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Notes on Complex Analysis

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Contents

Chapter 1

The complex plane.

1.1 The complex plane.

We are familiar with the set $\mathbb{C}$ of all complex numbers

$$z = x + iy, \qquad x, y \in \mathbb{R},$$

which we add and multiply as follows:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

In particular:

$$i^2 = -1.$$

With these operations of addition and multiplication, $\mathbb{C}$ is an algebraic field and $\mathbb{R}$ is a subfield of $\mathbb{C}$. We shall prove later that, besides the polynomial equation $z^2 + 1$ which has as solutions the complex numbers $\pm i$, every polynomial equation with coefficients in $\mathbb{C}$ is solvable in $\mathbb{C}$. In other words, we shall prove that $\mathbb{C}$ is an *algebraically closed field*.

In the following we shall only review a few basic things and fix some terminology and notation.

We identify the complex number $z = x + iy$ with the pair (x, y) of $\mathbb{R}^2$ and write

$$z = x + iy = (x, y).$$

It is customary to use symbols like x, y, u, v, t, ξ, η for real numbers and symbols like z, w, ζ for complex numbers. For instance, we write $z = x + iy$, $w = u + iv$, $\zeta = \xi + i\eta$.

For every $z = (x, y) = x + iy$ we write

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y, \quad \overline{z} = (x, -y) = x - iy, \quad |z| = \sqrt{x^2 + y^2}$$

for the *real part*, the *imaginary part*, the *conjugate* and the *absolute value* (or *modulus*) of z, respectively.

The geometrical model for $\mathbb{C}$ is the cartesian plane with two perpendicular axes: every $z = (x, y) = x + iy$ corresponds to the point of the plane with abscissa x and ordinate y. The horizontal axis of all points $(x, 0)$ is the *real axis* and, through the identification $x = (x, 0)$, it represents $\mathbb{R}$ as a subset of $\mathbb{C}$. The vertical axis of all points $iy = (0, y)$ is the *imaginary axis*.

We recall that the cartesian equation of the general line in the plane is

$$ax + by = c,$$

where $a, b, c \in \mathbb{R}$, $a^2 + b^2 \neq 0$. If we set $z = x + iy$ and $w = a + ib \neq 0$, then the above equation takes the form

$$\operatorname{Re}(\overline{w}z) = c.$$

Similarly, the defining inequalities $ax + bc < c$ and $ax + bc > c$ of the two halfplanes on the two sides of the line with equation $ax+by = c$ become $\mathrm{Re}(\overline{w}z) < c$ and $\mathrm{Re}(\overline{w}z) > c$, respectively.

We shall denote
$$[z_1, z_2] = \{(1-t)z_1 + tz_2 \,|\, 0 \le t \le 1\}$$
the *linear segment* joining the points z_1, z_2.

When we say *interval* we mean a linear segment on the real line: $[a, b] \subseteq \mathbb{R}$.

The *euclidean distance* between the points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |z_2 - z_1|.$$
Therefore, the *circle*, the *open disc* and the *closed disc* with center $z = (x, y)$ and radius $r \ge 0$ are the sets
$$C_z(r) = \{w \,|\, |w - z| = r\}, \quad D_z(r) = \{w \,|\, |w - z| < r\}, \quad \overline{D}_z(r) = \{w \,|\, |w - z| \le r\}.$$
For the unit circle, the open unit disc and the closed unit disc with center 0 we have the special symbols:
$$\mathbb{T} = C_0(1), \qquad \mathbb{D} = D_0(1), \qquad \overline{\mathbb{D}} = \overline{D}_0(1).$$
The *real part* and the *imaginary part* of a complex function $f : A \to \mathbb{C}$ are the functions $u = \mathrm{Re}\, f : A \to \mathbb{R}$ and $v = \mathrm{Im}\, f : A \to \mathbb{R}$, respectively, defined by
$$u(z) = \mathrm{Re}\, f(z) = \frac{1}{2}(f(z) + \overline{f(z)}), \qquad v(z) = \mathrm{Im}\, f(z) = \frac{1}{2i}(f(z) - \overline{f(z)}).$$
Of course, we have
$$f(z) = u(z) + iv(z) = (u(z), v(z)), \qquad z \in A.$$
We attach one extra element (not a complex number) to $\mathbb{C}$, which we call **infinity** and denote ∞, and we form the set
$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$
The set $\widehat{\mathbb{C}}$ is called **extended** $\mathbb{C}$ or **extended complex plane**.

We extend in $\widehat{\mathbb{C}}$ the usual algebraic operations between complex numbers, as follows:
$$z + \infty = \infty + z = \infty, \quad -\infty = \infty, \quad z - \infty = \infty - z = \infty.$$
$$z\,\infty = \infty\, z = \infty \ \text{ if } z \ne 0, \quad \infty\,\infty = \infty, \quad \frac{1}{\infty} = 0, \quad \frac{1}{0} = \infty, \quad \frac{z}{\infty} = 0, \quad \frac{\infty}{z} = \infty.$$
$$\overline{\infty} = \infty, \quad |\infty| = +\infty.$$
The expressions
$$\infty + \infty, \quad \infty - \infty, \quad 0\,\infty, \quad \infty\, 0, \quad \frac{\infty}{\infty}, \quad \frac{0}{0}$$
are not defined and they are called **indeterminate forms**.

1.2 Argument and polar representation.

The trigonometric functions sin and cos are defined and their properties are studied in the theory of functions of a real variable. In particular, we know that sin and cos are periodic with smallest positive period 2π:
$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta.$$
We also know the following result. Let I be any interval of length 2π which contains only one of its endpoints, e.g. $[0, 2\pi)$ or $(-\pi, \pi]$. Then for every $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ there exists a

unique $\theta \in I$ so that $\cos\theta = a$ and $\sin\theta = b$. Equivalently, for every $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ there exists a unique $\theta \in I$ so that $\zeta = \cos\theta + i\sin\theta$.

Therefore, the function

$$\cos + i\sin : \mathbb{R} \to \mathbb{T}$$

is periodic with 2π as smallest positive period and its restriction

$$\cos + i\sin : I \to \mathbb{T}$$

to any interval I of length 2π which contains only one of its endpoints is one-to-one and onto $\mathbb{T}$. Thus, for every $\zeta \in \mathbb{T}$ the equation $\cos\theta + i\sin\theta = \zeta$ has infinitely many solutions in $\mathbb{R}$ and exactly one solution in each interval I of length 2π which contains only one of its endpoints. If θ is any of these solutions, then the set of all solutions is $\{\theta + k2\pi \,|\, k \in \mathbb{Z}\}$.

Definition. *For every $z \in \mathbb{C}$, $z \neq 0$, we have $\frac{z}{|z|} \in \mathbb{T}$ and then the set of all solutions of the equation* $\cos\theta + i\sin\theta = \frac{z}{|z|}$ *is called* **argument** *or* **angle** *of z and is denoted* $\arg z$*:*

$$\arg z = \left\{\theta \in \mathbb{R} \,\middle|\, \cos\theta + i\sin\theta = \tfrac{z}{|z|}\right\}.$$

The unique solution of this equation in the interval $(-\pi, \pi]$ is called **principal argument** *or* **principal angle** *of z and it is denoted* $\operatorname{Arg} z$*:*

$$\theta = \operatorname{Arg} z \quad \Leftrightarrow \quad \cos\theta + i\sin\theta = \tfrac{z}{|z|}, \ -\pi < \theta \leq \pi.$$

Each of the elements of $\arg z$ *is called* **a value of the argument** *of z.*

We must be careful: $\arg z$ is a set while $\operatorname{Arg} z$ is a number, one of the elements of $\arg z$. In fact

$$\arg z = \{\operatorname{Arg} z + k2\pi \,|\, k \in \mathbb{Z}\}.$$

Examples. (i) $\operatorname{Arg} 3 = 0$, $\arg 3 = \{k2\pi \,|\, k \in \mathbb{Z}\}$,

(ii) $\operatorname{Arg}(4i) = \frac{\pi}{2}$, $\arg(4i) = \left\{\frac{\pi}{2} + k2\pi \,|\, k \in \mathbb{Z}\right\}$,

(iii) $\operatorname{Arg}(-2) = \pi$, $\arg(-2) = \{\pi + k2\pi \,|\, k \in \mathbb{Z}\}$,

(iv) $\operatorname{Arg}(1+i) = \frac{\pi}{4}$, $\arg(1+i) = \left\{\frac{\pi}{4} + k2\pi \,|\, k \in \mathbb{Z}\right\}$,

(v) $\operatorname{Arg}(-1 - i\sqrt{3}) = \frac{4\pi}{3}$, $\arg(-1 - i\sqrt{3}) = \{\frac{4\pi}{3} + k2\pi \,|\, k \in \mathbb{Z}\}$.

It is obvious that the argument of any nonzero z is a (two-sided) arithmetical progression of step 2π. Therefore, it is also obvious that, if we have any nonzero z_1 and z_2, then their arguments are either identical sets or disjoint sets. Equivalently, either the arguments of z_1 and z_2 have exactly the same values or their arguments have no common value. More precisely, any nonzero z_1 and z_2 have the same argument if and only if each of them is a multiple of the other by a positive number or, equivalently, if and only if z_1 and z_2 belong to the same halfline with vertex 0. Moreover, if z_1 and z_2 belong to different halflines with vertex 0, their arguments have no common value.

The following identity is equivalent to the addition formulas of sin and cos:

$$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2).$$

A direct consequence by induction is the familiar *formula of de Moivre*:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

for all $n \in \mathbb{Z}$.

For any two nonempty subsets A and B of $\mathbb{C}$ we define

$$A + B = \{a + b \,|\, a \in A, b \in B\}, \quad AB = \{ab \,|\, a \in A, b \in B\}.$$

We also write

$$a + B = \{a + b \,|\, b \in B\}, \quad aB = \{ab \,|\, b \in B\}, \quad -B = \{-b \,|\, b \in B\}.$$

Proposition 1.1. *For every nonzero z_1 and z_2 we have*

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

Proof. Take any $\theta \in \arg z_1 + \arg z_2$. Then there are $\theta_1 \in \arg z_1$ and $\theta_2 \in \arg z_2$ so that $\theta = \theta_1 + \theta_2$. By the addition formulas,

$$\cos\theta + i\sin\theta = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \frac{z_1}{|z_1|}\frac{z_2}{|z_2|} = \frac{z_1 z_2}{|z_1 z_2|}.$$

Therefore, $\theta \in \arg(z_1 z_2)$.
Conversely, take any $\theta \in \arg(z_1 z_2)$. We consider any $\theta_1 \in \arg z_1$ and set $\theta_2 = \theta - \theta_1$. Then

$$\cos\theta_2 + i\sin\theta_2 = \frac{\cos\theta + i\sin\theta}{\cos\theta_1 + i\sin\theta_1} = \frac{z_1 z_2}{|z_1 z_2|}\Big/\frac{z_1}{|z_1|} = \frac{z_2}{|z_2|}.$$

Therefore, $\theta_2 \in \arg z_2$ and, thus, $\theta = \theta_1 + \theta_2 \in \arg z_1 + \arg z_2$. $\square$

We must stress that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ is an equality between sets. The similar equality between numbers, $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$, is not true in general.

Example 1.2.1. $\mathrm{Arg}(-1) + \mathrm{Arg}(-1) = \pi + \pi = 2\pi$, while $\mathrm{Arg}((-1)(-1)) = \mathrm{Arg}\, 1 = 0$.

The equalities $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ express the well-known rule: *when two complex numbers are multiplied, their absolute values are multiplied and their arguments (or angles) are added.*

Definition. *It is clear by now that for any nonzero z we may write*

$$z = r(\cos\theta + i\sin\theta), \qquad r = |z|,\ \theta \in \arg z.$$

This is called **a polar representation** *of z. There are infinitely many polar representations of z, one for each value θ of its argument. The polar representation with $\theta = \mathrm{Arg}\, z$ is called* **principal polar representation** *of z.*

Remark. We do not define argument or angle or polar representation for the number 0.

Exercises.

1.2.1. Which are all the possible values of $\mathrm{Arg}(z_1 z_2) - \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2$?

1.2.2. Prove that $\arg \frac{1}{z} = \arg \overline{z} = -\arg z$ and $\arg(-z) = \pi + \arg z$. (Note that these are equalities between sets.)

1.2.3. Prove the following statement for any nonzero z, z_1 and z_2. It is true that $z = z_1 z_2$ if and only if the triangle $T(0, 1, z_1)$ with vertices $0, 1, z_1$ is *similar* to the triangle $T(0, z_2, z)$ with vertices $0, z_2, z$ (0 corresponding to 0, 1 corresponding to z_2 and z_1 corresponding to z). This expresses the geometric visualization of the operation of multiplication in $\mathbb{C}$.

1.3 Sequences, neighborhoods, open sets, closed sets.

Definition. *We say that the sequence (z_n) in $\mathbb{C}$* **converges** *to $z \in \mathbb{C}$ if for every $\epsilon > 0$ there is n_0 so that $|z_n - z| < \epsilon$ for all $n \geq n_0$. We denote this by*

$$\lim_{n\to+\infty} z_n = z \qquad \text{or} \qquad z_n \to z \text{ when } n \to +\infty.$$

We say that the sequence (z_n) in $\mathbb{C}$ **diverges** *to ∞ if for every $R > 0$ there is n_0 so that $|z_n| > R$ for all $n \geq n_0$. We denote this by*

$$\lim_{n\to+\infty} z_n = \infty \qquad \text{or} \qquad z_n \to \infty \text{ when } n \to +\infty.$$

The definition of convergence is formally identical to the analogous definition for sequences in $\mathbb{R}$. We shall make a comment regarding the case of divergence to ∞ and, specifically, on the difference between the use of ∞ in the framework of complex analysis and the use of $\pm\infty$ in the framework of real analysis. The terms of a sequence (x_n) on the real line move unboundedly away from 0 in exactly two distinct directions: either to the right or to the left and then we say that $x_n \to +\infty$ or $x_n \to -\infty$, respectively. On the complex plane there aren't any two particularly prefered directions. The term z_n of a complex sequence can move away from 0 either on halflines (i.e. in infinitely many directions) or on spiral-like curves or in any other arbitrary manner. Therefore, we simply say that $z_n \to \infty$. We shall come back at this point when we comment on the equality $\frac{1}{0} = \infty$.

Relating the notions of convergence and divergence for complex sequences to the similar notions for real sequences, we observe that

$$z_n \to z \ \Leftrightarrow\ |z_n - z| \to 0, \qquad z_n \to \infty \ \Leftrightarrow\ |z_n| \to +\infty.$$

This is clear from the corresponding definitions.

Example 1.3.1. The sequence $((-2)^n)$ does not have a limit as a real sequence since its subsequences of the odd and the even indices have the different limits $-\infty$ and $+\infty$. But as a complex sequence $((-2)^n)$ tends to ∞, because $|(-2)^n| = 2^n \to +\infty$.

Proposition 1.2. *Let $z_n \to z$ and $w_n \to w$, where $z, w \in \widehat{\mathbb{C}}$. Then, provided the result in each case is not an indeterminate form, we have*

$$z_n \pm w_n \to z \pm w, \quad z_n w_n \to zw, \quad \frac{z_n}{w_n} \to \frac{z}{w}, \quad \overline{z_n} \to \overline{z}, \quad |z_n| \to |z|.$$

Moreover, if $z_n = x_n + iy_n$ and $z = x + iy$, then

$$z_n \to z \quad \Leftrightarrow \quad x_n \to x,\ y_n \to y.$$

Proof. The proofs of the first three properties are identical to the proofs of the analogous properties for real sequences. For the fourth and fifth properties we write

$$|\overline{z_n} - \overline{z}| = |z_n - z| \to 0, \qquad ||z_n| - |z|| \le |z_n - z| \to 0.$$

Moreover, from

$$|z_n - z| \le |x_n - x| + |y_n - y|, \qquad |x_n - x| \le |z_n - z|, \quad |y_n - y| \le |z_n - z|$$

we get the last equivalence. □

Let us comment on the equality $\frac{1}{0} = \infty$. In $\mathbb{R}$ the expression $\frac{1}{0}$ is an indeterminate form, since $\frac{1}{x_n} \to +\infty$ when $x_n \to 0+$ and $\frac{1}{x_n} \to -\infty$ when $x_n \to 0-$. But in $\mathbb{C}$ signs do not play the same role as in $\mathbb{R}$. In $\mathbb{C}$ only the absolute value of $\frac{1}{z_n}$ is significant and we see that, when $z_n \to 0$, then $|\frac{1}{z_n}| = \frac{1}{|z_n|} \to +\infty$ and hence $\frac{1}{z_n} \to \infty$.

Example 1.3.2. Let us consider the geometric progression (z^n).
If $|z| < 1$, then $|z^n - 0| = |z|^n \to 0$ and hence $z^n \to 0$.
If $|z| > 1$, then $|z^n| = |z|^n \to +\infty$ and hence $z^n \to \infty$.
If $z = 1$, then $z^n = 1 \to 1$.
Finally, let $|z| = 1$, $z \ne 1$ and assume that $z^n \to w$. Since $|z^n| = |z|^n = 1$ for every n, we find that $|w| = 1$. From $z^n \to w$ we have $z = \frac{z^{n+1}}{z^n} \to \frac{w}{w} = 1$ and we arrive at a contradiction.
Thus:

$$z^n \begin{cases} \to 0, & \text{if } |z| < 1 \\ \to 1, & \text{if } z = 1 \\ \to \infty, & \text{if } |z| > 1 \\ \text{has no limit}, & \text{if } |z| = 1, z \ne 1 \end{cases}$$

The open disc $D_z(r)$ is also called r-**neighborhood** of z. It is useful to have a similar notation to take care of points which are "close" to ∞. We say that the set

$$D_\infty(r) = \{w \,|\, |w| > 1/r\} \cup \{\infty\}$$

is an r-neighborhhod of ∞.

Proposition 1.3. *$z \in \widehat{\mathbb{C}}$ is the limit of a sequence (z_n) if and only if every neighborhood $D_z(\epsilon)$ of z contains all terms of the sequence after some index.*

Proof. Trivial. □

Definition. *Let $A \subseteq \mathbb{C}$ and $z \in \mathbb{C}$.*
We say that z is an **interior point** *of A if some neighborhood of z is contained in A.*
We say that z is a **boundary point** *of A if every neighborhood of z intersects both A and A^c.*
We say that z is a **limit point** *of A if every neighborhood of z intersects A.*
We say that z is an **accumulation point** *of A if every neighborhood of z intersects A at a point different from z.*

Definition. *Let $A \subseteq \mathbb{C}$. We define*

$$\begin{aligned} A^\circ &= \{z \in \mathbb{C} \,|\, z \text{ is an interior point of } A\}, \\ \partial A &= \{z \in \mathbb{C} \,|\, z \text{ is a boundary point of } A\}, \\ \overline{A} &= \{z \in \mathbb{C} \,|\, z \text{ is a limit point of } A\}. \end{aligned}$$

The sets A°, ∂A and $\overline{A}$ are called **interior**, **boundary** *and* **closure** *of A, respectively.*

Here is a comment regarding ∞. We say that ∞ is a limit point of a set A if every neighborhood of ∞ intersects A. If we look at the exact shape of the neighborhoods of ∞, we realize that ∞ is a limit point of A if and only if there are points of A arbitrarily far away from 0, i.e. if and only if A is an unbounded set. Also, since any neighborhood of ∞ can intersect A only at points different from ∞ (since $A \subseteq \mathbb{C}$), we realize that ∞ is a limit point of A if and only if it is an accumulation point of A. Moreover, we may accept that every neighborhood of ∞ intersects A^c since it (the neighborhood) contains ∞. After these thoughts we conclude that *(i) an unbounded set has ∞ as a limit point, as an accumulation point and as a boundary point (ii) a bounded set does not have ∞ either as a limit point or as an accumulation point or as a boundary point*. Nevertheless, when we talk about limit points, boundary points, accumulation points of a set A we consider only complex numbers and when we write $\overline{A}$, ∂A we do not include ∞ in these sets even if the set A is unbounded. If in some particular statement we want to consider ∞ as a limit point or accumulation point or boundary point of a particular set A, then we have to state this clearly.

If $A \subseteq \mathbb{C}$, the complement of A with respect to $\mathbb{C}$ is denoted A^c.

Proposition 1.4. *Let $A \subseteq \mathbb{C}$. Then*
(i) $\partial A = \partial(A^c)$.
(ii) $A^\circ \subseteq A \subseteq \overline{A}$.
(iii) $\overline{A} \setminus A^\circ = \partial A$.
(iv) $A^\circ = A \setminus \partial A$.
(v) $\overline{A} = A \cup \partial A$.

Proof. (i) From the definition of a boundary point it is clear that the boundary points of A are the same as the boundary points of A^c. In other words, the sets ∂A and $\partial(A^c)$ have the same elements.
(ii) If $z \in A^\circ$, then there is a neighborhood of z which is contained in A and hence $z \in A$ (since z is the center of its neighborhood). Therefore, $A^\circ \subseteq A$.
If $z \in A$, then every neighborhood of z intersects A and hence $z \in \overline{A}$. Therefore, $A \subseteq \overline{A}$.
(iii) Let $z \in \overline{A} \setminus A^\circ$. Since $z \in \overline{A}$, every neighborhood of z interects A. Since $z \notin A^\circ$, there is

no neighborhood of z which is contained in A and hence every neighborhood of z intersects A^c. Therefore, $z \in \partial A$.
Conversely, let $z \in \partial A$. Then every neighborhood of z intersects A and hence $z \in \overline{A}$. Also every neighborhood of z intersects A^c which means that there is no neighborhood of z which is contained in A and hence $z \notin A^\circ$.
(iv) and (v) are straightforward corollaries of (ii) and (iii). □

Part (iv) of proposition 1.4 says that A° *results from* A *when we take away from it the boundary points of* A *which belong to* A. Also, (v) says that $\overline{A}$ *results from* A *when we attach to it the boundary points of* A *which do not belong to* A. In other words, the set $A \setminus A^\circ$ consists of the boundary points of A which belong to A and the set $\overline{A} \setminus A$ consists of the boundary points of A which do not belong to A.

Example 1.3.3. We consider a relatively simple curve C which divides the plane in three subsets: the set A_1 of the points on one side of C, the set A_2 of points on the other side of C and the set of points of C. For instance C can be a circle or an ellipse or a line or a closed polygonal line (the circumference of a rectangle, for instance). Just looking at these shapes on the plane, we understand that $A_1^\circ = A_1$, $\partial A_1 = C$ and $\overline{A_1} = A_1 \cup C$. We have analogous results for A_2 and also $C^\circ = \emptyset$, $\partial C = C$ and $\overline{C} = C$.

Proposition 1.5. *Let* $A \subseteq \mathbb{C}$. *Then* z *is a limit point of* A *or, equivalently,* $z \in \overline{A}$ *if and only if there is a sequence* (z_n) *in* A *so that* $z_n \to z$.

Proof. Let $z \in \overline{A}$. Then every neighborhhod of z intersects A and hence for every $n \in \mathbb{N}$ there is some $z_n \in D_z(\frac{1}{n}) \cap A$. Then the sequence (z_n) is in A and also $|z_n - z| < \frac{1}{n} \to 0$.
Conversely, if (z_n) is in A and $z_n \to z$, then for every $\epsilon > 0$ the neighborhood $D_z(\epsilon)$ contains all terms of (z_n) after some index. Thus, every neighborhood of z intersects A and hence $z \in \overline{A}$. □

Definition. *Let* $A \subseteq \mathbb{C}$.
We say that A *is* **open** *if it consists only of its interior points.*
We say that A *is* **closed** *if it contains all its limit points.*

In other words, A is open if and only if $A = A^\circ$, and A is closed if and only if $A = \overline{A}$

Proposition 1.6. *Let* $A \subseteq \mathbb{C}$.
(i) A *is open if and only if it contains none of its boundary points.*
(ii) A *is closed if it contains all its boundary points.*

Proof. (i) Immediate from (iv) of proposition 1.4.
(ii) Immediate from (v) of proposition 1.4. □

Example 1.3.4. In example 1.3.3 the sets A_1, A_2 are open and the sets $A_1 \cup C$, $A_2 \cup C$ and C are closed.

