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THE STEINHAUS TILING PROBLEM AND THE RANGE OF CERTAIN QUADRATIC FORMS

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Dedicated to the memory of Tom Wolff

Abstract. We give a short proof of the fact that there are no measurable subsets of Euclidean space (in dimension $d \geq 3$) which, no matter how translated and rotated, always contain exactly one integer lattice point. In dimension $d = 2$ (the original Steinhaus problem) the question remains open.

1. The Steinhaus problem

Steinhaus (1957) asked if there exists a subset of the plane which, no matter how translated and rotated, always contains exactly one point with integer coordinates. This question remains unanswered.

In this paper we deal only with the measurable version of the Steinhaus problem in dimension $d \geq 2$, in which a *measurable* subset E of $\mathbb{R}^d$ is sought with the property that for almost every $x \in \mathbb{R}^d$ and for almost every isometry $\mathcal{S} : \mathbb{R}^d \to \mathbb{R}^d$

$$\left|(\mathcal{S}E + x) \cap \mathbb{Z}^d\right| = 1.$$

The *Steinhaus property* may also be written as follows. For almost all isometries $\mathcal{S}$

$$\sum_{n \in \mathbb{Z}^d} \mathbf{1}_{\mathcal{S}E}(x - n) = 1, \quad \text{a.e.}(x).$$

In other words, the set E must be such that almost all of its rotations $\mathcal{S}E$ must *tile* when translated at the locations $\mathbb{Z}^d$.

In a recent paper [2] Wolff showed that there are no Steinhaus sets in dimension $d \geq 3$. He showed much more: if $f \in L^1(\mathbb{R}^d)$, $d \geq 3$, is a *Steinhaus function*, i.e., if

$$\sum_{n \in \mathbb{Z}^d} f(x - \mathcal{S}n) = 1, \quad \text{a.e.}(x),$$

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for a dense set of isometries $\mathcal{S}$, then f is almost everywhere equal to a continuous function. This, of course, implies that no Steinhaus *sets* exist for $d \geq 3$. We refer the reader to [2] for additional results regarding properties of Steinhaus sets in dimension 2 (if they exist) and references to other work.

Suppose that $\Lambda = A\mathbb{Z}^d \subset \mathbb{R}^d$, $A \in GL(d, \mathbb{R})$, is a lattice and let $\Lambda^* = A^{-\top}\mathbb{Z}^d$ be its dual lattice. By elementary harmonic analysis one can see that, for an L^1 function f, we have

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = C, \quad \text{a.e.}(x),$$

if and only if its Fourier Transform $\widehat{f}$ vanishes on $\Lambda^* \setminus \{0\}$. Integrating over a large region it is easy to see that the constant C is equal to the density of the lattice Λ times the integral of f.

It follows that for E to be a Steinhaus set it is necessary and sufficient that $|E| = 1$ and that $\widehat{\mathbf{1}_E}$ vanishes on all rotations of the (self-dual) lattice $\mathbb{Z}^d$, except at 0, i.e., it is necessary and sufficient that $\widehat{\mathbf{1}_E}$ vanishes on all spheres with positive radius centered at the origin which go through at least one integer lattice point.

In this paper we will show that there are no Steinhaus sets in dimension $d \geq 3$. The method relies on some arithmetic properties of certain quadratic forms in d variables and is overall much simpler than the method used in [2]. There, of course, much stronger results were proved, using advanced methods of harmonic analysis, about Steinhaus *functions*. As mentioned above, these results have as a corollary the non-existence of Steinhaus sets for $d \geq 3$. Our method does not seem capable of giving any interesting results about Steinhaus functions. (These do exist: take any L^1 function whose Fourier Transform vanishes on all spheres centered at the origin that go through a lattice point.)

Our method is not applicable for the $d = 2$, and we include a proof of this.

The case $d \geq 4$ is presented separately from $d = 3$ (from which it follows) since it is much simpler.

