Dirichlet’s theorem on prime numbers

Alexios Terezakis

Department of Mathematics, National and Kapodistrian University of Athens

Advisor: *Michael Papadimitrakis*, Department of Mathematics and Applied Mathematics, University of Crete

We shall study the celebrated theorem of Dirichlet:

If $m, k \in \mathbb{N}$ and $(m, k) = 1$, there are infinitely many primes of the form $m + kn$, $n = 1, 2, 3, \ldots$.

We shall denote p the general prime, ϕ is the well known Euler function and we shall denote $U(R)$ the group of the invertible elements of a ring R. For example, $U(\mathbb{Z}/k\mathbb{Z})$ consists of all equivalence classes mod k of the form $[n]_k$ with $n \in \mathbb{Z}$ and $(n, k) = 1$.

Chapter 1

Characters

We consider the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Definition. *Let G be a finite Abelian group. Every group homomorphism $\chi : G \to \mathbb{C}^*$ is called* ***character*** *of G*

Proposition 1.1. *If $|G| = n$ and $\chi : G \to \mathbb{C}^*$ is a character of G, then $\chi(g)$ is a n-th root of unity for every $g \in G$.*

Proof. $\chi(g)^n = \chi(g^n) = \chi(e) = 1$, where e is the unit element of G. □

For example, if χ is a character of $U(\mathbb{Z}/k\mathbb{Z})$, then $\chi([n]_k)$ is a $\phi(k)$-th root of unity for every $[n]_k \in U(\mathbb{Z}/k\mathbb{Z})$.

We observe that $\chi(G) \subseteq \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ for every character χ of G.

Definition. *We denote $\widehat{G}$ the set of all characters of the group G*

$\widehat{G}$ is an abelian group with multiplication $(\chi_1, \chi_2) \mapsto \chi_1\chi_2$, where $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for every $g \in G$.

Definition. *We denote χ_0 the character $\chi_0 : G \to \mathbb{C}^*$ defined by $\chi_0(g) = 1$ for every $g \in G$.*

The character χ_0 is the unit element of $\widehat{G}$.

Theorem 1.1. $G \simeq \widehat{G}$.

Proof. It is enough to prove the result for finite cyclic groups G, since the structure theorem of finite Abelian groups will allow us to extend the result from the finite cyclic groups to all finite Abelian groups.
Let $|G| = n$ and assume that G is generated by g_0. If Λ_n is the cyclic group of all n-th roots of unity, then $G \simeq \Lambda_n$.
We define $f : \widehat{G} \to \Lambda_n$ by

$$f(\chi) = \chi(g_0) \qquad \text{for every } \chi \in \widehat{G}.$$

Clearly f is a homomorphism.
Now take $\chi \in \widehat{G}$ such that $f(\chi) = 1$. Then for every $g \in G$ we have $g = g_0^m$ for some $m \in \mathbb{N}$ and then $\chi(g) = \chi(g_0^m) = \chi(g_0)^m = f(\chi)^m = 1$. Hence $\chi = \chi_0$ and thus f is one-to-one.
Finally, f is onto since for every $\omega \in \Lambda_n$ there is a specific $\chi \in \widehat{G}$ such that $\chi(g_0) = \omega$ and hence $f(\chi) = \omega$. □

Proposition 1.2. *For every $\chi_1, \chi_2 \in \widehat{G}$ we have*

$$\sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \begin{cases} 0, & \text{if } \chi_1 \neq \chi_2 \\ |G|, & \text{if } \chi_1 = \chi_2 \end{cases}$$

Proof. If $\chi_1 = \chi_2$, then

$$\sum_{g\in G} \chi_1(g)\overline{\chi_1(g)} = \sum_{g\in G} |\chi_1(g)|^2 = \sum_{g\in G} 1 = |G|.$$

If $\chi_1 \neq \chi_2$, we choose $g_0 \in G$ such that $\chi_1(g_0) \neq \chi_2(g_0)$ or, equivalently, $\chi_1(g_0)\overline{\chi_2(g_0)} \neq 1$ and we get

$$\chi_1(g_0)\overline{\chi_2(g_0)} \sum_{g\in G} \chi_1(g)\overline{\chi_2(g)} = \sum_{g\in G} \chi_1(gg_0)\overline{\chi_2(gg_0)} = \sum_{g\in G} \chi_1(g)\overline{\chi_2(g)}.$$

Thus $\sum_{g\in G} \chi_1(g)\overline{\chi_2(g)} = 0$. ☐

Proposition 1.3. *For every $g_1, g_2 \in G$ we have*

$$\sum_{\chi\in\widehat{G}} \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0, & \text{if } g_1 \neq g_2 \\ |G|, & \text{if } g_1 = g_2 \end{cases}$$

Proof. The proof is similar to the previous one. ☐

Definition. *For every character χ of the group $U(\mathbb{Z}/k\mathbb{Z})$ we define $\chi : \mathbb{Z} \to \mathbb{C}$ by:*

$$\chi(n) = \begin{cases} \chi([n]_k), & \text{if } (n,k) = 1 \\ 0, & \text{otherwise} \end{cases}$$

*The function $\chi : \mathbb{Z} \to \mathbb{C}$ is called **character** **mod k**.*

The function $\chi : \mathbb{Z} \to \mathbb{C}$ has the same symbol as the character χ from which it is derived, but this should cause no confusion.

