda\|_{H^p}}$ is its normalised version, and, for $0 < p \leq 1$, we have $K_\lambda(z) = \frac{1-|\lambda|^2}{(1-\overline{\lambda}z)^{(p+1)/p}}$.

(iii) There exists $C > 0$ such that for $0 < h < 1$ and $\zeta \in \mathbb{T}$ we have

$$\mu(W(\zeta,h)) \geq Ch.$$

(iv) There exists $C > 0$ such that the Radon-Nikodym derivative of $\mu|_{\mathbb{T}}$ (the restriction of μ on $\mathbb{T}$) with respect to m is bounded below from C.

Remark 1. As we have already mentioned, the case $p > 1$ of theorem 9.1.5 is proved in [26]. For the case $0 < p \leq 1$, we have just to observe that, with the choice of $K_\lambda(z) = \frac{1-|\lambda|^2}{(1-\bar{\lambda}z)^{(p+1)/p}}$ in assertion (ii) of theorem 9.1.5, then the proof of all assertions of the theorem, as described in [26], works also for the case $0 < p \leq 1$.

In addition, one more theorem from theory of Hardy spaces is going to be used. Next, we formulate this result which characterizes the functions of H^p spaces in terms of their boundary values. For a proof, see [24] (theorem 4.4, page 66). We will use this theorem in the study of composition operators C_φ in H^p and in $BMOA$.

Theorem 9.1.6. *Let $0 < p < \infty$ and $h(t)$ a non-negative function in $L^p(\mathbb{T})$. Then, there exists $f(z) \in H^p$ such that $|f(t)| = h(t)$, for $m - a.e.\ e^{it} \in \mathbb{T}$, if and only if*

$$\int_0^{2\pi} \log h(t) dt > -\infty \tag{9.5}$$

Remark 2. From the proof of theorem 9.1.6 it is implied that if (9.5) holds then the corresponding function $f(z) \in H^p$ has the form

$$f(z) = e^{u(z)+iv(z)}, \tag{9.6}$$

where $u(z)$ is the Poisson integral (defined in (2.12)) of $\log h(t)$. More specifically

$$u(z) = \int_0^{2\pi} P_z(t) \log h(t) dt,$$

and $P_z(t)$ is the Poisson kernel defined in (2.11). In (9.6), $v(z)$ is a harmonic conjugate of $u(z)$. We have to remind here that the harmonic conjugates of $u(z)$ differ by a positive constant.

Proof of theorem 9.1.2. Lemmas 8.0.1 and 8.0.4 allow us, without loss of generality, to assume that $\varphi(0) = 0$.

$(i) \Rightarrow (iii)$ If C_φ has closed range then there exist $C > 0$ (we may suppose

$C < 2$) such that for every $f \in H^p$ we have

$$\|C_\varphi f\|_{H^p}^p \geq C\|f\|_{H^p}^p$$

i.e.

$$|f(0)|^p + \iint_{\mathbb{D}} |f(\varphi(z))|^{p-2}|f'(\varphi(z))|^2|\varphi'(z)|^2 \log\frac{1}{|z|}dA(z) \geq C\iint_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}dA(z).$$

By making change of variables as in proposition 2.5.1 we have

$$|f(0)|^p + \iint_{\mathbb{D}} |f(w)|^{p-2}|f'(w)|^2 N_\varphi(w)dA(w) \geq C\iint_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}dA(z)$$

i.e.

$$|f(0)|^p + \iint_{\mathbb{D}} |f(w)|^{p-2}|f'(w)|^2 \tau_\varphi(w) \log\frac{1}{|w|}dA(w) \geq C\iint_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2 \log\frac{1}{|z|}dA(z). \tag{9.7}$$

Let $f \in H^p$ with $f(z) \neq 0$ for every $z \in \mathbb{D}$. We define the analytic function

$$g(z) = f(z)^{p/2}.$$

Obviously, $g \in H^2$ and $g(z) \neq 0$ for every $z \in \mathbb{D}$. Then from 9.7 we have

$$|g(0)|^2 + \iint_{\mathbb{D}} |g'(w)|^2 \tau_\varphi(w) \log\frac{1}{|w|}dA(w) \geq C\iint_{\mathbb{D}} |g'(z)|^2 \log\frac{1}{|z|}dA(z). \tag{9.8}$$

Because τ_φ is a bounded function (see lemma 2.4.1), there exists $c > 0$ (used for set G_c) such that (9.8) implies

$$|g(0)|^2 + \iint\limits_{G_c} |g'(w)|^2 \log \frac{1}{|w|} dA(w) \geq C \iint\limits_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{|z|} dA(z)$$

or, equivalently,

$$|g(0)|^2 + \iint\limits_{G_c} |g'(w)|^2 (1-|w|^2) dA(w) \geq C \iint\limits_{\mathbb{D}} |g'(z)|^2 (1-|z|^2) dA(z). \quad (9.9)$$

for every $g \in H^2$ with $g(z) \neq 0$ for every $z \in \mathbb{D}$.

Let $a \in \mathbb{D}$. We choose $g \in H^2$ such that $|g'(z)|^2 = \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6}$ and $g(0) = (1-|\alpha|^2)^{\frac{3}{2}}$. It's clear that $G_c \subseteq (G_c \cap D_\eta(a)) \cup (\mathbb{D} \setminus D_\eta(a))$, so we have

$$\begin{aligned}\iint\limits_{G_c \cap D_\eta(a)} |g'(z)|^2 (1-|z|^2) dA(z) + &\iint\limits_{\mathbb{D} \setminus D_\eta(a)} |g'(z)|^2 (1-|z|^2) dA(z) \\ &\geq \iint\limits_{G_c} |g'(z)|^2 (1-|z|^2) dA(z)\end{aligned}$$

and

$$\begin{aligned}\iint\limits_{G_c \cap D_\eta(a)} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6} (1-|z|^2) dA(z) \geq &\iint\limits_{G_c} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6} (1-|z|^2) dA(z) \\ &- \iint\limits_{\mathbb{D} \setminus D_\eta(a)} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6} (1-|z|^2) dA(z).\end{aligned} \quad (9.10)$$

From 9.9 we get

$$\begin{aligned}\iint\limits_{G_c} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6} (1-|z|^2) dA(z) &\geq C \iint\limits_{\mathbb{D}} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6} (1-|z|^2) dA(z) - (1-|\alpha|^3)^2 \\ &= C - (1-|\alpha|^2)^3. \end{aligned} \quad (9.11)$$

In addition, by making the change of variable $z = \psi_\alpha(w)$, we get

$$\iint\limits_{\mathbb{D}\setminus D_\eta(a)} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6}(1-|z|^2)dA(z)$$
$$= \iint\limits_{\mathbb{D}\setminus D_\eta(0)} \frac{(1-|a|^2)^3}{|1-\bar{a}\psi_\alpha(w)|^6}(1-|\psi_\alpha(w)|^2)|\psi'_\alpha(w)|^2dA(z) \leq 1-\eta^2. \quad (9.12)$$

Combining (9.10), (9.11) and (9.12), we get

$$\iint\limits_{G_c\cap D_\eta(a)} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6}(1-|z|^2)dA(z) \geq C-(1-|\alpha|^2)^3-(1-\eta^2).$$

Choosing η close enough to 1 and, also, $|\alpha|$ close enough to 1 (let $|\alpha| \geq R_0$ for some R_0) so that

$$C-(1-|\alpha|^2)^3-(1-\eta^2) > \frac{C}{2}$$

we get

$$\iint\limits_{G_c\cap D_\eta(a)} \frac{(1-|a|^2)^3}{|1-\bar{a}z|^6}(1-|z|^2)dA(z) \geq \frac{C}{2}$$

Using (2.19) and the simple relation $|1-\bar{a}z| \geq 1-|a|^2$ we get

$$\iint\limits_{G_c\cap D_\eta(a)} \frac{1}{(1-|\alpha|^2)^2}dA(z) \geq \frac{C}{2},$$

thus

$$\frac{C'A(G_c\cap D_\eta(a))}{(1-|a|^2)^2} \geq \frac{C}{2}$$

and, finally,

$$A(G_c\cap D_\eta(a)) \geq \delta A(D_\eta(a)).$$

The case $0 \leq |\alpha| \leq R_0$ can be handled with similar arguments as in the proof of (4.19).

$(iii) \Rightarrow (i)$ Lemma 8.0.3 allows us, without loss of generality, to assume that $f(0) = 0$ for all $f \in H^p$. We consider the measure m_φ as defined in 9.4

and we will show that (iii) of theorem 9.1.2 implies (iii) of theorem 9.1.5 with $\mu = m_\varphi$.

Now we consider $\zeta \in \mathbb{T}$ and $0 < h < 1$ and the corresponding Carleson square $W(\zeta, h)$. Having in mind to apply theorem 9.1.4, we take $c = \frac{1}{16}$ and $h' = \frac{h}{8}$. Then there exists $a \in W(\zeta, h')$ so that

$$D_\eta(a) \subset W(\zeta, h') \subset W(\zeta, (1+c)h') \subset W(\zeta, h), \qquad 1 - |a|^2 \geq Ch,$$

where C depends upon η. We have $A(G_c \cap D_\eta(a)) > \delta A(D_\eta(a))$ and hence $G_c \cap D_\eta(a) \neq \emptyset$. Let $b \in G_c \cap D_\eta(a)$. Then $1 - |b|^2 \geq Ch$ and $N_\varphi(b) \geq c \log \frac{1}{|b|}$ (since $b \in G_c$). Applying theorem 9.1.4 (recalling that $\varphi(0) = 0$), we find

$$\begin{aligned} m_\varphi(W(\zeta, h)) \geq m_\varphi(W(\zeta, (1+c)h')) &\geq C \sup_{z \in W(\zeta,h) \cap \mathbb{D}} N_\varphi(z) \geq C N_\varphi(b) \\ &\geq C \log \frac{1}{|b|} \geq C(1 - |b|^2) \geq Ch. \end{aligned}$$

Therefore we get (iii) of theorem 9.1.5 which is equivalent to (i) of the same theorem, with $\mu = m_\varphi$. Now we take any f which is analytic in a disk larger than $\mathbb{D}$ and so that $f(0) = 0$ and we use (i) of theorem 9.1.5 together with lemma 9.1.3 to find

$$\begin{aligned} \|C_\varphi f\|_{H^p}^p &= \iint_{\mathbb{D}} |f(\varphi(z))|^{p-2} |f'(\varphi(z))|^2 |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) \\ &= \iint_{\mathbb{D}} |f(w)|^{p-2} |f'(w)|^2 N_\varphi(w) dA(w) \\ &= \frac{1}{p} \iint_{\mathbb{D}} \Delta(|f|^p) N_\varphi(w) dA(w) \\ &= C \iint_{\overline{\mathbb{D}}} |f(w)|^p dm_\varphi(w) \\ &\geq C \|f\|_{H^p}^p, \end{aligned}$$

Now, if f is the general function in H^p with $f(0) = 0$, we apply the result to the functions f_r, $0 < r < 1$, defined by $f_r(z) = f(rz)$, $z \in \mathbb{D}$, and we take the limit as $r \to 1-$. Therefore C_φ has closed range.