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2. The key observation

In any dimension d write $\mathcal{B}$ for the union of all spheres centered at the origin that go through at least one lattice point. The point 0 is included in $\mathcal{B}$.

Assume from now on that the set E is a Steinhaus set in dimension d.

Suppose now that we can find a lattice $\Lambda^* \subset \mathcal{B}$ with $\det \Lambda^*$ not an integer. Since $\widehat{\mathbf{1}_E}$ vanishes on $\Lambda^* \setminus \{0\}$ it follows that $E + \Lambda$ is a tiling at level $C = |E| \times \operatorname{dens} \Lambda = 1 \times \det \Lambda^*$, which is not an integer. This is a contradiction as, obviously, any set may only tile at an integral level.

Hence there are no Steinhaus sets in dimension d if one can find a lattice of non-integral volume which is contained in $\mathcal{B}$. Since a point $x \in \mathbb{R}^d$ belongs to $\mathcal{B}$ if and only if $|x|^2$ is a sum of d integer squares, we obtain the following theorem, by looking at the quadratic form $\langle A^\top Ax, x\rangle$ for each lattice $\Lambda^* = A\mathbb{Z}^d$.

Theorem 1. *If there exists a positive definite quadratic form $Q(x) = Q(x_1, \ldots, x_d) = \langle Bx, x\rangle$ such that for all integral $x_1, \ldots, x_d$ its value is the sum of d integer squares, and the determinant of Q, $\det B$, is not the square of an integer, then there are no Steinhaus sets in dimension d.*

3. Dimension $d \geq 4$

Consider the 4×4 matrix B with 1 on the diagonal and $1/2$ everywhere else. The matrix B is positive definite (its eigenvalues are $1/2$, $1/2$, $1/2$ and $5/2$) and its determinant is $5/16$. It defines the quadratic form

$$Q(x) = Q(x_1, \ldots, x_4) = \langle Bx, x\rangle = \sum_{i=1}^{4} x_i^2 + \sum_{i>j} x_i x_j,$$

which is obviously integer valued and has non-square determinant. Furthermore, every non-negative integer may be written as a sum of four squares (Lagrange). It follows from Theorem 1 that there are no Steinhaus sets for $d = 4$.

The same is true for dimension $d > 4$ as one may consider the matrix which has B in its upper left 4×4 corner and is equal to the identity matrix elsewhere.

4. Dimension $d = 3$

The determinant of the form that appears in the following Theorem is $2{\cdot}11{\cdot}6$, which is not a square. Hence there are no Steinhaus sets in dimension 3.

Theorem 2. *For each $x, y, z \in \mathbb{Z}$ the number*

$$Q(x, y, z) = 2x^2 + 11y^2 + 6z^2$$

is a sum of three integer squares.

Proof. Suppose this is false and that there are $(x_0, y_0, z_0) \neq (0, 0, 0)$ such that

(a) $Q(x_0, y_0, z_0)$ is not a sum of three squares, and
(b) $x_0^2 + y_0^2 + z_0^2$ is minimal.

From (a) and the well known characterization of those natural numbers that cannot be written as a sum of three squares, we have that

$$Q(x_0, y_0, z_0) = 4^\nu(8k + 7), \quad \nu \geq 0,\ k \geq 0.$$

If all x_0, y_0, z_0 are even, we have $\nu \geq 1$, and, setting $x_0 = 2x_1$, $y_0 = 2y_1$ and $z_0 = 2z_1$, we obtain that $Q(x_1, y_1, z_1)$ is not a sum of three squares, which contradicts the minimality of the initial triple (x_0, y_0, z_0). We conclude that at least one of x_0, y_0, z_0 is odd.

Case 1: $\nu = 0$.

Then $Q(x_0, y_0, z_0) = 7 \bmod 8$. But the quadratic residues mod 8 are 0, 1 and 4, and one checks by examining all the possibilities that Q is never $7 \bmod 8$.

Case 2: $\nu = 1$.