The new function χ is substantially an extension of the original χ from the numbers $n \in \mathbb{Z}$ with $(n,k) = 1$ to all numbers $n \in \mathbb{Z}$. We observe that the new function is multiplicative, i.e.

$$\chi(nm) = \chi(n)\chi(m) \qquad \text{for every } n, m \in \mathbb{Z}.$$

In the case of characters mod k Propositions 1.2 and 1.3 take the following forms.

Proposition 1.4. *If χ_1, χ_2 are characters* mod k *and $A \subseteq \mathbb{Z}$ consists of k numbers from k different equivalence classes* mod k*, then*

$$\sum_{n\in A} \chi_1(n)\overline{\chi_2(n)} = \begin{cases} 0, & \text{if } \chi_1 \neq \chi_2 \\ \phi(k), & \text{if } \chi_1 = \chi_2 \end{cases}$$

Proposition 1.5. *For every $n, m \in \mathbb{Z}$ we have*

$$\sum_{\chi \text{ char. } \bmod k} \chi(n)\overline{\chi(m)} = \begin{cases} 0, & \text{if } n \not\equiv m (\bmod k) \\ \phi(k), & \text{if } n \equiv m (\bmod k) \end{cases}$$

Proposition 1.6. *Let χ be a character* mod k*. If $n \in \mathbb{N}$ and $n = p_1^{a_1} \cdots p_m^{a_m}$ is the representation of n as a product of primes, then*

$$\sum_{d|n} \chi(d) = \frac{1 - \chi(p_1)^{a_1+1}}{1 - \chi(p_1)} \cdots \frac{1 - \chi(p_m)^{a_m+1}}{1 - \chi(p_m)},$$

where the expression $\frac{1-t^{a+1}}{1-t}$ is taken to be equal to $a + 1$ when $t = 1$.

Proof. The divisors of n are the numbers $d = p_1^{b_1} \cdots p_m^{b_m}$ with $0 \le b_1 \le a_1, \ldots, 0 \le b_m \le a_m$. Hence

$$\sum_{d|n} \chi(d) = \sum_{0 \le b_1 \le a_1, \ldots, 0 \le b_m \le a_m} \chi(p_1^{b_1} \cdots p_m^{b_m}) = \sum_{0 \le b_1 \le a_1, \ldots, 0 \le b_m \le a_m} \chi(p_1)^{b_1} \cdots \chi(p_m)^{b_m}$$
$$= \sum_{b_1=0}^{a_1} \chi(p_1)^{b_1} \cdots \sum_{b_m=0}^{a_m} \chi(p_m)^{b_m} = \frac{1-\chi(p_1)^{a_1+1}}{1-\chi(p_1)} \cdots \frac{1-\chi(p_m)^{a_m+1}}{1-\chi(p_m)}$$

and the proof is complete. □

Chapter 2

The zeta-function of Riemann

Definition. *The **zeta-function of Riemann**, $\zeta : \{s \in \mathbb{R} \mid s > 1\} \to \mathbb{R}$, is defined by*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \qquad \textit{for every } s > 1.$$

When we write $\sum_p a(p)$ we mean the series of numbers $a(p)$ over all primes p. Thus, if $p_1 < p_2 < \ldots < p_n < \ldots$ are the primes in increasing order, we define

$$\sum_p a(p) = \sum_{n=1}^{+\infty} a(p_n).$$

The same can be said of the product:

$$\prod_p a(p) = \prod_{n=1}^{+\infty} a(p_n).$$

Proposition 2.1. *For every $s > 1$ we have*

$$\zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}.$$

Proof. We observe that $\zeta(s)\,\frac{1}{p^s} = \sum_{n \equiv 0 (\mathrm{mod}\, p)} \frac{1}{n^s}$ and thus

$$\zeta(s)\Big(1 - \frac{1}{p^s}\Big) = \sum_{n \not\equiv 0 (\mathrm{mod}\, p)} \frac{1}{n^s}. \tag{2.1}$$

If $p_1 < p_2 < \ldots < p_n < \ldots$ are the primes in increasing order, we define

$$a_n = \zeta(s) \prod_{m=1}^{n} \Big(1 - \frac{1}{p_m^s}\Big).$$

Using (2.1) with $p = p_1$ and applying induction, we can easily prove that

$$1 \le a_n = \sum_{m \not\equiv 0 (\mathrm{mod}(p_1 \cdots p_n))} \frac{1}{m^s} \le 1 + \sum_{m = p_{n+1}}^{+\infty} \frac{1}{m^s}$$

Therefore $a_n \to 1$ and hence $\prod_p (1 - \frac{1}{p^s})^{-1}$ converges to $\zeta(s)$. □