$(i) \Rightarrow (ii)$ Let's suppose that C_φ has closed range on H^p and $E \subset \mathbb{T}$. For $n \in \mathbb{N}$, using theorem 9.1.6, we choose $f_n \in H^p$ such that

$$|f_n(\zeta)|^p = \begin{cases} 1, & \text{if } \zeta \in E \\ \frac{1}{2^n}, & \text{if } \zeta \in \mathbb{T} \setminus E \end{cases}$$

Then $\|C_\varphi f_n\|_{H^p}^p \geq C\|f_n\|_{H^p}^p$ and hence

$$m(\varphi^{-1}(E)) + \frac{1}{2^n} m(\mathbb{T} \setminus \varphi^{-1}(E)) \geq Cm(E) + C\frac{1}{2^n} m(\mathbb{T} \setminus E).$$

Taking limit as $n \to +\infty$, we get $m(\varphi^{-1}(E)) \geq Cm(E)$, i.e.

$$\nu_\varphi(E) \geq Cm(E).$$

Thus the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is bounded below from C.

$(ii) \Rightarrow (i)$ Let's suppose that the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is bounded below from C. For $\lambda > 0$ we consider the set

$$E_f(\lambda) = \{e^{i\theta} : |f(e^{i\theta})| > \lambda\}.$$

Then,

$$m\big(E_{f\circ\varphi}(\lambda)\big) = \nu_\varphi\big(E_f(\lambda)\big) \geq Cm\big(E_f(\lambda)\big)$$

for all $\lambda > 0$, and finally

$$\begin{aligned} \|C_\varphi f\|_{H^p}^p &= \int_0^{2\pi} |f(\varphi(e^{i\theta}))|^p dm(\theta) = \int_0^{+\infty} p\lambda^{p-1} m\big(E_{f\circ\varphi}(\lambda)\big) d\lambda \\ &\geq C\int_0^{+\infty} p\lambda^{p-1} m\big(E_f(\lambda)\big) d\lambda = C\int_0^{2\pi} |f(e^{i\theta})|^p dm(\theta) = C\|f\|_{H^p}^p. \end{aligned}$$

Hence C_φ has closed range. □

9.2 Applications of the main theorem

9.2.1 Regarding inner functions

In this section, using theorem 9.1.2, we prove a result, in the next lemma, for the operator $C_\varphi : H^p \to H^p$ to have closed range when φ is an inner function.

Lemma 9.2.1. *Let $0 < p < \infty$. If $\varphi : \mathbb{D} \to \mathbb{D}$ is inner then $C_\varphi : H^p \to H^p$ has closed range.*

We derive the result of lemma 9.2.1 by following two different approaches. In the first case, we use a statement already known from [38] and conclude easily the desired result. In the second case, we prove lemma 9.2.1 by following a different, and actually, more complicated way, using Alexandrov-Clark measures.

Proof of lemma 9.2.1. In [38], we have that, in case that φ is an inner function, then the Radon-Nikodym derivative $\frac{d\nu_\varphi}{dm}$ is given by

$$\frac{d\nu_\varphi}{dm}(\zeta) = \frac{1 - |\varphi(0)|}{1 + |\varphi(0)|} \tag{9.13}$$

for all $\zeta \in \mathbb{T}$. So, we have that $\frac{d\nu_\varphi}{dm} > 0$ and lemma 9.2.1 is an immediate consequence of part (ii) of theorem 9.1.2. □

Using Alexandrov-Clark measures

Alternative proof of lemma 9.2.1. From proposition 2.7.3, relation (2.32) and the fact that $|\varphi(\zeta)| = 1$ for $m - a.e.\ \zeta \in \mathbb{T}$ when φ is an inner function, we get

$$\begin{aligned}\frac{d\nu_\varphi}{dm}(\alpha) &= \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2} - \int_{\mathbb{T}} \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2} dm(\zeta) \\ &= \frac{1 - |\varphi(0)|^2}{|\alpha - \varphi(0)|^2} \geq \frac{1 - |\varphi(0)|^2}{4} > 0 \end{aligned} \tag{9.14}$$

for $m-a.e.\ \zeta \in \mathbb{T}$. Part (ii) of theorem 9.1.2 implies that C_φ has closed range.

□

9.2.2 An application to Besov type spaces

Now we are going to show that a result regarding Besov type spaces (for definition see section 2.1.4), due to M. Tjani [49], can be extended by using theorem 9.1.2. If $\varepsilon > 0$, the set $G_{\varepsilon,p,\alpha}$ is defined as

$$G_{\varepsilon,p,\alpha} = \left\{ w \in \mathbb{D} : \frac{N_{p,\alpha}(w,\varphi)}{(1-|w|^2)^\alpha} > \varepsilon \right\}$$

where

$$N_{p,\alpha}(w,\varphi) = \sum_{\varphi(z)=w} |\varphi'(z)|^{p-2}(1-|z|^2)^\alpha$$

In [49] the following results are proved.

Theorem 9.2.2. *(Theorem 5.2 in [49]). For $p > 2$, the operator $C_\varphi : B_{p,p-1} \to B_{p,p-1}$ has closed range if and only if there exists an $\varepsilon > 0$ so that $G_{\varepsilon,p,p-1}$ satisfies a reverse Carleson condition, which means that there exist $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that $A(G_{\varepsilon,p,p-1} \cap D_\eta(z)) > \delta A(D_\eta(z))$ for every $z \in \mathbb{D}$.*

Corollary 9.2.3. *(Corollary 5.3 in [49]). Let $p > 2$. If C_φ has closed range on $B_{p,p-1}$ then C_φ has closed range on H^2.*

By using theorem 9.1.2 we can extend corollary 9.2.3 and get the following result.

Corollary 9.2.4. *Let $p > 2$. If C_φ has closed range on $B_{p,p-1}$ then C_φ has closed range on every H^q, $0 < q < \infty$.*

Proof. If C_φ has closed range on $B_{p,p-1}$ then, by theorem 9.2.2, we have that there exist $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that $A(G_{\varepsilon,p,p-1} \cap D_\eta(z)) > \delta A(D_\eta(z))$ for every $z \in \mathbb{D}$. Moreover, we have for the set G_ε, as defined in

(9.2), that $G_\varepsilon = G_{\varepsilon,2,1}$ and also $G_{\varepsilon,p,p-1} \subset G_{\varepsilon,2,1}$. Finally we have that

$$A(G_\varepsilon \cap D_\eta(z)) > A(G_{\varepsilon,p,p-1} \cap D_\eta(z)) > \delta A(D_\eta(z))$$

so, by theorem 9.1.2, C_φ is closed range on every H^q, $0 < q < \infty$. □

9.2.3 Other examples: When C_φ doesn't have closed range

The following two examples and their proofs are in [53] regarding the H^2 space. We include them here to show that these examples can be extended to all H^p spaces, as an application of theorem 9.1.2.

- **Example 1:** Let $\Omega \subset \overline{\mathbb{D}}$ be the lens domain bounded by the upper semicircle and by a circular arc (in the lower semidisc) with endpoints -1 and 1 and making an angle α with the real interval [-1,1] $(0 < \alpha < \frac{\pi}{2})$. Let's suppose ψ be analytic and 1-1 from $\mathbb{D}$ onto Ω, continuous on $\overline{\mathbb{D}}$, $\psi(-1) = -1$ and $\psi(1) = 1$. Consider the function $\varphi = \psi^2$. Then $\varphi(\mathbb{D}) = \mathbb{D}$ and also $n_\varphi(w) \leq 2$ for all $w \in \mathbb{D}$. In addition, φ is continuous on $\overline{\mathbb{D}}$, $\varphi^{-1}(1) = \{-1, 1\}$ and φ doesn't possess a finite angular derivative either at -1 or at 1. As it is proved in [53], C_φ doesn't have closed range on H^2, and from theorem 9.1.2, the same holds for all H^p, $0 < p < \infty$ spaces.

- **Example 2:** If the range $\varphi(\mathbb{D})$ of φ doesn't intersects some Carleson square $S_{h,\theta}$ (defined in (3.2)) or some euclidean disk $E(\beta; r)$ (defined in (2.22)) which is internally tangent to $\mathbb{T}$, then C_φ doesn't have closed range on H^p. This happens because condition (iii) of theorem 9.1.2 is violated, since for α close enough to the boundary we can have $D_\eta(\alpha) \subset S_{h,\theta}$ and $D_\eta(\alpha) \subset E(\beta; r)$. Moreover, if $\varphi(\mathbb{D})$ misses a neighbourhood N_ξ of a point $\xi \in \mathbb{T}$ then, again, part (iii) of 9.1.2 is violated and C_φ doesn't have closed range. This can be seen if we take some $\alpha \in \mathbb{D}$ close enough to ξ so that $D_\eta(\alpha) \subset N_\xi$.

