ON BEST UNIFORM APPROXIMATION BY BOUNDED ANALYTIC FUNCTIONS

M. PAPADIMITRAKIS

$C(T)$ is the space of continuous functions on the unit circle T with the supremum norm $\|\cdot\|_\infty \cdot H^\infty(T)$ is the space of nontangential limits of bounded analytic functions in the unit disc D. Also, $A(T) = H^\infty(T) \cap C(T)$. Let $\mathscr{F}\ell^1$ be the subspace of $C(T)$ of all functions whose Fourier series is absolutely convergent with norm

$$\|f\|_{\mathscr{F}\ell^1} = \sum |\hat{f}(n)|, \quad \hat{f}(n) = \int f(e^{i\theta})\, e^{-in\theta} \frac{d\theta}{2\pi}.$$

$H^1(T)$ is the Hardy space of nontangential limits of functions F analytic in D such that

$$\|F\|_1 = \sup_{0<r<1} \int |F(re^{i\theta})| \frac{d\theta}{2\pi} < +\infty,$$

and $H^1_0(T) = \{F \in H^1(T) \colon \hat{F}(0) = 0\}$.

It is known (see [2]) that any $f \in C(T)$ has a unique best approximation $g \in H^\infty(T)$ in the sense

$$d = d(f, H^\infty) = \inf_{h \in H^\infty(T)} \|f-h\|_\infty = \|f-g\|_\infty,$$

and that, by duality,

$$d = \sup \int f(e^{i\theta})\, F(e^{i\theta}) \frac{d\theta}{2\pi}, \quad F \in H^1_0(T), \quad \|F\|_1 = 1. \qquad (*)$$

There is also (at least) one F for which the sup $(*)$ is attained. f, g and any of those maximizing F are connected by

$$f(e^{i\theta}) - g(e^{i\theta}) = d \cdot \frac{\overline{F(e^{i\theta})}}{|F(e^{i\theta})|} \quad \text{a.e. } (d\theta).$$

We agree to write $g = T(f)$.

Generally, differentiability properties of f are preserved by g; see **[1, 2]**.

It is also known (see [3] for more information) that $f \in \mathscr{F}\ell^1$ implies $g \in \mathscr{F}\ell^1$. In [3], the following question is raised (with expectation of a negative answer). Is it true that

$$\|g\|_{\mathscr{F}\ell^1} \leqslant c \cdot \|f\|_{\mathscr{F}\ell^1}$$

for some absolute constant c? The answer is negative indeed!

THEOREM. *There is no absolute constant c such that $\|g\|_{\mathscr{F}\ell^1} \leqslant c \cdot \|f\|_{\mathscr{F}\ell^1}$, where g is the best approximation of $f \in \mathscr{F}\ell^1$ in $H^\infty(T)$.*

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REMARK. In the same context, one should note that the operator of best approximation is bounded on the Besov classes $B_p^{1/p}$ and on VMO (see [4]), and unbounded on the Besov classes B_p^s with $s > 1/p$ and on the Hölder classes (see [3]).

Proof. We start with a simple observation. Suppose $f \in A(T)$, $\hat{f}(0) = 0$, $g \in H^\infty(T)$, $F \in H^1(T)$ and $\bar{f} + g = e^{-i\theta} \cdot \bar{F}/|F|$ a.e. in T. Then $-g = T(\bar{f})$ and $d(\bar{f}, H^\infty) = 1$.

The proof is trivial and uses (*).

Now consider any nice F in $H^1(T)$, with $\bar{F}/|F|$ having an absolutely convergent Fourier series. $F(z) = e^{iz}$ with $\bar{F}/|F| = e^{-i\cos\theta}$ is good.

Let $\bar{F}/|F| = \sum a_n e^{in\theta}$, $\sum |a_n| < +\infty$.

Let $R > 1$. We shall investigate the behaviour of the Fourier series of $\bar{G}/|G|$, where $G = (z-R)^2 \cdot F$, when $R \to 1+$:

$$\frac{\bar{G}}{|G|} = \sum a_n e^{in\theta} \cdot \left[-\frac{1}{R} e^{-i\theta} + \sum_0^\infty \left(1 - \frac{1}{R^2}\right) \frac{1}{R^n} e^{in\theta} \right]$$

$$= -\sum \frac{a_{n+1}}{R} e^{in\theta} + \sum a_n e^{in\theta} \cdot \sum_0^\infty \left(1 - \frac{1}{R^2}\right) \frac{1}{R^n} e^{in\theta}. \qquad (**)$$

The first series, if $R \to 1+$, is almost the series of $\bar{F}/|F|$ shifted one step to the left. Now we need the following.

LEMMA. *Suppose that* $h = \sum a_n e^{in\theta}$ *with* $\sum |a_n| < \infty, \sum a_n = 0$. *Let*

$$h \cdot \sum \beta_n(r) e^{in\theta} = \sum c_n(r) e^{in\theta},$$

where we assume

$$\sup_{0<r<1} \sum |\beta_n(r)| < \infty, \quad \lim_{r\to 1} \sum |\beta_n(r) - \beta_{n-1}(r)| = 0.$$

Then $\lim_{r\to 1} \sum |c_n(r)| = 0$.

The proof of the lemma is contained in the proof of Wiener's theorem (about absolutely convergent Fourier series) which is in [6, Chapter VI] and in [5].

The third series in (**) behaves like the $\sum \beta_n(r) e^{in\theta}$ of the Lemma, as a trivial calculation shows. We cannot immediately apply the Lemma to the second term of (**) since

$$A = \sum a_n = \frac{\bar{F}}{|F|}(0) \neq 0.$$

In fact, $|A| = 1$.