Proposition 2.2. $\zeta(s) = \frac{1}{s-1} + \mathrm{O}(1)$ *when $s \to 1+$.*

Proof. For all $s > 1$ we have

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \geq \sum_{n=1}^{+\infty} \int_n^{n+1} \frac{1}{t^s}\, dt = \int_1^{+\infty} \frac{1}{t^s}\, dt = \frac{1}{s-1}$$

and

$$\zeta(s) = 1 + \sum_{n=2}^{+\infty} \frac{1}{n^s} \leq 1 + \sum_{n=2}^{+\infty} \int_{n-1}^{n} \frac{1}{t^s}\, dt = 1 + \int_1^{+\infty} \frac{1}{t^s}\, dt = 1 + \frac{1}{s-1}.$$

Hence $\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}$ for every $s > 1$. □

Theorem 2.1. $\sum_p \frac{1}{p^s} = \log \frac{1}{s-1} + \mathrm{O}(1)$ *when* $s \to 1+$.

Proof. From $\log(1-z)^{-1} = \sum_{n=1}^{+\infty} \frac{z^n}{n}$ and Proposition 2.2 we get

$$\sum_p \sum_{n=1}^{+\infty} \frac{1}{np^{ns}} = \sum_p \log\left(1 - \frac{1}{p^s}\right)^{-1} = \log \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \log \zeta(s) = \log \frac{1}{s-1} + \mathrm{O}(1)$$

when $s \to 1+$. Moreover, for every $s > 1$ we have

$$0 \leq \sum_p \sum_{n=2}^{+\infty} \frac{1}{np^{ns}} \leq \sum_p \sum_{n=2}^{+\infty} \frac{1}{p^n} = \sum_p \frac{1}{p(p-1)} < +\infty$$

These two relations and

$$\sum_p \frac{1}{p^s} = \sum_p \sum_{n=1}^{+\infty} \frac{1}{np^{ns}} - \sum_p \sum_{n=2}^{+\infty} \frac{1}{np^{ns}}$$

imply the result. □

Chapter 3

Dirichlet's theorem

Dirichlet's Theorem. *If $m, k \in \mathbb{N}$ and $(m,k) = 1$, there are infinitely many primes p such that $p \equiv m (\operatorname{mod} k)$.*

Proof. We take $m' \in \mathbb{Z}$ so that $mm' \equiv 1 (\operatorname{mod} k)$. Proposition 1.5 implies

$$\sum_{\chi \text{ char. } \operatorname{mod} k} \chi(pm') = \begin{cases} 0, & \text{if } p \not\equiv m (\operatorname{mod} k) \\ \phi(k), & \text{if } p \equiv m (\operatorname{mod} k) \end{cases}$$

Hence

$$\phi(k) \sum_{p \equiv m (\operatorname{mod} k)} \frac{1}{p^s} = \sum_p \frac{1}{p^s} \sum_{\chi \text{ char. } \operatorname{mod} k} \chi(pm') = \sum_{\chi \text{ char. } \operatorname{mod} k} \chi(m') \sum_p \frac{\chi(p)}{p^s}. \tag{3.1}$$

For the term of the last sum corresponding to $\chi = \chi_0$ we observe that:

$$\chi_0(m') \sum_p \frac{\chi_0(p)}{p^s} = \sum_{(p,k)=1} \frac{1}{p^s} = \sum_p \frac{1}{p^s} - \sum_{p|k} \frac{1}{p^s}.$$

Since $\sum_{p|k} \frac{1}{p^s}$ is a finite sum, Theorem 2.1 implies that the right side of the last identity diverges to $+\infty$ when $s \to 1+$.
The only thing left for us to show is that, if $\chi \neq \chi_0$, then $\sum_p \frac{\chi(p)}{p^s} = \mathrm{O}(1)$ when $s \to 1+$. Indeed, if we show this, then (3.1) will imply that

$$\sum_{p \equiv m (\operatorname{mod} k)} \frac{1}{p} = \lim_{s \to 1+} \sum_{p \equiv m (\operatorname{mod} k)} \frac{1}{p^s} = +\infty$$

and thus there will be infinitely many primes p such that $p \equiv m (\operatorname{mod} k)$.
That $\sum_p \frac{\chi(p)}{p^s} = \mathrm{O}(1)$ when $s \to 1+$ is the content of Proposition 4.3 at the end of this work. □

It is worthwhile to note that up to now we have used no complex analysis.

Chapter 4

Dirichlet's L-functions

We use the halfplane notation

$$\mathbb{H}_\sigma^+ = \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) > \sigma\}$$

for every $\sigma \in \mathbb{R}$.

Now we extend the zeta-function on the halfplane $\mathbb{H}_1^+$ in the natural manner:

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

The series defining $\zeta(s)$ converges absolutely when Re$(s) > 1$.

Definition. *If (a_n) is a complex sequence, the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad s \in \mathbb{C}$$

*is called **Dirichlet series**.*

For instance, the series defining the zeta-function is a Dirichlet series.