Proposition 1.7. *Let* $A \subseteq \mathbb{C}$. *Then* A° *is open and* $\overline{A}, \partial A$ *are closed.*

Proof. Let $z \in A^\circ$, i.e. there is $r > 0$ so that $D_z(r) \subseteq A$. Now we take any $w \in D_z(r)$. It is geometrically clear that there is some $s > 0$ so that $D_w(s) \subseteq D_z(r)$ and hence $D_w(s) \subseteq A$. Therefore w is an interior point of A, i.e. $w \in A^\circ$. We proved that $D_z(r) \subseteq A^\circ$ and hence z is an interior point of A°. Therefore, every point of A° is an interior point of A° and A° is open.
Now let z be a limit point of $\overline{A}$. We take any $r > 0$ and then $D_z(r)$ intersects $\overline{A}$. We consider any $w \in D_z(r) \cap \overline{A}$. Again, there is some $s > 0$ so that $D_w(s) \subseteq D_z(r)$. Since $w \in \overline{A}$, $D_w(s)$ intersects A and hence $D_z(r)$ also intersects A. Therefore, every neighborhood of z intersects A and hence z is a limit point of A, i.e. $z \in \overline{A}$. We proved that every limit point of $\overline{A}$ belongs to $\overline{A}$ and $\overline{A}$ is closed.
Finally, let z be a limit point of ∂A. We take any $r > 0$ and then $D_z(r)$ intersects ∂A. We consider

any $w \in D_z(r) \cap \partial A$. Again, there is some $s > 0$ so that $D_w(s) \subseteq D_z(r)$. Since $w \in \partial A$, $D_w(s)$ intersects both A and A^c and hence $D_z(r)$ also intersects both A and A^c. Thus every neighborhood of z intersects both A and A^c and hence z is a boundary point of A, i.e. $z \in \partial A$. We proved that every limit point of ∂A belongs to ∂A and ∂A is closed. □

Proposition 1.8. *Let $A \subseteq \mathbb{C}$. Then A is closed if and only if it contains the limit of every convergent sequence in A.*

Proof. Let A be closed. If (z_n) is in A and $z_n \to z$, then $z \in \overline{A}$ (proposition 1.5) and hence $z \in A$. Conversely, assume that A contains the limit of every convergent sequence in A. If $z \in \overline{A}$, there is a sequence (z_n) in A so that $z_n \to z$ and from our assumption we get that $z \in A$. Therefore, A is closed. □

Proposition 1.9. *Let $A \subseteq \mathbb{C}$. Then A is closed if and only if A^c is open.*

Proof. Based on proposition 1.6 and since A and A^c have the same boundary points, we have the following successive equivalent statements: A is closed if and only if A contains all boundary points of A if and only if A contains all boundary points of A^c if and only if A^c contains no boundary point of A^c if and only if A^c is closed. □

The complement of the complement of a set is the set itself and hence: *A is open if and only if A^c is closed.*

Exercises.

1.3.1. Prove that the limit in $\widehat{\mathbb{C}}$ of every sequence is unique.

1.3.2. Prove formally, using neighborhoods, that open discs are open and that closed discs and circles are closed.

1.3.3. Prove that $\{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is not a closed set, while $\{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$ is a closed set.

1.3.4. Prove formally, using sequences, that closed discs, circles, lines and closed halfplanes are closed sets.

1.3.5. Is the open segment (a, b) an open set?

1.3.6. Prove that both $\mathbb{C}$ and $\emptyset$ are open and closed.

1.3.7. Prove that $\overline{A}$ is the smallest closed set which contains A, and that A° is the largest open set which is contained in A.

1.3.8. Prove that the union of any open sets is open, that the intersection of any closed sets is closed, that the intersection of finitely many open sets is open, and that the union of finitely many closed sets is closed.

1.3.9. We define the diameter of A to be $\operatorname{diam} A = \sup\{|z - w| \,|\, z, w \in A\}$. Prove that $\operatorname{diam} A = \operatorname{diam} \overline{A}$.

1.3.10. We define the distance of z from A to be $d(z, A) = \inf\{|z - w| \,|\, w \in A\}$. Prove that
(i) $d(z, A) = d(z, \overline{A})$.
(ii) $d(z, A) = 0 \;\Leftrightarrow\; z \in \overline{A}$.
(iii) $|d(z, A) - d(w, A)| \leq |z - w|$.

1.3.11. Let A, B be closed and disjoint. Prove that there are U, V open and disjoint so that $A \subseteq U$ and $B \subseteq V$.

1.4 Limits and continuity of functions.

Definition. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, $z_0 \in \widehat{\mathbb{C}}$ be an accumulation point of A. We say that $w_0 \in \widehat{\mathbb{C}}$ is a* **limit** *of f at z_0, and denote*

$$\lim_{z \to z_0} f(z) = w_0,$$

if for every $\epsilon > 0$ there is $\delta > 0$ so that $f(z) \in D_{w_0}(\epsilon)$ for every $z \in D_{z_0}(\delta) \cap A$, $z \neq z_0$.

There are four cases, depending on whether z_0, w_0 are complex numbers or ∞ and we have corresponding formulations of the above definition of limit:
(i) $z_0, w_0 \in \mathbb{C}$. Then $\lim_{z \to z_0} f(z) = w_0$ if for every $\epsilon > 0$ there is $\delta > 0$ so that $|f(z) - w_0| < \epsilon$ for every $z \in A$ with $0 < |z - z_0| < \delta$.
(ii) $z_0 \in \mathbb{C}, w_0 = \infty$. Then $\lim_{z \to z_0} f(z) = \infty$ if for every $R > 0$ there is $\delta > 0$ so that $|f(z)| > R$ for every $z \in A$ with $0 < |z - z_0| < \delta$.
(iii) $z_0 = \infty$, $w_0 \in \mathbb{C}$. Then $\lim_{z \to \infty} f(z) = w_0$ if for every $\epsilon > 0$ there is $r > 0$ so that $|f(z) - w_0| < \epsilon$ for every $z \in A$ with $|z| > r$.
(iv) $z_0 = w_0 = \infty$. Then $\lim_{z \to \infty} f(z) = \infty$ if for every $R > 0$ there is $r > 0$ so that $|f(z)| > R$ for every $z \in A$ with $|z| > r$.

Definition. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ and $z_0 \in A$. We say that f is* **continuous** *at z_0 if for every $\epsilon > 0$ there is $\delta > 0$ so that $f(z) \in D_{f(z_0)}(\epsilon)$ for every $z \in D_{z_0}(\delta) \cap A$ or, equivalently, if for every $\epsilon > 0$ there is $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for every $z \in A$ with $|z - z_0| < \delta$.*

If $z_0 \in A$ is not an accumulation point of A (i.e. it is an **isolated point** of A), then we may easily see that f is automatically continuous at z_0. On the other hand, if $z_0 \in A$ is an accumulation point of A, then f is continuous at z_0 if and only if $\lim_{z \to z_0} f(x) = f(z_0)$.

Definition. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$. We say that f is* **continuous** *in A if it is continuous at every point of A.*

Proposition 1.10. *Let $A, B \subseteq \mathbb{C}$, $z_0 \in A$, $f : A \to B$ and $g : B \to \mathbb{C}$. If f is continuous at z_0 and g is continuous at $w_0 = f(z_0)$, then $g \circ f : A \to \mathbb{C}$ is continuous at z_0.*

Proof. The proof is exactly the same as the proof of the analogous result for real functions of a real variable. ☐

All simple algebraic properties of limits and of continuity which hold for real functions of a real variable also hold for complex functions of a complex variable. (Look back at proposition 1.2 for the case of sequences.) For instance, the limit of the sum is the sum of the limits (except in the case of an indeterminate form). We do not bother to repeat the formal arguments. The proofs are identical with the proofs in the real case.

Nevertheless, we mention the two results which restate the notions of limit and continuity of a function in terms of sequences. Again, the proofs are identical with the proofs in the real case and we omit them.

Proposition 1.11. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$, $z_0 \in \widehat{\mathbb{C}}$ be an accumulation point of A and $w_0 \in \widehat{\mathbb{C}}$. The following are equivalent.*
(i) $\lim_{z \to z_0} f(z) = w_0$.
(ii) For every (z_n) in $A \setminus \{z_0\}$ with $z_n \to z_0$ we have $f(z_n) \to w_0$.

Proposition 1.12. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ and $z_0 \in A$. The following are equivalent.*
(i) f is continuous at z_0.
(ii) For every (z_n) in A with $z_n \to z_0$ we have $f(z_n) \to f(z_0)$.

Example 1.4.1. Let us consider any polynomial function

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. The domain of definition of p is $\mathbb{C}$.
For every $z_0 \in \mathbb{C}$ we have

$$\lim_{z \to z_0} p(z) = p(z_0).$$

To prove it we use the well-known algebraic rules of limits and the trivial limits: $\lim_{z \to z_0} c = c$ and $\lim_{z \to z_0} z = z_0$.
Therefore, p is continuous in $\mathbb{C}$.
If the degree of p is ≥ 1, i.e. $n \geq 1$ and $a_n \neq 0$, then

$$\lim_{z \to \infty} p(z) = \infty$$

since $p(z) = z^n (a_n + a_{n-1} \frac{1}{z} + \cdots + a_0 \frac{1}{z^n}) \to \infty \, a_n = \infty$.
If the degree of p is 0, then the function is constant: $p(z) = a_0$ for all z. Hence

$$\lim_{z \to \infty} p(z) = a_0.$$

Example 1.4.2. Now we consider a rational function

$$r(z) = \frac{p(z)}{q(z)} = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0},$$

where $a_0, \ldots, a_n, b_0, \ldots, b_m \in \mathbb{C}$ and $a_n \neq 0$ and $b_m \neq 0$. The domain of definition of r is $\mathbb{C} \setminus \{z_1, \ldots, z_s\}$, where $z_1, \ldots, z_s$ are the roots of the polynomial q. We know that $0 \leq s \leq m$.
If $z_0 \in \mathbb{C}$ and $q(z_0) \neq 0$, then using the algebraic rules of limits, we get:

$$\lim_{z \to z_0} r(z) = r(z_0).$$

Therefore r is continuous in its domain of definition.
Writing r in the form $r(z) = z^{n-m} (a_n + a_{n-1} \frac{1}{z} + \cdots + a_0 \frac{1}{z^n}) / \left(b_m + b_{m-1} \frac{1}{z} + \cdots + b_0 \frac{1}{z^m}\right)$, we can prove that

$$\lim_{z \to \infty} r(z) = \begin{cases} \infty, & \text{if } n > m \\ \frac{a_n}{b_n}, & \text{if } n = m \\ 0, & \text{if } n < m \end{cases}$$

Finally, let $z_0 \in \mathbb{C}$ and $q(z_0) = 0$. Thus z_0 is any of the roots $z_1, \ldots, z_s$ of q. Then $z - z_0$ divides $q(z)$, and there is $k \geq 1$ and a polynomial $q_1(z)$ so that $q(z) = (z - z_0)^k q_1(z)$ and $q_1(z_0) \neq 0$. This means that the *multiplicity* of the root z_0 of $q(z)$ is k. There is also $l \geq 0$ and a polynomial $p_1(z)$ so that $p(z) = (z - z_0)^l p_1(z)$ and $p_1(z_0) \neq 0$. Indeed, if $p(z_0) = 0$, then $l \geq 1$ is the multiplicity of z_0 as a root of $p(z)$ and, if $p(z_0) \neq 0$, we take $l = 0$ (and say that the multiplicity of z_0 as a root of $p(z)$ is zero) and $p_1(z) = p(z)$. Thus for every z different from the roots of $q(z)$ we have

$$r(z) = (z - z_0)^{l-k} \frac{p_1(z)}{q_1(z)} \qquad \text{and} \qquad p_1(z_0) \neq 0, \; q_1(z_0) \neq 0.$$

Now $\frac{p_1(z_0)}{q_1(z_0)}$ is neither ∞ nor 0, and hence

$$\lim_{z \to z_0} r(z) = \begin{cases} \infty, & \text{if } k > l \\ \frac{p_1(z_0)}{q_1(z_0)}, & \text{if } k = l \\ 0, & \text{if } k < l \end{cases}$$

Definition. *Let $A \subseteq \mathbb{C}$ and $f : A \cup \{\infty\} \to \mathbb{C}$. We say that f is continuous at ∞ if for every $\epsilon > 0$ there is $R > 0$ so that $|f(z) - f(\infty)| < \epsilon$ for all $z \in A$ with $|z| > R$.*

Therefore, if A is unbounded, i.e. if ∞ is an accumulation point of A, then f is continuous at ∞ if and only if $\lim_{z\to\infty} f(z) = f(\infty)$. If A is bounded, then it is easy to see that f is automatically continuous at ∞. It has to be stressed that for f to be continuous at ∞ it is necessary that its value $f(\infty)$ be a complex number.

Example 1.4.3. If p is a polynomial as in example 1.4.1, then p is continuous at ∞ only if it is a constant polynomial $p(z) = a_0$ and provided we define its value at ∞ to be $p(\infty) = a_0$.
Similarly, if r is a rational function as in example 1.4.2, then r is continuous at ∞ only if $n \leq m$ and provided we define $r(\infty) = \frac{a_n}{b_n}$ or $r(\infty) = 0$ depending on whether $n = m$ or $n < m$, respectively.

Exercises.

1.4.1. Prove that the limit of a function is unique.

1.4.2. Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$. Prove that the following are equivalent.
(i) f is continuous in A.
(ii) For every open set W there is an open set U so that $f^{-1}(W) = U \cap A$.
(iii) For every closed set F there is a closed set G so that $f^{-1}(F) = G \cap A$.

1.5 Compactness.

Definition. *We say that $M \subseteq \mathbb{C}$ is* **compact** *if every sequence in M has at least one subsequence converging to a point of M.*

Example 1.5.1. Take $M = (a, b]$ and the sequence $z_n = a + \frac{b-a}{n}$. Since $z_n \to a$, every subsequence of (z_n) converges to a. Hence (z_n) is contained in M but has no subsequence converging to a point of M. Therefore, M is not a compact set.

Example 1.5.2. Take $M = \{z \,|\, |z| \geq 1\}$ and the sequence $z_n = 2^n$. Since $z_n \to \infty$, every subsequence of (z_n) diverges to ∞. Thus (z_n) is in M but has no subsequence converging to an element of M. Therefore, M is not compact.

In general, to prove that a set M *is not* compact is a relatively easy problem: it is enough to find *a specific* sequence in M which has no subsequence converging to a point of M. But to prove that a set M *is* compact is usually a harder problem: we have to take the *general* sequence in M and prove that it has a subsequence converging to an element of M.

Example 1.5.3. Let $M \subseteq \mathbb{C}$ be finite, i.e. $M = \{w_1, \ldots, w_m\}$.
We consider an arbitrary sequence (z_n) in M. Then at least one of the elements of M appears infinitely often as a term of the sequence. I.e. there is a subsequence (z_{n_k}) of (z_n) with all its terms equal to the same w_j. This subsequence is constant $z_{n_k} = w_j$ and hence converges to w_j.
Thus, every sequence in M has at least one subsequence converging to an element of M and M is compact.

Proposition 1.13. *If $M \subseteq \mathbb{C}$ is compact, then it is bounded and closed.*

Proof. Assume that M is not bounded. Then for every $n \in \mathbb{N}$ there is $z_n \in M$ so that $|z_n| \geq n$. Then the sequence (z_n) is in M and, since M is compact, there is a subsequence (z_{n_k}) of (z_n) so that $z_{n_k} \to z$ for some $z \in M$. This implies $|z_{n_k}| \to |z|$. But $|z_{n_k}| \geq n_k$ for every k and hence $|z_{n_k}| \to +\infty$. We arrive at a contradiction and we conclude that M is bounded.
Now, take any sequence (z_n) in M so that $z_n \to z$. Since M is compact, there is a subsequence (z_{n_k}) of (z_n) so that $z_{n_k} \to z'$ for some $z' \in M$. From $z_n \to z$ we get $z_{n_k} \to z$ and, due to the uniqueness of limit, we get $z' = z$. Thus $z \in M$. Therefore, the limit of every convergent sequence in M belongs to M and M is closed. $\square$

Proposition 1.14. *Let $N \subseteq M \subseteq \mathbb{C}$. If M is compact and N is closed, then N is compact.*

Proof. Take any sequence (z_n) in N. Then (z_n) is in M and, since M is compact, there is a subsequence (z_{n_k}) of (z_n) so that $z_{n_k} \to z$ for some $z \in M$. Since (z_{n_k}) is in N and N is closed, we have $z \in N$. Therefore, every sequence in N has a subsequence converging to an element of N and N is compact. $\square$

Proposition 1.15 says that if two sets, one of them compact and the other closed, are disjoint, then there is a *positive* distance between them.

Proposition 1.15. *Let $M, N \subseteq \mathbb{C}$ with $M \cap N = \emptyset$. If M is compact and N is closed, there is $\epsilon > 0$ so that $|z - w| \geq \epsilon$ for every $z \in M$ and $w \in N$.*

Proof. Assume that there is no $\epsilon > 0$ so that $|z - w| \geq \epsilon$ for every $z \in M$ and $w \in N$. Then for every $n \in \mathbb{N}$ there are $z_n \in M$ and $w_n \in N$ so that $|z_n - w_n| < \frac{1}{n}$. Since M is compact there is a subsequence (z_{n_k}) of (z_n) so that $z_{n_k} \to z$ for some $z \in M$. From

$$|w_{n_k} - z| \leq |z_{n_k} - w_{n_k}| + |z_{n_k} - z| < \frac{1}{n_k} + |z_{n_k} - z| \to 0$$

we get $w_{n_k} \to z$. Since (w_{n_k}) is in N and N is closed, we find $z \in N$. This is impossible, because $M \cap N = \emptyset$, and we arrive at a contradiction. $\square$

Proposition 1.16 is a generalization of the well known result for sequences of nested closed and bounded intervals in $\mathbb{R}$: *if $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$, there is x which belongs to all $[a_n, b_n]$ and, if moreover $b_n - a_n \to 0$, then x is unique.*

Definition. *We define the* **diameter** *of $M \subseteq \mathbb{C}$ to be*

$$\operatorname{diam} M = \sup\{|z - w| \mid z, w \in M\}.$$

Proposition 1.16. *Let (K_n) be a sequence of non-empty compact sets in $\mathbb{C}$ so that $K_{n+1} \subseteq K_n$ for every n. Then there is some point which belongs to all K_n. If moreover* diam $K_n \to 0$*, then the common element of K_n is unique.*

Proof. For each n we take any $z_n \in K_n$. Since K_1 is compact and the sequence (z_n) is in K_1, there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some $z \in K_1$. We observe that, for each m, (z_n) is in K_m after the value of the index $n = m$. Thus (z_{n_k}) is, after some value of the index k, in K_m. Since K_m is closed, we get $z \in K_m$. Therefore, z is in every K_m.
Now, let diam $K_n \to 0$. If z, w belong to every K_n, then $0 \leq |z - w| \leq$ diam K_n for every n. This implies $|z - w| = 0$ and hence $z = w$. $\square$

Bolzano-Weierstrass theorem. *Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Proof. Let (z_n) be a bounded sequence with $z_n = x_n + iy_n$. Then (z_n) is contained in some rectangle $M = [a, b] \times [c, d]$.
Taking the midpoints $\frac{a+b}{2}$ and $\frac{c+d}{2}$ of $[a, b]$ and $[c, d]$, we can split M in four equal subrectangles. The size of each of them is $\frac{1}{2}$ of the size of M. Since (z_n) is contained in M, at least one of the four subrectangles contains *infinitely many* terms of (z_n). We take one of the subrectangles with this property and denote it $M_1 = [a_1, b_1] \times [c_1, d_1]$. We repeat with the rectangle M_1. We split it in four equal subrectangles with size equal to $\frac{1}{2}$ of the size of M_1 and denote $M_2 = [a_2, b_2] \times [c_2, d_2]$ whichever of these subrectangles contains infinitely many terms of (z_n). Continuing inductively, we produce a sequence of rectangles $M_l = [a_l, b_l] \times [c_l, d_l]$ with the following properties:
(i) every M_l contains infinitely many terms of (z_n).
(ii) $a_{l-1} \leq a_l \leq b_l \leq b_{l-1}$ and $c_{l-1} \leq c_l \leq d_l \leq d_{l-1}$ for every l.
(iii) $b_l - a_l = \frac{b-a}{2^l} \to 0$ and $d_l - c_l = \frac{d-c}{2^l} \to 0$.

Since M_1 contains infinitely many terms of (z_n) there is a $z_{n_1} \in M_1$. Since M_2 contains infinitely many terms of (z_n) there is a $z_{n_2} \in M_2$ with $n_2 > n_1$. Since M_3 contains infinitely many terms of (z_n) there is a $z_{n_3} \in M_3$ with $n_3 > n_2$. Continuing inductively, we get a subsequence (z_{n_l}) of (z_n) so that $z_{n_l} \in M_l$ for every $l \geq 1$. I.e.

$$a_l \leq x_{n_l} \leq b_l,\ c_l \leq y_{n_l} \leq d_l \qquad \text{for every } l. \tag{1.1}$$

From (ii) we get that (a_l) is increasing and bounded and that (b_l) is decreasing and bounded and hence the two sequences converge to two limits, which, due to (iii), coincide. The same is true for the sequences (c_l) and (d_l). We set

$$x = \lim_{l\to+\infty} a_l = \lim_{l\to+\infty} b_l, \qquad y = \lim_{l\to+\infty} c_l = \lim_{l\to+\infty} d_l.$$

From (1.1) we get $x_{n_l} \to x$ and $y_{n_l} \to y$ and hence $z_{n_l} \to z = x + iy$. □

Definition. *We say that the sequence (z_n) is a* **Cauchy sequence** *if for every $\epsilon > 0$ there is n_0 so that $|z_n - z_m| < \epsilon$ for every $n, m \geq n_0$.*

Proposition 1.17. *Every Cauchy sequence converges.*

Proof. Let (z_n) be a Cauchy sequence. Then we easily see that (z_n) is bounded. Indeed, there is n_0 so that $|z_n - z_m| < 1$ for every $n, m \geq n_0$. This implies that $|z_n - z_{n_0}| < 1$ for every $n \geq n_0$ and hence $|z_n| \leq |z_{n_0}| + 1$ for every $n \geq n_0$. Therefore,

$$|z_n| \leq \max\{|z_1|, \ldots, |z_{n_0-1}|, |z_{n_0}| + 1\} \qquad \text{for every } n.$$

The Bolzano-Weierstrass theorem implies that there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some z. Now, we have that $|z_k - z_{n_k}| \to 0$, because (z_n) is a Cauchy sequence, and hence

$$|z_k - z| \leq |z_k - z_{n_k}| + |z_{n_k} - z| \to 0.$$

Therefore, $z_k \to z$. □

This property of $\mathbb{C}$, i.e. that every Cauchy sequence in $\mathbb{C}$ converges to some point of $\mathbb{C}$, is called **completeness** of $\mathbb{C}$.

Theorem 1.1 is very useful for the determination of compact sets. Theorem 1.1 complements proposition 1.13.

Theorem 1.1. *$M \subseteq \mathbb{C}$ is compact if and only if it is bounded and closed.*

Proof. Proposition 1.13 proves one direction.
Assume that M is bounded and closed. We take any sequence (z_n) in M. Since M is bounded, (z_n) is also bounded and the Bolzano-Weierstrass theorem implies that there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some z. Since (z_{n_k}) is in M and M is closed, we get $z \in M$.
Thus, every sequence in M has a subsequence converging to a point in M and M is compact. □

Example 1.5.4. All closed rectangles and all closed discs are compact.

Proposition 1.18. *Let $M \subseteq \mathbb{C}$ and $f : M \to \mathbb{C}$. If f is continuous in M and M is compact, then $f(M)$ is compact.*

Proof. Let (w_n) be an arbitrary sequence in $f(M)$. It is enough to prove that (w_n) has a subsequence converging to a point of $f(M)$.
For each n there is $z_n \in M$ so that $f(z_n) = w_n$. Then (z_n) is in M, and, since M is compact, there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some $z \in M$. Since f is continuous in M, we get $w_{n_k} = f(z_{n_k}) \to f(z) \in f(M)$. □

Proposition 1.19. *Every non-empty compact subset of $\mathbb{R}$ has a maximal and a minimal element.*

Proof. Let $M \subseteq \mathbb{R}$ be non-empty and compact. Since M is non-empty and bounded, $u = \sup M$ is in $\mathbb{R}$. Then for every $\epsilon > 0$ there is $x \in M$ so that $u - \epsilon < x \le u$ and hence $x \in D_u(\epsilon)$. Thus u is a limit point of M and, since M is closed, $u \in M$. Therefore u is the maximal element of M. The proof for the existence of a minimal element is similar. □

Theorem 1.2 generalizes the familiar analogous theorem for continuous $f : [a, b] \to \mathbb{R}$.

Theorem 1.2. *Let $M \subseteq \mathbb{C}$ and $f : M \to \mathbb{R}$. If f is continuous in M and M is compact, then f is bounded and has a maximum and a minimum value.*

Proof. Proposition 1.18 implies that $f(M) \subseteq \mathbb{R}$ is compact. Now proposition 1.19 says that $f(M)$ is bounded and has a maximal and a minimal element. □

Exercises.

1.5.1. Let $M_1, \ldots, M_n \subseteq \mathbb{C}$. If $M_1, \ldots, M_n$ are compact, prove that $M_1 \cup \cdots \cup M_n$ is compact.

1.5.2. Let $A, B \subseteq \mathbb{C}$. If A is compact and B is closed, prove that $A \cap B$ is compact.

1.5.3. Let $z_0 \in \mathbb{C}$, $M \subseteq \mathbb{C}$ be non-empty and closed and $N \subseteq \mathbb{C}$ be non-empty and compact.
Prove that there is $z_1 \in M$ so that $|z_1 - z_0| = \inf\{|z - z_0| \mid z \in M\}$.
Prove that there are $z_1 \in M$ and $w_1 \in N$ so that $|z_1 - w_1| = \inf\{|z - w| \mid z \in M, w \in N\}$.

1.5.4. Let $M \subseteq \mathbb{C}$ be bounded. Prove that $\overline{M}$ and ∂M are compact.

1.5.5. Let $M \subseteq \mathbb{C}$ and $f : M \to \mathbb{C}$. If f is continuous in M and M is compact, prove that f is uniformly continuous in M.

1.5.6. Let A be bounded and $f : A \to \mathbb{C}$ be continuous. Prove that there is a continuous $F : \overline{A} \to \mathbb{C}$ so that $F = f$ in A if and only if f is uniformly continuous in A.

1.5.7. Prove the following restatement of the Bolzano-Weierstrass theorem: every bounded infinite set has at least one accumulation point.