Write

$$\frac{\bar{G}}{|G|} = -\sum \frac{a_{n+1}}{R} e^{in\theta} + \left(\sum a_n e^{in\theta} - A\right) \cdot \sum_0^\infty \left(1 - \frac{1}{R^2}\right) \frac{1}{R^n} e^{in\theta}$$

$$+ A \sum_0^\infty \left(1 - \frac{1}{R^2}\right) \frac{1}{R^n} e^{in\theta}.$$

The $\mathscr{F}\ell^1$ norm of the middle term approaches 0 as $R \to 1$. Hence the series of $\bar{G}/|G|$, if R is close to 1, looks like

$$-\sum_{-\infty}^{-1} \frac{a_{n+1}}{R} e^{in\theta} - \sum_0^\infty \left\{ \frac{a_{n+1}}{R} - \left(1 - \frac{1}{R^2}\right) \frac{A}{R^n} \right\} e^{in\theta}.$$

In particular, any initial coefficients of the nonnegative (frequency) part are arbitrarily close to $-a_1, -a_2, \ldots$.

Obviously, $\lim_{R\to 1} \sum_{-\infty}^{-1} |a_{n+1}/R| = \sum_{-\infty}^{-1} |a_{n+1}|$.

I claim that

$$\lim_{R\to 1} \sum_{0}^{\infty} \left| \frac{a_{n+1}}{R} - \left(1 - \frac{1}{R^2}\right) \frac{A}{R^n} \right| = 2 + \sum_{0}^{\infty} |a_{n+1}|.$$

To prove the claim, we choose N so large that

$$\sum_{N+1}^{\infty} |a_{n+1}| < \varepsilon.$$

Then

$$\lim_{R\to 1} \sum_{0}^{N} \left| \frac{a_{n+1}}{R} - \left(1 - \frac{1}{R^2}\right) \frac{A}{R^n} \right| = \sum_{0}^{N} |a_{n+1}|.$$

Also,

$$\sum_{N+1}^{\infty} \left| \frac{a_{n+1}}{R} - \left(1 - \frac{1}{R^2}\right) \frac{A}{R^n} \right|$$

differs from

$$\sum_{N+1}^{\infty} \left(1 - \frac{1}{R^2}\right) \frac{|A|}{R^n} = \frac{1 + 1/R}{R^{N+1}}$$

by at most ε in absolute value. Thus

$$\limsup_{R\to 1} \left| \sum_{0}^{\infty} \left| \frac{a_{n+1}}{R} - \left(1 - \frac{1}{R^2}\right) \frac{A}{R^n} \right| - 2 - \sum_{0}^{\infty} |a_{n+1}| \right| \leqslant 2\varepsilon, \quad \text{for all } \varepsilon > 0.$$

We summarize. If R is close enough to 1, then:

(α) the new series is almost the old one shifted one step to the left;
(β) the new negative part has $\mathscr{F}\ell^1$ norm almost the $\mathscr{F}\ell^1$ norm of the old negative part plus the old $|a_0|$;
(γ) the new nonnegative part has $\mathscr{F}\ell^1$ norm almost the $\mathscr{F}\ell^1$ norm of the old nonnegative part minus $|a_0|$ plus 2.

Fixing M in advance and choosing $R_1, \ldots, R_M$ close to 1, we may perform the previous procedure M times, and the result on $\bar{F}/|F|$ will be to shift (with any degree of accuracy) $|a_0| + \ldots + |a_{M-1}|$ from the $\mathscr{F}\ell^1$ norm of the nonnegative part to the $\mathscr{F}\ell^1$ norm of the negative part, simultaneously adding $2M$ to the last norm.

Hence the $\mathscr{F}\ell^1$ norm of the nonnegative part will increase by $2M - |a_0| - \ldots - |a_{M-1}|$, while the norm of the negative part will increase by $|a_0| + \ldots + |a_{M-1}|$, a quantity which is bounded uniformly in M.

Now start with our initial $\bar{F}(e^{i\theta})/|F|$, and suppose that N is the $\mathscr{F}\ell^1$ norm of the negative part of its Fourier series, P is the norm of the positive part, and a_0 is the constant term.

Now if $R_1, \ldots, R_M$ are close enough to 1, then $\bar{G}_M/|G_M|$, where

$$G_M = (z - R_1)^2 \ldots (z - R_M)^2 F,$$

will have

$$N_M < N + |a_0| + P + 1,$$
$$P_M > 2M - P - 1.$$

Letting $M \to \infty$, we obtain $N_M/P_M \to 0$.

Writing now $\bar{f}_M + g_M = e^{-i\theta} \cdot \bar{G}_M/|G_M|$ and using the initial observation, we obtain

$$\|f_M\|_{\mathscr{F}\ell^1}/\|g_M\|_{\mathscr{F}\ell^1}$$

arbitrarily small.

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References

1. L. Carleson and S. Jacobs, 'Best uniform approximation by analytic functions', *Ark. Mat.* 10 (1972) 219–229.
2. J. Garnett, *Bounded analytic functions* (Academic Press, New York, 1981).
3. V. V. Peller, 'Boundedness properties of the operators of best approximation by analytic and meromorphic functions', *Ark. Mat.* 30 (1992) 331–343.
4. V. V. Peller and S. Khrushchev, 'Hankel operators, best approximations and stationary Gaussian processes', *Russian Math. Surveys* 37: 1 (1982) 61–144.
5. N. Wiener, *The Fourier integral and certain of its applications* (Dover, New York, 1958).
6. A. Zygmund, *Trigonometrical series* (Dover, New York, 1955).

Department of Mathematics
University of Crete
Iraklion
Crete
Greece