Lemma 4.1. *If the Dirichlet series $\sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ converges when $s = s_0 \in \mathbb{C}$, then for every $\theta \in (0, \pi/2)$ it converges uniformly on the angular set $\Gamma(s_0, \theta) = \{s \in \mathbb{C} \,|\, |\mathrm{Arg}(s - s_0)| < \theta\}$.*

Proof. Let $s = \sigma + i\tau \in \Gamma(s_0, \theta)$ and $s_0 = \sigma_0 + i\tau_0$. We set $b_n = \frac{a_n}{n^{s_0}}$ and then we have

$$r_n := \sum_{l=n}^{+\infty} b_l \to 0 \qquad \text{when } n \to +\infty.$$

We consider an arbitrary $\epsilon > 0$ and then there is some n_0 so that $|r_n| < \epsilon$ for every $n \geq n_0$. Moreover, since $s \in \Gamma(s_0, \theta)$, we have that $\sigma - \sigma_0 > 0$ and thus $|n^{s-s_0}| = n^{\sigma-\sigma_0} \geq 1$ for every $n \in \mathbb{N}$.

Now, if $n, m \in \mathbb{N}$ and $n_0 \leq n < m$, then:

$$
\begin{aligned}
\Big|\sum_{l=n}^{m} \frac{a_l}{l^s}\Big| &= \Big|\sum_{l=n}^{m} \frac{b_l}{l^{s-s_0}}\Big| = \Big|\sum_{l=n}^{m} \frac{r_l - r_{l+1}}{l^{s-s_0}}\Big| = \Big|\sum_{l=n}^{m} \frac{r_l}{l^{s-s_0}} - \sum_{l=n+1}^{m+1} \frac{r_l}{(l-1)^{s-s_0}}\Big| \\
&\leq \Big|\frac{r_n}{n^{s-s_0}}\Big| + \Big|\frac{r_{m+1}}{m^{s-s_0}}\Big| + \Big|\sum_{l=n+1}^{m} r_l\Big(\frac{1}{l^{s-s_0}} - \frac{1}{(l-1)^{s-s_0}}\Big)\Big| \\
&\leq 2\epsilon + \Big|\sum_{l=n+1}^{m} r_l(s-s_0)\int_{l-1}^{l} \frac{1}{t^{s-s_0+1}}\,dt\Big| \\
&\leq 2\epsilon + \epsilon|s-s_0|\sum_{l=n+1}^{m}\int_{l-1}^{l} \frac{1}{t^{\sigma-\sigma_0+1}}\,dt = 2\epsilon + \epsilon|s-s_0|\int_{n}^{m} \frac{1}{t^{\sigma-\sigma_0+1}}\,dt \\
&= 2\epsilon + \epsilon\frac{|s-s_0|}{\sigma-\sigma_0}\Big(\frac{1}{n^{\sigma-\sigma_0}} - \frac{1}{m^{\sigma-\sigma_0}}\Big) \leq 2\epsilon + \frac{\epsilon}{\cos\theta}\,\frac{1}{n^{\sigma-\sigma_0}} \leq 2\epsilon + \frac{\epsilon}{\cos\theta}.
\end{aligned}
$$

Cauchy's criterion implies the result. □

Lemma 4.2. *For every r, R with $0 < r < R < +\infty$ the partial sums of the series $\sum_{n=1}^{+\infty} \frac{1}{n^s}$ are uniformly bounded on the halfring $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1, r \leq |s-1| \leq R\}$.*

Proof. Take $s = \sigma + i\tau$ such that $\sigma \geq 1$ and $r \leq |s-1| \leq R$. Then

$$
\begin{aligned}
\Big|\sum_{n=1}^{N} \frac{1}{n^s}\Big| &\leq \Big|\sum_{n=1}^{N} \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{1}{x^s}\,dx\Big)\Big| + \Big|\sum_{n=1}^{N}\int_n^{n+1} \frac{1}{x^s}\,dx\Big| \\
&= \Big|\sum_{n=1}^{N}\int_n^{n+1} \Big(\frac{1}{n^s} - \frac{1}{x^s}\Big)\,dx\Big| + \Big|\int_1^{N+1} \frac{1}{x^s}\,dx\Big| \\
&= |s|\Big|\sum_{n=1}^{N}\int_n^{n+1}\int_n^{x} \frac{1}{t^{s+1}}\,dt\,dx\Big| + \frac{1}{|s-1|}\Big|1 - \frac{1}{(N+1)^{s-1}}\Big| \\
&\leq (1+R)\sum_{n=1}^{N}\int_n^{n+1}\int_n^{x} \frac{1}{t^{\sigma+1}}\,dt\,dx + \frac{1}{r}\Big(1 + \frac{1}{(N+1)^{\sigma-1}}\Big) \\
&\leq (1+R)\sum_{n=1}^{N}\int_n^{n+1}\int_n^{x} \frac{1}{t^2}\,dt\,dx + \frac{2}{r} \leq (1+R)\sum_{n=1}^{N}\int_n^{n+1}\int_n^{x} \frac{1}{n^2}\,dt\,dx + \frac{2}{r} \\
&\leq (1+R)\sum_{n=1}^{N}\frac{1}{n^2} + \frac{2}{r} \leq (1+R)\sum_{n=1}^{+\infty}\frac{1}{n^2} + \frac{2}{r}
\end{aligned}
$$

and the result has been proved. □

Lemma 4.3. *If $b_1 \geq \ldots \geq b_N \geq 0$, then*

$$
\Big|\sum_{n=1}^{N} b_n x_n\Big| \leq b_1 \max_{1\leq n\leq N}\Big|\sum_{l=1}^{n} x_l\Big|.
$$

Proof. Let $s_0 = 0$ and $s_n = \sum_{l=1}^{n} x_l$ for all n with $1 \leq n \leq N$.