Chapter 10

Closed range composition operators on BMOA space

$BMOA$ space and the corresponding equivalent norms in this space were defined in section 2.1.3. $C_\varphi : BMOA \to BMOA$ is bounded for every analytic $\varphi : \mathbb{D} \to \mathbb{D}$. This result is proved in [8] (theorem 12). In this chapter two necessary conditions (theorems 10.1.2 and 10.1.3) and one sufficient (theorem 10.2.2) are proved for C_φ to have closed range on $BMOA$. Next, using these conditions, we prove a result for C_φ to have closed range in $BMOA$ in case that φ is an inner function (proposition 10.3.3).

By defining the measure

$$d\nu_{z_0}(z) = \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} \log \frac{1}{|z|} dA(z) \tag{10.1}$$

then, the norm of $f \in BMOA$ (see section 2.1.3) can be written as

$$\|f\|_*^2 = |f(0)|^2 + \sup_{z_0 \in \mathbb{D}} \iint_{\mathbb{D}} |f'(z)|^2 d\nu_{z_0}(z).$$

The corresponding norm of the composition operator C_φ is

$$\|f \circ \varphi\|_*^2 = |f(\varphi(0))|^2 + \sup_{z_0 \in \mathbb{D}} \iint_{\mathbb{D}} \frac{1-|z_0|^2}{|1-\overline{z_0}z|^2} |f'(\varphi(z))|^2 |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z).$$

By the change of variables $w = \varphi(z)$ and use of Proposition 2.5.2 we have

$$\begin{aligned}\|f\circ\varphi\|_*^2 &= |f(\varphi(0))|^2 + \sup_{z_0\in\mathbb{D}} \iint_{\mathbb{D}} |f'(w)N_{\varphi,z_0}(w)dA(w)\\ &= |f(\varphi(0))|^2 + \sup_{z_0\in\mathbb{D}} \iint_{\mathbb{D}} |f'(w)|^2 d\mu_{z_0}(w),\end{aligned}$$

where

$$d\mu_{z_0}(w) = \sum_{w=\varphi(z)} \left\{\frac{1-|z_0|^2}{|1-\overline{z_0}z|^2}\log\frac{1}{|z|}\right\}dA(w) = N_{\varphi,z_0}(w)dA(w) \tag{10.2}$$

and

$$N_{\varphi,z_0}(w) = \sum_{w=\varphi(z)} \left\{\frac{1-|z_0|^2}{|1-\overline{z_0}z|^2}\log\frac{1}{|z|}\right\}.$$

As in the case of Hardy spaces (see chapter 9.1) and because of lemmas 8.0.1, 8.0.3 and 8.0.4, we can, without loss of generality, suppose that $\varphi(0) = 0$ and $f(0) = 0$ for all functions f in $BMOA$.

10.1 Two necessary conditions

First, we prove an auxiliary result.

Proposition 10.1.1. *Let $r \in (0,1)$ and the measures μ_{z_0}, $z_0 \in \mathbb{D}$, as they defined in (10.2). We have that there is $C > 0$, depending only on r, such that*

$$C(1-|\alpha|^2)^2 \geq \mu_{z_0}(D_r(\alpha)), \tag{10.3}$$

for all $\alpha \in \mathbb{D}$ and for all $z_0 \in \mathbb{D}$.

Proof. Since C_φ is always bounded on $BMOA$, there is $C > 0$ such that $C\|f\|_* \geq \|C_\varphi f\|_*$. Let $z_0 \in \mathbb{D}$ arbitrary. Let $\alpha \in \mathbb{D}$ and set $f = \psi_\alpha$, so we have $C\|\psi_\alpha\|_* \geq \|C_\varphi \psi_\alpha\|_*$. But $\|\psi_\alpha\|_* \asymp 1$ so

$$
\begin{aligned}
C \geq \|C_\varphi \psi_\alpha\|_*^2 &\geq \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} d\mu_\beta(z) \\
&\geq \iint_{\mathbb{D}} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} d\mu_{z_0}(z) \\
&\geq \iint_{D_r(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} d\mu_{z_0}(z) \\
&\geq \iint_{D_r(\alpha)} \frac{1}{(1-|\alpha|^2)^2} d\mu_{z_0}(z),
\end{aligned}
$$

where the last inequality is justified because of (2.19). Hence,

$$
C \geq \frac{\mu_{z_0}(D_r(\alpha))}{(1-|\alpha|^2)^2}
$$

and the proof of the proposition is complete. □

Next, we prove a necessary condition for C_φ to have closed range on $BMOA$.

Theorem 10.1.2. *If $C_\varphi : BMOA \to BMOA$ has closed range then, there exist $\delta > 0$ and $\eta \in (0,1)$ such that*

$$
\sup_{z_0\in\mathbb{D}} \mu_{z_0}(D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha))
$$

for all $\alpha \in \mathbb{D}$.

Proof. Let $C_\varphi : BMOA \to BMOA$ has closed range.

If $z_0 \in \mathbb{D}$, we define the set

$$
E_{\mu_{z_0}} = \left\{ z \in \mathbb{D} : \text{there is } \alpha \text{ in support of } \mu_{z_0} \text{ such that } \rho(z,\alpha) < \tfrac{1}{2} \right\}.
$$

If $\alpha \in supp(\mu_{z_0})$, applying (5.1) for $p = 2$, $\gamma = 0$ and $r = 1/2$, we have

$$
\begin{aligned}
|f'(\alpha)|^2 &\leq \frac{C}{(1-|\alpha|^2)^2} \iint\limits_{D_{\frac{1}{2}}(\alpha)} |f'(z)|^2 dA(z) \\
&\leq C \iint\limits_{D_{\frac{1}{2}}(\alpha)} |f'(z)|^2 (1-|\alpha|^2)^{-2} dA(z) \\
&\leq C \iint\limits_{D_{\frac{1}{2}}(\alpha)} |f'(z)|^2 (1-|z|^2)^{-2} dA(z),
\end{aligned}
$$

where, in the last relation, we used (2.19). But $D_{\frac{1}{2}}(\alpha) \subset E_{\mu_{z_0}}$ so,

$$
|f'(\alpha)|^2 \leq C \iint\limits_{E_{\mu_{z_0}}} \chi_{D_{\frac{1}{2}}(\alpha)}(z) |f'(z)|^2 (1-|z|^2)^{-2} dA(z).
$$

Integrating with respect to measure μ_{z_0} we get

$$
\iint\limits_{\mathbb{D}} |f'(\alpha)|^2 d\mu_{z_0}(\alpha) \leq C \iint\limits_{\mathbb{D}} \iint\limits_{E_{\mu_{z_0}}} \chi_{D_{\frac{1}{2}}(\alpha)}(z) |f'(z)|^p (1-|z|^2)^{-2} dA(z) d\mu_{z_0}(\alpha).
$$

Observe that $\chi_{D_{\frac{1}{2}}(\alpha)}(z) = \chi_{D_{\frac{1}{2}}(z)}(\alpha)$ and apply Fubini's theorem to get

$$
\begin{aligned}
\iint\limits_{\mathbb{D}} |f'(\alpha)|^2 d\mu_{z_0}(\alpha) &\leq C \iint\limits_{E_{\mu_{z_0}}} \iint\limits_{\mathbb{D}} \chi_{D_{\frac{1}{2}}(z)}(\alpha) |f'(z)|^p (1-|z|^2)^{-2} d\mu_{z_0}(\alpha) dA(z) \\
&= C \iint\limits_{E_{\mu_{z_0}}} \mu_{z_0}(D_{\frac{1}{2}}(z)) |f'(z)|^p (1-|z|^2)^{-2} dA(z).
\end{aligned}
$$

Proposition 10.1.1 implies

$$
\mu_{z_0}(D_{\frac{1}{2}}(z)) \leq C(1-|z|^2)^2, \tag{10.4}
$$

hence

$$
\iint\limits_{\mathbb{D}} |f'(\alpha)|^2 d\mu_{z_0}(\alpha) \leq C \iint\limits_{E_{\mu_{z_0}}} |f'(z)|^2 (1-|z|^2)^2 (1-|z|^2)^{-2} dA(z)
$$

$$\leq C_0 \iint\limits_{E_{\mu_{z_0}}} |f'(z)|^2 dA(z).$$

So we have

$$\iint\limits_{\mathbb{D}} |f'(z)|^2 d\mu_{z_0}(z) \leq C_0 \iint\limits_{E_{\mu_{z_0}}} |f'(z)|^2 dA(z). \tag{10.5}$$

For a fixed $\alpha \in \mathbb{D}$ and η close to 1, set $\gamma = \frac{\eta - \frac{1}{2}}{1 - \frac{1}{2}\eta}$ and consider the measure $\tilde{\mu}_{z_0} = \chi_{\mathbb{D} \setminus D_\eta(\alpha)} \mu_{z_0}$. It's obvious that $\tilde{\mu}_{z_0}$ is dominated by μ_{z_0}, so (10.4) holds also for $\tilde{\mu}_{z_0}$, hence we can derive (10.5) with $\tilde{\mu}_{z_0}$ in position of μ_{z_0}. Hence

$$\iint\limits_{\mathbb{D}} |f'(z)|^2 d\tilde{\mu}_{z_0}(z) \leq C_0 \iint\limits_{E_{\tilde{\mu}_{z_0}}} |f'(z)|^2 dA(z). \tag{10.6}$$

It holds that $E_{\tilde{\mu}_{z_0}} \subset \mathbb{D} \setminus D_\gamma(\alpha)$, so using (10.6) we get

$$\iint\limits_{\mathbb{D} \setminus D_\eta(\alpha)} |f'(z)|^2 d\mu_{z_0}(z) \leq C_0 \iint\limits_{\mathbb{D} \setminus D_\gamma(\alpha)} |f'(z)|^2 dA(z).$$

Thus

$$\sup_{z_0 \in \mathbb{D}} \iint\limits_{\mathbb{D} \setminus D_\eta(\alpha)} |f'(z)|^2 d\mu_{z_0}(z) \leq C_0 \iint\limits_{\mathbb{D} \setminus D_\gamma(\alpha)} |f'(z)|^2 dA(z). \tag{10.7}$$