1.6 Connectedness.

Definition. *Let $A \subseteq \mathbb{C}$. We say that B, C form a* **decomposition** *of A if (i) $B \cup C = A$, (ii) $B \cap C = \emptyset$, (iii) $B \neq \emptyset$, $C \neq \emptyset$, (iv) none of B, C contains a limit point of the other.*

When (i), (ii), (iii) hold we say that B, C form a *partition* of A. We may restate (iv) as follows: $B \cap \overline{C} = \overline{B} \cap C = \emptyset$.

Example 1.6.1. We consider the closed discs $B = \overline{D}_0(1)$, $C = \overline{D}_3(1)$ and their union $A = B \cup C$. It is clear that B, C form a decomposition of A.
If we consider the open discs $B = D_0(1)$, $C = D_2(1)$ and $A = B \cup C$, then the discs B, C are tangent but, again, they form a decomposition of A.
If we take the closed disc $B = \overline{D}_0(1)$, the open disc $C = D_2(1)$ and $A = B \cup C$, then the discs B, C are tangent and they *do not* form a decomposition of A: B contains the limit point 1 of C.

Definition. *Let $A \subseteq \mathbb{C}$. We say that A is* **connected** *if there is no decomposition of A, i.e. there is no pair of sets B, C with the properties (i)-(iv) of the above definition.*

Example 1.6.2. The first two sets A of example 1.6.1 are not connected since each admits a specific decomposition. But we cannot decide at this moment if the third set A of example 1.6.1 is connected or not. We know that the specific B, C related to this A do not form a decomposition of A. *To decide that A is connected we must prove that, not only the specific pair, but an arbitrary pair does not form a decomposition of A.*

Example 1.6.3. It is obvious that $\emptyset$ as well as any $\{z\}$ is a connected set. These sets do not even have a partition, since for a set to have a partition it is necessary that it has at least two elements.

Lemma 1.1. *Let $A, B, C \subseteq \mathbb{C}$ with $B \cap C = \emptyset$ and assume that none of B, C contains a limit point of the other. If A is connected and $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.*

Proof. We define

$$B_1 = A \cap B, \qquad C_1 = A \cap C.$$

Clearly, $B_1 \cup C_1 = A$ and $B_1 \cap C_1 = \emptyset$.
Now let $z \in B_1$. Then $z \in B$, and hence z is not a limit point of C. Then there is $r > 0$ so that $D_z(r) \cap C = \emptyset$ and, since $C_1 \subseteq C$, we get $D_z(r) \cap C_1 = \emptyset$. Thus, z is not a limit point of C_1. We conclude that B_1 does not contain any limit point of C_1. Symmetrically, C_1 does not contain any limit point of B_1.
If $B_1 \neq \emptyset$ and $C_1 \neq \emptyset$, then B_1, C_1 form a decomposition of A and this contradicts the connectedness of A. Hence, either $B_1 = \emptyset$ or $C_1 = \emptyset$ and thus either $A \subseteq C$ or $A \subseteq B$, respectively □

Proposition 1.20. *Let Σ be a collection of connected subsets of $\mathbb{C}$ all of which have a common point. Then $\bigcup_{A \in \Sigma} A$ is connected.*

Proof. We set $U = \bigcup_{A \in \Sigma} A$ and we shall prove that U is connected.
Let z_0 be a common point of all $A \in \Sigma$.
We assume that U is not connected. Then there are B, C which form a decomposition of U. Since $z_0 \in U$, we have that $z_0 \in B$ or $z_0 \in C$. Assume that $z_0 \in B$ (the proof is the same if $z_0 \in C$). For every $A \in \Sigma$ we have $A \subseteq U$ and hence $A \subseteq B \cup C$. According to lemma 1.1, every $A \in \Sigma$ is contained either in B or in C. But if any $A \in \Sigma$ is contained in C, it cannot contain z_0 which is in B. Therefore every $A \in \Sigma$ is contained in B. Thus $U = \bigcup_{A \in \Sigma} A$ is contained in B, i.e. $U \subseteq B$ and we arrive at a contradiction since $C \neq \emptyset$.
Hence U is connected. □

Proposition 1.21. *Let $A \subseteq \mathbb{C}$. If A is connected, then $\overline{A}$ is connected.*

Proof. Let $\overline{A}$ not be connected. Then there are B, C which form a decomposition of $\overline{A}$.
Since $A \subseteq \overline{A}$, we have $A \subseteq B \cup C$. Lemma 1.1 implies that $A \subseteq B$ or $A \subseteq C$. Let $A \subseteq B$. (The proof is similar if $A \subseteq C$.)
Every point of $\overline{A}$ is a limit point of A and hence a limit point of B (since $A \subseteq B$). Therefore no point of $\overline{A}$ belongs to C (since C does not contain limit points of B). This is wrong since $C \neq \emptyset$.
Hence $\overline{A}$ is connected. □

Proposition 1.22. *Let $A \subseteq \mathbb{C}$ and let $f : A \to \mathbb{C}$ be continuous in A. If A is connected, then $f(A)$ is connected.*

Proof. Assume that $f(A)$ is not connected. Then there are B', C' which form a decomposition of $f(A)$. We consider the inverse images of B', C', i.e. the sets

$$B = f^{-1}(B') = \{b \in A \,|\, f(b) \in B'\}, \qquad C = f^{-1}(C') = \{c \in A \,|\, f(c) \in C'\}.$$

It is clear that $B \cup C = A$, $B \cap C = \emptyset$, $B \neq \emptyset$, $C \neq \emptyset$.
Now, let B contain a limit point b of C. Then there is a sequence (c_n) in C so that $c_n \to b$. Since f is continuous at b, we get $f(c_n) \to f(b)$. The sequence $(f(c_n))$ is in C' and thus $f(b)$ is a limit point of C'. But $f(b) \in B'$ and we arrive at a contradiction, because B' does not contain any limit point of C'. Hence B does not contain any limit point of C. Symmetrically, C does not contain any limit point of B.
Therefore B, C form a decomposition of A. This is wrong since A is connected and hence $f(A)$ is connected. □

Definition. *Let $a, b \in \mathbb{C}$ and $r > 0$. Every finite set $\{z_0, \dots, z_n\} \subseteq \mathbb{C}$ with $z_0 = a$, $z_n = b$ and $|z_{k-1} - z_k| < r$ for every $k = 1, \dots, n$ is called r-***succession of points*** which joins a, b. If, moreover, $z_k \in A$ for every $k = 0, \dots, n$, we say that the r-succession of points is in A.*

Theorem 1.3. *Let $K \subseteq \mathbb{C}$ be compact. Then K is connected if and only if for every $z, w \in K$ and every $r > 0$ there is an r-succession of points in K which joins z, w.*

Proof. Assume K is connected. We take any $z, w \in K$ and any $r > 0$ and let there be no r-succession of points in K which joins z, w.
We define the sets

$$B = \{b \in K \mid \text{there is an } r\text{-succession of points in } K \text{ which joins } z, b\},$$
$$C = \{c \in K \mid \text{there is no } r\text{-succession of points in } K \text{ which joins } z, c\}.$$

It is clear that $B \cup C = K$, $B \cap C = \emptyset$, $B \neq \emptyset$ (since $z \in B$) and $C \neq \emptyset$ (since $w \in C$).
Assume that B contains a limit point b of C. Then (since $b \in B$) there is an r-succession of points in K which joins z, b and, also, (since b is a limit point of C) there is $c \in C$ so that $|b - c| < r$. If to the r-succession of points of K which joins z, b we attach c (as a final point after b), then we get an r-succession of points in K which joins z, c. This is wrong since $c \in C$. Hence B *does not contain any limit point of* C.
Now assume that C contains a limit point c of B. Then (since c is a limit point of B) there is $b \in B$ so that $|b - c| < r$ and (since $b \in B$) there is an r-succession of points in K which joins z, b. If to the r-succession of points in K which joins z, b we attach c (as a final point after b), then we get an r-succession of points in K which joins z, c. This is wrong since $c \in C$. Hence C *does not contain any limit point of* B.
We conclude that B, C form a decomposition of K and this is wrong since K is connected. Therefore there is an r-succession of points in K which joins z, w.
Conversely, assume that for every $z, w \in K$ and every $r > 0$ there is an r-succession of points in K which joins z, w.
We assume that K is not connected. Then there are B, C which form a decomposition of K.
Let z be a limit point of B. Since $B \subseteq K$, z is a limit point of K and, since K is closed, we get $z \in K$. Now, $z \notin C$ (because C does not contain any limit point of B) and we get that $z \in B$. Thus B contains all its limit points and it is closed. Finally, since $B \subseteq K$ and K is compact, B is also compact. Symmetrically, C is also compact.
Now B, C are compact and disjoint and proposition 1.15 implies that there is $r > 0$ so that $|b - c| \geq r$ for every $b \in B$ and $c \in C$. Since $B \neq \emptyset$, $C \neq \emptyset$, we consider $b' \in B$ and $c' \in C$. Then it is easy to see that there is no r-succession of points in K which joins b', c', and we arrive at a contradiction. Indeed, assume that there is an r-succession $\{z_0, \dots, z_n\}$ in K so that $z_0 = b'$, $z_n = c'$ and $|z_{k-1} - z_k| < r$ for every $k = 1, \dots, n$. Since $z_0 \in B$, $z_n \in C$, it is clear that there is k so that $z_{k-1} \in B$, $z_k \in C$. Then $|z_{k-1} - z_k| < r$ contradicts that we have $|b - c| \geq r$ for every $b \in B, c \in C$. $\square$

Example 1.6.4. Every polygonal line is connected.

Proposition 1.23. *A set $I \subseteq \mathbb{R}$ is connected if and only if it is an interval.*

Proof. Let I be connected. If I is not an interval, then there are $x_1, x_2 \in I$ and $x \notin I$ so that $x_1 < x < x_2$. Then the sets $B = I \cap (-\infty, x)$ and $C = I \cap (x, +\infty)$ form a decomposition of I and we have a contradiction. Thus I is an interval.
Conversely, let I be an interval.
If I has only one element, then it is connected.
If $I = [a, b]$ with $a < b$, then $[a, b]$ is compact and if we take any x, y in $[a, b]$ and any $r > 0$, it is clear that we can find an r-succession of points in $[a, b]$ which joins x and y. Thus $[a, b]$ is connected.

If I is an interval of any other type, we can find a sequence of intervals $I_n = [a_n, b_n]$ which increase and their union is I. Then each I_n is connected and proposition 1.20 implies that I is also connected. $\square$

Proposition 1.24. *Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{R}$ be continuous in A. If A is connected, then f has the intermediate value property in A.*

Proof. $f(A)$ is a connected subset of $\mathbb{R}$ and hence it is an interval. Now, let u_1, u_2 be values of f in A, i.e. u_1, u_2 are in the interval $f(A)$. Then every u with $u_1 < u < u_2$ is also in the interval $f(A)$. Thus, every number between the values u_1, u_2 of f in A is also a value of f in A. $\square$

A special case of proposition 1.24 is the well known *intermediate value theorem* saying that *if $f : I \to \mathbb{R}$ is continuous in the interval $I \subseteq R$, then it has the intermediate value property in I.*

Definition. *Let $A \subseteq \mathbb{C}$. We say that A is* **polygonally connected** *if for every two points of A there is a polygonal line in A which joins those two points.*

Proposition 1.25. *Let $A \subseteq \mathbb{C}$. If A is polygonally connected, then it is connected.*

Proof. We fix any $z_0 \in A$. For every $z \in A$ there is a polygonal line l_z in A which joins z_0 and z. Then $l_z \subseteq A$ for every $z \in A$ and hence $\bigcup_{z\in A} l_z \subseteq A$. Conversely, since every $z \in A$ is a point of l_z, we have that $A \subseteq \bigcup_{z\in A} l_z$. Therefore $A = \bigcup_{z\in A} l_z$. Now, every l_z is connected and since all l_z have the point z_0 in common, we conclude that A is connected. $\square$

Example 1.6.5. Every ring between two circles is a connected set.

Example 1.6.6. Every convex set $A \subseteq \mathbb{C}$ is polygonally connected and hence connected. Indeed, if we take any two points in A the linear segment which joins them is contained in A.
For instance, all discs and all rectangles are connected sets.

Example 1.6.7. The set $A = \overline{D}_0(1) \cup D_2(1)$ in examples 1.6.1 and 1.6.2 is connected, since it is polygonally connected.

Theorem 1.4. *Let $A \subseteq \mathbb{C}$ be open. Then A is connected if and only if it is polygonally connected.*

Proof. If A is polygonally connected, proposition 1.25 implies that it is connected.
Conversely, let A be connected. We take $z, w \in A$ and we assume that there is no polygonal line in A which joins z, w.
We define the sets

$$B = \{\text{b} \in A \,|\, \text{there is a polygonal line in } A \text{ which joins } z, b\},$$
$$C = \{\text{c} \in A \,|\, \text{there is no polygonal line in } A \text{ which joins } z, c\}.$$

It is clear that $B \cup C = A$, $B \cap C = \emptyset$, $B \neq \emptyset$ (since $z \in B$) and $C \neq \emptyset$ (since $w \in C$).
We assume that B contains a limit point b of C. Then (since $b \in B$) there is a polygonal line in A which joins z, b. Since A is open, there is $r > 0$ so that $D_b(r) \subseteq A$ and (since b is a limit point of C) there is $c \in D_b(r) \cap C$. If to the polygonal line in A which joins z, b we attach (as last) the linear segment $[b, c]$ (which is contained in $D_b(r)$ and hence in A), we get a polygonal line in A which joins z, c. This is wrong, since $c \in C$. Therefore, *B does not contain any limit point of C.*
Now we assume that C contains a limit point c of B. Since A is open, there is $r > 0$ so that $D_c(r) \subseteq A$. Then (since c is a limit point of B) there is $b \in D_c(r) \cap B$. As before, (since $b \in B$) there is a polygonal line in A which joins z, b and, if to this we attach the linear segment $[b, c]$ (which is contained in $D_c(r)$ and hence in A), we get a polygonal line in A which joins z, c. This is wrong, since $c \in C$. Therefore, *C does not contain any limit point of B.*
We conclude that B, C form a decomposition of A and we arrive at a contradiction because A is connected.
Therefore, there is a polygonal line in A which joins z, w. $\square$

Definition. *An open and connected $A \subseteq \mathbb{C}$ is called* **region**. *The closure $\overline{A}$ of a region A is called* **closed region**.

Definition. *Let $A \subseteq \mathbb{C}$. We say that $C \subseteq A$ is a* **connected component** *of A if C is connected and has the following property: if $C \subseteq C' \subseteq A$ and C' is connected, then $C = C'$.*

In other words, C is a connected component of A if *it is a connected subset of A and there is no strictly larger connected subset of A.*

Let us see a characteristic property of connected components. Let C be a connected component of A and let B be any connected subset of A so that $C \cap B \neq \emptyset$. Then $C \cup B$ is connected (being the union of connected sets with a common point) and $C \subseteq C \cup B \subseteq A$. Since C is a connected component of A, we get $C \cup B = C$ and hence $B \subseteq C$. In oher words, *a connected component of A swallows every connected subset of A intersecting it.*

Let C_1, C_2 be *different* connected components of A and assume that $C_1 \cap C_2 \neq \emptyset$. Since C_1 is a connected subset of A which intersects the connected component C_2 of A, we get $C_1 \subseteq C_2$. Symmetrically, $C_2 \subseteq C_1$ and hence $C_1 = C_2$. This is a contradiction and we get $C_1 \cap C_2 = \emptyset$. We conclude that *different connected components of A are disjoint*.

Proposition 1.26. *Let $A \subseteq \mathbb{C}$. Then A is the union of its (mutually disjoint) connected components.*

Proof. We shall prove that every point of A belongs to a connected component of A.
We take $z \in A$ and define C_z to be the union of all connected subsets B of A which contain z. (For instance, such a set is $\{z\}$.) I.e.

$$C_z = \bigcup\{B \mid B \text{ is connected } \subseteq A \text{ and } z \in B\}.$$

Now C_z is a subset of A and it contains z. It is also connected, since it is the union of connected sets B with z as a common point. If $C_z \subseteq C' \subseteq A$ and C' is connected, then C' is one of the connected subsets B of A which contain z and hence $C' \subseteq C_z$. Thus $C_z = C'$. Therefore C_z is a connected component of A and it contains z. □

It is obvious that A is connected if and only if A is the only connected component of A.

Example 1.6.8. We take the set $A = D_0(1) \cup D_3(1)$. The discs $D_0(1)$ and $D_3(1)$ are connected subsets of A. Applying lemma 1.1 with $B = D_0(1)$ and $C = D_3(1)$, we see that any connected subset of A is contained either in $D_0(1)$ or in $D_3(1)$. I.e. there is no connected subset of A strictly larger than either $D_0(1)$ or $D_3(1)$.
Therefore the discs $D_0(1)$ and $D_3(1)$ are the connected components of A.

Example 1.6.9. We take the set $\mathbb{Z}$ and any $n \in \mathbb{Z}$. Then $\{n\}$ is a connected set. Let $\{n\} \subseteq C' \subseteq \mathbb{Z}$ and $C' \neq \{n\}$. Then $C' = \{n\} \cup \big(C' \setminus \{n\}\big)$ and it is clear that the sets $\{n\}$ and $C' \setminus \{n\}$ form a decomposition of C'. Thus C' is not connected and hence $\{n\}$ is a connected component of $\mathbb{Z}$.

Proposition 1.27. *Let $A \subseteq \mathbb{C}$. If A is closed, then every connected component of A is closed.*

Proof. Let C be a connected component of A. Since $C \subseteq A$ and A is closed, we get $C \subseteq \overline{C} \subseteq A$. Proposition 1.21 implies that $\overline{C}$ is connected and, since C is a connected component of A, we get that $C = \overline{C}$. Therefore C is closed. □

Proposition 1.28. *Let $A \subseteq \mathbb{C}$. If A is open, then every connected component of A is open.*

Proof. Let C be a connected component of A and let $z \in C$. Then $z \in A$ and, since A is open, there is $r > 0$ so that $D_z(r) \subseteq A$. Since $D_z(r)$ is a connected subset of A and intersects the connected component C of A, we see that $D_z(r) \subseteq C$. Thus, z is an interior point of C.
Therefore C is open. □

Propositions 1.26 and 1.28 imply that *every open set is the union of disjoint regions*.

Exercises.

1.6.1. Find the connected components of the complements of a circle, of a triangle and of a linear segment. Also of: $\{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$, $[0,1] \cup \bigcup_{n=1}^{+\infty}[\frac{i}{n}, 1 + \frac{i}{n}]$, $\bigcup_{n=1}^{+\infty} C_0(1 + \frac{1}{n})$, $\mathbb{Q} \times \mathbb{Q}$.

1.6.2. Prove that the following sets are connected. $\{x + i \sin x \,|\, x \in \mathbb{R}\}$, $\{x + i \sin \frac{1}{x} \,|\, 0 < x \leq 1\}$, $\{x + i \sin \frac{1}{x} \,|\, 0 < x \leq 1\} \cup [-i, i]$.

1.6.3. Find a simple example of (i) two connected sets whose intersection is not connected, (ii) a connected set A such that ∂A is not connected, (iii) a connected set A such that A° is not connected.

1.6.4. Let A be a region and $z_1, \ldots, z_n \in A$. Prove that $A \setminus \{z_1, \ldots, z_n\}$ is a region.

1.6.5. Let $A \subseteq \mathbb{C}$ and $A \subseteq D \subseteq \overline{A}$. If A is connected, prove that D is connected.

1.6.6. $A \subseteq \mathbb{C}$ is called **star-shaped** if there is a specific point $z_0 \in A$ so that for every $z \in A$ the linear segment $[z_0, z]$ is contained in A. Prove that a star-shaped A is polygonally connected and hence connected.

1.6.7. Let $A_n \subseteq \mathbb{C}$ be connected and $A_n \cap A_{n+1} \neq \emptyset$ for all n. Prove that $\bigcup_{n=1}^{+\infty} A_n$ is connected.

1.6.8. If $B \subseteq \mathbb{C}$ is open and closed, prove that either $B = \emptyset$ or $B = \mathbb{C}$.

1.6.9. Let $A \subseteq \mathbb{C}$ be connected (not necessarily compact). Prove that for every $r > 0$ and every $z, w \in A$ there is an r-succession of points in A which joins z, w.

1.6.10. (i) Let A be closed. Prove that A is connected if and only if there are no closed B, C such that $B \cup C = A$, $B \cap C = \emptyset$, $B \neq \emptyset$, $C \neq \emptyset$.
(ii) Let A be open. Prove that A is connected if and only if there are no open B, C such that $B \cup C = A$, $B \cap C = \emptyset$, $B \neq \emptyset$, $C \neq \emptyset$.

1.6.11. Prove that A is connected if and only if the only continuous functions $f : A \to \mathbb{Z}$ are the constant functions.

1.6.12. Let A be a region and let every point of $B \subseteq A$ be an isolated point of B. Prove that $A \setminus B$ is connected.

1.6.13. (i) Let A_n be compact so that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$ and so that every two points of A_n can be joined by some $\frac{1}{n}$-succession of points in A_n. Prove that $\bigcap_{n=1}^{+\infty} A_n$ is connected.
(ii) Let F be compact and let $z, w \in F$ belong to different connected components of F. Prove that there is a decomposition B, C of F so that $z \in B$ and $w \in C$.

Chapter 2

Series.

2.1 Series of numbers.

Definition. *If (z_n) is a sequence of complex numbers, the expression*

$$z_1 + z_2 + \cdots + z_n + \cdots \qquad \textit{or} \qquad \sum_{n=1}^{+\infty} z_n$$

is called **series of complex numbers** *or, simply,* **complex series**. *If all numbers z_n are real, we say* **series of real numbers** *or* **real series**.
The $s_n = z_1 + \cdots + z_n$ are the **partial sums** *of the series $\sum_{n=1}^{+\infty} z_n$.*
We say that the series $\sum_{n=1}^{+\infty} z_n$ **converges** *if the sequence (s_n) converges and then the limit s of (s_n) is called* **sum** *of the series and we write*

$$\sum_{n=1}^{+\infty} z_n = s.$$

We say that the series $\sum_{n=1}^{+\infty} z_n$ **diverges** *if (s_n) diverges. If (s_n) diverges to ∞, then we say that $\sum_{n=1}^{+\infty} z_n$* **diverges to** *∞ and that ∞ is the* **sum** *of the series and we write*

$$\sum_{n=1}^{+\infty} z_n = \infty.$$

We note that the sum of a complex series can be either a complex number or ∞. Only a real series can have sum equal to $+\infty$ or $-\infty$. Therefore, when we write $\sum_{n=1}^{+\infty} z_n = +\infty$ or $-\infty$, we accept that all z_n are real and that the series diverges to $+\infty$ or $-\infty$ as a real series. Of course, if a real series diverges to $+\infty$ or $-\infty$, then as a complex series it diverges to ∞.

Example 2.1.1. We have $\sum_{n=1}^{+\infty} c = \begin{cases} 0, & \text{if } c = 0 \\ \infty, & \text{if } c \neq 0 \end{cases}$

Example 2.1.2. To examine the geometric series $\sum_{n=0}^{+\infty} z^n$, we use the formula $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ for its partial sums, and we find that its sum is

$$\sum_{n=0}^{+\infty} z^n \begin{cases} = \frac{1}{1-z}, & \text{if } |z| < 1 \\ = \infty, & \text{if } |z| > 1 \text{ or } z = 1 \\ \text{it does not exist}, & \text{if } |z| = 1,\ z \neq 1 \end{cases}$$

The usual simple algebraic rules, which hold for real series, hold also for complex series. We mention them without proofs. The proofs for the complex case are identical with the proofs in the real case.

Proposition 2.1. *If* $\sum_{n=1}^{+\infty} z_n$ *converges, then* $z_n \to 0$.

Proposition 2.2. *Provided the right sides of the following formulas exist and they are not indeterminate forms, we have*

$$\sum_{n=1}^{+\infty}(z_n + w_n) = \sum_{n=1}^{+\infty} z_n + \sum_{n=1}^{+\infty} w_n, \qquad \sum_{n=1}^{+\infty} \lambda z_n = \lambda \sum_{n=1}^{+\infty} z_n, \qquad \sum_{n=1}^{+\infty} \overline{z_n} = \overline{\sum_{n=1}^{+\infty} z_n}.$$

Moreover, if $z_n = x_n + iy_n$*, then* $\sum_{n=1}^{+\infty} z_n$ *converges if and only if* $\sum_{n=1}^{+\infty} x_n$ *and* $\sum_{n=1}^{+\infty} y_n$ *converge, and*

$$\sum_{n=1}^{+\infty} z_n = \sum_{n=1}^{+\infty} x_n + i \sum_{n=1}^{+\infty} y_n.$$

Regarding the *comparison theorems*, we may say that, since these are based on order relations which can be expressed only between real numbers, when we write $\sum_{n=1}^{+\infty} z_n \leq \sum_{n=1}^{+\infty} w_n$ as a consequence of $z_n \leq w_n$, we accept that all z_n, w_n are real and then we just apply the well-known comparison theorems for real series.

Cauchy criterion. *The series* $\sum_{n=1}^{+\infty} z_n$ *converges if and only if for every* $\epsilon > 0$ *there is* n_0 *so that* $|\sum_{k=m+1}^{n} z_k| = |z_{m+1} + \cdots + z_n| < \epsilon$ *for every* m, n *with* $n > m \geq n_0$.