If $M = \max_{1\le n\le N} |\sum_{l=1}^{n} x_l|$, then

$$\begin{aligned}\Big|\sum_{n=1}^{N} b_n x_n\Big| &= \Big|\sum_{n=1}^{N} b_n(s_n - s_{n-1})\Big| = \Big|\sum_{n=1}^{N} b_n s_n - \sum_{n=0}^{N-1} b_{n+1}s_n\Big| \\ &= \Big|\sum_{n=1}^{N-1}(b_n - b_{n+1})s_n + b_N s_N\Big| \le \sum_{n=1}^{N-1}(b_n - b_{n+1})|s_n| + b_N|s_N| \\ &\le \sum_{n=1}^{N-1}(b_n - b_{n+1})M + b_N M = b_1 M.\end{aligned}$$

and the proof is complete. □

Lemma 4.4. *Suppose that for some $s_0 \in \mathbb{C}$ the sequence $(\frac{a_n}{n^{s_0-1}})$ is non-negative and decreasing. Then for every r, R with $0 < r < R < +\infty$ the partial sums of the Dirichlet series $\sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ are uniformly bounded on the halfring $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \ge \mathrm{Re}(s_0), r \le |s - s_0| \le R\}$.*

Proof. Assume that $\mathrm{Re}(s) \ge \mathrm{Re}(s_0)$ and $r \le |s - s_0| \le R$. We set $b_n = \frac{a_n}{n^{s_0-1}}$ and $x_n = \frac{1}{n^{s-s_0+1}}$ and we have that $b_1 \ge \ldots \ge b_n \ge \ldots \ge 0$ and the Dirichlet series takes the form $\sum_{n=1}^{+\infty} b_n x_n$. Lemma 4.2 implies that there is $M = M(r, R)$ so that $|\sum_{l=1}^{n} x_l| \le M$ for every n and Lemma 4.3 implies the result: $|\sum_{n=1}^{N} b_n x_n| \le b_1 M = a_1 M$. □

Theorem 4.1. *Let (f_n) be a sequence of analytic functions on the open $\Omega \subseteq \mathbb{C}$. If $f_n \to f$ uniformly on the compact subsets of Ω, then f is analytic on Ω.*

Proof. f is continuous on Ω since the convergence is uniform on every closed disc contained in Ω and since every f_n is continuous. Now we take any triangle Δ in Ω and the uniform convergence on $\partial\Delta$ together with Cauchy's theorem imply

$$\int_{\partial\Delta} f(z)\,dz = \lim_{n\to+\infty} \int_{\partial\Delta} f_n(z)\,dz = 0.$$

Morera's theorem implies the result. □

Theorem 4.2. *Let $\sum_{n=1}^{+\infty} \frac{a_n}{n^s}$ be a Dirichlet series.*
(i) If the series converges when $s = s_0 = \sigma_0 + i\tau_0 \in \mathbb{C}$, then the series defines an analytic function on the open halfplane $\mathbb{H}^+_{\sigma_0}$.
(ii) If for some $s_0 = \sigma_0 + i\tau_0 \in \mathbb{C}$ the sequence $(\frac{a_n}{n^{s_0-1}})$ is non-negative and decreasing, then the series defines an analytic function on $\mathbb{H}^+_{\sigma_0}$ and for every r, R with $0 < r < R < +\infty$ this analytic function is bounded on the halfring $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \sigma_0, r \le |s - s_0| \le R\}$.

Proof. (i) Every compact subset of $\mathbb{H}^+_{\sigma_0}$ is contained in $\Gamma(s_0, \theta)$ for some $\theta \in (0, \pi/2)$. Thus the result is a corollary of Lemma 4.1 and Theorem 4.1.
(ii) Take $s_0' = \sigma_0' + i\tau_0$ with $\sigma_0' > \sigma_0$. Then the series

$$\sum_{n=1}^{+\infty} \frac{a_n}{n^{s_0'}} = \sum_{n=1}^{+\infty} \frac{a_n}{n^{s_0-1}} \frac{1}{n^{\sigma_0'-\sigma_0+1}}$$

converges absolutely and, according to (i), defines an analytic function on $\mathbb{H}^+_{\sigma_0'}$. Therefore the series defines an analytic function on $\mathbb{H}^+_{\sigma_0}$. The rest is a consequence of Lemma 4.4. □

Corollary 4.1. *The zeta-function is analytic on $\mathbb{H}^+_1$ and for every r, R with $0 < r < R < +\infty$ it is bounded on the halfring $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1, r \le |s - 1| \le R\}$.*