Then $Q(x_0, y_0, z_0) = 32k + 28$. Hence y_0 is even, say $y_0 = 2y_1$. We get

$$x_0^2 + 22y_1^2 + 3z_0^2 = 16k + 14,$$

from which we conclude that x_0 and z_0 are odd, say $x_0 = 2x_1+1$, $z_0 = 2z_1+1$. Substitution gives

$$\begin{aligned} 4x_1^2 + 4x_1 + 1 + 22y_1^2 + 12z_1^2 + 12z_1 + 3 &= 16k + 14, \\ 2x_1(x_1+1) + 11y_1^2 + 6z_1(z_1+1) + 2 &= 8k + 7, \\ 2x_1(x_1+1) + 11y_1^2 + 6z_1(z_1+1) &= 5 \bmod 8. \end{aligned}$$

But for all ξ, $\xi^2 + \xi = 0$ or 2 or 4 or 6 mod 8. Applying this to the first and last term in the above sum and checking all possibilities we get a contradiction.

Case 3: $\nu \geq 2$.

As in Case 2 we have $y_0 = 2y_1$, $z_0 = 2z_1 + 1$, $x_0 = 2x_1 + 1$. Hence

$$2x_1(x_1+1) + 11y_1^2 + 6z_1(z_1+1) + 2 = 4^{\nu-1}(8k+7), \quad \nu - 1 \geq 1.$$

So y_1 is even, say $y_1 = 2y_2$, which gives

$$x_1(x_1+1) + 22y_2^2 + 3z_1(z_1+1) + 1 = 2 \cdot 4^{\nu-2}(8k+7),$$

a contradiction as the left hand side is odd while the right hand side is even. □

5. Dimension $d = 2$

Our method cannot give any results in dimension 2, in view of the following theorem:

Theorem 3. *Any positive definite binary quadratic form whose values are always sums of two integer squares must have a determinant which is the square of an integer.*

Proof. Let $Q(x, y)$ be such a quadratic form, which we may write as

$$Q(x, y) = \left| A \begin{pmatrix} x \\ y \end{pmatrix} \right|^2,$$

where $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. In this notation the determinant of Q is $(\det A)^2$. We now use the following theorem.

THEOREM 4 (Davenport, Lewis, and Schinzel [1]). *Suppose that $f \in \mathbb{Z}[t]$ is such that for each arithmetic progression S there is $t \in S$ such that $f(t)$ is a sum of two squares. Then there are $x, y \in \mathbb{Z}[t]$, such that*

$$f(t) = x^2(t) + y^2(t).$$

REMARKS.

(1) The assumptions of Theorem 4 are very weak and in proving Theorem 3 they are easily guaranteed to hold.

(2) Theorem 4 was proved to answer a question raised by LeVeque, who had asked if any polynomial in $\mathbb{Z}[t]$ whose values are always sums of two squares must be a sum of squares of two linear forms with integer coefficients.

Let $f(t) = Q(t, 1)$, which is a polynomial with integer coefficients. It follows that there are integers $\alpha, \beta, \gamma, \delta$ such that, for all $t \in \mathbb{Z}$,

$$(at + c)^2 + (bt + d)^2 = (\alpha t + \gamma)^2 + (\beta t + \delta)^2.$$

Expanding, and identifying the coefficients we obtain

$$a^2 + b^2 = \alpha^2 + \beta^2, \ ac + bd = \alpha\gamma + \beta\delta, \text{ and } c^2 + d^2 = \gamma^2 + \delta^2.$$

We have for the determinant of Q:

$$\begin{aligned} \det Q &= (ad - bc)^2 \\ &= \begin{vmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{vmatrix} \\ &= \begin{vmatrix} \alpha^2 + \beta^2 & \alpha\gamma + \beta\delta \\ \alpha\gamma + \beta\delta & \gamma^2 + \delta^2 \end{vmatrix} \\ &= (\alpha\delta - \beta\gamma)^2, \end{aligned}$$

which is the square of an integer. □

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