Because C_φ has closed range, it is bounded below, so

$$\|C_\varphi(f)\|_* > C\|f\|_*$$

which means that, for all $\alpha \in \mathbb{D}$, we have

$$\begin{aligned}
\sup_{z_1 \in \mathbb{D}} \iint\limits_{\mathbb{D}} |f'(z)|^2 d\mu_{z_1}(z) &\geq C \sup_{z_2 \in \mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1 - |z_2|^2}{|1 - \overline{z_2} z|^2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) \\
&\geq C \iint\limits_{\mathbb{D}} \frac{1 - |\alpha|^2}{|1 - \bar{\alpha} z|^2} |f'(z)|^2 \log \frac{1}{|z|} dA(z).
\end{aligned}$$

Consequently

$$\sup_{z_1\in\mathbb{D}}\iint_{\mathbb{D}}|f'(z)|^2d\mu_{z_1}(z)\geq C\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)$$

and we have

$$\sup_{z_1\in\mathbb{D}}\iint_{D_\eta(\alpha)}|f'(z)|^2d\mu_{z_1}(z)+\sup_{z_1\in\mathbb{D}}\iint_{\mathbb{D}\setminus D_\eta(\alpha)}|f'(z)|^2d\mu_{z_1}(z)$$
$$\geq C\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z).$$

From (10.7) we get

$$\sup_{z_1\in\mathbb{D}}\iint_{D_\eta(\alpha)}|f'(z)|^2d\mu_{z_1}(z)+C_0\iint_{\mathbb{D}\setminus D_\gamma(\alpha)}|f'(z)|^2dA(z)$$
$$\geq C\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z).$$

Hence

$$\sup_{z_1\in\mathbb{D}}\iint_{D_\eta(\alpha)}|f'(z)|^2d\mu_{z_1}(z)$$
$$\geq C\iint_{\mathbb{D}}\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)-C_0\iint_{\mathbb{D}\setminus D_\gamma(\alpha)}|f'(z)|^2dA(z).$$

Setting in the last relation $f=f_\alpha=\psi_\alpha(z)=\frac{\alpha-z}{1-\bar{\alpha}z}$, we have

$$|\psi_\alpha'(z)|=\frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2}$$

and

$$\sup_{z_1\in\mathbb{D}}\iint_{D_\eta(\alpha)}\frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4}d\mu_{z_1}(z)$$

$$\geq C \iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} \log \frac{1}{|z|} dA(z) - C_0 \iint\limits_{\mathbb{D}\setminus D_\gamma(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} dA(z). \tag{10.8}$$

But

$$\sup_{z_1\in\mathbb{D}} \iint\limits_{D_\eta(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} d\mu_{z_1}(z) \leq \sup_{z_1\in\mathbb{D}} \frac{\mu_{z_1}(D_\eta(\alpha))}{(1-|\alpha|^2)^2} \tag{10.9}$$

and

$$\begin{aligned}
\iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} \log \frac{1}{|z|} dA(z) &\geq C \iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} (1-|z|^2) dA(z) \\
&= C \iint\limits_{\mathbb{D}} (1-|\psi_\alpha(z)|^2)|\psi_\alpha'(z)|^2 dA(z) \\
&= C \iint\limits_{\mathbb{D}} (1-|w|^2) dA(w) \\
&\geq C \iint\limits_{D_{\frac{1}{2}}(0)} (1-|w|^2) dA(w) \geq C',
\end{aligned} \tag{10.10}$$

where we used proposition 2.6.2, relation (2.16) and the change of variable $w = \psi_\alpha(z)$. Moreover

$$\begin{aligned}
C_0 \iint\limits_{\mathbb{D}\setminus D_\gamma(\alpha)} \frac{(1-|\alpha|^2)^2}{|1-\bar{\alpha}z|^4} dA(z) &= C_0 \iint\limits_{\mathbb{D}\setminus D_\gamma(\alpha)} |\psi_\alpha'(z)|^2 dA(z) \\
&= C_0 \iint\limits_{\mathbb{D}\setminus D_\gamma(0)} dA(w) \\
&= C_0 A(\mathbb{D} \setminus D_\gamma(0)) \\
&= C_0(1-\gamma^2).
\end{aligned} \tag{10.11}$$

Using (10.9), (10.10) and (10.11) in relation (10.8) we get

$$\sup_{z_1 \in \mathbb{D}} \frac{\mu_{z_1}(D_\eta(\alpha))}{(1-|\alpha|^2)^2} \geq C' - C_0(1-\gamma^2).$$

If η approaches 1, then γ also approaches 1. So, we can choose η close enough to 1, (and hence γ close enough to 1) so that

$$C' - C_0(1-\gamma^2) > 0.$$

Therefore

$$\sup_{z_1 \in \mathbb{D}} \mu_{z_1}(D_\eta(\alpha)) \geq C(1-|\alpha|^2)^2$$

and finally, using (2.20),

$$\sup_{z_1 \in \mathbb{D}} \mu_{z_1}(D_\eta(\alpha)) \geq \delta A(D_\eta(\alpha)).$$

□

Next, we will prove a second necessary condition for the C_φ to have closed range on $BMOA$. For $\alpha \in \mathbb{D}$ and $E \subset \overline{\mathbb{D}}$ Borel, we define the measure

$$\rho_{\varphi,\alpha}(E) = m(\zeta \in \mathbb{T} : \varphi \circ \psi_\alpha(\zeta) \in E) \tag{10.12}$$

and, for $b \in \mathbb{D}$, the measure ν_b on $\mathbb{T}$ as

$$d\nu_b(\zeta) = P_b(\zeta) dm(\zeta), \tag{10.13}$$

where $P_b(\zeta)$ is the Poisson kernel defined in (2.11). The following is the result we have to prove.

Theorem 10.1.3. *Let $C_\varphi : BMOA \to BMOA$ has closed range. Then, there exists $C > 0$ such that*

$$\sup_{\alpha \in \mathbb{D}} m(\psi_\alpha \circ \varphi^{-1}(E)) > C \sup_{\beta \in \mathbb{D}} \nu_\beta(E)$$

for all E Borel subsets of $\mathbb{T}$.

Alternatively, we can restate the above theorem as:

Let $C_\varphi : BMOA \to BMOA$ has closed range. Then, there exists $C > 0$ such that

$$\sup_{\alpha\in\mathbb{D}} \rho_{\varphi,\alpha}|_{\mathbb{T}}(E) > C \sup_{\beta\in\mathbb{D}} \nu_\beta(E),$$

for all E Borel subsets of $\mathbb{T}$.

Proof. Let C_φ have closed range. Then, there exists $C > 0$ (we can suppose that $C < 1$) such that

$$\|f \circ \varphi\|_* > C\|f\|_* \tag{10.14}$$

for all $f \in BMOA$. Let E be a Borel subset of $\mathbb{T}$. Using theorem 9.1.6, we choose functions $f_n \in H^p, n \in \mathbb{N}$, such that

$$|f_n(\zeta)|^2 = \begin{cases} 1, & \text{if } \zeta \in E \\ \frac{1}{2^n}, & \text{if } \zeta \in \mathbb{T} \setminus E. \end{cases}$$

Since $f_n \in H^\infty$ it's true that $f_n \in BMOA$, so we can use them in (10.14) to get

$$\|C_\varphi f_n\|_*^2 \geq C\|f_n\|_*^2.$$

Using the norms defined in (2.13) and (2.14) we have

$$\sup_{\alpha\in\mathbb{D}} \int_{\mathbb{T}} |f_n \circ \varphi \circ \psi_\alpha(\zeta) - f_n(\varphi(\alpha))|^2 dm(\zeta) > C \sup_{\beta\in\mathbb{D}} \int_{\mathbb{T}} |f_n(\zeta) - f_n(\beta)|^2 P_\beta(\zeta) dm(\zeta)$$

and, consequently,

$$\sup_{\alpha\in\mathbb{D}} \Big[\int_{\mathbb{T}} |f_n \circ \varphi \circ \psi_\alpha(\zeta)|^2 dm(\zeta) - |f_n(\varphi(\alpha))|^2 \Big] > C \sup_{\beta\in\mathbb{D}} \Big[\int_{\mathbb{T}} |f_n(\zeta)|^2 P_\beta(\zeta) dm(\zeta) - |f_n(\beta)|^2 \Big].$$

So, for arbitrary $\beta \in \mathbb{D}$, we have

$$\sup_{\alpha\in\mathbb{D}} \Big[\int\limits_{(\varphi\circ\psi_\alpha)^{-1}(E)} |f_n \circ \varphi \circ \psi_\alpha(\zeta)|^2 dm(\zeta) \Big] + \sup_{\alpha\in\mathbb{D}} \Big[\int\limits_{\mathbb{T}\setminus(\varphi\circ\psi_\alpha)^{-1}(E)} |f_n \circ \varphi \circ \psi_\alpha(\zeta)|^2 dm(\zeta) - |f_n(\varphi(\alpha))|^2 \Big]$$
$$> C \int\limits_E |f_n(\zeta)|^2 P_\beta(\zeta) dm(\zeta) + C \int\limits_{\mathbb{T}\setminus E} |f_n(\zeta)|^2 P_\beta(\zeta) dm(\zeta) - C|f_n(\beta)|^2.$$

Hence,

$$\sup_{\alpha\in\mathbb{D}} \Big[\int\limits_{(\varphi\circ\psi_\alpha)^{-1}(E)} |f_n \circ \varphi \circ \psi_\alpha(\zeta)|^2 dm(\zeta) \Big] + \sup_{\alpha\in\mathbb{D}} \Big[\int\limits_{\mathbb{T}\setminus(\varphi\circ\psi_\alpha)^{-1}(E)} |f_n \circ \varphi \circ \psi_\alpha(\zeta)|^2 dm(\zeta) \Big] - \inf_{\alpha\in\mathbb{D}} \Big[|f_n(\varphi(\alpha))|^2 \Big]$$
$$> C \int\limits_E |f_n(\zeta)|^2 P_\beta(\zeta) dm(\zeta) + C \int\limits_{\mathbb{T}\setminus E} |f_n(\zeta)|^2 P_\beta(\zeta) dm(\zeta) - C|f_n(\beta)|^2. \quad (10.15)$$