We observe that

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} - \sum_{n=1}^{+\infty} \frac{2}{(2n)^s} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s}, \qquad \text{Re}(s) > 1. \tag{4.1}$$

The series at the right side of (4.1) converges for every $s > 0$, and Theorem 4.2 implies that it defines a function, say f, analytic on $\mathbb{H}_0^+$. Hence the function $h(s) = \frac{f(s)}{1-2^{1-s}}$ is analytic on $\mathbb{H}_0^+$ except for possible poles at the points $s = 1 + m\,\frac{2\pi}{\log 2}\,i$ $(m \in \mathbb{Z})$ which are the roots of order one of the function $1 - 2^{1-s}$ and all of which lie on the line $\text{Re}(s) = 1$. Since h is identical to the zeta-function on $\mathbb{H}_1^+$ and due to Corollary 4.1, all the above points, except $s = 1$, are regular points of h. Moreover, Proposition 2.2 implies that $s = 1$ is a pole of order one of h. We can now extend the zeta-function on $\mathbb{H}_0^+$ defining it as being identical to the function h. Therefore we can think of the zeta-function as a meromorphic function on $\mathbb{H}_0^+$ with a single pole of order one at $s = 1$.

Definition. *Let χ be a character* mod k, *$\chi \neq \chi_0$. The function $L(\cdot,\chi) : \mathbb{H}_0^+ \to \mathbb{C}$ defined by*

$$L(s,\chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}, \qquad \text{Re}(s) > 0,$$

is called ***Dirichlet's L-function****.*
Especially for χ_0, we define Dirichlet's L-function $L(\cdot,\chi_0)$ in the same way but with $\mathbb{H}_1^+$ as its domain of definition.

Lemma 4.5. *If χ is a character* mod k, *$\chi \neq \chi_0$, then $|\sum_{l=n}^{m} \chi(l)| \leq \phi(k)$ for every $n, m \in \mathbb{Z}$ with $n \leq m$.*

Proof. We group the natural numbers l from n up to m in subsets each of which consists of k successive numbers and at most one of which consists of at most $k-1$ successive numbers. Proposition 1.4 implies that the sum $\sum_l \chi(l)$ over each of the complete subsets consisting of k successive numbers equals 0. Moreover, the sum over the last subset contains at most $\phi(k)$ non-zero terms and each of them satisfies $|\chi(l)| \leq 1$. □

Proposition 4.1. *If χ is a character* mod k, *$\chi \neq \chi_0$, then the Dirichlet series $\sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ converges on $\mathbb{H}_0^+$ and the corresponding $L(\cdot,\chi)$ is analytic on $\mathbb{H}_0^+$. If $\chi = \chi_0$, we have the same result but with $\mathbb{H}_1^+$ instead of $\mathbb{H}_0^+$.*

Proof. Take $\chi \neq \chi_0$ and $s > 0$. Then for every n, m with $n \leq m$ Lemmas 4.3 and 4.5 imply

$$\Big|\sum_{l=n}^{m} \frac{\chi(l)}{l^s}\Big| \leq \frac{\phi(k)}{n^s}.$$

By Cauchy's criterion the Dirichlet series $\sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ converges. Since this is true for every $s > 0$, the Dirichlet series converges on $\mathbb{H}_0^+$ and $L(\cdot,\chi)$ is analytic on $\mathbb{H}_0^+$.
If $\chi = \chi_0$, then we simply observe that $|\chi_0(n)| \leq 1$ for every n, and hence $\sum_{n=1}^{+\infty} \frac{\chi_0(n)}{n^s}$ converges for every $s > 1$. □

Proposition 4.2. *If χ is any character* mod k, *then*

$$L(s,\chi) = \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} = \prod_{p \not\equiv 0 (\text{mod}\, k)} \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} \neq 0, \qquad \text{Re}(s) > 1.$$

Especially when $\chi = \chi_0$,

$$L(s,\chi_0) = \prod_{p \not\equiv 0 (\text{mod}\, k)} \Big(1 - \frac{1}{p^s}\Big)^{-1} = \zeta(s) \prod_{p \equiv 0 (\text{mod}\, k)} \Big(1 - \frac{1}{p^s}\Big) \neq 0, \qquad \text{Re}(s) > 1.$$

Proof. Let $\mathrm{Re}(s) > 1$. We define

$$a_n = L(s,\chi)\prod_{m=1}^{n}\Big(1-\frac{\chi(p_m)}{p_m^s}\Big).$$

With the method of the proof of Proposition 2.1, we show that $a_n \to 1$. Thus $L(s,\chi) \neq 0$ and $\prod_p (1-\frac{\chi(p)}{p^s})^{-1}$ converges to $L(s,\chi)$. □

If k is not a prime, the product $\prod_{p\equiv 0(\mathrm{mod}\, k)}(1-\frac{1}{p^s})$ contains no terms and it is equal to 1. In this case we have $L(s,\chi_0) = \zeta(s)$ when $\mathrm{Re}(s) > 1$. If k is prime, then $\prod_{p\equiv 0(\mathrm{mod}\, k)}(1-\frac{1}{p^s}) = 1-\frac{1}{k^s}$ and hence $L(s,\chi_0) = \zeta(s)(1-\frac{1}{k^s})$ when $\mathrm{Re}(s) > 1$. In every case we can extend the function $L(s,\chi_0)$ on $\mathbb{H}_0^+$ with a single pole of order one at $s = 1$.