Proof. We consider the partial sums $s_n = z_1 + \cdots + z_n$. The series $\sum_{n=1}^{+\infty} z_n$ converges if and only if (s_n) converges or, equivalently, if (s_n) is a Cauchy sequence. That (s_n) is a Cauchy sequence means that for every $\epsilon > 0$ there is n_0 so that

$$|z_{m+1} + \cdots + z_n| = |(z_1 + \cdots + z_n) - (z_1 + \cdots + z_m)| = |s_n - s_m| < \epsilon$$

for every n, m with $n > m \geq n_0$. □

Definition. *We say that* $\sum_{n=1}^{+\infty} z_n$ **converges absolutely** *if the (real) series* $\sum_{n=1}^{+\infty} |z_n|$ *converges, i.e. if* $\sum_{n=1}^{+\infty} |z_n| < +\infty$.

Criterion of absolute convergence. *If* $\sum_{n=1}^{+\infty} z_n$ *converges absolutely, then it converges and*

$$\Big|\sum_{n=1}^{+\infty} z_n\Big| \leq \sum_{n=1}^{+\infty} |z_n|.$$

Proof. Let $\sum_{n=1}^{+\infty} |z_n|$ converge and take any $\epsilon > 0$. From the Cauchy criterion we have that there is n_0 so that $|z_{m+1}| + \cdots + |z_n| < \epsilon$ and hence $|z_{m+1} + \cdots + z_n| < \epsilon$ for every m, n with $n > m \geq n_0$. The Cauchy criterion, again, implies that $\sum_{n=1}^{+\infty} z_n$ converges.
Now we take the partial sums $s_n = z_1 + \cdots + z_n$ and $S_n = |z_1| + \cdots + |z_n|$. We have $|s_n| \leq S_n$ for all n and, taking the limit of this as $n \to +\infty$, we finish the proof. □

Ratio test of d' Alembert. *Let* $z_n \neq 0$ *for all* n.
(i) If $\overline{\lim} \left|\frac{z_{n+1}}{z_n}\right| < 1$, *then* $\sum_{n=1}^{+\infty} z_n$ *converges absolutely.*
(ii) If $\underline{\lim} \left|\frac{z_{n+1}}{z_n}\right| > 1$, *then* $\sum_{n=1}^{+\infty} z_n$ *diverges.*
(iii) If $\underline{\lim} \left|\frac{z_{n+1}}{z_n}\right| \leq 1 \leq \overline{\lim} \left|\frac{z_{n+1}}{z_n}\right|$, *then there is no general conclusion.*

Proof. (i) We consider any a such that $\overline{\lim} \left|\frac{z_{n+1}}{z_n}\right| < a < 1$. Then there is n_0 so that $\left|\frac{z_{n+1}}{z_n}\right| \leq a$ for every $n \geq n_0$. Therefore, for every $n \geq n_0 + 1$ we have

$$|z_n| = \left|\frac{z_n}{z_{n-1}}\right| \left|\frac{z_{n-1}}{z_{n-2}}\right| \cdots \left|\frac{z_{n_0+1}}{z_{n_0}}\right| |z_{n_0}| \leq a^{n-n_0} |z_{n_0}| = c\, a^n,$$

where $c = |z_{n_0}|/a^{n_0}$. Since $0 \le a < 1$, the geometric series $\sum_{n=1}^{+\infty} a^n$ converges and, by comparison, $\sum_{n=1}^{+\infty} |z_n|$ also converges.
(ii) There is n_0 so that $\left|\frac{z_{n+1}}{z_n}\right| \ge 1$ for every $n \ge n_0$. Therefore, for every $n \ge n_0 + 1$ we have

$$|z_n| \ge |z_{n-1}| \ge \cdots \ge |z_{n_0}| > 0.$$

This implies that $z_n \not\to 0$ and $\sum_{n=1}^{+\infty} z_n$ diverges.
(iii) For the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ and $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ we have that $\left|\frac{1/(n+1)}{1/n}\right| \to 1$ and $\left|\frac{1/(n+1)^2}{1/n^2}\right| \to 1$. The first series diverges and the second converges. $\square$

Root test of Cauchy. *(i) If* $\overline{\lim} \sqrt[n]{|z_n|} < 1$*, then* $\sum_{n=1}^{+\infty} z_n$ *converges absolutely.*
(ii) If $\overline{\lim} \sqrt[n]{|z_n|} > 1$*, then* $\sum_{n=1}^{+\infty} z_n$ *diverges.*
(iii) If $\overline{\lim} \sqrt[n]{|z_n|} = 1$*, then there is no general conclusion.*

Proof. (i) We consider any a such that $\overline{\lim} \sqrt[n]{|z_n|} < a < 1$. Then there is n_0 so that $\sqrt[n]{|z_n|} \le a$ and hence $|z_n| \le a^n$ for every $n \ge n_0$. Since $0 \le a < 1$, the geometric series $\sum_{n=1}^{+\infty} a^n$ converges and, by comparison, $\sum_{n=1}^{+\infty} |z_n|$ also converges.
(ii) We have $\sqrt[n]{|z_n|} \ge 1$ for infinitely many n. Therefore, $|z_n| \ge 1$ for infinitely many n and hence $z_n \not\to 0$. Thus, $\sum_{n=1}^{+\infty} z_n$ diverges.
(iii) For the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ and $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ we have $\sqrt[n]{\left|\frac{1}{n}\right|} \to 1$ and $\sqrt[n]{\left|\frac{1}{n^2}\right|} \to 1$. The first series diverges and the second converges. $\square$

Applying the ratio test and the root test to specific series $\sum_{n=1}^{+\infty} z_n$, we find very often that the limits $\lim_{n\to+\infty} \left|\frac{z_{n+1}}{z_n}\right|$ and $\lim_{n\to+\infty} \sqrt[n]{|z_n|}$ exist. We know (and we used it in the proofs of parts (iii) of both tests) that in this case: $\underline{\lim} = \overline{\lim} = \lim$.

Example 2.1.3. To the series $\sum_{n=1}^{+\infty} \frac{z^n}{n!}$ we apply the ratio test. If $z = 0$, the series obviously converges absolutely. If $z \ne 0$, then $\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0 < 1$. Hence the series converges absolutely for every z.
Now we apply the root test. We have $\sqrt[n]{\left|\frac{z^n}{n!}\right|} = \frac{|z|}{\sqrt[n]{n!}} \to \frac{|z|}{+\infty} = 0 < 1$ and we arrive at the same conclusion as before.

Example 2.1.4. We consider $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ and we apply the ratio test. If $z = 0$, the series obviously converges absolutely. If $z \ne 0$, then $\left|\frac{z^{n+1}/(n+1)^2}{z^n/n^2}\right| \to |z|$. Hence, if $0 < |z| < 1$, the series converges absolutely and, if $|z| > 1$, the series diverges.
Now we apply the root test. We have $\sqrt[n]{\left|\frac{z^n}{n^2}\right|} \to |z|$. Therefore, if $|z| < 1$, the series converges absolutely and, if $|z| > 1$, the series diverges.
If $|z| = 1$, none of the two tests applies. But we observe that $\sum_{n=1}^{+\infty} \left|\frac{z^n}{n^2}\right| = \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$ in this case, and $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ converges absolutely.
Conclusion: $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ converges absolutely if $|z| \le 1$, and diverges if $|z| > 1$.

Example 2.1.5. We consider $\sum_{n=1}^{+\infty} \frac{z^n}{n}$ and we apply the ratio test. If $z = 0$, the series obviously converges absolutely. If $z \ne 0$, then $\left|\frac{z^{n+1}/(n+1)}{z^n/n}\right| \to |z|$. Hence, if $0 < |z| < 1$, the series converges absolutely and, if $|z| > 1$, the series diverges.
Now we apply the root test. We have $\sqrt[n]{\left|\frac{z^n}{n}\right|} \to |z|$. Therefore, if $|z| < 1$, the series converges absolutely and, if $|z| > 1$, the series diverges.
If $|z| = 1$, none of the two tests applies. If $z = 1$, the series becomes $\sum_{n=1}^{+\infty} \frac{1}{n}$ and diverges. If $|z| = 1$, $z \ne 1$, then $\sum_{n=1}^{+\infty} \left|\frac{z^n}{n}\right| = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$, and $\sum_{n=1}^{+\infty} \frac{z^n}{n}$ does not converge absolutely. In fact, exercise 2.1.10 (iv) shows that the series converges in this case. In general, when a series is convergent but not absolutely convergent we say that it is **conditionally convergent**.
Conclusion: $\sum_{n=1}^{+\infty} \frac{z^n}{n}$ converges absolutely if $|z| < 1$, diverges if $|z| > 1$ or $z = 1$, and converges conditionally if $|z| = 1$, $z \ne 1$.

Exercises.

2.1.1. Which of the series $\sum_{n=1}^{+\infty}(\frac{1}{n}+\frac{i}{n^2})$, $\sum_{n=1}^{+\infty}(\frac{n}{2^n}+\frac{i}{n^3})$, $\sum_{n=1}^{+\infty}\frac{1+i^n}{n^2}$, $\sum_{n=1}^{+\infty}\frac{1}{2+i^n}$, $\sum_{n=1}^{+\infty}\frac{1}{n+i}$, $\sum_{n=1}^{+\infty}\frac{1}{n^2+in}$ converge?

2.1.2. Find the sum of the series $\sum_{n=1}^{+\infty} n(-1)^{n-1}$ if we consider it as a complex series and also if we consider it as a real series.

2.1.3. Apply the ratio test whenever possible: $\sum_{n=1}^{+\infty} n^3 i^n$, $\sum_{n=1}^{+\infty}\frac{n!}{i^n}$, $\sum_{n=1}^{+\infty}\frac{(1+i)^n}{n!}$, $\sum_{n=1}^{+\infty}\frac{(2i)^n n!}{n^n}$, $\sum_{n=1}^{+\infty}\frac{(2+i)^n n!}{n^n}$, $\sum_{n=1}^{+\infty}\frac{e^n n!}{n^n}$, $\sum_{n=1}^{+\infty}\frac{(n!)^2}{(2n)!}$, $\sum_{n=1}^{+\infty}\frac{(4i)^n (n!)^2}{(2n)!}$, $\sum_{n=1}^{+\infty}\frac{(3+i)(6+i)(9+i)\cdots(3n+i)}{(3+4i)(3+8i)(3+12i)\cdots(3+4ni)}$.
Apply the root test whenever possible: $\sum_{n=1}^{+\infty} n^n i^n$, $\sum_{n=1}^{+\infty}(\frac{n+i}{2n-i})^n$, $\sum_{n=1}^{+\infty}(\frac{n+i}{n-i})^{2n}$, $\sum_{n=1}^{+\infty}\frac{n^3}{(1+2i)^n}$, $\sum_{n=1}^{+\infty} n^3(1-i)^n$, $\sum_{n=1}^{+\infty}\frac{(2+3i)^n}{n^n}$, $\sum_{n=1}^{+\infty}\frac{n+i}{(\sqrt[n]{n}+i)^n}$.

2.1.4. If $\sum_{n=1}^{+\infty}|z_n|<+\infty$, prove that $\sum_{n=1}^{+\infty} z_n(\cos n\theta+i\sin n\theta)$ converges.

2.1.5. Let $z_n=x_n+iy_n$ for all n. Prove that $\sum_{n=1}^{+\infty} z_n$ converges absolutely if and only if $\sum_{n=1}^{+\infty} x_n$, $\sum_{n=1}^{+\infty} y_n$ converge absolutely.

2.1.6. Let $|a_n|r^n\le Mn^k$ for all n. Prove that $\sum_{n=1}^{+\infty} a_n z^n$ converges for every z with $|z|<r$.

2.1.7. Find all z for which $\sum_{n=1}^{+\infty}\frac{z^n}{2+z^n}$ converges.

2.1.8. Let $0\le\theta_0<\frac{\pi}{2}$ and assume for every n that $\arg z_n$ has a value in $[-\theta_0,\theta_0]$. Prove that $\sum_{n=1}^{+\infty} z_n$ converges if and only if it converges absolutely. Prove that $\sum_{n=1}^{+\infty} z_n=\infty$ if and only if $\sum_{n=1}^{+\infty}|z_n|=+\infty$.

2.1.9. Find a series $\sum_{n=1}^{+\infty} z_n$ which converges and is such that $\sum_{n=1}^{+\infty} z_n^2$ diverges.

2.1.10. Consider the sequences (a_n), (z_n) and the partial sums $s_n=z_1+\cdots+z_n$.
(i) Prove that $\sum_{k=m+1}^{n} a_k z_k=\sum_{k=m+1}^{n}(a_k-a_{k+1})s_k+a_{n+1}s_n-a_{m+1}s_m$ for every n,m with $n>m$. This is the **summation by parts** formula due to Abel.
(ii) Prove the **Dirichlet test**: if (a_n) is real and decreasing and $a_n\to 0$ and if (s_n) is bounded, then $\sum_{n=1}^{+\infty} a_n z_n$ converges.
(iii) Prove the **Abel test**: if (a_n) is real and decreasing and bounded below and if (s_n) converges, i.e. if $\sum_{n=1}^{+\infty} z_n$ converges, then $\sum_{n=1}^{+\infty} a_n z_n$ converges.
(iv) If (a_n) is real and decreasing and $a_n\to 0$, prove that $\sum_{n=0}^{+\infty} a_n z^n$ converges for every z with $|z|\le 1$, $z\ne 1$.
(v) Check the conditional convergence and the absolute convergence of the series: $\sum_{n=1}^{+\infty}\frac{i^n}{n}$, $\sum_{n=2}^{+\infty}\frac{i^n}{n\log n}$, $\sum_{n=2}^{+\infty}\frac{i^n}{n(\log n)^2}$, $\sum_{n=1}^{+\infty} i^{n-1}\sin\frac{1}{n}$, $\sum_{n=1}^{+\infty} i^{n-1}(1-\cos\frac{1}{n})$.

2.1.11. Let $s_n=z_1+\cdots+z_n$ for all n. If $(a_{n+1}s_n)$ converges and if $\sum_{n=1}^{+\infty}(a_n-a_{n+1})s_n$ converges, prove that $\sum_{n=1}^{+\infty} a_n z_n$ converges. In particular: if (s_n) is bounded, if $a_n\to 0$ and if $\sum_{n=1}^{+\infty}|a_n-a_{n+1}|<+\infty$, prove that $\sum_{n=1}^{+\infty} a_n z_n$ converges.
What is the relation of all these with the tests of Dirichlet and Abel in the previous exercise?

2.2 Sequences and series of functions.

Definition. *Let $f_n:A\to\mathbb{C}$ for every n and $f:A\to\mathbb{C}$. We say that the sequence of functions (f_n)* **converges** *to the function f* **uniformly** *in A if* $\sup\{|f_n(z)-f(z)|\,|\,z\in A\}\to 0$. *We denote*

$$f_n\xrightarrow{u} f\quad\text{on } A.$$

Equivalently, $f_n\xrightarrow{u} f$ in A if for every $\epsilon>0$ there is n_0 so that $|f_n(z)-f(z)|\le\epsilon$ for every $n\ge n_0$ and every $z\in A$.

It is easy to see that, if (f_n) converges to f uniformly in A, then $f_n(z) \to f(z)$ for every $z \in A$, i.e. (f_n) converges to f **pointwise** in A. Indeed, for every $z \in A$ we have

$$0 \le |f_n(z) - f(z)| \le \sup\{|f_n(w) - f(w)| \,|\, w \in A\} \to 0.$$

Proposition 2.3. *Let (f_n) converge to f uniformly in A and let $z_0 \in A$. If every f_n is continuous at z_0, then f is continuous at z_0. In particular, if every f_n is continuous in A, then f is continuous in A.*

Proof. Let $\epsilon > 0$. Then there is n_0 so that $|f_n(z) - f(z)| \le \frac{\epsilon}{3}$ for every $n \ge n_0$ and every $z \in A$. In particular, we have $|f_{n_0}(z) - f(z)| \le \frac{\epsilon}{3}$ for every $z \in A$. Since f_{n_0} is continuous at z_0, there is $\delta > 0$ so that $|f_{n_0}(z) - f_{n_0}(z_0)| \le \frac{\epsilon}{3}$ for every $z \in A$ with $|z - z_0| < \delta$.
Hence, for every $z \in A$ with $|z - z_0| < \delta$ we get

$$|f(z) - f(z_0)| \le |f(z) - f_{n_0}(z)| + |f_{n_0}(z) - f_{n_0}(z_0)| + |f_{n_0}(z_0) - f(z_0)| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

and f is continuous at z_0. □

From the notion of uniform convergence of a sequence of functions we move to the notion of uniform convergence of a series of functions (through the sequence of partial sums).

Definition. *Let $f_n : A \to \mathbb{C}$ for every n. We take the partial sums $s_n : A \to \mathbb{C}$, where $s_n(z) = f_1(z) + \cdots + f_n(z)$ for every $z \in A$. Let also $s : A \to \mathbb{C}$. We say that the series of functions $\sum_{n=1}^{+\infty} f_n$* **converges** *to its* **sum** *s* **uniformly** *in A if the sequence of functions (s_n) converges to the function s uniformly in A.*
We denote

$$\sum_{n=1}^{+\infty} f_n \overset{u}{=} s \quad \textit{on } A.$$

As in the case of a sequence of functions, we have that, if $\sum_{n=1}^{+\infty} f_n$ converges to its sum s uniformly in A, then $\sum_{n=1}^{+\infty} f_n(z) = s(z)$ for every $z \in A$, i.e. $\sum_{n=1}^{+\infty} f_n$ converges to its sum s **pointwise** in A.

Proposition 2.4. *Let $\sum_{n=1}^{+\infty} f_n$ converge to its sum s uniformly in A and let $z_0 \in A$. If every f_n is continuous at z_0, then s is continuous at z_0. In particular, if every f_n is continuous in A, then s is continuous in A.*

Proof. We consider the partial sums $s_n = f_1 + \cdots + f_n$. Then every s_n is continuous at z_0 and proposition 2.3 implies that s is continuous at z_0. □

Finally, we have a basic criterion for uniform convergence of a series of functions.

Weierstrass test. *Let $|f_n(z)| \le M_n$ for every n and every $z \in A$. If the series (of non-negative terms) $\sum_{n=1}^{+\infty} M_n$ converges, i.e. if $\sum_{n=1}^{+\infty} M_n < +\infty$, then $\sum_{n=1}^{+\infty} f_n$ converges uniformly in A.*

Proof. For every $z \in A$ we have $\sum_{n=1}^{+\infty} |f_n(z)| \le \sum_{n=1}^{+\infty} M_n < +\infty$ and hence $\sum_{n=1}^{+\infty} f_n(z)$ converges (as a series of complex numbers). Therefore, we may define the function $s : A \to \mathbb{C}$ with $s(z) = \sum_{n=1}^{+\infty} f_n(z)$ for every $z \in A$. Now we consider the partial sums $s_n = f_1 + \cdots + f_n$ and then for every $z \in A$ we have

$$|s_n(z) - s(z)| = \Big|\sum_{k=1}^{n} f_k(z) - \sum_{k=1}^{+\infty} f_k(z)\Big| = \Big|\sum_{k=n+1}^{+\infty} f_k(z)\Big| \le \sum_{k=n+1}^{+\infty} |f_k(z)| \le \sum_{k=n+1}^{+\infty} M_k.$$

Since this is true for every $z \in A$, we get

$$\sup\{|s_n(z) - s(z)| \,|\, z \in A\} \le \sum_{k=n+1}^{+\infty} M_k \to 0 \qquad \text{when } n \to +\infty,$$

because $\sum_{n=1}^{+\infty} M_n < +\infty$. Therefore, (s_n) converges to s uniformly in A and hence $\sum_{n=1}^{+\infty} f_n$ converges to its sum s uniformly in A. $\square$

Exercises.

2.2.1. Prove that $\sum_{n=-\infty}^{+\infty} \frac{1}{(z+n)^2}$ converges for every $z \in \mathbb{C} \setminus \mathbb{Z}$ and that for every compact set K the series converges uniformly in $K \setminus \mathbb{Z}$.

2.2.2. (i) If $K \subseteq \mathbb{C} \setminus \mathbb{T}$ is compact, prove that there is r with $0 < r < 1$ (r depends on K) so that for every $z \in K$ either $|z| \leq r$ or $|z| \geq \frac{1}{r}$ holds.
(ii) Prove that $\sum_{n=0}^{+\infty} \frac{z^n}{z^{2n}+1}$ converges uniformly in every compact $K \subseteq \mathbb{C} \setminus \mathbb{T}$.

2.2.3. (i) If $\mathrm{Re}\, z > -\frac{1}{2}$, prove that $\left|\frac{z}{z+1}\right| < 1$. If $K \subseteq \{z \mid \mathrm{Re}\, z > -\frac{1}{2}\}$ is compact, prove that there is r with $0 < r < 1$ (r depends on K) so that $\left|\frac{z}{z+1}\right| \leq r$ for every $z \in K$.
(ii) Prove that $\sum_{n=0}^{+\infty} (\frac{z}{z+1})^n$ converges for every z in the halfplane $\{z \mid \mathrm{Re}\, z > -\frac{1}{2}\}$ and uniformly in every compact subset of this halfplane.

Chapter 3

Curvilinear integrals.

3.1 Integrals of complex functions over an interval.

We shall extend the notion of integral of a *real* function over an interval to the notion of integral of a *complex* function over an interval.

Definition. *Let* $f : [a, b] \to \mathbb{C}$ *and let* $u = \operatorname{Re} f : [a, b] \to \mathbb{R}$ *και* $v = \operatorname{Im} f : [a, b] \to \mathbb{R}$ *be the real and imaginary parts of* f. *We say that* f *is* **(Riemann) integrable** *over* $[a, b]$ *if* u, v *are both (Riemann) integrable over* $[a, b]$ *and in this case we define the* **(Riemann) integral** *of* f *over* $[a, b]$ *to be*

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt. \tag{3.1}$$

Since the numbers $\int_a^b u(t)\,dt$ and $\int_a^b v(t)\,dt$ are real, we see that

$$\operatorname{Re}\int_a^b f(t)\,dt = \int_a^b \operatorname{Re} f(t)\,dt, \qquad \operatorname{Im}\int_a^b f(t)\,dt = \int_a^b \operatorname{Im} f(t)\,dt.$$

Now let us take any subdivision $\Delta = \{t_0, \ldots, t_n\}$ of $[a, b]$ and any choice $\Xi = \{\xi_1, \ldots, \xi_n\}$ of intermediate points $\xi_k \in [t_{k-1}, t_k]$ and the corresponding Riemann sum $\sum_{k=1}^n f(\xi_k)(t_k - t_{k-1})$. If $w(\Delta) = \max_{1\le k\le n}(t_k - t_{k-1})$ is the width of the subdivision Δ, then we know that

$$\lim_{w(\Delta)\to 0}\sum_{k=1}^n u(\xi_k)(t_k - t_{k-1}) = \int_a^b u(t)\,dt, \qquad \lim_{w(\Delta)\to 0}\sum_{k=1}^n v(\xi_k)(t_k - t_{k-1}) = \int_a^b v(t)\,dt.$$

Multiplying the second relation with i, adding and using (3.1), we find

$$\lim_{w(\Delta)\to 0}\sum_{k=1}^n f(\xi_k)(t_k - t_{k-1}) = \int_a^b f(t)\,dt.$$

Example 3.1.1. If f is piecewise-continuous in $[a, b]$, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are also piecewise-continuous in $[a, b]$. Hence u, v are integrable, and f is also integrable over $[a, b]$.

The following propositions are analogous to similar well-known propositions about integrals of real functions and can be proved easily by the reader. One should decompose every complex function into its real and imaginary parts and use the analogous properties for real functions together with (3.1).

Proposition 3.1. *Let* $f_1, f_2 : [a, b] \to \mathbb{C}$ *be integrable over* $[a, b]$ *and* $\lambda_1, \lambda_2 \in \mathbb{C}$. *Then* $\lambda_1 f_1 + \lambda_2 f_2 : [a, b] \to \mathbb{C}$ *is integrable over* $[a, b]$ *and*

$$\int_a^b (\lambda_1 f_1(t) + \lambda_2 f_2(t))\,dt = \lambda_1\int_a^b f_1(t)\,dt + \lambda_2\int_a^b f_2(t)\,dt.$$

Proposition 3.2. *Let $f : [a, c] \to \mathbb{C}$ and $a < b < c$. If f is integrable over $[a, b]$ and in $[b, c]$, then f is integrable over $[a, c]$ and*

$$\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt.$$

Proposition 3.3. *If $f_1, f_2 : [a, b] \to \mathbb{C}$ are integrable over $[a, b]$, then $f_1 f_2 : [a, b] \to \mathbb{C}$ is integrable over $[a, b]$.*

The proof of the next proposition is not entirely trivial.

Proposition 3.4. *Let $f : [a, b] \to \mathbb{C}$ be integrable over $[a, b]$. Then $|f| : [a, b] \to \mathbb{R}$ is integrable over $[a, b]$ and*

$$\Big|\int_a^b f(t)\,dt\Big| \le \int_a^b |f(t)|\,dt.$$

Equality $|\int_a^b f(t)\,dt| = \int_a^b |f(t)|\,dt$ holds if and only if there is some halfline l with vertex 0 so that $f(t) \in l$ for every continuity point t of f.