From remark 2, after theorem 9.1.6, we conclude that the functions f_n have the form

$$f_n(z) = e^{u(z)+iv(z)},$$

where $u(z)$ is the Poisson integral (defined in (2.12)) of the function $\log|f_n(e^{i\theta})|$ and $v(z)$ is a harmonic conjugate of $u(z)$. So we have

$$|f_n(z)| = e^{u(z)} = \exp\Big(\int\limits_{\mathbb{T}} \log|f_n(e^{i\theta})| P_z(e^{i\theta}) d\theta \Big). \quad (10.16)$$

Because of (10.16), the relation (10.15) gives

$$\sup_{\alpha\in\mathbb{D}} m((\varphi\circ\psi_\alpha)^{-1}(E)) + \frac{1}{2^n} \sup_{\alpha\in\mathbb{D}} m(\mathbb{T} \setminus (\varphi\circ\psi_\alpha)^{-1}(E))$$
$$- \inf_{\alpha\in\mathbb{D}} \exp\Big(-\log 2^n \int\limits_{\mathbb{T}\setminus E} P_{\varphi(\alpha)}(e^{i\theta}) d\theta \Big)$$

$$> C\nu_\beta(E) + \frac{C}{2^n}\nu_\beta(\mathbb{T}\setminus E) - C\exp\Big(-\log 2^n \int\limits_{\mathbb{T}\setminus E} P_\beta(e^{i\theta})d\theta\Big).$$

Therefore

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) + \frac{1}{2^n}\sup_{\alpha\in\mathbb{D}} m(\mathbb{T}\setminus\psi_\alpha\circ\varphi^{-1}(E)) - \inf_{\alpha\in\mathbb{D}}\frac{1}{2^{nm(\psi_{\varphi(\alpha)}(\mathbb{T}\setminus E))}}$$
$$> C\nu_\beta(E) + \frac{C}{2^n}\nu_\beta(\mathbb{T}\setminus E) - \frac{C}{2^{n\nu_\beta(\mathbb{T}\setminus E))}}. \tag{10.17}$$

Let's suppose that $0 < m(\mathbb{T}\setminus E) \leq 1$. Then, (10.17) gives

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) + \frac{1}{2^n} - \frac{1}{2^n} > C\nu_\beta(E) + \frac{C}{2^n}\nu_\beta(\mathbb{T}\setminus E) - \frac{C}{2^{n\nu_\beta(\mathbb{T}\setminus E))}}.$$

Taking limit as $n \to +\infty$ in the last relation, we get

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) > C\nu_\beta(E). \tag{10.18}$$

Now, if we suppose $m(\mathbb{T}\setminus E) = 0$ then, it's also true that $m(\psi_{\varphi(\alpha)}(\mathbb{T}\setminus E)) = 0$ and $\nu_\beta(\mathbb{T}\setminus E)) = 0$, so (10.17) gives

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) + \frac{1}{2^n} - 1 > C\nu_\beta(E) - C.$$

i.e.

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) + \frac{1}{2^n} > C\nu_\beta(E) + 1 - C.$$

But we have supposed $C < 1$ so, by cancelling $1 - C$ from the right side of the last relation, it follows that

$$\sup_{\alpha\in\mathbb{D}} m(\psi_\alpha\circ\varphi^{-1}(E)) + \frac{1}{2^n} > C\nu_\beta(E).$$

Taking limit as $n \to +\infty$ in the last relation, we get

$$\sup_{\alpha \in \mathbb{D}} m(\psi_\alpha \circ \varphi^{-1}(E)) > C\nu_\beta(E). \tag{10.19}$$

Taking supremum over $\beta \in \mathbb{D}$ in (10.18) and (10.19) we get the desired result and the proof of the theorem is complete. $\square$

10.2 A sufficient condition

For $\alpha \in \mathbb{D}$ and $E \subset \overline{\mathbb{D}}$ Borel, define the measure

$$\mu_{\varphi,\alpha}(E) = m(\zeta \in \mathbb{T} : \psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha(\zeta) \in E). \tag{10.20}$$

Next, we will prove a lemma, which will be used in the proof of theorem 10.2.2.

Lemma 10.2.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $f \in BMOA$. If there is $r \in (0,1)$ such that $\varphi(\mathbb{D}) \cap D_r(\alpha) \neq \emptyset$ for all $\alpha \in \mathbb{D}$, then*

$$\|f\|_*^2 \asymp \sup_{\alpha \in \mathbb{D}} \int_{\mathbb{T}} |f(e^{i\theta}) - f(\varphi(\alpha))|^2 P_{\varphi(\alpha)}(e^{i\theta}) d\theta$$

Proof. Let $f \in BMOA$, fixed. Then, there exists sequence $z_n \in \mathbb{D}, n \in \mathbb{N}$, such that

$$\|f\|_*^2 = \lim_{n \to \infty} \iint_{\mathbb{D}} \frac{1 - |z_n|^2}{|1 - \overline{z_n} z|^2} |f'(z)|^2 \log \frac{1}{|z|} dA(z).$$

Because $\varphi(\mathbb{D}) \cap D_r(z_n) \neq \emptyset$, we have that there exist $\alpha_n \in \mathbb{D}$ such that $\varphi(\alpha_n) \in D_r(z_n)$, for all $n \in \mathbb{N}$. It holds that

$$|1 - \overline{z_n} z| \asymp |1 - \overline{\varphi(\alpha_n)} z|$$

for every $n \in \mathbb{N}$ and for all $z \in \mathbb{D}$. In addition, from (2.19), it follows that

$$1 - |z_n|^2 \asymp 1 - |\varphi(\alpha_n)|^2.$$

Hence,

$$\frac{1-|z_n|^2}{|1-\overline{z_n}z|^2}\log\frac{1}{|z|} \asymp \frac{1-|\varphi(\alpha_n)|^2}{|1-\overline{\varphi(\alpha_n)}z|^2}\log\frac{1}{|z|}$$

for all $z \in \mathbb{D}$. The underlying constants in the last relation depend only on r. So, there exists $C_r > 0$ such that

$$\begin{aligned}
\|f\|_*^2 &= \lim_{n\to\infty}\iint\limits_{\mathbb{D}} \frac{1-|z_n|^2}{|1-\overline{z_n}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)\\
&\le \sup_{n\in\mathbb{N}}\iint\limits_{\mathbb{D}} \frac{1-|z_n|^2}{|1-\overline{z_n}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)\\
&\le C_r\sup_{n\in\mathbb{N}}\iint\limits_{\mathbb{D}} \frac{1-|\varphi(\alpha_n)|^2}{|1-\overline{\varphi(\alpha_n)}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)\\
&\le C_r\sup_{\alpha\in\mathbb{D}}\iint\limits_{\mathbb{D}} \frac{1-|\varphi(\alpha)|^2}{|1-\overline{\varphi(\alpha)}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)\\
&\le C_r\sup_{\alpha\in\mathbb{D}}\iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z) = C_r\|f\|_*^2.
\end{aligned}$$

We proved that

$$\|f\|_*^2 \asymp \sup_{\beta\in\mathbb{D}}\iint\limits_{\mathbb{D}} \frac{1-|\varphi(\beta)|^2}{|1-\overline{\varphi(\beta)}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z). \tag{10.21}$$

Let $\alpha \in \mathbb{D}$. Using the identity (see [24], relation (3.3), page 230)

$$\int\limits_{\mathbb{T}} |f(e^{i\theta}) - f(\alpha)|^2 P_\alpha(e^{i\theta})d\theta = \iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}z|^2}|f'(z)|^2\log\frac{1}{|z|}dA(z)$$

in combination with (10.21), we finally get

$$\|f\|_*^2 \asymp \sup_{\beta\in\mathbb{D}} \int_{\mathbb{T}} |f(e^{i\theta}) - f(\varphi(\beta))|^2 P_{\varphi(\beta)}(e^{i\theta})d\theta.$$

□

Remark 3. If the assumptions of lemma 10.2.1 hold then, we can calculate the $BMOA$ norm of f by taking supremum over $\varphi(\mathbb{D})$ and, not necessarily, over $\mathbb{D}$. This, of course, can be applied to all equivalent norms in BMOA.

Next, we will use measure $\mu_{\varphi,\alpha}$ as defined in (10.20).

Theorem 10.2.2. *If there exist $r \in (0,1)$ and $C > 0$ such that $\varphi(\mathbb{D}) \cap D_r(\alpha) \neq \emptyset$ and $m(\psi_\alpha \circ \varphi^{-1} \circ \psi_{\varphi(\alpha)}(E)) > Cm(E)$ for all $\alpha \in \mathbb{D}$ and for all Borel $E \subset \mathbb{T}$, then, $C_\varphi : BMOA \to BMOA$ has closed range.*

Alternatively, we can restate the above theorem as:

If there exists $r \in (0,1)$ and $C > 0$ such that, for all $\alpha \in \mathbb{D}$, $\varphi(\mathbb{D}) \cap D_r(\alpha) \neq \emptyset$ and if the Radon-Nikodym derivative $\frac{d\mu_{\varphi,\alpha}|_{\mathbb{T}}}{dm}$ is bounded below from a positive constant, then $C_\varphi : BMOA \to BMOA$ has closed range.

Proof. Let $\alpha \in \mathbb{D}$. We have

$$\begin{aligned}
\|f\circ\varphi\|_*^2 &= \sup_{\beta\in\mathbb{D}} \int_{\mathbb{T}} |f\circ\varphi\circ\psi_\beta(\zeta) - f(\varphi(\beta))|^2 dm(\zeta) \\
&= \sup_{\beta\in\mathbb{D}} \iint_{\overline{\mathbb{D}}} |f(\psi_{\varphi(\beta)}(z)) - f(\varphi(\beta))|^2 d\mu_{\varphi,\beta}(z) \\
&\geq \iint_{\overline{\mathbb{D}}} |f(\psi_{\varphi(\alpha)}(z)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(z) \\
&\geq \int_{\mathbb{T}} |f(\psi_{\varphi(\alpha)}(\zeta)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(\zeta) \\
&\geq C\int_{\mathbb{T}} |f(\psi_{\varphi(\alpha)}(\zeta)) - f(\varphi(\alpha))|^2 dm(\zeta),
\end{aligned}$$

where the last inequality is justified by the fact that the Radon-Nikodym derivative $\frac{d\mu_{\varphi,\alpha}|_{\mathbb{T}}}{dm}$ is bounded below from a positive constant C. So

$$\|f \circ \varphi\|_*^2 \geq C \int_{\mathbb{T}} |f(e^{i\theta}) - f(\varphi(\alpha))|^2 P_{\varphi(\alpha)}(e^{i\theta}) d\theta.$$

According to lemma 10.2.1, if we take supremum in the last relation over $\varphi(\alpha) \in \varphi(\mathbb{D})$, we finally get

$$\|f \circ \varphi\|_* > C\|f\|_*.$$

So C_φ has closed range. □

10.3 Regarding inner functions

In this section, we investigate the case $C_\varphi : BMOA \to BMOA$ to have closed range, when φ is an inner function. Actually, it is known that if φ is an inner function then C_φ is an isometry and so it has closed range (see [10], [27], [42]). Here, we shall give another proof that C_φ is an isometry if φ is inner. First, we prove a lemma using the following, well known theorem, due to Frostman. For a proof, see theorem 6.4 in [24].