Theorem 4.3. *If χ is a character* $\mathrm{mod}\, k$*, $\chi \neq \chi_0$, then $L(1,\chi) \neq 0$*

Proof. We consider two cases.
(i) Assume that there is at least one *complex* (i.e. having at least one non-real value) character χ_1 $\mathrm{mod}\, k$ such that $L(1,\chi_1) = 0$. Then $\chi_2 = \overline{\chi_1}$ is a second complex character $\mathrm{mod}\, k$ such that $L(1,\chi_2) = 0$. We define the function ζ_k by

$$\zeta_k(s) = \prod_{\chi \text{ char. mod } k} L(s,\chi).$$

Then ζ_k is analytic on $\mathbb{H}_0^+$ except for at most one pole of order one at $s = 1$. Since $L(s,\chi_0)$ has a pole of order one at $s = 1$ and the product defining ζ_k contains at least two functions having $s = 1$ as a root, we get $\zeta_k(1) = 0$. But when $s > 1$ we have:

$$\begin{aligned}
\log\zeta_k(s) &= \log\prod_{\chi \text{ char. mod } k}\prod_p\Big(1-\frac{\chi(p)}{p^s}\Big)^{-1} = \sum_{\chi \text{ char. mod } k}\sum_p \log\Big(1-\frac{\chi(p)}{p^s}\Big)^{-1}\\
&= \sum_{\chi \text{ char. mod } k}\sum_p\sum_{n=1}^{+\infty}\frac{\chi(p^n)}{np^{ns}} = \sum_p\sum_{n=1}^{+\infty}\frac{1}{np^{ns}}\sum_{\chi \text{ char. mod } k}\chi(p^n)\\
&= \phi(k)\sum_p\sum_{n:p^n\equiv 1(\mathrm{mod}\, k)}\frac{1}{np^{ns}} \geq 0.
\end{aligned}$$

Hence $\zeta_k(s) \geq 1$ for all $s > 1$ and we arrive at a contradiction.
(ii) If every character $\mathrm{mod}\, k$ is real, the previous argument does not work.
Now for every real character χ $\mathrm{mod}\, k$ we define

$$f(n) = \sum_{d|n}\chi(d).$$

Since χ is real, its only possible values are 0 and ± 1. Proposition 1.6 easily implies that $f(n) \geq 0$ and $f(n^2) \geq 1$ for every $n \in \mathbb{N}$. Therefore

$$\sum_{n=1}^{+\infty}\frac{f(n)}{\sqrt{n}} \geq \sum_{n=1}^{+\infty}\frac{f(n^2)}{\sqrt{n^2}} \geq \sum_{n=1}^{+\infty}\frac{1}{n} = +\infty.$$

Now it is enough to show that $\sum_{n=1}^{N^2}\frac{f(n)}{\sqrt{n}} = 2NL(1,\chi) + \mathrm{O}(1)$. This contradicts $L(1,\chi) = 0$.
In the following, the symbol 1_M denotes the characteristic function of the set M: the function which equals 1 on M and 0 on the complement of M.
We consider the following subsets of $\mathbb{N}\times\mathbb{N}$:

$$A = \{(d,b)\,|\,d \leq N, b \leq N^2/d\}, \qquad B = \{(d,b)\,|\,b \leq N, N < d \leq N^2/b\}.$$

Now, considering that all variables of the following sums are natural numbers, we have:

$$\begin{aligned}
&\sum_{d\le N}\frac{\chi(d)}{\sqrt{d}}\Big(\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}\Big)+\sum_{b\le N}\frac{1}{\sqrt{b}}\Big(\sum_{N<d\le N^2/b}\frac{\chi(d)}{\sqrt{d}}\Big)\\
&\quad=\sum_{d,b}\frac{\chi(d)}{\sqrt{db}}1_A(d,b)+\sum_{d,b}\frac{\chi(d)}{\sqrt{db}}1_B(d,b)=\sum_{d,b}\frac{\chi(d)}{\sqrt{db}}\big(1_A(d,b)+1_B(d,b)\big)\\
&\quad=\sum_{d,b}\frac{\chi(d)}{\sqrt{db}}1_{A\cup B}(d,b),
\end{aligned}\tag{4.2}$$

The last equality is true because $A\cap B=\emptyset$.
Now we observe that $A\cup B=\{(d,b)\,|\,db\le N^2\}$ and we make the change of variables: $d=d$, $n=db$. Then $1_{A\cup B}=1_C$, where $C=\{(n,d)\,|\,n\le N^2, d\mid n\}$ and thus

$$\sum_{d,b}\frac{\chi(d)}{\sqrt{db}}1_{A\cup B}(d,b)=\sum_{n,d}\frac{\chi(d)}{\sqrt{n}}1_C(n,d)=\sum_{n=1}^{N^2}\frac{1}{\sqrt{n}}\sum_{d|n}\chi(d)=\sum_{n=1}^{N^2}\frac{f(n)}{\sqrt{n}}.$$