Proof. Let $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. Then u, v are integrable over $[a, b]$ hence $|f| = \sqrt{u^2 + v^2}$ is integrable over $[a, b]$. Now we have two cases.
(i) If $\int_a^b f(t)\,dt = 0$, then $|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|\,dt$ becomes $0 \le \int_a^b |f(t)|\,dt$ and it is clearly true.
(ii) Let $\int_a^b f(t)\,dt \ne 0$. We consider any polar representation of the number $\int_a^b f(t)\,dt$, i.e.

$$\int_a^b f(t)\,dt = \Big|\int_a^b f(t)\,dt\Big|(\cos\theta + i\sin\theta) = \Big|\int_a^b f(t)\,dt\Big|\,z,$$

where θ is any value of the argument of $\int_a^b f(t)\,dt$ and where we set $z = \cos\theta + i\sin\theta$. We observe that

$$|z| = |\cos\theta + i\sin\theta| = 1.$$

Now,

$$\Big|\int_a^b f(t)\,dt\Big| = z^{-1}\int_a^b f(t)\,dt = \int_a^b (z^{-1} f(t))\,dt. \tag{3.2}$$

The left side of (3.2) is real and hence its right side is also real and thus equal to its real part! Hence

$$\begin{aligned}\Big|\int_a^b f(t)\,dt\Big| &= \operatorname{Re}\int_a^b (z^{-1} f(t))\,dt = \int_a^b \operatorname{Re}(z^{-1} f(t))\,dt \le \int_a^b |z^{-1} f(t)|\,dt \\ &= \int_a^b |f(t)|\,dt.\end{aligned} \tag{3.3}$$

Now assume $|\int_a^b f(t)\,dt| = \int_a^b |f(t)|\,dt$.
In case (i), we have $\int_a^b |f(t)|\,dt = 0$ and this is equivalent to $f(t) = 0$ at every continuity point t of f.
In case (ii), we see from (3.3) that $|\int_a^b f(t)\,dt| = \int_a^b |f(t)|\,dt$ is equivalent to $\int_a^b \operatorname{Re}(z^{-1} f(t))\,dt = \int_a^b |z^{-1} f(t)|\,dt$. This is equivalent to $\operatorname{Re}(z^{-1} f(t)) = |z^{-1} f(t)|$ at every continuity point t of f. The last equality is equivalent to $\operatorname{Re}(z^{-1} f(t)) \ge 0$ and $\operatorname{Im}(z^{-1} f(t)) = 0$ and this is equivalent to $z^{-1} f(t) \ge 0$ and this is equivalent to $f(t)$ being a non-negative multiple of the fixed z (with $|z| = 1$).
Thus, in both cases we get that $|\int_a^b f(t)\,dt| = \int_a^b |f(t)|\,dt$ if and only there is a halfline l with vertex 0 so that $f(t) \in l$ for every continuity point t of f. $\square$

3.2 Curvilinear integrals of complex functions.

We recall that every *continuous* complex function $\gamma : [a, b] \to \mathbb{C}$, where $[a, b]$ is any interval, is called *curve* in the complex plane.

The set of the values of a curve γ, i.e. the set $\gamma^* = \{\gamma(t) \mid t \in [a, b]\} \subseteq \mathbb{C}$ is the *trajectory* of the curve and it is a compact and connected subset of $\mathbb{C}$, since γ is continuous and $[a, b]$ is compact and connected. The points $\gamma(a)$ and $\gamma(b)$ are the endpoints, the initial and the final endpoint, respectively, of the curve. The variable $t \in [a, b]$ is the *parameter* and $[a, b]$ is the *parametric interval* of the curve. When the parameter t increases in $[a, b]$, the variable point $\gamma(t)$ moves on the trajectory γ^* in a definite direction (from the initial to the final endpoint) which is the so-called *direction* of the curve. Finally,

$$z = \gamma(t), \qquad t \in [a, b],$$

is the *parametric equation* of the curve γ.

If the endpoints of the curve γ coincide, i.e. $\gamma(a) = \gamma(b)$, then we say that the curve is *closed*. If $\gamma : [a, b] \to A$ is a curve, where $A \subseteq \mathbb{C}$, then $\gamma(t) \in A$ for all $t \in [a, b]$ or, equivalently, the trajectory γ^* is contained in A. Then we say that *the curve is in* A.

The term *curve* for the continuous function γ is justified by the fact that the shape of its trajectory γ^* is, usually, what in everyday language we call *curve in the plane*. Sometimes we use the term *curve* for the trajectory γ^* even though this is not typically correct. The problem is that there are cases of different curves γ_1, γ_2 with the same trajectory ${\gamma_1}^* = {\gamma_2}^*$.

Example 3.2.1. If $z_0, z_1 \in \mathbb{C}$, then $\gamma : [a, b] \to \mathbb{C}$ with the parametric equation

$$z = \gamma(t) = \frac{t-a}{b-a} z_1 + \frac{b-t}{b-a} z_0, \qquad t \in [a, b],$$

is a curve whose trajectory γ^* is the linear segment $[z_0, z_1]$. Its initial and final endpoints are z_0 and z_1, respectively.
The *same* linear segment $[z_0, z_1]$ is the trajectory of another curve $\gamma_0 : [0, 1] \to \mathbb{C}$ with the parametric equation

$$z = \gamma_0(t) = tz_1 + (1-t)z_0, \qquad t \in [0, 1].$$

Example 3.2.2. If $r > 0$, then $\gamma : [0, 2\pi] \to \mathbb{C}$ with parametric equation

$$z = \gamma(t) = z_0 + r(\cos t + i \sin t), \qquad t \in [0, 2\pi],$$

is a closed curve whose trajectory γ^* is the circle $C_{z_0}(r)$. The direction of this curve is the so-called *positive direction of rotation* around z_0: the counterclockwise rotation.
If we consider $\gamma_1 : [0, 2\pi] \to \mathbb{C}$ with parametric equation

$$z = \gamma_1(t) = z_0 + r(\cos(2t) + i \sin(2t)), \qquad t \in [0, 2\pi],$$

then we get a different curve. But the trajectories of the two curves, γ and γ_1, coincide: the circle $C_{z_0}(r)$. The direction of the two curves is the same: the positive direction of rotation around z_0. But the first curve goes around z_0 only once, while the second curve goes around z_0 twice.

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and let $x = \operatorname{Re} \gamma$ and $y = \operatorname{Im} \gamma$ be the real and imaginary parts of γ. I.e.

$$\gamma(t) = x(t) + iy(t) = (x(t), y(t)), \qquad t \in [a, b].$$

If γ is differentiable at $t_0 \in [a, b]$ or, equivalently, if x, y are differentiable at t_0, then

$$\gamma'(t_0) = x'(t_0) + iy'(t_0) = (x'(t_0), y'(t_0))$$

is the *tangent vector* of the trajectory γ^* at its point $\gamma(t_0)$. If $\gamma'(t_0) \neq 0$, then the vector $\gamma'(t_0)$ determines the *tangent line* of the trajectory γ^* at its point $\gamma(t_0)$ and its direction is the same as the direction of the curve.

Remark. Strictly speaking, at its endpoints, $\gamma(a), \gamma(b)$, the curve can only have *tangent halflines*; not tangent lines. If $t_0 = a$ and $\gamma'(a) \neq 0$, then the vector $\gamma'(a)$ determines the *tangent halfline* of the tracectory at the endpoint $\gamma(a)$ with direction coinciding with the direction of the curve. If $t_0 = b$ and $\gamma'(b) \neq 0$, then the vector $-\gamma'(b)$ determines the *tangent halfline* of the tracectory at the endpoint $\gamma(b)$ with direction opposite to the direction of the curve.

If at some $t_0 \in (a, b)$ the one-sided derivatives $\gamma'_-(t_0) \neq 0$ and $\gamma'_+(t_0) \neq 0$ exist but they are not equal, then the tangent halflines of the trajectory at its point $\gamma(t_0)$ may not be opposite and so there may be no tangent line of the trajectory at this point: one of the halflines is determined by $\gamma'_+(t_0)$ and the other by $-\gamma'_-(t_0)$.

We know that, if the curve $\gamma : [a, b] \to \mathbb{C}$ is continuously differentiable, i.e. if $\gamma' : [a, b] \to \mathbb{C}$ is continuous in $[a, b]$, then the *length* of the curve, denoted $l(\gamma)$, is equal to

$$l(\gamma) = \int_a^b |\gamma'(t)|\, dt. \tag{3.4}$$

Example 3.2.3. If $\gamma : [a, b] \to \mathbb{C}$ has parametric equation

$$z = \gamma(t) = \frac{b-t}{b-a} z_0 + \frac{t-a}{b-a} z_1, \qquad t \in [a, b],$$

then its length is equal to

$$l(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_a^b \Big|\frac{z_1 - z_0}{b-a}\Big|\, dt = \Big|\frac{z_1 - z_0}{b-a}\Big| \int_a^b dt = |z_1 - z_0|.$$

Example 3.2.4. If $r > 0$ and $\gamma : [0, 2\pi] \to \mathbb{C}$ has parametric equation

$$z = \gamma(t) = z_0 + r(\cos t + i \sin t), \qquad t \in [0, 2\pi],$$

then its length is equal to

$$l(\gamma) = \int_a^b |\gamma'(t)|\, dt = \int_0^{2\pi} |r(-\sin t + i \cos t)|\, dt = \int_0^{2\pi} r\, dt = 2\pi r.$$

The same formula (3.4) gives the length of the curve γ if it is *piecewise continuously differentiable*. This means that there is a subdivision $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the parametric interval $[a, b]$ so that the restriction of γ in every $[t_{k-1}, t_k]$ is continuously differentiable. (Strictly speaking, at the division points t_k the derivative of γ may not exist; the two one-sided derivatives should exist at these points.)

Another useful terminology is the following. A curve $\gamma : [a, b] \to \mathbb{C}$ is called *regular* if it is continuously differentiable and $\gamma'(t) \neq 0$ for every $t \in [a, b]$. This means that, when t increases in $[a, b]$, the tangent line at the point $\gamma(t)$ of the trajectory moves continuously. We also have the *piecewise regular* curves. The meaning is obvious.

At this point we shall make the following convention for the rest of this course:

Al our curves will be piecewise continuously differentiable.

Now let $\gamma_1 : [a, b] \to \mathbb{C}$ be a curve. We consider any $\sigma : [c, d] \to [a, b]$ which is one-to-one in the interval $[c, d]$ and onto $[a, b]$, has continuous derivative in $[c, d]$ and has $\sigma'(s) > 0$ for every $s \in [c, d]$. Thus, σ is strictly increasing in $[c, d]$ and $\sigma(c) = a$, $\sigma(d) = b$. Every such σ is called *change of parameter*. Then $\gamma_2 = \gamma_1 \circ \sigma : [c, d] \to \mathbb{C}$ is continuous in $[c, d]$ and hence it is a new curve. We say that γ_2 is a *reparametrization* of γ_1: the parameter of γ_1 is $t \in [a, b]$ and the parameter of γ_2 is $s \in [c, d]$. The curves γ_1, γ_2 have the same trajectory, the same endpoints and the same direction. Since σ' is continuous and > 0, the two curves are simultaneously (piecewise) continuously differentiable and simultaneously (piecewise) regular. The lengths of γ_1, γ_2 are equal:

$$\begin{aligned} l(\gamma_2) &= \int_c^d |\gamma_2{}'(s)|\, ds = \int_c^d |\gamma_1{}'(\sigma(s))||\sigma'(s)|\, ds \\ &= \int_c^d |\gamma_1{}'(\sigma(s))|\sigma'(s)\, ds = \int_a^b |\gamma_1{}'(t)|\, dt = l(\gamma_1). \end{aligned}$$

We may define the following binary relation between curves: $\gamma_1 \sim \gamma_2$ if γ_2 is a reparametrization of γ_1. It is not difficult to prove that this binary relation between curves is an *equivalence relation*, i.e. it satisfies the following three properties:
(i) $\gamma \sim \gamma$.
(ii) $\gamma_1 \sim \gamma_2 \ \Rightarrow\ \gamma_2 \sim \gamma_1$.
(iii) $\gamma_1 \sim \gamma_2,\ \gamma_2 \sim \gamma_3 \ \Rightarrow\ \gamma_1 \sim \gamma_3$.
Indeed: (i) Let $\gamma : [a,b] \to \mathbb{C}$ be any curve. We consider the change of parameter $id : [a,b] \to [a,b]$, defined by $id(t) = t$, and then $\gamma = \gamma \circ id : [a,b] \to \mathbb{C}$. Thus, $\gamma \sim \gamma$. (ii) Let $\gamma_1 \sim \gamma_2$. Then $\gamma_2 = \gamma_1 \circ \sigma$ where $\sigma : [c,d] \to [a,b]$ is a change of parameter. But then $\sigma^{-1} : [a,b] \to [c,d]$ is also a change of parameter and $\gamma_1 = \gamma_2 \circ \sigma^{-1}$. Therefore $\gamma_2 \sim \gamma_1$. (iii) Let $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. Then $\gamma_2 = \gamma_1 \circ \sigma$ and $\gamma_3 = \gamma_2 \circ \tau$, where $\sigma : [c,d] \to [a,b]$ and $\tau : [e,f] \to [c,d]$ are changes of parameter. But then $\chi = \sigma \circ \tau : [e,f] \to [a,b]$ is a change of parameter and $\gamma_3 = \gamma_2 \circ \tau = (\gamma_1 \circ \sigma) \circ \tau = \gamma_1 \circ \chi$. Therefore $\gamma_1 \sim \gamma_3$.

It is of some value to note that if we have a curve $\gamma : [a,b] \to \mathbb{C}$ with parametric interval $[a,b]$ and we are given an arbitrary interval $[c,d]$, then there is a reparametrization of γ with parametric interval $[c,d]$ instead of $[a,b]$. We can do this if we can find an appropriate change of parameter $\sigma : [c,d] \to [a,b]$. There are many such σ, but a simple one is

$$t = \sigma(s) = \frac{d-s}{d-c}a + \frac{s-c}{d-c}b, \qquad s \in [c,d].$$

Therefore, if for some reason (and we shall presently see that there is such a reason) we do not distinguish curves which are reparametrizations of each other, then the parametric interval of a curve is of no particular importance: we may consider a reparametrization of a given curve changing the given parametric interval to any other which we might prefer.

For every curve $\gamma : [a,b] \to \mathbb{C}$ we consider the curve $\neg\,\gamma : [a,b] \to \mathbb{C}$ given by

$$(\neg\,\gamma)(t) = \gamma(a+b-t), \qquad t \in [a,b].$$

Then $\neg\,\gamma$ is called *opposite* of γ. The curves γ and $\neg\,\gamma$ have the same trajectory but opposite directions. Their lengths are equal:

$$l(\neg\,\gamma) = \int_a^b |(\neg\,\gamma)'(t)|\, dt = \int_a^b |\gamma'(a+b-t)|\, dt = -\int_b^a |\gamma'(s)|\, ds = \int_a^b |\gamma'(s)|\, ds = l(\gamma).$$

If the curves $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ have $\gamma_1(b) = \gamma_2(b)$, then we say that γ_1, γ_2 (in this order) are *successive* and then we may define the curve $\gamma_1 \dot{+} \gamma_2 : [a,c] \to \mathbb{C}$ by

$$(\gamma_1 \dot{+} \gamma_2)(t) = \begin{cases} \gamma_1(t), & \text{if } a \leq t \leq b \\ \gamma_2(t), & \text{if } b \leq t \leq c \end{cases}$$

The curve $\gamma_1 \dot{+} \gamma_2$ is called *sum* of γ_1 and γ_2. Since γ_1 and γ_2 are (piecewise) continuously differentiable, $\gamma_1 \dot{+} \gamma_2$ is also piecewise continuously differentiable. The trajectory $(\gamma_1 \dot{+} \gamma_2)^*$ is the union of the trajectories $\gamma_1{}^*$ and $\gamma_2{}^*$.

Of course, the sum of two curves can be generalized to the sum of more than two curves provided that these are successive.

Example 3.2.5. Every polygonal line can be considered as the trajectory of a piecewise regular curve. This curve is the sum of successive curves each of which has as its trajectory a corresponding linear segment of the polygonal line.

Through the operation of summation of successive curves, we may consider successive curves as one curve (synthesis) and, conversely, we may consider one curve as a sum of successive curves (analysis).

The length of the sum of successive curves equals the sum of their lengths:

$$\begin{aligned} l(\gamma_1 \dot{+} \gamma_2) &= \int_a^c |(\gamma_1 \dot{+} \gamma_2)'(t)|dt = \int_a^b |(\gamma_1 \dot{+} \gamma_2)'(t)|dt + \int_b^c |(\gamma_1 \dot{+} \gamma_2)'(t)|dt \\ &= \int_a^b |{\gamma_1}'(t)|dt + \int_b^c |{\gamma_2}'(t)|dt = l(\gamma_1) + l(\gamma_2). \end{aligned}$$

Now we shall extend the notion of integral of a complex function over an *interval* to the notion of integral of a complex function over a *curve*.

Definition. *Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and let $f : \gamma^* \to \mathbb{C}$ be continuous in the trajectory $\gamma^* = \{\gamma(t) \,|\, t \in [a, b]\}$. Then $f \circ \gamma : [a, b] \to \mathbb{C}$ is continuous in $[a, b]$. Thus, $(f \circ \gamma)\gamma'$ is piecewise continuous in $[a, b]$ and hence integrable over $[a, b]$. We define the* **curvilinear integral** *of f over γ by*

$$\int_\gamma f(z)\, dz = \int_a^b (f \circ \gamma)(t)\gamma'(t)\, dt = \int_a^b f(\gamma(t))\gamma'(t)\, dt$$

We shall usually write

$$\oint_\gamma f(z)\, dz$$

when γ is closed.

Example 3.2.6. Let $\gamma : [a, b] \to \mathbb{C}$ be the curve with parametric equation $z = \gamma(t) = (1 - t)z_0 + tz_1$, $t \in [0, 1]$. The trajectory of γ is the linear segment $[z_0, z_1]$ having direction from z_0 to z_1. If $f : [z_0, z_1] \to \mathbb{C}$ is continuous in $[z_0, z_1]$, then the curvilinear integral $\int_\gamma f(z)\, dz$ is denoted $\int_{[z_0,z_1]} f(z)\, dz$. I.e.

$$\int_{[z_0,z_1]} f(z)\, dz = \int_\gamma f(z)\, dz = (z_1 - z_0) \int_0^1 f((1 - t)z_0 + tz_1)\, dt.$$

This is the *curvilinear integral of f over the linear segment $[z_0, z_1]$ from z_0 to z_1.*

Example 3.2.7. Let $r > 0$ and $\gamma : [0, 2\pi] \to \mathbb{C}$ be the curve with parametric equation $z = \gamma(t) = z_0 + r(\cos t + i \sin t)$, $t \in [0, 2\pi]$. The trajectory of γ is the circle $C_{z_0}(r)$ with the positive direction of rotation around z_0. If $f : C_{z_0}(r) \to \mathbb{C}$ is continuous in the circle $C_{z_0}(r)$, then the curvilinear integral $\oint_\gamma f(z)\, dz$ is denoted $\oint_{C_{z_0}(r)} f(z)\, dz$. I.e.,

$$\oint_{C_{z_0}(r)} f(z)\, dz = \oint_\gamma f(z)\, dz = \int_0^{2\pi} f\big(z_0 + r(\cos t + i \sin t)\big) r(-\sin t + i \cos t)\, dt.$$

This is the *curvilinear integral of f over the circle $C_{z_0}(r)$ with the positive direction of rotation.*

An important concrete instance of the previous example is the following.

Example 3.2.8. If $n \in \mathbb{Z}$, we know that

$$\int_0^{2\pi} \sin(nt)\, dt = 0, \qquad \int_0^{2\pi} \cos(nt)\, dt = \begin{cases} 2\pi, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

Therefore, if $n \in \mathbb{Z}$, we get

$$\begin{aligned}\oint_{C_{z_0}(r)} (z-z_0)^n \, dz &= \int_0^{2\pi} r^n(\cos t + i\sin t)^n r(-\sin t + i\cos t)\, dt \\ &= ir^{n+1}\int_0^{2\pi} (\cos t + i\sin t)^n(\cos t + i\sin t)\, dt \\ &= ir^{n+1}\int_0^{2\pi} \big(\cos((n+1)t) + i\sin((n+1)t)\big)\, dt \\ &= \begin{cases} 2\pi i, & \text{if } n=-1 \\ 0, & \text{if } n \neq -1 \end{cases}\end{aligned}$$

The following propositions are easy to prove.

Proposition 3.5. *Let $\gamma : [a,b] \to \mathbb{C}$ be a curve, $f_1, f_2 : \gamma^* \to \mathbb{C}$ be continuous in γ^* and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then*

$$\int_\gamma (\lambda_1 f_1(z) + \lambda_2 f_2(z))\, dz = \lambda_1 \int_\gamma f_1(z)\, dz + \lambda_2 \int_\gamma f_2(z)\, dz.$$

Proof. An application of proposition 3.1 and of the definition of the curvilinear integral. □

Proposition 3.6. *Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $f : \gamma^* \to \mathbb{C}$ be continuous in γ^*. If $|f(z)| \leq M$ for every $z \in \gamma^*$, then*

$$\Big|\int_\gamma f(z)\, dz\Big| \leq Ml(\gamma).$$

Proof.

$$\Big|\int_\gamma f(z)\, dz\Big| = \Big|\int_a^b f(\gamma(t))\gamma'(t)\, dt\Big| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leq M\int_a^b |\gamma'(t)|\, dt = Ml(\gamma).$$

The first inequality uses proposition 3.4. □

Proposition 3.7. *Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $f_n, \phi : \gamma^* \to \mathbb{C}$ be continuous in γ^* and let $f : \gamma^* \to \mathbb{C}$. If $f_n \to f$ uniformly in γ^*, then*

$$\int_\gamma f_n(z)\phi(z)\, dz \to \int_\gamma f(z)\phi(z)\, dz. \tag{3.5}$$

Proof. Because of uniform convergence, f is continuous in γ^*. Therefore, the existence of the integrals $\int_\gamma f_n(z)\phi(z)\, dz$ and $\int_\gamma f(z)\phi(z)\, dz$ is assured.
Since ϕ is continuous in the compact set γ^*, there is M so that $|\phi(z)| \leq M$ for every $z \in \gamma^*$. If we set $M_n = \sup_{z\in\gamma^*} |f_n(z) - f(z)|$, then proposition 3.6 implies

$$\Big|\int_\gamma f_n(z)\phi(z)\, dz - \int_\gamma f(z)\phi(z)\, dz\Big| = \Big|\int_\gamma (f_n(z) - f(z))\phi(z)\, dz\Big| \leq M_n M l(\gamma).$$

Since $M_n \to 0$, we get that $\int_\gamma f_n(z)\phi(z)\, dz \to \int_\gamma f(z)\phi(z)\, dz$. □

We may rewrite (3.5) in the form

$$\lim_{n\to+\infty}\int_\gamma f_n(z)\phi(z)\, dz = \int_\gamma \lim_{n\to+\infty} f_n(z)\phi(z)\, dz$$

of an interchange between the symbols $\lim_{n\to+\infty}$ and $\int_\gamma$. This interchange under the assumption of uniform convergence is the content of proposition 3.7.

Proposition 3.8. *Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and $f_n, \phi : \gamma^* \to \mathbb{C}$ be continuous in γ^* and let $s : \gamma^* \to \mathbb{C}$. If $\sum_{n=1}^{+\infty} f_n = s$ uniformly in γ^*, then*

$$\sum_{n=1}^{+\infty} \int_\gamma f_n(z)\phi(z)\, dz = \int_\gamma s(z)\phi(z)\, dz. \tag{3.6}$$

Proof. We consider the partial sums $s_n = f_1 + \cdots + f_n$ and apply proposition 3.7 to them. Then

$$\sum_{k=1}^{n} \int_\gamma f_k(z)\phi(z)\, dz = \int_\gamma \sum_{k=1}^{n} f_k(z)\phi(z)\, dz = \int_\gamma s_n(z)\phi(z)\, dz \to \int_\gamma s(z)\phi(z)\, dz. \tag{3.7}$$

I.e. the series (of numbers) $\sum_{n=1}^{+\infty} \int_\gamma f_n(z)\phi(z)\, dz$ converges to (the number) $\int_\gamma s(z)\phi(z)\, dz$. □

As in the case of (3.5), we may rewrite (3.6) in the form

$$\sum_{n=1}^{+\infty} \int_\gamma f_n(z)\phi(z)\, dz = \int_\gamma \sum_{n=1}^{+\infty} f_n(z)\phi(z)\, dz,$$

since $\sum_{n=1}^{+\infty} f_n(z) = s(z)$ for every $z \in \gamma^*$. Again, this interchange between the symbols $\sum_{n=1}^{+\infty}$ and $\int_\gamma$ under the assumption of uniform convergence is the content of proposition 3.8.

Proposition 3.9. *Consider the curves $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ and let γ_2 be a reparametrization of γ_1. Let also $f : {\gamma_1}^* = {\gamma_2}^* \to \mathbb{C}$ be continuous. Then*

$$\int_{\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

Proof. There is a change of parameter $\sigma : [c, d] \to [a, b]$ so that $\gamma_2(s) = \gamma_1(\sigma(s))$ for all $s \in [c, d]$. Then

$$\begin{aligned}\int_{\gamma_2} f(z)\, dz &= \int_c^d f(\gamma_2(s)){\gamma_2}'(s)\, ds = \int_c^d f(\gamma_1(\sigma(s))){\gamma_1}'(\sigma(s))\sigma'(s)\, ds \\ &= \int_a^b f(\gamma_1(t)){\gamma_1}'(t)\, dt = \int_{\gamma_1} f(z)\, dz\end{aligned}$$

after a change of parameter in the third integral. □

At this point we observe that replacing a curve γ_1 with a reparametrization γ_2 of it does not alter certain quantities related to the curve: its trajectory, its endpoints, its direction, its length, the number of times it covers its trajectory and, more important, the curvilinear integrals of continuous functions defined over its trajectory. Since in this course we shall use curves only to evaluate curvilinear integrals, we conclude that there is no reason to actually distinguish between a curve and its reparametrizations. Therefore, when we have a geometric object C which we would call, in everyday language, *curve in the plane*, e.g. a linear segment or a circle or a polygonal line, and a continuous function $f : C \to \mathbb{C}$, we can give a meaning to

$$\int_C f(z)\, dz$$

by specifying a continuous $\gamma : [a, b] \to \mathbb{C}$, i.e. a curve, with trajectory γ^* coinciding with C, with endpoints coinciding with the endpoints of C and a specific assigned direction. The use of different curves, which are reparametrizations of the particular γ we have chosen, will not alter the value of the integral. In fact we have already seen two examples of this situation. One is the

$$\int_{[z_0,z_1]} f(z)\, dz$$

for which we use any parametric equation with trajectory equal to the linear segment $[z_0, z_1]$ and direction from z_0 to z_1. The simplest such parametric equation is $z = \gamma(t) = (1-t)z_0 + tz_1$, $t \in [0,1]$. The second example is the

$$\oint_{C_{z_0}(r)} f(z)\, dz$$

for which we use any parametric equation with trajectory equal to the circle $C_{z_0}(r)$ and which covers this circle once and in the positive direction of rotation around z_0. The simplest such parametric equation is $z = \gamma(t) = z_0 + r(\cos t + i \sin t)$, $t \in [0, 2\pi]$.