Theorem 10.3.1 (Frostman)**.** *Let φ be a nonconstant inner function on $\mathbb{D}$. Then for all $w \in \mathbb{D}$, except possibly for a set of capacity zero, the function*

$$B_w(z) = \frac{\varphi(z) - w}{1 - \overline{w}\varphi(z)} \tag{10.22}$$

is a Blaschke product.

Lemma 10.3.2. *Let φ be an inner function. Then φ takes every value in the unit disk $\mathbb{D}$, except possibly of the values in a set of capacity zero.*

Proof. Let φ be inner function and Γ the set of capacity zero mentioned in theorem 10.3.1. Take an arbitrary but fixed $w \in \mathbb{D} \setminus \Gamma$ and let B_w be the

Blaschke product associating with φ as in theorem 10.3.1. Since B_w maps $\mathbb{D}$ onto $\mathbb{D}$, we have that there is at least one $z_1 \in \mathbb{D}$ such that $B_w(z_1) = 0$. From (10.22), we have that $\varphi(z) = \frac{B_w(z)+w}{1+\overline{w}B_w(z)}$, so $\varphi(z_1) = w$ and the proof of the lemma is complete. □

The following is the main result of this section.

Proposition 10.3.3. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is inner then $C_\varphi : BMOA \to BMOA$ is an isometry.*

Proof. Let $\alpha \in \mathbb{D}$ and $E \subset \mathbb{T}$ Borel, then we have that

$$\mu_{\varphi,\alpha}|_{\mathbb{T}}(E) = m((\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha)^{-1}(E)).$$

We observe that the function $\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha$ is inner and, in addition, $\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha(0) = 0$, so from (9.13), applied for the inner function $\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha$, we get

$$\frac{d\mu_{\varphi,\alpha}|_{\mathbb{T}}}{dm}(\zeta) = \frac{1 - |\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha(0)|}{1 + |\psi_{\varphi(\alpha)} \circ \varphi \circ \psi_\alpha(0)|} = 1 \tag{10.23}$$

for all $\zeta \in \mathbb{T}$.

We will show that $\mu_{\varphi,\alpha}(\mathbb{D}) = 0$. It's true that $\psi_\alpha(\zeta) \in \mathbb{T}$, when $\zeta \in \mathbb{T}$ and, since φ is inner, $\varphi(\psi_\alpha(\zeta)) \in \mathbb{T}$ for m-a.e $\zeta \in \mathbb{T}$ and obviously, $\psi_{\varphi(\alpha)}(\varphi(\psi_\alpha(\zeta))) \in \mathbb{T}$ for m-a.e $\zeta \in \mathbb{T}$. Hence, the implication: if $\zeta \in \mathbb{T}$ then $\psi_{\varphi(\alpha)}(\varphi(\psi_\alpha(\zeta))) \in \mathbb{T}$ is true just for m-a.e $\zeta \in \mathbb{T}$ and, consequently, we have $\mu_{\varphi,\alpha}(\mathbb{D}) = 0$. Therefore,

$$\iint_{\overline{\mathbb{D}}} |f(\psi_{\varphi(\alpha)}(z)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(z) = \int_{\mathbb{T}} |f(\psi_{\varphi(\alpha)}(\zeta)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(\zeta) \tag{10.24}$$

for all $f \in BMOA$ and for all $\alpha \in \mathbb{D}$.

Let $f \in BMOA$. Using (10.24), we get

$$\int_{\mathbb{T}} |f \circ \varphi \circ \psi_\alpha(\zeta) - f(\varphi(\alpha))|^2 dm(\zeta) = \iint_{\overline{\mathbb{D}}} |f(\psi_{\varphi(\alpha)}(z)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(z)$$

$$= \int_{\mathbb{T}} |f(\psi_{\varphi(\alpha)}(\zeta)) - f(\varphi(\alpha))|^2 d\mu_{\varphi,\alpha}(\zeta)$$
$$= \int_{\mathbb{T}} |f(\psi_{\varphi(\alpha)}(\zeta)) - f(\varphi(\alpha))|^2 dm(\zeta)$$

where the last equality is justified by (10.23) and by the absolute continuity of $\mu_{\varphi,\alpha}$ with respect to m.

Lemma 10.3.2 ensures that lemma 10.2.1 and remark 3 can be used, so taking supremum over $\alpha \in \mathbb{D}$ in the last relation we get

$$\|f \circ \varphi\|_* = \|f\|_*.$$

So C_φ is an isometry. □

10.3.1 Using Alexandrov-Clark measures

We can also prove relation (10.23) by using Alexandrov-Clark measures (see section 2.7). Let $\beta \in \mathbb{D}$. Then, according to the notation of (2.33), we have that

$$\mu_{\varphi,\beta}|_{\mathbb{T}} = \nu_{\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta}. \tag{10.25}$$

From (2.32), applied for the function $\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta$, we get

$$\|\mu_\lambda^s\| = \frac{1 - |\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(0)|^2}{|\lambda - \psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(0)|^2} - \int_{\mathbb{T}} \frac{1 - |\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(\zeta)|^2}{|\lambda - \psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(\zeta)|^2} dm(\zeta),$$

for $m - a.e.\ \lambda \in \mathbb{T}$.

Since φ is an inner function we have that $\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta$ is also an inner function, so $|\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(\zeta)| = 1$ for $m - a.e.\ \zeta \in \mathbb{T}$. Moreover $\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(0) = 0$. Therefore

$$\|\mu_\lambda^s\| = \frac{1 - |\psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(0)|^2}{|\lambda - \psi_{\varphi(\beta)} \circ \varphi \circ \psi_\beta(0)|^2} = 1,$$

for $m - a.e.$ $\lambda \in \mathbb{T}$.

From proposition 2.7.3 we get

$$\frac{d\mu_{\varphi,\beta}|_{\mathbb{T}}}{dm}(\lambda) = \|\mu_\lambda^s\| = 1$$

for $m - a.e.$ $\lambda \in \mathbb{T}$ and for all $\beta \in \mathbb{D}$.

Chapter 11

Closed range composition operators on Bergman spaces

The Bergman spaces A^p and the weighted Bergman spaces $A^p_\gamma, \gamma > -1$, as well as some equivalent norms in these spaces were defined in section 2.1.2. This chapter doesn't contain any new result. We just show here that an already known proof of Akeroyd and Fulmer (2008) for C_φ to have closed range on Bergman space A^2 works for $A^p_\gamma, \gamma > -1$ spaces $(1 \leq p < \infty)$.

$C_\varphi : A^p_\gamma \to A^p_\gamma$ is bounded, for every analytic $\varphi : \mathbb{D} \to \mathbb{D}$ as it is implied by a theorem due to Littlewood (see [51], Theorem 11.6).

Theorem 11.0.1 (Littlewood's subordination theorem)**.** *If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, $p > 0$ and $\gamma > -1$, then*

$$\iint\limits_{\mathbb{D}} |f(\varphi(z))|^p (1-|z|^2)^\gamma dA(z) \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{2+\gamma} \iint\limits_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\gamma dA(z),$$

for all $f \in A^p_\gamma$.

In 2008, Akeroyd and Ghatage (see [2]), proved a necessary and sufficient condition for C_φ to have closed range in A^2. In 2011, Akeroyd and Fulmer (see [3]) gave a (more complicated) proof that condition for A^2 space holds also for all weighted Bergman spaces $A^p_\gamma, 1 \leq p < \infty, \gamma > -1$. Here, we present a simpler proof of the same result for A^p_γ, similar to the original proof of Akeroyd and Ghatage for A^2.

Let $\varepsilon > 0$. Set

$$\Omega_\varepsilon = \left\{ z \in \mathbb{D} : \frac{(1-|z|^2)^{p+\gamma-1}|\varphi'(z)|^{p-2}}{(1-|\varphi(z)|^2)^{p+\gamma-1}} > \varepsilon \right\} \tag{11.1}$$

and $G_\varepsilon = \varphi(\Omega_\varepsilon)$.

Theorem 11.0.2. *Let $1 \leq p < \infty$, $\gamma > -1$. The following are equivalent:*

(i) $C_\varphi : A^p_\gamma \to A^p_\gamma$ has closed range.

(ii) There exist $\varepsilon > 0$, $\delta > 0$ and $\eta \in (0,1)$ such that the set G_ε to satisfy the condition

$$A(G_\varepsilon \cap D_\eta(a)) \geq \delta A(D_\eta(a)) \tag{11.2}$$

for all $a \in \mathbb{D}$.

Proof. We will use the norm defined in (2.9). Because of lemmas 8.0.1, 8.0.3 and 8.0.4, we can, without loss of generality, suppose $\varphi(0) = 0$ and $f(0) = 0$ for all $f \in A^p_\gamma$. If $f \in A^p_\gamma$ then $f' \in A^p_{p+\gamma}$, so we will make use of theorem 5.2.2 with $G = G_\varepsilon$.