The last equality together with (4.2) imply

$$\sum_{n=1}^{N^2}\frac{f(n)}{\sqrt{n}}=\sum_{d\le N}\frac{\chi(d)}{\sqrt{d}}\Big(\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}\Big)+\sum_{b\le N}\frac{1}{\sqrt{b}}\Big(\sum_{N<d\le N^2/b}\frac{\chi(d)}{\sqrt{d}}\Big)=P+Q.\tag{4.3}$$

We shall first prove that $P=2NL(1,\chi)+\mathrm{O}(1)$. We have:

$$\begin{aligned}
P&=\sum_{d=1}^{N}\frac{\chi(d)}{\sqrt{d}}\Big(\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}\Big)\\
&=\sum_{d=1}^{N}\frac{\chi(d)}{\sqrt{d}}\Big(\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}-\frac{2N}{\sqrt{d}}\Big)+2N\Big(\sum_{d=1}^{N}\frac{\chi(d)}{d}-L(1,\chi)\Big)+2NL(1,\chi)\\
&=I+II+III.
\end{aligned}$$

The integral criterion for series implies:

$$\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}=2\sqrt{\frac{N^2}{d}}+\mathrm{O}(1)=\frac{2N}{\sqrt{d}}+\mathrm{O}(1).$$

Hence:

$$|I|=\Big|\sum_{d=1}^{N}\frac{\chi(d)}{\sqrt{d}}\Big(\sum_{b\le N^2/d}\frac{1}{\sqrt{b}}-\frac{2N}{\sqrt{d}}\Big)\Big|=\mathrm{O}(1)\Big|\sum_{d=1}^{N}\frac{\chi(d)}{\sqrt{d}}\Big|=\mathrm{O}(1).$$

The last equality is due to the convergence of $\sum_{d=1}^{+\infty}\frac{\chi(d)}{\sqrt{d}}$.
Regarding II we observe the following. Since $\sum_{d=1}^{+\infty}\frac{\chi(d)}{d}=L(1,\chi)$, there is some $M\ge N+1$ such that $|\sum_{d=M+1}^{+\infty}\frac{\chi(d)}{d}|<\frac{\epsilon}{2N}$. Then:

$$|II|=2N\Big|\sum_{d=N+1}^{+\infty}\frac{\chi(d)}{d}\Big|\le 2N\Big|\sum_{d=N}^{M}\frac{\chi(d)}{d}\Big|+2N\Big|\sum_{d=M+1}^{+\infty}\frac{\chi(d)}{d}\Big|\le 2N\frac{\phi(k)}{N}+\epsilon=\mathrm{O}(1)$$

For the second inequality we used Lemmas 4.3 and 4.5.
We have thus proved $P=2NL(1,\chi)+\mathrm{O}(1)$ and we shall now show that $Q=\mathrm{O}(1)$.
Lemmas 4.3 and 4.5 imply:

$$\Big|\sum_{N<d\le N^2/b}\frac{\chi(d)}{\sqrt{d}}\Big|\le\frac{\phi(k)}{\sqrt{N}}.$$

Hence

$$|Q| = \Big| \sum_{b=1}^{N} \frac{1}{\sqrt{b}} \sum_{N<d\le N^2/b} \frac{\chi(d)}{\sqrt{d}} \Big| \le \frac{\phi(k)}{\sqrt{N}} \sum_{b=1}^{N} \frac{1}{\sqrt{b}} = \mathrm{O}(1).$$

Finally, (4.3) implies $\sum_{n=1}^{N^2} \frac{f(n)}{\sqrt{n}} = 2NL(1,\chi) + \mathrm{O}(1)$ and the proof is complete. □

Proposition 4.3. *If* χ *is a character* $\bmod k$, $\chi \neq \chi_0$, *then* $\sum_p \frac{\chi(p)}{p^s} = \mathrm{O}(1)$ *when* $s \to 1+$.

Proof. Using

$$\sum_p \sum_{m=2}^{+\infty} \Big| \frac{\chi(p^m)}{mp^{ms}} \Big| \le \sum_p \sum_{m=2}^{+\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} < +\infty$$

for every $s > 1$, we get

$$\begin{aligned} \sum_p \frac{\chi(p)}{p^s} &= \sum_p \sum_{m=1}^{+\infty} \frac{\chi(p^m)}{mp^{ms}} - \sum_p \sum_{m=2}^{+\infty} \frac{\chi(p^m)}{mp^{ms}} = \sum_p \sum_{m=1}^{+\infty} \frac{\chi(p^m)}{mp^{ms}} + \mathrm{O}(1) \\ &= \sum_p \log \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1} + \mathrm{O}(1) = \log L(s,\chi) + \mathrm{O}(1) \end{aligned}$$

for every $s > 1$. Theorem 4.3 implies that the function $\log L(s,\chi)$ is well defined and analytic in a neighborhood of 1. □

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