Proposition 3.10. *Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [b,c] \to \mathbb{C}$ be two curves so that $\gamma_1(b) = \gamma_2(b)$ and let $f : {\gamma_1}^* \cup {\gamma_2}^* \to \mathbb{C}$ be continuous. Then*

$$\int_{\gamma_1 \dot{+} \gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$$

Proof. f is continuous in the trajectory $(\gamma_1 \dot{+} \gamma_2)^* = {\gamma_1}^* \cup {\gamma_2}^*$ of $\gamma_1 \dot{+} \gamma_2$. Hence

$$\begin{aligned}\int_{\gamma_1 \dot{+} \gamma_2} f(z)\, dz &= \int_a^c f\big((\gamma_1 \dot{+} \gamma_2)(t)\big)(\gamma_1 \dot{+} \gamma_2)'(t)\, dt \\ &= \int_a^b f(\gamma_1(t)){\gamma_1}'(t)\, dt + \int_b^c f(\gamma_2(t)){\gamma_2}'(t)\, dt = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.\end{aligned}$$

The second equality uses proposition 3.2. ☐

Proposition 3.11. *Consider the curve $\gamma : [a,b] \to \mathbb{C}$ and let $f : \gamma^* \to \mathbb{C}$ be continuous. Then*

$$\int_{\neg\gamma} f(z)\, dz = -\int_{\gamma} f(z)\, dz.$$

Proof. f is continuous in the trajectory $(\neg\gamma)^* = \gamma^*$. Hence

$$\begin{aligned}\int_{\neg\gamma} f(z)\, dz &= \int_a^b f((\neg\gamma)(t))(\neg\gamma)'(t)\, dt = -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)\, dt \\ &= \int_b^a f(\gamma(s))\gamma'(s)\, ds = -\int_a^b f(\gamma(s))\gamma'(s)\, ds = -\int_{\gamma} f(z)\, dz.\end{aligned}$$

after a simple change of parameter in the third integral. ☐

Example 3.2.9. Let γ be the curve describing the linear segment $[z_0, z_1]$ from z_0 to z_1. Then $\neg\gamma$ describes the same segment from z_1 to z_0. Therefore,

$$\int_{[z_0,z_1]} f(z)\, dz = \int_{\gamma} f(z)\, dz, \qquad \int_{[z_1,z_0]} f(z)\, dz = \int_{\neg\gamma} f(z)\, dz.$$

Hence

$$\int_{[z_1,z_0]} f(z)\, dz = -\int_{[z_0,z_1]} f(z)\, dz.$$

Exercises.

3.2.1. Consider an open set $\Omega \subseteq \mathbb{C}$ and a curve $\gamma : [a,b] \to \Omega$ and prove that there is $\delta > 0$ so that $|\gamma(t) - z| \geq \delta$ for every $t \in [a,b]$ and every $z \notin \Omega$.

3.2.2. Calculate $\int_\gamma |z|\,dz$, where γ is each of the following curves with initial endpoint $-i$ and final endpoint i. (i) $\gamma(t) = it$ for $t \in [-1,1]$. (ii) $\gamma(t) = \cos t + i\sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. (iii) $\gamma(t) = -\cos t + i\sin t$ for $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

3.2.3. (i) If $n \in \mathbb{Z}$, $n \geq 0$, prove that $\int_\gamma z^n\,dz = \frac{z_1^{n+1} - z_0^{n+1}}{n+1}$, where z_0, z_1 are the initial and the final endpoint of γ.
(ii) Are there polynomials $p_n(z)$ so that $p_n(z) \to \frac{1}{z}$ uniformly in the circle $C_0(1)$? Think in terms of curvilinear integrals over the circle $C_0(1)$.

3.2.4. Let f be continuous in the ring $\{z \,|\, 0 < |z| < r_0\}$ or in the ring $\{z \,|\, r_0 < |z| < +\infty\}$. We define $M(r) = \max\{|f(z)| \,|\, |z| = r\}$ and assume that $rM(r) \to 0$ when $r \to 0$ or $r \to +\infty$, respectively. If $\gamma_r(t) = r(\cos t + i\sin t)$ for $t_1 \leq t \leq t_2$, then prove that $\int_{\gamma_r} f(z)\,dz \to 0$ when $r \to 0$ or $r \to +\infty$, respectively.

3.2.5. Let $f : D_{z_0}(R) \to \mathbb{C}$ be continuous. Prove that $\lim_{r\to 0} \oint_{C_{z_0}(r)} \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$.

3.2.6. Let $\gamma : [a,b] \to \mathbb{C}$ be a curve and $f : \gamma^* \to \mathbb{C}$ be continuous in γ^*. Consider any subdivision $\Delta = \{t_0, \ldots, t_n\}$ of $[a,b]$ and any choice $\Xi = \{\xi_1, \ldots, \xi_n\}$ of intermediate points $\xi_k \in [t_{k-1}, t_k]$. Then we say that $\Delta^* = \{z_0, \ldots, z_n\}$, where $z_k = \gamma(t_k)$, is a subdivision of the trajectory γ^* and that $\Xi^* = \{\eta_1, \ldots, \eta_n\}$, where $\eta_k = \gamma(\xi_k)$, is a choice of intermediate points on the trajectory (η_k is between z_{k-1} and z_k on the trajectory). We say that $\sum_{k=1}^n f(z_k)(\eta_k - \eta_{k-1})$ is the corresponding Riemann sum. If $w(\Delta^*) = \max_{1\leq k\leq n} |z_k - z_{k-1}|$ is the width of the subdivision Δ^*, then prove that $\lim_{w(\Delta^*)\to 0} \sum_{k=1}^n f(z_k)(\eta_k - \eta_{k-1}) = \int_\gamma f(z)\,dz$.

3.2.7. Let $f : \Omega \to \mathbb{C}$ be continuous in the open set Ω and let $[a_n, b_n], [a,b] \subseteq \Omega$ for every n. If $a_n \to a$ and $b_n \to b$, prove that $\int_{[a_n,b_n]} f(z)\,dz \to \int_{[a,b]} f(z)\,dz$.

3.2.8. Let $f : \Omega \to \mathbb{C}$ be continuous in the open set Ω and γ be a curve in Ω. Prove that for every $\epsilon > 0$ there is a polygonal curve σ in Ω so that $|\int_\sigma f(z)\,dz - \int_\gamma f(z)\,dz| < \epsilon$.

Chapter 4

Holomorphic functions.

4.1 Differentiability and holomorphy.

Definition. *Let $f : A \to \mathbb{C}$ and z_0 be an interior point of A. We say that f is* **differentiable** *at z_0 if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and is a complex number. We call this limit* **derivative** *of f at z_0 and denote it*

$$f'(z_0) = \frac{df}{dz}(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}.$$

Example 4.1.1. The constant function c is differentiable at every point of $\mathbb{C}$ and its derivative is the constant function 0. Indeed, for every z_0 we have $\frac{dc}{dz}(z_0) = \lim_{z\to z_0} \frac{c-c}{z-z_0} = \lim_{z\to z_0} 0 = 0$.

Example 4.1.2. The function z is differentiable at every point of $\mathbb{C}$ and its derivative is the constant function 1: for every z_0 we have $\frac{dz}{dz}(z_0) = \lim_{z\to z_0} \frac{z-z_0}{z-z_0} = \lim_{z\to z_0} 1 = 1$.

Example 4.1.3. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = \overline{z}$. We take an arbitrary z_0 and we shall prove that the $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{\overline{z}-\overline{z_0}}{z-z_0}$ does not exist, i.e. f is not differentiable at z_0. Let $z_0 = x_0 + iy_0$. The limit of $\frac{f(z)-f(z_0)}{z-z_0}$ when $z \to z_0$ on the horizontal line containing z_0 is

$$\lim_{x\to x_0} \frac{\overline{(x+iy_0)} - \overline{(x_0+iy_0)}}{(x+iy_0)-(x_0+iy_0)} = \lim_{x\to x_0} \frac{x-x_0}{x-x_0} = \lim_{x\to x_0} 1 = 1$$

and the limit of $\frac{f(z)-f(z_0)}{z-z_0}$ when $z \to z_0$ on the vertical line containing z_0 is

$$\lim_{y\to y_0} \frac{\overline{(x_0+iy)} - \overline{(x_0+iy_0)}}{(x_0+iy)-(x_0+iy_0)} = \lim_{y\to y_0} \frac{-iy+iy_0}{iy-iy_0} = \lim_{y\to y_0} (-1) = -1.$$

Since these two limits are different, the $\lim_{z\to z_0} \frac{\overline{z}-\overline{z_0}}{z-z_0}$ does not exist.

The proofs of the following four propositions are identical with the proofs of the well-known analogous propositions for real functions of a real variable and we omit them.

Proposition 4.1. *If $f : A \to \mathbb{C}$ is differentiable at the interior point z_0 of A, then f is continuous at z_0.*

Proposition 4.2. *If $f, g : A \to \mathbb{C}$ are differentiable at the interior point z_0 of A, then $f+g, f-g, fg : A \to \mathbb{C}$ are also differentiable at z_0. Furthermore, if $g(z) \neq 0$ for all $z \in A$, then $\frac{f}{g} : A \to \mathbb{C}$ is differentiable at z_0. Finally,*

$$(f+g)'(z_0) = f'(z_0) + g'(z_0), \qquad (f-g)'(z_0) = f'(z_0) - g'(z_0),$$

$$(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0), \quad \Big(\frac{f}{g}\Big)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}.$$

Proposition 4.3. *If $f : A \to B$ is differentiable at the interior point z_0 of A and $g : B \to \mathbb{C}$ is differentiable at the interior point $w_0 = f(z_0)$ of B, then $g \circ f : A \to \mathbb{C}$ is differentiable at z_0. Also,*

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0).$$

Proposition 4.4. *Let $f : A \to B$ be one-to-one from A onto B and let $f^{-1} : B \to A$ be the inverse function. Let also z_0 be an interior point of A and $w_0 = f(z_0)$ be an interior point of B. If f is differentiable at z_0 and $f'(z_0) \neq 0$ and f^{-1} is continuous at w_0, then f^{-1} is differentiable at w_0 and*

$$(f^{-1})'(w_0) = \frac{1}{f'(z_0)}.$$

Example 4.1.4. Starting with the derivatives of the constant function c and the function z and using the usual algebraic rules for derivatives, we get that every polynomial function is differentiable at every point of $\mathbb{C}$ and that its derivative is another polynomial function: if $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, then $p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$.

Example 4.1.5. Every rational function is differentiable at every point of its domain of definition and its derivative is another rational function.

Example 4.1.6. If $h(z) = (z^2 - 3z + 2)^{15} - 3(z^2 - 3z + 2)^2$, then by the chain rule we get $h'(z) = 15(z^2 - 3z + 2)^{14}(2z - 3) - 6(z^2 - 3z + 2)(2z - 3)$.

Definition. *Let $f : A \to \mathbb{C}$ and z_0 be an interior point of A. We say that f is* **holomorphic** *(or* **analytic***) at z_0 if there is $r > 0$ so that $D_{z_0}(r) \subseteq A$ and f is differentiable at every point of $D_{z_0}(r)$.*

The notion of holomorphy is stronger than the notion of differentiability: for a function to be holomorphic at a point it is necessary for it to be differentiable at this point *and* at all nearby points.

Example 4.1.7. Every polynomial function is holomorphic at every point of $\mathbb{C}$.

Example 4.1.8. Every rational function is holomorphic at every point of its domain of definition.

Example 4.1.9. Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = |z|^2$.
We have $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \overline{z} = 0$ and f is differentiable at 0 with $f'(0) = 0$.
We take an arbitrary $z_0 \neq 0$ and we shall prove that the $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}$ does not exist and therefore f is not differentiable at z_0.
Let $z_0 = x_0 + iy_0$. The limit of $\frac{f(z)-f(z_0)}{z-z_0}$ when $z \to z_0$ on the horizontal line containing z_0 is

$$\lim_{x\to x_0} \frac{|x+iy_0|^2 - |x_0+iy_0|^2}{(x+iy_0)-(x_0+iy_0)} = \lim_{x\to x_0} \frac{x^2 - {x_0}^2}{x - x_0} = \lim_{x\to x_0} (x + x_0) = 2x_0$$

and the limit of $\frac{f(z)-f(z_0)}{z-z_0}$ when $z \to z_0$ on the vertical line containing z_0 is

$$\lim_{y\to y_0} \frac{|x_0+iy|^2 - |x_0+iy_0|^2}{(x_0+iy)-(x_0+iy_0)} = \lim_{y\to y_0} \frac{y^2 - {y_0}^2}{iy - iy_0} = -i \lim_{y\to y_0} (y + y_0) = -2iy_0.$$

Since $z_0 \neq 0$, these two limits are different and the $\lim_{z\to z_0} \frac{|z|^2-|z_0|^2}{z-z_0}$ does not exist.
We conclude that f is differentiable only at 0 and that it is nowhere holomorphic.

Definition. *The set of points at which f is holomorphic is called* **domain of holomorphy** *of f.*

Proposition 4.5. *Let $f : A \to \mathbb{C}$ and $B \subseteq A$ be the set of the points at which f is differentiable. Then the domain of holomorphy of f is the interior of B. In particular, the domain of holomorphy of f is an open set.*

Proof. Let U be the domain of holomorphy of f. If $z \in U$, there is $r > 0$ so that f is differentiable at every point of $D_z(r)$ and hence $D_z(r) \subseteq B$. Thus z is an interior point of B, i.e. $z \in B^\circ$. Conversely, let $z \in B^\circ$. Then there is $r > 0$ so that $D_z(r) \subseteq B$, and so f is differentiable at every point of $D_z(r)$. Therefore f is holomorphic at z, i.e. $z \in U$. □

Example 4.1.10. The domain of holomorphy of any polynomial function is $\mathbb{C}$.

Example 4.1.11. The domain of holomorphy of any rational function is its domain of definition.

Example 4.1.12. The domain of holomorphy of both functions $f(z) = \overline{z}$ and $f(z) = |z|^2$ is the empty set.

Definition. *Let $f : A \to \mathbb{C}$ and $\Omega \subseteq A$ be an open set. We say that f is holomorphic (or analytic) in Ω if it is holomorphic at every point of Ω or, equivalently, if Ω is a subset of the domain of holomorphy of f.*

Clearly, the largest open set Ω in which f is holomorphic is its domain of holomorphy. It is also clear that *if f is differentiable at every point of an open set Ω, then f is holomorphic in Ω.*

Definition. *Let $f : D_\infty(r) \to \mathbb{C}$. We consider the function $g : D_0(r) \to \mathbb{C}$ defined as*

$$g(w) = f(1/w), \qquad \textit{for every } w \textit{ with } |w| < r.$$

We say that f is differentiable or holomorphic at ∞ if g is differentiable or holomorphic, respectively, at 0.

We observe that $g(0) = f(\infty)$ and that the inverse functions $w = \frac{1}{z}$ and $z = \frac{1}{w}$ map each of the neighborhoods $D_\infty(r) = \{z \,|\, |z| > \frac{1}{r}\} \cup \{\infty\}$ and $D_0(r) = \{w \,|\, |w| < r\}$ onto the other. Now we shall see that the condition of differentiability of f at ∞, i.e. the differentiability of g at 0, can be translated into an equivalent condition in terms of f itself.

Proposition 4.6. *Let $f : D_\infty(r) \to \mathbb{C}$. Then f is differentiable at ∞ if and only if*

$$\lim_{z\to\infty} z(f(z) - f(\infty)) \in \mathbb{C}. \tag{4.1}$$

Moreover, f is holomorphic at ∞ if and only if, besides (4.1), f is differentiable at every complex number in a neighborhood of ∞.

Proof. Let $g(w) = f(\frac{1}{z})$ be the function considered in the above definition. Through the change of variable $w = \frac{1}{z}$, we have $\frac{g(w)-g(0)}{w-0} = z(f(z) - f(\infty))$. Thus, the existence of $\lim_{w\to 0} \frac{g(w)-g(0)}{w-0}$ is equivalent to the existence of $\lim_{z\to\infty} z(f(z) - f(\infty))$. In fact the two limits are equal. □

It is easy to see that differentiability of f at ∞ implies continuity of f at ∞.

Example 4.1.13. We shall check the differentiability (and hence holomorphy) of polynomial and rational functions. We recall the notation and the results of examples 1.4.1, 1.4.2 and 1.4.3.
A polynomial function p is continuous only if it is a constant $p(z) = a_0$ and provided we define $p(\infty) = a_0$. In this case it is also differentiable at ∞, since

$$\lim_{z\to\infty} z(p(z) - p(\infty)) = \lim_{z\to\infty} 0 = 0.$$

A rational function r is continuous only if $n \le m$, where n and m are the degrees of its numerator and denominator. If $n = m$, then we define $r(\infty) = \frac{a_n}{b_n}$ and then, after some algebraic manipulations, we get

$$\lim_{z\to\infty} z(r(z) - r(\infty)) = \lim_{z\to\infty} z\Big(\frac{a_n z^n + \cdots + a_1 z + a_0}{b_n z^n + \cdots + b_1 z + b_0} - \frac{a_n}{b_n}\Big) = \frac{a_{n-1}b_n - a_n b_{n-1}}{b_n^2}.$$

If $n < m$, then we define $r(\infty) = 0$ and then we get

$$\lim_{z\to\infty} z(r(z) - r(\infty)) = \lim_{z\to\infty} z\,\frac{a_nz^n + \cdots + a_1z + a_0}{b_mz^m + \cdots + b_1z + b_0} = \begin{cases} \frac{a_n}{b_{n+1}}, & \text{if } n+1 = m \\ 0, & \text{if } n+1 < m \end{cases}$$

Thus, if polynomial or rational functions are continuous at ∞, they are also holomorphic at ∞.

Exercises.

4.1.1. Check the differentiability of the functions $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$.

4.1.2. Let Ω be open and $f : \Omega \to \mathbb{C}$. We take $\Omega^* = \{z \,|\, \overline{z} \in \Omega\}$ and $f^* : \Omega^* \to \mathbb{C}$ given by $f^*(z) = \overline{f(\overline{z})}$ for every $z \in \Omega^*$. Prove that Ω^* is open and that, if f is differentiable at $z_0 \in \Omega$, then f^* is differentiable at $\overline{z_0} \in \Omega^*$.

4.1.3. Consider open sets U, V and $f : V \to U$, $g : U \to \mathbb{C}$, $h : V \to \mathbb{C}$ so that h is one-to-one and $h = g \circ f$. If h is differentiable at $w_0 \in V$, g is differentiable at $z_0 = f(w_0)$, $g'(z_0) \neq 0$ and f is continuous at w_0, prove that f is differentiable at w_0 and $f'(w_0) = \frac{h'(w_0)}{g'(z_0)}$.

4.1.4. (i) If p is a polynomial of degree n with roots $z_1, \ldots, z_n$, prove $\frac{p'(z)}{p(z)} = \frac{1}{z-z_1} + \cdots + \frac{1}{z-z_n}$ for every $z \neq z_1, \ldots, z_n$. Then prove that, if the roots of p are contained in a closed halfplane, then the roots of p' are contained in the same halfplane. Conclude that the roots of p' are contained in the smallest convex polygon which contains the roots of p.
(ii) For every a and every $n \in \mathbb{N}$, $n \geq 2$ prove that the equation $1 + z + az^n = 0$ has at least one root $z \in \overline{D}_0(2)$.

4.1.5. (i) Let $a_1, \ldots, a_n$ be distinct and $q(z) = (z - a_1) \cdots (z - a_n)$. If the polynomial p has degree $< n$, prove $\frac{p(z)}{q(z)} = \sum_{k=1}^n \frac{p(a_k)}{q'(a_k)(z-a_k)}$ for every $z \neq a_1, \ldots, a_n$.
(ii) Let $a_1, \ldots, a_n$ be distinct. Prove that for every $c_1, \ldots, c_n$ there is a unique polynomial p of degree $< n$ so that $p(a_k) = c_k$ for every $k = 1, \ldots, n$.

4.1.6. Let f have continuous derivative in a neighborhood of z_0. Prove that $\frac{f(z_n)-f(z'_n)}{z_n-z'_n} \to f'(z_0)$ if $z_n \to z_0$, $z'_n \to z_0$ and $z_n \neq z'_n$ for every n.

4.2 The Cauchy-Riemann equations.

Now we shall relate the differentiability of $f : A \to \mathbb{C}$, as a function of $z = x + iy$, at some interior point $z_0 = x_0 + iy_0$ of A with the partial derivatives of $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ as functions of (x, y) at the same point (x_0, y_0).

Theorem 4.1. *Let $f : A \to \mathbb{C}$ and $z_0 = (x_0, y_0)$ be an interior point of A and let u, v be the real and imaginary part of f. If f is differentiable at z_0, then u, v have partial derivatives with respect to x and y at (x_0, y_0) and*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{4.2}$$

Proof. We assume

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = \mu + i\nu, \qquad \mu, \nu \in \mathbb{R}. \tag{4.3}$$

Since the limit of $\frac{f(z)-f(z_0)}{z-z_0}$ exists when z tends to z_0, the limits of the same expression when z tends to z_0 on the horizontal line containing z_0 as well as on the vertical line containing z_0 also exist and have the same value:

$$\lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \mu + i\nu, \qquad \lim_{y\to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{iy - iy_0} = \mu + i\nu. \tag{4.4}$$

From the first limit in (4.4) we get $\lim_{x\to x_0} \frac{u(x,y_0)+iv(x,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{x-x_0} = \mu + i\nu$, and hence

$$\begin{aligned}\frac{\partial u}{\partial x}(x_0,y_0) &= \lim_{x\to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} = \mu,\\ \frac{\partial v}{\partial x}(x_0,y_0) &= \lim_{x\to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0} = \nu.\end{aligned} \tag{4.5}$$

From the second limit in (4.4) we find $\lim_{y\to y_0} \frac{u(x_0,y)+iv(x_0,y)-u(x_0,y_0)-iv(x_0,y_0)}{iy-iy_0} = \mu + i\nu$, and hence

$$\begin{aligned}\frac{\partial v}{\partial y}(x_0,y_0) &= \lim_{y\to y_0} \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0} = \mu,\\ \frac{\partial u}{\partial y}(x_0,y_0) &= \lim_{x\to x_0} \frac{u(x_0,y)-u(x_0,y_0)}{y-y_0} = -\nu.\end{aligned} \tag{4.6}$$

Comparing (4.5) and (4.6) we get (4.2). □

The equalities (4.2) are called **(system of) Cauchy-Riemann equations** at the point (x_0, y_0). We observe that, if f is differentiable at z_0, then (4.3), (4.5) and (4.6) imply

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).$$

The next result is the converse of theorem 4.1 but with extra assumptions.