Proof $(i) \Rightarrow (ii)$. Let's suppose that C_φ has closed range and (11.2) doesn't hold. Then, since (11.2) is the same with (5.4), from theorem (5.2.2) we have that (5.3) doesn't hold, too. This means that, for $k \in \mathbb{N}$, there is sequence $f_k \in A^p_\gamma$ with $\|f_k\|_{A^p_\gamma} = 1$ and

$$\iint\limits_{G_{1/k}} |f_k'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \to 0, \;\; k \to +\infty. \tag{11.3}$$

For contradiction, we will show that $\|f_k \circ \varphi\|_{A^p_\gamma} \to 0$ as $k \to \infty$. We have:

$$\begin{aligned} \|f_k \circ \varphi\|^p_{A^p_\gamma} &\asymp \iint\limits_{\mathbb{D}} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z) \\ &= \iint\limits_{\mathbb{D}\setminus\Omega_{1/k}} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z) \end{aligned}$$

$$+\iint\limits_{\Omega_{1/k}} |f_k'(z)|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z).$$

Next, the relation $N_\varphi(w) \leq 1-|w|^2$, for all $w \in \mathbb{D}$ (see lemma 2.4.1) will be used, as well as the relations (2.21), $1-|w|^2 \leq 1-|\varphi(w)|^2$ (see lemma 2.2.1), $|\varphi'(w)|(1-|w|^2) \leq 1-|\varphi(w)|^2$ (see lemma 2.2.2) and the change of variable $w=\varphi(z)$ according to proposition 2.5.1. Hence,

$$\begin{aligned}
&\iint\limits_{\mathbb{D}\setminus\Omega_{1/k}} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z) \\
&= \iint\limits_{\mathbb{D}\setminus\Omega_{1/k}} |f_k'(\varphi(z))|^p \Big\{ |\varphi'(z)|^{p-2} (1-|z|^2)^{p+\gamma-1} \Big\} (1-|z|^2) |\varphi'(z)|^2 dA(z) \\
&\leq \frac{C}{k} \iint\limits_{\mathbb{D}\setminus\Omega_{1/k}} |f_k'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p+\gamma-1} \log\frac{1}{|z|} |\varphi'(z)|^2 dA(z) \\
&= \frac{C}{k} \iint\limits_{\varphi\left(\mathbb{D}\setminus\Omega_{1/k}\right)} |f_k'(w)|^p (1-|w|^2)^{p+\gamma-1} N_\varphi(w) dA(w) \\
&= \frac{C}{k} \iint\limits_{\varphi\left(\mathbb{D}\setminus\Omega_{1/k}\right)} |f_k'(w)|^p (1-|w|^2)^{p+\gamma-1} (1-|w|^2) dA(w) \\
&\leq \frac{C}{k} \iint\limits_{\varphi\left(\mathbb{D}\setminus\Omega_{1/k}\right)} |f_k'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \\
&\leq \frac{C}{k} \iint\limits_{\mathbb{D}} |f_k'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \\
&\leq \frac{C}{k} \|f_k\|^p_{A^p_\gamma} = \frac{C}{k} \to 0,
\end{aligned}$$

as $k \to \infty$. Moreover,

$$\begin{aligned}
&\iint\limits_{\Omega_{1/k}} |f_k'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z) \\
&= \iint\limits_{\Omega_{1/k}} |f_k'(\varphi(z))|^p \Big\{ |\varphi'(z)|(1-|z|^2) \Big\}^{p+\gamma-2} (1-|z|^2)(1-|z|^2) |\varphi'(z)|^2 dA(z)
\end{aligned}$$

$$\le C \iint\limits_{\Omega_{1/k}} |f_k'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p+\gamma-2}(1-|\varphi(z)|^2)\log\frac{1}{|z|}|\varphi'(z)|^2 dA(z)$$
$$= C \iint\limits_{\Omega_{1/k}} |f_k'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p+\gamma-1}\log\frac{1}{|z|}|\varphi'(z)|^2 dA(z)$$
$$\le C \iint\limits_{G_{1/k}} |f_k'(w)|^p (1-|w|^2)^{p+\gamma-1} N_\varphi(w) dA(w)$$
$$\le C \iint\limits_{G_{1/k}} |f_k'(w)|^p (1-|w|^2)^{p+\gamma-1}(1-|w|^2) dA(w)$$
$$\le C \iint\limits_{G_{1/k}} |f_k'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \to 0,$$

as $k \to \infty$.

So, we have proved that $\|f_k \circ \varphi\|_{A^p_\gamma} \to 0, k \to \infty$, which contradicts (11.3) and the proof of necessity is complete .

Proof $(ii) \Rightarrow (i)$. We have

$$\|f\circ\varphi\|^p_{A^p_\gamma} = \iint\limits_{\mathbb{D}} |f'(\varphi(z))|^p |\varphi'(z)|^p (1-|z|^2)^{p+\gamma} dA(z)$$
$$= \iint\limits_{\mathbb{D}} |f'(\varphi(z))|^p |\varphi'(z)|^{p-2}(1-|z|^2)^{p+\gamma-1}(1-|z|^2)|\varphi'(z)|^2 dA(z)$$
$$\ge \iint\limits_{\Omega_\varepsilon} |f'(\varphi(z))|^p \left\{|\varphi'(z)|^{p-2}(1-|z|^2)^{p+\gamma-1}\right\}(1-|z|^2)|\varphi'(z)|^2 dA(z).$$

Let $z \in \Omega_\varepsilon$. Using lemma 2.2.2, we get

$$\varepsilon < \frac{|\varphi'(z)|^{p-2}(1-|z|^2)^{p+\gamma-1}}{(1-|\varphi(z)|^2)^{p+\gamma-1}} < \frac{(1-|\varphi(z)|^2)^{p-2}}{(1-|z|^2)^{p-2}}\frac{(1-|z|^2)^{p+\gamma-1}}{(1-|\varphi(z)|^2)^{p+\gamma-1}}$$
$$\le \frac{(1-|z|^2)^{\gamma+1}}{(1-|\varphi(z)|^2)^{\gamma+1}}.$$

Thus

$$1-|z|^2 \ge \varepsilon^{\frac{1}{\gamma+1}}(1-|\varphi(z)|^2),\ z\in\Omega_\varepsilon. \tag{11.4}$$

Our assumption is that (11.2) holds, which, as already said is the same with (5.4), so from theorem (5.2.2) we have that (5.3) holds, too. Hence, continuing from the last integral and using (11.4), we get

$$
\begin{aligned}
&\iint\limits_{\Omega_\varepsilon} |f'(\varphi(z))|^p \Big\{ |\varphi'(z)|^{p-2}(1-|z|^2)^{p+\gamma-1} \Big\} (1-|z|^2)|\varphi'(z)|^2 dA(z) \\
&\geq \varepsilon\varepsilon^{\frac{1}{\gamma+1}} \iint\limits_{\Omega_\varepsilon} |f'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p+\gamma-1}(1-|\varphi(z)|^2)|\varphi'(z)|^2 dA(z) \\
&= \varepsilon^{\frac{\gamma+2}{\gamma+1}} \iint\limits_{G_\varepsilon} |f'(w)|^p (1-|w|^2)^{p+\gamma} n_\varphi(w) dA(w) \\
&\geq \varepsilon^{\frac{\gamma+2}{\gamma+1}} \iint\limits_{G_\varepsilon} |f'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \\
&\geq \varepsilon^{\frac{\gamma+2}{\gamma+1}} C \iint\limits_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p+\gamma} dA(w) \\
&= C\|f\|^p_{A^p_\gamma},
\end{aligned}
$$

where we used the fact that when $w \in G_\varepsilon \subset \varphi(\mathbb{D})$, then $n_\varphi(w) \geq 1$. The last inequality is justified by (5.3).

We proved that $\|f \circ \varphi\|_{A^p_\gamma} \geq C\|f\|_{A^p_\gamma}$, so C_φ has closed range. $\square$

Appendix A

Some norms' estimations

Lemma A.0.1. *Let $\alpha \in \mathbb{D}$ and $1 \leq p < \infty$. Then,*

$$\|\psi_\alpha - \alpha\|_{H^p}^p \asymp 1 - |\alpha|.$$

Proof. Without loss of generality, we may suppose that $\alpha \in [0,1)$, let $\alpha = r$. Then

$$\begin{aligned}
\|\psi_\alpha - \alpha\|_{H^p}^p &= \int_0^{2\pi} |(\psi_\alpha - \alpha)(e^{i\theta})|^p d\theta \\
&= \int_0^{2\pi} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}e^{i\theta}|^p} d\theta \\
&\asymp \int_0^{\pi} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}e^{i\theta}|^p} d\theta.
\end{aligned}$$

If $r \in [0, \frac{1}{2}]$ then we have

$$C_1 \leq \frac{(1-|\alpha|^2)^{p-1}}{|1-\overline{\alpha}e^{i\theta}|^p} \leq C_2,$$

where C_1 and C_2 are absolute positive constants. Hence, $|(\psi_\alpha - \alpha)(e^{i\theta})|^p \asymp 1 - |\alpha|$ and we have nothing more to prove.