Theorem 4.2. *Let $f : A \to \mathbb{C}$ and $z_0 = (x_0, y_0)$ be an interior point of A and let u, v be the real and the imaginary part of f. If u, v have partial derivatives with respect to x and y at every point of some neighborhood of (x_0, y_0) and if these partial derivatives are continuous at (x_0, y_0) and if they satisfy the system of C-R equations at (x_0, y_0), then f is differentiable at z_0.*

Proof. Using the C-R equations, we define the real numbers μ and ν by:

$$\mu = \frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0), \qquad \nu = -\frac{\partial u}{\partial y}(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0). \tag{4.7}$$

Now take an arbitrary $\epsilon > 0$. Since $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ are continuous at (x_0, y_0), there is $r > 0$ so that

$$\Big|\frac{\partial u}{\partial x}(x,y) - \mu\Big| < \frac{\epsilon}{4}, \quad \Big|\frac{\partial u}{\partial y}(x,y) + \nu\Big| < \frac{\epsilon}{4} \qquad \text{for every } (x,y) \in D_{(x_0,y_0)}(r). \tag{4.8}$$

We take any $(x, y) \in D_{(x_0,y_0)}(r)$ and we write

$$u(x,y) - u(x_0,y_0) = u(x,y) - u(x_0,y) + u(x_0,y) - u(x_0,y_0). \tag{4.9}$$

By the mean value theorem, there is x' between x and x_0 so that

$$u(x,y) - u(x_0,y) = \frac{\partial u}{\partial x}(x',y)(x-x_0) \tag{4.10}$$

and y' between y and y_0 so that

$$u(x_0,y) - u(x_0,y_0) = \frac{\partial u}{\partial y}(x_0,y')(y-y_0). \tag{4.11}$$

The x', y' depend on x, y, but the points (x', y), (x_0, y') belong to $D_{(x_0,y_0)}(r)$. Therefore, (4.8) implies

$$\Big|\frac{\partial u}{\partial x}(x',y) - \mu\Big| < \frac{\epsilon}{4}, \qquad \Big|\frac{\partial u}{\partial y}(x_0,y') + \nu\Big| < \frac{\epsilon}{4}. \tag{4.12}$$

Combining (4.9), (4.10) and (4.11), we find

$$\begin{aligned}
&u(x,y)-u(x_0,y_0)-\big(\mu(x-x_0)-\nu(y-y_0)\big)\\
&\qquad=\big(u(x,y)-u(x_0,y)-\mu(x-x_0)\big)+\big(u(x_0,y)-u(x_0,y_0)+\nu(y-y_0)\big)\\
&\qquad=\Big(\frac{\partial u}{\partial x}(x',y)-\mu\Big)(x-x_0)+\Big(\frac{\partial u}{\partial y}(x_0,y')+\nu\Big)(y-y_0)
\end{aligned}$$

and, because of (4.12),

$$\begin{aligned}
&\big|u(x,y)-u(x_0,y_0)-\big(\mu(x-x_0)-\nu(y-y_0)\big)\big|\\
&\qquad\le\Big|\frac{\partial u}{\partial x}(x',y)-\mu\Big||x-x_0|+\Big|\frac{\partial u}{\partial y}(x_0,y')+\nu\Big||y-y_0|\\
&\qquad<\frac{\epsilon}{4}|x-x_0|+\frac{\epsilon}{4}|y-y_0|<\frac{\epsilon}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}.
\end{aligned}\tag{4.13}$$

In the same manner, for the function v we get

$$\big|v(x,y)-v(x_0,y_0)-\big(\nu(x-x_0)+\mu(y-y_0)\big)\big|<\frac{\epsilon}{2}\sqrt{(x-x_0)^2+(y-y_0)^2}.\tag{4.14}$$

The inequalities (4.13) and (4.14) hold at every $(x,y)\in D_{(x_0,y_0)}(r)$.
We observe that the expressions inside the absolute values of the left sides of (4.13) and (4.14) are, respectively, the real and the imaginary part of the number

$$f(z)-f(z_0)-(\mu+i\nu)(z-z_0)=f(x,y)-f(x_0,y_0)-(\mu+i\nu)\big((x-x_0)+i(y-y_0)\big).$$

Therefore, (4.13) and (4.14) imply

$$|f(z)-f(z_0)-(\mu+i\nu)(z-z_0)|<\epsilon\sqrt{(x-x_0)^2+(y-y_0)^2}=\epsilon|z-z_0|$$

for every $z\in D_{z_0}(r)$ and hence $|\frac{f(z)-f(z_0)}{z-z_0}-(\mu+i\nu)|<\epsilon$ for every $z\in D_{z_0}(r)$, $z\neq z_0$. Thus, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\mu+i\nu$, and f is differentiable at z_0 with $f'(z_0)=\mu+i\nu$. □

Example 4.2.1. The real and the imaginary parts of the function $f(z)=z^2$ are $u(x,y)=x^2-y^2$ and $v(x,y)=2xy$. We find $\frac{\partial u}{\partial x}(x,y)=2x$, $\frac{\partial u}{\partial y}(x,y)=-2y$, $\frac{\partial v}{\partial x}(x,y)=2y$ and $\frac{\partial v}{\partial y}(x,y)=2x$ and we see that the partial derivatives are continuous in the whole plane and they satisfy the C-R equations at every point. Theorem 4.2 implies that $f(z)=z^2$ is differentiable at every point and $f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)=2x+i2y=2z$.

Example 4.2.2. We reconsider the function $f(z)=\overline{z}$ of example 4.1.3. Its real and imaginary parts are $u(x,y)=x$ and $v(x,y)=-y$. The partial derivatives $\frac{\partial u}{\partial x}(x,y)=1$, $\frac{\partial u}{\partial y}(x,y)=0$, $\frac{\partial v}{\partial x}(x,y)=0$ and $\frac{\partial v}{\partial y}(x,y)=-1$ do not satisfy the C-R equations at any point (x,y). Theorem 4.1 implies that f is not differentiable at any point.

Example 4.2.3. We reconsider the function $f(z)=|z|^2$ of example 4.1.9. Its real and imaginary parts are $u(x,y)=x^2+y^2$ and $v(x,y)=0$. The partial derivatives are $\frac{\partial u}{\partial x}(x,y)=2x$, $\frac{\partial u}{\partial y}(x,y)=2y$, $\frac{\partial v}{\partial x}(x,y)=0$ and $\frac{\partial v}{\partial y}(x,y)=0$ and they satisfy the C-R equations only at the point $(0,0)$. Theorem 4.1 implies that f is not differentiable at any point besides, perhaps, the point $(0,0)$. Now, since the partial derivatives are continuous and satisfy the C-R equations at $(0,0)$, theorem 4.2 implies that f is differentiable at 0 and $f'(0)=\frac{\partial u}{\partial x}(0,0)+i\frac{\partial v}{\partial x}(0,0)=0+i0=0$.

Example 4.2.4. We shall see that the assumption of continuity of the partial derivatives of u,v at (x_0,y_0) in theorem 4.2 is crucial. We consider the function

$$f(z)=f(x,y)=\begin{cases}\frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$$

Then its real and imaginary parts are

$$u(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \qquad v(x,y) = 0.$$

It is clear that $\frac{\partial v}{\partial x}(x,y) = 0$ and $\frac{\partial v}{\partial y}(x,y) = 0$ and the partial derivatives of v are continuous at every (x,y). Moreover,

$$\frac{\partial u}{\partial x}(x,y) = \begin{cases} \frac{y^3}{(x^2+y^2)^{3/2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases} \qquad \frac{\partial u}{\partial y}(x,y) = \begin{cases} \frac{x^3}{(x^2+y^2)^{3/2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

The partial derivatives of u are continuous at every $(x,y) \neq (0,0)$ but they are *not* continuous at $(0,0)$. For instance, the limit of $\frac{y^3}{(x^2+y^2)^{3/2}}$ when (x,y) tends to $(0,0)$ on the line with equation $y = x$ does not exist: $\lim_{x\to\pm 0} \frac{x^3}{(x^2+x^2)^{3/2}} = \pm 1/\sqrt{8}$.
We will see now that f is *not* differentiable at 0, *even though* u, v *do satisfy the C-R equations at* 0. In fact the limit of $\frac{f(z)-f(0)}{z-0} = \frac{(xy)/\sqrt{x^2+y^2}}{x+iy}$ when z tends to 0 on the line with equation $y = x$ is $\lim_{x\to 0} \frac{x^2/\sqrt{x^2+x^2}}{x+ix} = \frac{1}{(1+i)\sqrt{2}} \lim_{x\to 0} \frac{x}{|x|}$ and it does not exist.

The next proposition is a corollary of theorem 4.2. It is the form of theorem 4.2 in which this is usually applied.

Proposition 4.7. *Let* $f : A \to \mathbb{C}$*, let* u, v *be the real and the imaginary part of* f *and let* $\Omega \subseteq A$ *be open. If* u, v *have partial derivatives which are continuous and which satisfy the C-R equations at every point of* Ω*, then* f *is holomorphic in* Ω*.*

Proof. We take an arbitrary $z \in \Omega$ and a neighborhood of z which is contained in Ω. Theorem 4.2 implies that f is differentiable at z. Thus f is differentiable at every point of Ω and, since Ω is open, f is holomorphic in Ω. □

We recall that *region* means: open and connected.

Theorem 4.3. *Let* $f : \Omega \to \mathbb{C}$ *be holomorphic in the region* Ω*. If* $f'(z) = 0$ *for every* $z \in \Omega$*, then* f *is constant in* Ω*.*

First proof. Using $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, we find

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0 \qquad \text{on } \Omega. \tag{4.15}$$

We take any linear segment $[z_1, z_2]$ in Ω and its parametric equation $\gamma(t) = (1-t)z_1 + tz_2$, $t \in [0,1]$. By the mean value theorem, there is $t_0 \in (0,1)$ so that

$$\begin{aligned} u(z_2) - u(z_1) = (u\circ\gamma)(1) - (u\circ\gamma)(0) &= \frac{d(u\circ\gamma)}{dt}(t_0) \\ &= \frac{\partial u}{\partial x}(\gamma(t_0))(x_2 - x_1) + \frac{\partial u}{\partial y}(\gamma(t_0))(y_2 - y_1), \end{aligned}$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. From (4.15) we find $u(z_2) = u(z_1)$. Thus, the values of u at the endpoints of any line segment in Ω are equal. Now we take arbitrary $z', z'' \in \Omega$. Then there is a polygonal line inside Ω which connects the two points z' and z''. The values of u at the endpoints of every line segment of the polygonal line are equal and hence $u(z') = u(z'')$. Therefore u is constant in Ω. Clearly, the same is true for the function v and hence for $f = u + iv$.
Second proof. We take arbitrary $z, w \in \Omega$. Since Ω is a region, there is a curve $\gamma : [a,b] \to \Omega$ such

that $\gamma(a) = z$, $\gamma(b) = w$. In fact we may choose γ to have a polygonal line in Ω as its trajectory. Then we have

$$f(w) - f(z) = (f \circ \gamma)(b) - (f \circ \gamma)(a) = \int_a^b (f \circ \gamma)'(t)\, dt = \int_a^b f'(\gamma(t))\gamma'(t)\, dt = 0$$

because $f'(\gamma(t)) = 0$ for every $t \in [a, b]$. We conclude that $f(w) = f(z)$ for every $w, z \in \Omega$ and hence f is constant in Ω. □

Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ with $u = \mathrm{Re}\, f : A \to \mathbb{R}$ and $v = \mathrm{Im}\, f : A \to \mathbb{R}$. Then $f = u + iv$ in A. Let $z_0 = x_0 + iy_0 = (x_0, y_0)$ be an interior point of A. Then, if the functions u, v have partial derivatives with respect to x and y at the point z_0, is is trivial to prove that at the point z_0 we have

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{4.16}$$

Definition. *We define the following differential operators:*

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{4.17}$$

Applying the differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ we just defined to f of the discussion above and using (4.16), we have at the point z_0:

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big), \\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big). \end{aligned} \tag{4.18}$$

From the second of equations (4.18) we see that the system of C-R equations at the point z_0 is equivalent to the single equation

$$\frac{\partial f}{\partial \overline{z}} = 0$$

at z_0. Moreover, if the system of C-R equations is satisfied, then the first equation (4.18) implies

$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'$$

at z_0. We summarize.

Proposition 4.8. *If $f : A \to \mathbb{C}$ is differentiable at the interior point z_0 of A then $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$ and $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$. Conversely, if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in a neighborhood of the point z_0 and they are continuous at z_0 and if $\frac{\partial f}{\partial \overline{z}}(z_0) = 0$, then f is differentiable at z_0.*

Proof. Trivial. The converse is a restatement of theorem 4.2. □

Sometimes a function $f : A \to \mathbb{C}$, with $A \subseteq \mathbb{C}$, is given to us through an expression $f(x, y)$ as a function of two real variables and we are interested in finding an expression $f(z)$ of the function in terms of the single complex variable z. We then write $x = \frac{z+\overline{z}}{2}$, $y = \frac{z-\overline{z}}{2i}$ and hence

$$f(x, y) = f\Big(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\Big). \tag{4.19}$$

In general, even after performing various algebraic simplifications we end up with an expression in terms of *both* variables z and $\overline{z}$. In order to end up with the occurence of z only, it is reasonable to impose the condition that the derivative of $f(x, y)$ with respect to $\overline{z}$ vanishes. From (4.19) and a formal chain rule we get

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

This is exactly the second differential operator (4.17) applied to f and we saw that the condition $\frac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the system of C-R equations. We conclude that *the function $f(x, y)$ is a function of the single variable z if and only if its real and imaginary parts satisfy the C-R equations*.

Exercises.

4.2.1. Solve exercise 4.1.1 under the light of C-R equations.

4.2.2. (i) Prove that $F(x, y) = \sqrt{|xy|}$ satisfies the C-R equations at 0 but that it is not differentiable at 0.
(ii) Prove that the function with $G(x, y) = \frac{x^2 y}{x^4 + y^2}$ if $(x, y) \neq (0, 0)$ and with $G(0, 0) = 0$ satisfies the C-R equations at 0, that $\frac{G(z) - G(0)}{z - 0}$ has a limit when $z \to 0$ on every line which contains 0, but that G is not differentiable at 0.

4.2.3. Let $f : A \to \mathbb{C}$ and z_0 be an interior point of A, let u, v be the real and the imaginary part of f and let $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist in a neighborhood of z_0 and be continuous at z_0.
(i) If $\lim_{z \to z_0} \mathrm{Re}\, \frac{f(z) - f(z_0)}{z - z_0}$ exists and is a real number, prove that f is differentiable at z_0.
(ii) If $\lim_{z \to z_0} \left| \frac{f(z) - f(z_0)}{z - z_0} \right|$ exists and is a real number, prove that either f is differentiable at z_0 or $\overline{f}$ is differentiable at z_0.

4.2.4. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and let u, v be the real and the imaginary part of f.
(i) If either u or v is constant in Ω, prove that f is constant in Ω.
(ii) More generally, if for some line l it is true that $f(z) \in l$ for every $z \in \Omega$, prove that f is constant in Ω.
(iii) Consider (ii) using a circle C instead of a line l.

4.2.5. This exercise juxtaposes the notion of differentiability of a function of two real variables, which we learn in multivariable calculus, and the notion of differentiability of a function of one complex variable, which we learn in complex analysis: to distinguish between them we call the first $\mathbb{R}$-differentiability and the second $\mathbb{C}$-differentiability.
We recall from multivariable calculus that a *real valued* $u : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$, is $\mathbb{R}$-differentiable at the interior point (x_0, y_0) of A if there are $a, b \in \mathbb{R}$ so that

$$\lim_{(x,y) \to (x_0, y_0)} \frac{u(x, y) - u(x_0, y_0) - (a(x - x_0) + b(y - y_0))}{((x - x_0)^2 + (y - y_0)^2)^{1/2}} = 0.$$

In this case we have that $\frac{\partial u}{\partial x}(x_0, y_0) = a$ and $\frac{\partial u}{\partial y}(x_0, y_0) = b$.
We also recall that a *vector valued* $f = (u, v) : A \to \mathbb{R}^2$, where $A \subseteq \mathbb{R}^2$, is $\mathbb{R}$-differentiable at the interior point (x_0, y_0) of A if its real valued components u and v are both $\mathbb{R}$-differentiable at (x_0, y_0), i.e. if there are $a, b, c, d \in \mathbb{R}$ so that

$$\lim_{(x,y) \to (x_0, y_0)} \frac{u(x, y) - u(x_0, y_0) - (a(x - x_0) + b(y - y_0))}{((x - x_0)^2 + (y - y_0)^2)^{1/2}} = 0,$$

$$\lim_{(x,y) \to (x_0, y_0)} \frac{v(x, y) - v(x_0, y_0) - (c(x - x_0) + d(y - y_0))}{((x - x_0)^2 + (y - y_0)^2)^{1/2}} = 0.$$

In this case we have that $\frac{\partial u}{\partial x}(x_0, y_0) = a$, $\frac{\partial u}{\partial y}(x_0, y_0) = b$, $\frac{\partial v}{\partial x}(x_0, y_0) = c$, $\frac{\partial v}{\partial y}(x_0, y_0) = d$ and that the $\mathbb{R}$-derivative of f is the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Prove that $f = (u, v) = u + iv : A \to \mathbb{C} = \mathbb{R}^2$, where $A \subseteq \mathbb{C} = \mathbb{R}^2$, is $\mathbb{C}$-differentiable at the interior point $z_0 = (x_0, y_0)$ of A, i.e. that the $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists and is a complex number, if and only if f is $\mathbb{R}$-differentiable at $z_0 = (x_0, y_0)$ and its $\mathbb{R}$-derivative is an antisymmetric matrix: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. In this case the $\mathbb{C}$-derivative and the $\mathbb{R}$-derivative of f are related by $f'(z_0) = a + ib$.

4.2.6. Consider the functions $z^n, \overline{z}^n, |z|^2$ and, using the differential operator $\frac{\partial}{\partial \overline{z}}$, examine whether they are functions of z only or, equivalently, whether they are holomorphic.

4.2.7. Let $f : A \to \mathbb{C}$ and z_0 be an interior point of A. If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in a neighborhood of the point z_0 and are continuous at z_0, prove that $\lim_{r\to 0} \frac{1}{2\pi i r^2} \oint_{C_{z_0}(r)} f(z)\,dz = \frac{\partial f}{\partial \overline{z}}(z_0)$.

4.3 Conformality.

Definition. *Let $f : A \to \mathbb{C}$ be continuous in A and $\gamma : [a, b] \to A$ be a curve. Thus the trajectory of γ is contained in the domain of definition of f. We define the function*

$$f(\gamma) = f \circ \gamma : [a, b] \to \mathbb{C},$$

which is continuous in $[a, b]$. Then $f(\gamma)$ is a curve and we call it **image of γ through f**.

Now we take $f : A \to \mathbb{C}$ and an interior point z_0 of A. Let f be differentiable at z_0 and

$$f(z_0) = w_0, \qquad f'(z_0) \neq 0.$$

We also take any curve $\gamma : [t_0, b) \to A$ with $\gamma(t_0) = z_0$. Then γ has z_0 as its initial point and its trajectory is contained in A. We also assume that

$$\gamma'(t_0) \neq 0$$

i.e. that γ has a non-zero tangent vector at the point z_0. The image curve $f(\gamma) : [t_0, b) \to \mathbb{C}$ has $f(\gamma)(t_0) = (f \circ \gamma)(t_0) = f(\gamma(t_0)) = f(z_0) = w_0$ as its initial point and its tangent vector at w_0 is

$$f(\gamma)'(t_0) = (f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0) \neq 0. \tag{4.20}$$

From (4.20) we have two conclusions. The first is that

$$|f(\gamma)'(t_0)| = |f'(z_0)||\gamma'(t_0)|.$$

Thus, the length of the tangent vector of $f(\gamma)$ at its initial point w_0 equals the length of the tangent vector of γ at its initial point z_0 multiplied with the factor $|f'(z_0)| > 0$. We express this as:
f multiplies the lengths of tangent vectors at z_0 with the factor $|f'(z_0)| > 0$ or, in other words, *f expands the tangent vectors at z_0 by the factor $|f'(z_0)| > 0$.*

The second conclusion is that

$$\arg f(\gamma)'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0). \tag{4.21}$$

Thus, the argument of the tangent vector of $f(\gamma)$ at its initial point w_0 equals the argument of the tangent vector of γ at its initial point z_0 increased by the angle $\arg f'(z_0)$. We express this as:
f increases the arguments of the tangent vectors at z_0 by the angle $\arg f'(z_0)$ or, in other words, *f rotates the tangent vectors at z_0 through the angle* $\arg f'(z_0)$.

We observe that the expansion and the rotation of the tangent vectors at z_0 is uniform over all these vectors: *independently of their direction all these tangent vectors are expanded by the same factor and they are rotated through the same angle*. Since, any two of these tangent vectors are rotated by f through the same angle, we conclude that their relative angle remains unchanged!

Indeed, let us consider two of the above curves, γ_1 and γ_2. The angle between their tangent vectors at z_0 is $\arg {\gamma_2}'(t_0)) - \arg {\gamma_1}'(t_0))$ and the angle between the tangent vectors of $f(\gamma_1)$ and $f(\gamma_2)$ at w_0 is $\arg f(\gamma_2)'(t_0) - \arg f(\gamma_1)'(t_0)$. From (4.21) for γ_1 and γ_2 we get

$$\arg f(\gamma_2)'(t_0) - \arg f(\gamma_1)'(t_0) = \arg {\gamma_2}'(t_0) - \arg {\gamma_1}'(t_0).$$

Therefore, the angle between the tangent vectors of $f(\gamma_1)$ and $f(\gamma_2)$ at w_0 equals the angle between the tangent vectors of γ_1 and γ_2 at z_0. We express this as:

f preserves the angles between tangent vectors at z_0.

This last property of f is called **conformality** of f at z_0 and holds, as we just saw, under the assumption that f is differentiable at z_0 and $f'(z_0) \neq 0$.

Exercises.

4.3.1. Consider the holomorphic function $w = f(z) = az + b$ with $a \neq 0$.
(i) Prove that f is ono-to-one from $\mathbb{C}$ onto $\mathbb{C}$.
(ii) Prove that f maps lines and circles onto lines and circles, respectively.
(iii) Consider two lines with equations $kx + ly = m$ and $k'x + l'y = m'$. Which is the condition for the two lines to intersect? Under this condition, find their intersection point and the angle of the two lines at this point. Then find the equations of the images of the two lines through f and find their intersection point and their angle at this point. Confirm the conformality of f.

4.3.2. Consider the holomorphic function $w = z^2$.
(i) With any fixed $u_0, v_0 \in \mathbb{R}$, consider the hyperbolas with equations $x^2 - y^2 = u_0$ and $2xy = v_0$ on the z-plane ($z = x + iy$). Do they intersect and at which points? Find the angle of the two hyperbolas at their common points.
(ii) With any fixed $x_0, y_0 \in \mathbb{R}$, $x_0, y_0 \neq 0$, consider the parabolas with equations $u = \frac{1}{4{y_0}^2}v^2 - {y_0}^2$ and $u = -\frac{1}{4{x_0}^2}v^2 + {x_0}^2$ on the w-plane ($w = u + iv$). Do they intersect and at which points? Find the angle of the two parabolas at their common points.

4.3.3. Let $f : U \to \mathbb{C}$ be holomorphic in the open set U, let γ be a curve in U and $\Gamma = f(\gamma)$ be the image of γ through f. If $\phi : \Gamma^* \to \mathbb{C}$ is continuous in Γ^*, prove that $\int_\Gamma \phi(w)\, dw = \int_\gamma \phi(f(z)) f'(z)\, dz$.

Chapter 5

Examples of holomorphic functions.

5.1 Linear fractional transformations.

Definition. *Every rational function of the form* $T(z) = \frac{az+b}{cz+d}$ *is called* **linear fractional transformation**. *We assume that* $ad - bc \neq 0$.

It is easy to show that $ad - bc \neq 0$ if and only if the function T is not constant.

In order to have the full picture of the definition of a linear fractional transformation T, we have to say something about the values of T at the roots of the denominator and at ∞. There are two cases. If $c = 0$, then because of $ad - bc \neq 0$ we have $ad \neq 0$ and then $T(z) = \frac{a}{d}z + \frac{b}{d}$ for all $z \in \mathbb{C}$. Since $\frac{a}{d} \neq 0$, we have that $T(\infty) = \infty$. Thus

$$T(z) = \begin{cases} \frac{a}{d}z + \frac{b}{d}, & \text{if } z \in \mathbb{C} \\ \infty, & \text{if } z = \infty \end{cases} \qquad \text{if } c = 0. \tag{5.1}$$

If $c \neq 0$, then the denominator has $z = -\frac{d}{c}$ as its root, which, because of $ad - bc \neq 0$, is not a root of the numerator. Hence $T(-\frac{d}{c}) = \infty$. Also $T(\infty) = \frac{a}{c}$. Thus

$$T(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \in \mathbb{C}, z \neq -\frac{d}{c} \\ \infty, & \text{if } z = -\frac{d}{c} \\ \frac{a}{c}, & \text{αν } z = \infty \end{cases} \qquad \text{if } c \neq 0. \tag{5.2}$$

We conclude that every linear fractional transformation (l.f.t.) is a function $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and, even though we write $T(z) = \frac{az+b}{cz+d}$, we must have in mind the full formulas (5.1) and (5.2).

Proposition 5.1. *Every l.f.t. is one-to-one from* $\widehat{\mathbb{C}}$ *onto* $\widehat{\mathbb{C}}$.

Proof. Trivial. □

In the course of the proof of proposition 5.1 we find the formula of the inverse l.f.t. of T:

$$T^{-1}(z) = \frac{dz - b}{-cz + a}.$$

We also see easily that the composition of two l.f.t. is another l.f.t. Indeed, if $T(z) = \frac{az+b}{cz+d}$ and $S(z) = \frac{a'z+b'}{c'z+d'}$, then

$$(S \circ T)(z) = \frac{a'T(z) + b'}{c'T(z) + d'} = \frac{a'\frac{az+b}{cz+d} + b'}{c'\frac{az+b}{cz+d} + d'} = \frac{(a'a + b'c)z + (a'b + b'd)}{(c'a + d'c)z + (c'b + d'd)}.$$

We observe that

$$(a'a + b'c)(c'b + d'd) - (a'b + b'd)(c'a + d'c) = (a'd' - b'c')(ad - bc) \neq 0.$$

The identity function $I(z) = z$ is clearly a l.f.t. with $a = d = 1$, $b = c = 0$.