If $r \in [\frac{1}{2}, 1)$ then we have $|1 - \overline{\alpha}e^{i\theta}| \asymp |\frac{1}{r} - e^{i\theta}|$. If $0 \leq \theta \leq 1 - r$ then $|\frac{1}{r} - e^{i\theta}| \asymp 1 - r$ and if $1 - r \leq \theta \leq \pi$ then $|\frac{1}{r} - e^{i\theta}| \asymp \theta$. So,

$$\int_0^{\pi} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}e^{i\theta}|^p} d\theta = \int_0^{1-r} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}e^{i\theta}|^p} d\theta + \int_{1-r}^{\pi} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}e^{i\theta}|^p} d\theta$$

and

$$\int_0^{1-r} \frac{(1-|\alpha|^2)^p}{|1-\bar{\alpha}e^{i\theta}|^p} d\theta \asymp \int_0^{1-r} \frac{(1-r^2)^p}{(1-r)^p} d\theta \asymp 1-r$$

and

$$\int_{1-r}^{\pi} \frac{(1-|\alpha|^2)^p}{|1-\bar{\alpha}e^{i\theta}|^p} d\theta \asymp \int_{1-r}^{\pi} \frac{(1-r^2)^p}{\theta^p} d\theta \asymp \left[\frac{(1-r^2)^p}{\theta^{p-1}}\right]_{\pi}^{1-r} \asymp 1-r.$$

Therefore

$$\int_0^{\pi} \frac{(1-|\alpha|^2)^p}{|1-\bar{\alpha}e^{i\theta}|^p} d\theta \asymp 1-r,$$

which is what we had to show. $\square$

Lemma A.0.2. *Let $\alpha \in \mathbb{D}$. Then,*

$$\|\psi_\alpha - \alpha\|_* \asymp 1.$$

Proof. Using proposition 2.6.2, relation (2.16) and the change of variable $w = \psi_\alpha(z)$, we have

$$\begin{aligned}
\|\psi_\alpha - \alpha\|_*^2 &= \sup_{\beta\in\mathbb{D}} \iint\limits_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\bar{\beta}z|^2} |\psi_\alpha'(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\geq \iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} |\psi_\alpha'(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\geq C \iint\limits_{\mathbb{D}} \frac{1-|\alpha|^2}{|1-\bar{\alpha}z|^2} |\psi_\alpha'(z)|^2 (1-|z|^2) dA(z) \\
&= C \iint\limits_{\mathbb{D}} |\psi_\alpha'(z)|^2 (1-|\psi_\alpha(z)|^2) dA(z) \\
&= C \iint\limits_{\mathbb{D}} (1-|w|^2) dA(w) \geq C.
\end{aligned}$$

Moreover, using the fact that $\frac{(1-|\beta|^2)(1-|z|^2)}{|1-\overline{\beta}z|^2} \leq 1$, for all $z, \beta \in \mathbb{D}$, and the change of variable $w = \psi_\alpha(z)$, we have

$$\begin{aligned}
\|\psi_\alpha - \alpha\|_*^2 &= \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2}|\psi'_\alpha(z)|^2 \log\frac{1}{|z|} dA(z) \\
&\leq C \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{1-|\beta|^2}{|1-\overline{\beta}z|^2}|\psi'_\alpha(z)|^2 (1-|z|^2) dA(z) \\
&\leq C \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} |\psi'_\alpha(z)|^2 dA(z) \\
&= C \iint_{\mathbb{D}} dA(w) \leq C.
\end{aligned}$$

So, $\|\psi_\alpha - \alpha\|_* \asymp 1$. □

Lemma A.0.3. *Let $\alpha \in \mathbb{D}$. Then,*

$$\|\psi_\alpha - \alpha\|_{Q_p} \asymp 1.$$

Proof. Using (2.16) and the change of variable $w = \psi_\alpha(z)$, we have

$$\begin{aligned}
\|\psi_\alpha - \alpha\|_{Q_p}^2 &= \sup_{\beta\in\mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}}|\psi'_\alpha(z)|^2 (1-|z|^2)^p dA(z) \\
&\geq \iint_{\mathbb{D}} \frac{(1-|\alpha|^2)^p}{|1-\overline{\alpha}z|^{2p}}|\psi'_\alpha(z)|^2 (1-|z|^2)^p dA(z) \\
&= \iint_{\mathbb{D}} |\psi'_\alpha(z)|^2 (1-|\psi_\alpha(z)|^2)^p dA(z) \\
&= \iint_{\mathbb{D}} (1-|w|^2)^p dA(w) \geq C.
\end{aligned}$$

Moreover, using the fact that $\frac{(1-|\beta|^2)^p(1-|z|^2)^p}{|1-\overline{\beta}z|^{2p}} \leq 1$, for all $z, \beta \in \mathbb{D}$ and for all $p \in (0, \infty)$, and the change of variable $w = \psi_\alpha(z)$, we have

$$\begin{aligned}
\|\psi_\alpha - \alpha\|^2_{Q_p} &= \sup_{\beta\in\mathbb{D}} \iint\limits_{\mathbb{D}} \frac{(1-|\beta|^2)^p}{|1-\overline{\beta}z|^{2p}} |\psi'_\alpha(z)|^2(1-|z|^2)^p dA(z) \\
&\leq C \sup_{\beta\in\mathbb{D}} \iint\limits_{\mathbb{D}} |\psi'_\alpha(z)|^2 dA(z) \\
&= C \iint\limits_{\mathbb{D}} dA(w) = C.
\end{aligned}$$

So, $\|\psi_\alpha - \alpha\|_* \asymp 1$. □

Lemma A.0.4. *Let $\alpha \in \mathbb{D}$ and $f_\alpha(z) = (1-|\alpha|^2)^{\frac{2}{p}} / [\frac{2\overline{\alpha}}{p}(1-\overline{\alpha}z)^{\frac{2}{p}}] - (1-|\alpha|^2)^{\frac{2}{p}} / (\frac{2\overline{\alpha}}{p})$. Then,*

$$\|f_\alpha\|_{B^p_0} \asymp 1.$$

Proof. A simple calculation shows that

$$|f'_\alpha(z)| = \frac{(1-|\alpha|^2)^{\frac{2}{p}}}{|1-\overline{\alpha}z|^{\frac{2}{p}+1}},$$

hence, using the relation $|1-\overline{\alpha}\psi_\alpha(w)| = (1-|\alpha|^2)/|1-\overline{\alpha}w|$, the identity (2.16) and the change of variable $z = \psi_\alpha(w)$, we get

$$\begin{aligned}
\|f_\alpha\|^p_{B^p_0} &= \iint\limits_{\mathbb{D}} |f'_\alpha(z)|^p (1-|z|^2)^{p-2} dA(z) \\
&= \iint\limits_{\mathbb{D}} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}z|^{2+p}} (1-|z|^2)^{p-2} dA(z) \\
&= \iint\limits_{\mathbb{D}} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}\psi_\alpha(w)|^{2+p}} (1-|\psi_\alpha(w)|^2)^{p-2} |\psi'_\alpha(w)|^2 dA(w) \\
&= \iint\limits_{\mathbb{D}} \frac{(1-|w|^2)^{p-2}}{|1-\overline{\alpha}w|^{p-2}} dA(w).
\end{aligned}$$

But, using the simple relation $1 - |w| \leq |1 - \overline{\alpha} w|$,

$$\iint\limits_{\mathbb{D}} \frac{(1 - |w|^2)^{p-2}}{|1 - \overline{\alpha} w|^{p-2}} dA(w) \leq \iint\limits_{\mathbb{D}} \frac{(1 - |w|^2)^{p-2}}{(1 - |w|)^{p-2}} dA(w) \leq C$$

and also, using (2.19),

$$\iint\limits_{\mathbb{D}} \frac{(1 - |w|^2)^{p-2}}{|1 - \overline{\alpha} w|^{p-2}} dA(w) \geq \iint\limits_{D_{\frac{1}{2}}(0)} \frac{(1 - |w|^2)^{p-2}}{|1 - \overline{\alpha} w|^{p-2}} dA(w) \geq C \iint\limits_{D_{\frac{1}{2}}(0)} dA(w) = C.$$

So, $\|f_\alpha\|_{B_0^p} \asymp 1$. □

Appendix B

A calculus result

Lemma B.0.1. *If $F \in H_0^1$ then there exists $C > 0$ such that*

$$\|F\|_{H_0^1} \leq C \iint\limits_{\mathbb{D}} |F'(z)| dA(z).$$

Proof. Let $e^{i\theta} \in \mathbb{T}$. Then we have that

$$F(e^{i\theta}) = \int_0^{e^{i\theta}} F'(z) dz.$$

Set $z = re^{i\theta}$, $0 \leq r \leq 1$. It follows that

$$F(e^{i\theta}) = \int_0^1 F'(re^{i\theta}) e^{i\theta} dr.$$

Thus

$$|F(e^{i\theta})| \leq \int_0^1 |F'(re^{i\theta})| dr.$$

Integrating the last relation with respect to θ we get

$$\begin{aligned}
\|F\|_{H_0^1} &\leq \int_0^{2\pi} \int_0^1 |F'(re^{i\theta})| dr d\theta \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{r} |F'(re^{i\theta})| r dr d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{1}{2}} \frac{1}{r} |F'(re^{i\theta})| r dr d\theta + \int_0^{2\pi} \int_{\frac{1}{2}}^1 \frac{1}{r} |F'(re^{i\theta})| r dr d\theta \\
&= A + B.
\end{aligned}$$

Let $|z| \leq \frac{1}{2}$ and consider the euclidean disk $E(z;\frac{1}{4}) \subset \mathbb{D}$. Then, because of the subharmonicity of $|F'|$ we have

$$\begin{aligned} |F'(z)| &\leq \frac{1}{A(E(z;\frac{1}{4}))} \iint\limits_{E(z;\frac{1}{4})} |F'(w)|dA(w) \\ &= C \iint\limits_{E(z;\frac{1}{4})} |F'(w)|dA(w) \\ &\leq C \iint\limits_{\mathbb{D}} |F'(w)|dA(w) \end{aligned} \tag{B.1}$$

Using (B.1), we get

$$\begin{aligned} A &= \iint\limits_{0\leq|z|\leq\frac{1}{2}} \frac{1}{|z|}|F'(z)|dA(z) \\ &\leq C \iint\limits_{0\leq|z|\leq\frac{1}{2}} \frac{1}{|z|} \iint\limits_{\mathbb{D}} |F'(w)|dA(w)dA(z) \\ &\leq C \iint\limits_{\mathbb{D}} |F'(w)|dA(w) \iint\limits_{0\leq|z|\leq\frac{1}{2}} \frac{1}{|z|}dA(z) \\ &\leq C \iint\limits_{\mathbb{D}} |F'(w)|dA(w). \end{aligned} \tag{B.2}$$

In addition, we have

$$\begin{aligned} B &= \iint\limits_{\frac{1}{2}\leq|z|\leq 1} \frac{1}{|z|}|F'(z)|dA(z) \\ &\leq C \iint\limits_{\frac{1}{2}\leq|z|\leq 1} |F'(z)|dA(z). \end{aligned} \tag{B.3}$$

From (B.2) and (B.3) it follows that

$$\|F\|_{H_0^1} \leq A + B \leq C \iint\limits_{\mathbb{D}} |F'(z)|dA(z)$$

which is the desired result. □

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