Based on this discussion we have:

Proposition 5.2. *The set of all l.f.t. is a group with the binary operation of composition. The neutral element is the identity function.*

Proof. Trivial. □

Proposition 5.3. *Every l.f.t. is holomorphic in $\widehat{\mathbb{C}}$ except at the point at which it takes the value ∞.*

Proof. This is a special case of example 4.1.13. □

We shall make a comment on a interesting relation between circles and lines. We observe that the equations of circles and lines can be unified in the following manner: if $\alpha, \beta, \gamma \in \mathbb{R}$, $w \in \mathbb{C}$, $w \neq 0$, $\alpha^2 + \beta^2 \neq 0$ and $\beta^2|w|^2 \geq 4\alpha\gamma$, then the equation

$$\alpha|z|^2 + \beta\,\mathrm{Re}(\overline{w}z) + \gamma = 0$$

is the equation of a line, if $\alpha = 0$, and the equation of a circle, if $\alpha \neq 0$. In fact, if $\alpha = 0$, then $\beta \neq 0$ and the equation becomes $\mathrm{Re}(\overline{w}z) = -\frac{\gamma}{\beta}$ and this is the equation of a line. If $\alpha \neq 0$, the equation becomes $|z + \frac{\beta}{2\alpha}w|^2 = \frac{\beta^2|w|^2 - 4\alpha\gamma}{4\alpha^2}$. This is the equation of the circle with center $-\frac{\beta}{2\alpha}w$ and radius $\frac{\sqrt{\beta^2|w|^2 - 4\alpha\gamma}}{2|\alpha|}$. Conversely, every circle and every line have equations of this form. If, for instance, we take the equation $\mathrm{Re}(\overline{w}z) = c$ of a line, with $w \in \mathbb{C}$, $w \neq 0$, and $c \in \mathbb{R}$, we may write it in the form $\alpha|z|^2 + \beta\,\mathrm{Re}(\overline{w}z) + \gamma = 0$ by taking $\alpha = 0$, $\beta = 1$ and $\gamma = -c$. If we take the equation $|z - z_0| = r$ of a circle with $z_0 \in \mathbb{C}$ and $r \geq 0$, we may write it as $|z|^2 - 2\,\mathrm{Re}(\overline{z_0}z) + |z_0|^2 = r^2$. This becomes $\alpha|z|^2 + \beta\,\mathrm{Re}(\overline{w}z) + \gamma = 0$ by taking $\alpha = 1$, $\gamma = |z_0|^2 - r^2$ and: $\beta = -2$ and $w = z_0$, in case $z_0 \neq 0$, or $\beta = 0$ and $w = 1$, in case $z_0 = 0$. In all cases the choices of the parameters satisfy the restrictions: $\alpha, \beta, \gamma \in \mathbb{R}$, $w \in \mathbb{C}$, $w \neq 0$, $\alpha^2 + \beta^2 \neq 0$ and $\beta^2|w|^2 \geq 4\alpha\gamma$.

This consideration of the equations of a line and a circle as special cases of one equation permits us to unify the notions of circle and line into the single notion of **generalized circle** in $\mathbb{C}$. If we attach the point ∞ to any line (and leave circles unchanged), then we are talking about generalized circles in $\widehat{\mathbb{C}}$.

Now, an important property of every l.f.t. is that it maps generalized circles in $\widehat{\mathbb{C}}$ onto generalized circles in $\widehat{\mathbb{C}}$. To prove it we consider three special cases first.

Example 5.1.1. Every function $T(z) = z + b$ is a l.f.t. with $a = 1$, $c = 0$, $d = 1$ and, for an obvious reason, it is called **translation** by b.
Every such T is holomorphic in $\mathbb{C}$, one-to-one from $\mathbb{C}$ onto $\mathbb{C}$ and $T(\infty) = \infty$. It is trivial to prove that T maps lines in $\widehat{\mathbb{C}}$ onto lines in $\widehat{\mathbb{C}}$ and circles in $\mathbb{C}$ onto circles in $\mathbb{C}$.

Example 5.1.2. Every function $T(z) = az$ with $a \neq 0$ is a l.f.t. with $b = c = 0$, $d = 1$ and it is called **homothety** with center 0.
Evet such T rotates points around 0 through the fixed angle $\arg a$. Indeed, if $w = T(z) = az$, then $\arg w = \arg z + \arg a$. Moreover, T multiplies distances between points by the fixed factor $|a|$. Indeed, if $w_1 = T(z_1) = az_1$ and $w_2 = T(z_2) = az_2$, then $|w_1 - w_2| = |a||z_1 - z_2|$.
Also T is holomorphic in $\mathbb{C}$, one-to-one from $\mathbb{C}$ onto $\mathbb{C}$ and also $T(\infty) = \infty$ and it is easy to prove that T maps lines in $\widehat{\mathbb{C}}$ onto lines in $\widehat{\mathbb{C}}$ and circles in $\mathbb{C}$ onto circles in $\mathbb{C}$.

Example 5.1.3. The function $T(z) = \frac{1}{z}$ is a l.f.t. with $a = d = 0$, $c = b = 1$ and it is called **inversion** with respect to the circle $\mathbb{T} = C_0(1)$.
The inversion T is holomorphic in $\widehat{\mathbb{C}} \setminus \{0\}$, one-to-one from $\widehat{\mathbb{C}} \setminus \{0, \infty\}$ onto $\widehat{\mathbb{C}} \setminus \{0, \infty\}$ and also

$T(0)=\infty$ and $T(\infty)=0$. Moreover, it is easy to show that T maps (i) lines in $\widehat{\mathbb{C}}$ which do not contain 0 onto circles in $\mathbb{C}$ which contain 0, (ii) lines in $\widehat{\mathbb{C}}$ which contain 0 onto lines in $\widehat{\mathbb{C}}$ which contain 0, (iii) circles in $\mathbb{C}$ which contain 0 onto lines in $\widehat{\mathbb{C}}$ which do not contain 0 and (iv) circles in $\mathbb{C}$ which do not contain 0 onto circles in $\mathbb{C}$ which do not contain 0.

Proposition 5.4. *Every l.f.t. is a composition of finitely many translations, homotheties and inversions.*

Proof. Let $T(z)=\frac{az+b}{cz+d}$.
If $c=0$, then $T(z)=a'z+b'$, where $a'=\frac{a}{d}\neq 0$ and $b'=\frac{b}{d}$. If we consider the homothety $T_1(z)=a'z$ and the translation $T_2(z)=z+b'$, then $T=T_2\circ T_1$.
If $c\neq 0$, then

$$T(z)=\frac{\frac{a}{c}(cz+d)+(b-\frac{ad}{c})}{cz+d}=\frac{a}{c}+\frac{bc-ad}{c^2}\,\frac{1}{z+\frac{d}{c}}.$$

If we consider the translation $T_1(z)=z+\frac{d}{c}$, the inversion $T_2(z)=\frac{1}{z}$, the homothety $T_3(z)=\frac{bc-ad}{c^2}z$ and the translation $T_4(z)=z+\frac{a}{c}$, then $T=T_4\circ T_3\circ T_2\circ T_1$. $\square$

Proposition 5.5. *Every l.f.t. maps generalized circles in $\widehat{\mathbb{C}}$ onto generalized circles in $\widehat{\mathbb{C}}$.*

Proof. A corollary of proposition 5.4 and of the examples 5.1.1, 5.1.2 and 5.1.3. $\square$

Proposition 5.6. *Take the distinct $z_1,z_2,z_3\in\widehat{\mathbb{C}}$ and the distinct $w_1,w_2,w_3\in\widehat{\mathbb{C}}$. Then there is a unique l.f.t. T so that $T(z_j)=w_j$ for $j=1,2,3$.*

Proof. We consider the l.f.t. S which, depending on whether one of z_1,z_2,z_3 is ∞ or not, has the formula

$$S(z)=\begin{cases}\frac{z_2-z_3}{z_2-z_1}\frac{z-z_1}{z-z_3}, & \text{if } z_1,z_2,z_3\neq\infty\\ \frac{z-z_1}{z_2-z_1}, & \text{if } z_3=\infty\\ \frac{z-z_1}{z-z_3}, & \text{if } z_2=\infty\\ \frac{z_2-z_3}{z-z_3}, & \text{if } z_1=\infty\end{cases}$$

The l.f.t. S has values: $S(z_1)=0$, $S(z_2)=1$, $S(z_3)=\infty$.
There is a similar l.f.t. R with values: $R(w_1)=0$, $R(w_2)=1$, $R(w_3)=\infty$.
Then the l.f.t. $T=R^{-1}\circ S$ has values: $T(z_1)=w_1$, $T(z_2)=w_2$, $T(z_3)=w_3$.
To prove the uniqueness of T with $T(z_1)=w_1$, $T(z_2)=w_2$, $T(z_3)=w_3$ we consider the previous l.f.t S,R and then the l.f.t. $Q=R\circ T\circ S^{-1}$ has values: $Q(0)=0$, $Q(1)=1$, $Q(\infty)=\infty$. Since $Q(\infty)=\infty$, we get that Q has the form $Q(z)=az+b$ with $a\neq 0$. Now from $Q(0)=0$, $Q(1)=1$ we find $a=1$, $b=0$ and hence Q is the identity l.f.t. I with $I(z)=z$. Thus $R\circ T\circ S^{-1}=I$ and hence $T=R^{-1}\circ S$. $\square$

When we apply the previous results we should bear in mind that every three distinct points in $\widehat{\mathbb{C}}$ belong to a unique generalized circle in $\widehat{\mathbb{C}}$.

Example 5.1.4. The l.f.t. which maps the triple $i,2,1$ onto the triple $0,1,\infty$ is

$$w=T(z)=\frac{2-1}{2-i}\,\frac{z-i}{z-1}=\frac{2+i}{5}\,\frac{z-i}{z-1}=\frac{(2+i)z+(1-2i)}{5z-5}.$$

The points $i,2,1$ in the z-plane are not co-linear and hence belong to a circle A. The points $0,1$ in the w-plane belong to the real axis m. Thus the points $0,1,\infty$ belong to the line $B=m\cup\{\infty\}$ in $\widehat{\mathbb{C}}$. Now, T maps the circle A in the z-plane onto some generalized circle $T(A)$ in the w-plane. Since A contains $i,2,1$, $T(A)$ must contain the images of $i,2,1$, i.e. $0,1,\infty$. Thus $T(A)=B$.
If we want to determine the circle $A=C_{z_0}(r)$ which contains $i,2,1$, we have to find z_0,r so that $i,2,1$ satisfy the equation $|z-z_0|=r$: we just solve a system of three equations in three

real unknowns: x_0, y_0, r. But there is a second and probably easier way to find the equation of A. Indeed, w belongs to m if and only if $\operatorname{Im} w = 0$ if and only if $\operatorname{Im} \frac{(2+i)z+(1-2i)}{5z-5} = 0$ (and $z \neq 1$) if and only if $|z|^2 - 3\operatorname{Re}((1-i)z) = -2$ (and $z \neq 1$) if and only if $|z - \frac{3}{2}(1+i)|^2 = -2 + \frac{9}{4}|1+i|^2 = \frac{5}{2}$ (and $z \neq 1$) if and only if z belongs to $C_{3(1+i)/2}\big(\sqrt{5/2}\big)$ except 1. Since $z = 1$ is mapped onto $w = \infty$, we have that w belongs to B if and only if z belongs to the circle $C_{3(1+i)/2}\big(\sqrt{5/2}\big)$. We conclude that $A = C_{3(1+i)/2}\big(\sqrt{5/2}\big)$.

Exercises.

5.1.1. Find l.f.t. T so that $T(1) = i$, $T(i) = 0$, $T(-1) = -i$. Find $T(\mathbb{T})$ and $T(\mathbb{D})$.

5.1.2. Find l.f.t. T so that $T(\mathbb{D}) = \{z \mid \operatorname{Im} z > 0\}$, $T(i) = 1$, $T(1) = 0$, $T(a) = -1$, where $a \in \mathbb{T}$. Can a be an arbitrary point of $\mathbb{T}$?

5.1.3. (i) Let $T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ and $T_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$. Prove that T_1, T_2 are the same function if and only if there is $\lambda \neq 0$ so that $a_2 = \lambda a_1$, $b_2 = \lambda b_1$, $c_2 = \lambda c_1$, $d_2 = \lambda d_1$.
(ii) Prove that every l.f.t. T can take the form $T(z) = \frac{az+b}{cz+d}$ with $ad - bc = 1$.

5.1.4. Let A be a generalized circle of the z-plane $\widehat{\mathbb{C}}$ and B be a generalized circle B of the w-plane $\widehat{\mathbb{C}}$. Then, in an obvious way, A splits $\widehat{\mathbb{C}}$ into two disjoint sets A_+ and A_- and, similarly, B splits $\widehat{\mathbb{C}}$ into two disjoint sets B_+ and B_-. Now, let T be a l.f.t. and let $T(A) = B$. Assume that $z_0 \in A_+$ and $w_0 = T(z_0) \in B_+$. Prove that $T(A_+) = B_+$ and $T(A_-) = B_-$.

5.1.5. A point $z \in \widehat{\mathbb{C}}$ is called **fixed point** of the l.f.t. T if $T(z) = z$. If the l.f.t. T is not the identity (in which case T has infinitely many fixed points), prove that T has either one or two fixed points in $\widehat{\mathbb{C}}$. In each case, which are the images through T of the generalized circles which contain its fixed points?
Apply the above to each of: $T(z) = z + 2$, $T(z) = 2z - 1$, $T(z) = \frac{z-1}{z+1}$ and $T(z) = \frac{3z-4}{z-1}$.

5.1.6. (i) The points $a, b \in \widehat{\mathbb{C}}$ are called **symmetric** with respect to $C_{z_0}(r)$ if either $a = z_0$, $b = \infty$ or $a = \infty$, $b = z_0$ or $a, b \in \mathbb{C}$ are on the same halfline with vertex z_0 and $|a - z_0||b - z_0| = r^2$. Observe that either a, b coincide with one and the same point of $C_{z_0}(r)$ or a, b are on different sides of $C_{z_0}(r)$. Given $a \in \widehat{\mathbb{C}} \setminus \{z_0, \infty\}$, describe a geometric construction "with ruler and compass" of its symmetric point, $b \in \widehat{\mathbb{C}} \setminus \{z_0, \infty\}$, with respect to $C_{z_0}(r)$. Prove that a, b are symmetric with respect to $C_{z_0}(r)$ if and only if $b = z_0 + \frac{r^2}{\overline{a} - \overline{z_0}}$.
(ii) The points $a, b \in \widehat{\mathbb{C}}$ are called **symmetric** with respect to the line $\widehat{l} = l \cup \{\infty\}$ in $\widehat{\mathbb{C}}$ if either $a = b = \infty$ or $a, b \in \mathbb{C}$ are symmetric with respect to l. Prove that a, b are symmetric with respect to $\widehat{l}$ if and only if $b = z_1 + \frac{z_2 - z_1}{\overline{z_2} - \overline{z_1}}(\overline{a} - \overline{z_1})$, where z_1, z_2 are two distinct fixed points of the line l.
(iii) We take a l.f.t. $w = T(z)$ and generalized circles A in the z-plane $\widehat{\mathbb{C}}$ and B in the w-plane $\widehat{\mathbb{C}}$. Prove that, if T maps A onto B, then T maps symmetric points with respect to A onto symmetric points with respect to B.
(iv) Find l.f.t. T so that $T(C_0(1)) = C_i(3)$, $T(i) = 3 + i$, $T(\frac{1}{2}) = 0$.

5.1.7. The l.f.t. $w = T(z)$ is called **real** if it maps the real line (with ∞) in the z-plane $\widehat{\mathbb{C}}$ onto the real line (with ∞) in the w-plane $\widehat{\mathbb{C}}$.
(i) Prove that the l.f.t. T is real if and only if there are $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ so that $T(z) = \frac{az+b}{cz+d}$.
(ii) If the l.f.t. T is real and $T(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{R}$, $ad - bc \neq 0$, we define $\operatorname{sign} T$ to be the sign of $ad - bc$. Using exercise 5.1.3(i), prove that $\operatorname{sign} T$ is well defined.
(iii) Prove that, if the l.f.t. T is real, then T^{-1} is real, and that, if the l.f.t. S, T are real, then $S \circ T$ is real. Also prove that $\operatorname{sign} T^{-1} = \operatorname{sign} T$ and $\operatorname{sign}(S \circ T) = \operatorname{sign} S \ \operatorname{sign} T$.
(iv) Take a real l.f.t. T. Prove that T maps the upper halfplane onto the upper halfplane (and the lower onto the lower) if and only if $\operatorname{sign} T = +1$ and that T maps the upper halfplane onto the lower halfplane (and the lower onto the upper) if and only if $\operatorname{sign} T = -1$.

5.1.8. (i) Let $z_0 \in \mathbb{D}$ and $|\lambda| = 1$ and consider the l.f.t. $T(z) = \lambda \frac{z-z_0}{1-\overline{z_0}z}$. Prove that $T(\mathbb{T}) = \mathbb{T}$ and $T(z_0) = 0$. Find $T(\mathbb{D})$.
(ii) Let $z_0 \in \mathbb{D}$ and let T be a l.f.t. such that $T(\mathbb{T}) = \mathbb{T}$ and $T(z_0) = 0$. Prove that there is λ with $|\lambda| = 1$ so that $T(z) = \lambda \frac{z-z_0}{1-\overline{z_0}z}$.
(iii) Let $a, b \in \mathbb{D}$ and let T be a l.f.t. such that $T(\mathbb{T}) = \mathbb{T}$ and $T(a) = b$. Prove that there is λ with $|\lambda| = 1$ so that $\frac{T(z)-b}{1-\overline{b}T(z)} = \lambda \frac{z-b}{1-\overline{a}z}$.

5.1.9. Consider $H_+ = \{z \mid \operatorname{Im} z > 0\}$ and $H_- = \{z \mid \operatorname{Im} z < 0\}$.
(i) Let $z_0 \in H_+$ and $|\lambda| = 1$ and consider the l.f.t. $T(z) = \lambda \frac{z-z_0}{z-\overline{z_0}}$. Prove that $T(\mathbb{R} \cup \{\infty\}) = \mathbb{T}$ and $T(z_0) = 0$. Find $T(H_+)$.
(ii) Let $z_0 \in H_+$ and let T be a l.f.t. such that $T(\mathbb{R} \cup \{\infty\}) = \mathbb{T}$ and $T(z_0) = 0$. Prove that there is λ with $|\lambda| = 1$ so that $T(z) = \lambda \frac{z-z_0}{z-\overline{z_0}}$.

5.1.10. Consider distinct $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. We define the **double ratio** of z_1, z_2, z_3, z_4 (in this order) to be

$$(z_1, z_2, z_3, z_4) = \begin{cases} \frac{z_1-z_3}{z_1-z_4}\frac{z_2-z_4}{z_2-z_3}, & \text{if } z_1, z_2, z_3, z_4 \in \mathbb{C} \\ \frac{z_2-z_4}{z_2-z_3}, & \text{if } z_1 = \infty \\ \frac{z_1-z_3}{z_1-z_4}, & \text{if } z_2 = \infty \\ \frac{z_2-z_4}{z_1-z_4}, & \text{if } z_3 = \infty \\ \frac{z_1-z_3}{z_2-z_3}, & \text{if } z_4 = \infty \end{cases}$$

(i) Prove that $(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4)$ for every l.f.t. T and every distinct $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$.
(ii) Prove that the distinct $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ belong to the same generalized circle if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R} \setminus \{0\}$.
(iii) If $(z_1, z_2, z_3, z_4) = \lambda$, find all values (depending on λ) which result from this double ratio after all rearrangements of z_1, z_2, z_3, z_4.

5.1.11. Prove that the group of all l.f.t. is simple, i.e. that its only normal subgroups are itself and $\{I\}$, where I is the identity l.f.t.

5.2 The exponential function.

Definition. *We define the* **exponential function** $\exp : \mathbb{C} \to \mathbb{C}$ *by*

$$\exp z = e^x(\cos y + i \sin y), \qquad z = x + iy.$$

If $z \in \mathbb{R}$, i.e. $z = x + i0$, then $\exp z = e^x(\cos 0 + i \sin 0) = e^x = e^z$. This implies that we may use the symbol e^z instead of $\exp z$ without the danger of contradiction, in the case that z is real, between the symbol e^z as we just defined it and the symbol e^z as we know it from infinitesimal calculus. Therefore, we define

$$e^z = \exp z = e^x(\cos y + i \sin y), \qquad z = x + iy.$$

Proposition 5.7. *(i)* $|e^z| = e^{\operatorname{Re} z}$ *for all* z.
(ii) $\arg e^z = \{\operatorname{Im} z + k2\pi \mid k \in \mathbb{Z}\}$ *for all* z.
(iii) $\overline{e^z} = e^{\overline{z}}$ *for all* z.
(iv) For all z_1, z_2*:*

$$e^{z_1} e^{z_2} = e^{z_1+z_2}.$$

(v) $e^{z_2} = e^{z_1} \;\Leftrightarrow\; z_2 - z_1 = k2\pi i,\ k \in \mathbb{Z}$.
(vi) $e^z \neq 0$ *for all* z.
(vii) If $w \neq 0$*, then:*

$$e^z = w \quad \Leftrightarrow \quad z \in \ln|w| + i \arg w.$$

Proof. (i) If $z = x + iy$, then $|e^z| = |e^x||\cos y + i \sin y| = e^x$.
(ii) From $e^z = e^x(\cos y + i \sin y)$ and $|e^z| = e^x$ we get $e^z = |e^z|(\cos y + i \sin y)$.
(iii) $\overline{e^z} = e^x(\cos y - i \sin y) = e^x(\cos(-y) + i \sin(-y)) = e^{\overline{z}}$.
(iv) We have

$$\begin{aligned} e^{z_1} e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)) = e^{z_1+z_2}, \end{aligned}$$

since $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
(v) If $z_2 - z_1 = k2\pi i$ with $k \in \mathbb{Z}$, then $e^{z_2} = e^{z_1} e^{k2\pi i} = e^{z_1}(\cos(k2\pi) + i \sin(k2\pi)) = e^{z_1}$. Conversely, let $e^{z_2} = e^{z_1}$ and $z_2 - z_1 = x + iy$. Then $e^x(\cos y + i \sin y) = e^{z_2 - z_1} = \frac{e^{z_2}}{e^{z_1}} = 1$ and hence $e^x = 1$, $\cos y = 1$ and $\sin y = 0$. Therefore, $x = 0$ and $y = k2\pi$ with $k \in \mathbb{Z}$. Thus, $z_2 - z_1 = k2\pi i$ with $k \in \mathbb{Z}$.
(vi) For all $z = x + iy$ we have $|e^z| = e^x > 0$.
(vii) We take $w \neq 0$ and $z = x+iy$. Then the equality $w = e^z$ becomes $w = e^x(\cos y + i \sin y)$ and it just means that its right side is one of the polar representations of w. Hence, $w = e^z$ is equivalent to $e^x = |w|$ and $y \in \arg w$. Now, $e^x = |w|$ is equivalent to $x = \ln |w|$, where $\ln : (0, +\infty) \to \mathbb{R}$ is the usual logarithmic function from infinitesimal calculus. Hence, for every $w \neq 0$, the equation $e^z = w$ has infinitely many solutions: if $r = |w|$ and θ is any of the values of $\arg w$, then the solutions of $e^z = w$ are given by

$$z = \ln r + i(\theta + 2\pi k) \qquad \text{with } k \in \mathbb{Z}.$$

Since the set of all $\theta + 2\pi k$, $k \in \mathbb{Z}$, is $\arg w$, the set of all solutions is $\ln |w| + i \arg w$. □

Parts (v), (vi) and (vii) of proposition 5.7 imply that the exponential function $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is onto $\mathbb{C} \setminus \{0\}$ but not one-to-one. In fact exp is periodic with period $2\pi i$.

Based on the equality $e^{iy} = \cos y + i \sin y$, we may write the polar representations of any $z \neq 0$ in an equivalent form:

$$z = r(\cos\theta + i \sin\theta) \quad \Leftrightarrow \quad z = re^{i\theta}.$$

The second form is simpler and we shall use it extensively in the rest of the course. For instance, we may rewrite the examples 3.2.7 and 3.2.8 as follows.

Example 5.2.1. Using the parametric equation $z = \gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, for the circle $C_{z_0}(r)$, we have

$$\oint_{C_{z_0}(r)} f(z)\, dz = \oint_\gamma f(z)\, dz = \int_0^{2\pi} f(z_0 + re^{it}) ire^{it}\, dt.$$

Example 5.2.2. If $n \in \mathbb{Z}$, we have

$$\int_0^{2\pi} e^{int}\, dt = \int_0^{2\pi} (\cos(nt) + \sin(nt))\, dt = \begin{cases} 2\pi, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases}$$

Therefore, if $n \in \mathbb{Z}$, we get

$$\oint_{C_{z_0}(r)} (z - z_0)^n\, dz = \int_0^{2\pi} r^n e^{int} ire^{it}\, dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt = \begin{cases} 2\pi i, & \text{if } n = -1 \\ 0, & \text{if } n \neq -1 \end{cases}$$

Proposition 5.8. *The exponential function* $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ *is holomorphic in* $\mathbb{C}$. *Moreover*

$$\exp' = \exp.$$

Proof. The real and imaginary parts of $\exp z$ are $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. Therefore, u, v have partial derivatives

$$\frac{\partial u}{\partial x}(x,y) = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y,$$

which are continuous and satisfy the system of C-R equations in $\mathbb{C}$ and hence exp is holomorphic in $\mathbb{C}$. To calculate the derivative of exp we write

$$\exp' z = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = \exp z.$$

Thus, $\exp' = \exp$. □

We shall now examine the mapping properties of the function $w = e^z$. We write $z = x + iy$ and $w = u + iv$.

If $z = x + iy$ varies on the horizontal line h_y in the z-plane which intersects the y-axis at the fixed point iy, then $w = e^z = e^x(\cos y + i \sin y)$ varies on the halfline r_y in the w-plane with vertex 0 (without 0) which forms angle y with the positive u-semiaxis. Also, if z varies on the horizontal line h_y from left to right, i.e. when x increases from $-\infty$ to $+\infty$, then $w = e^z$ varies on the halfline r_y from 0 to ∞. If y increases by $\Delta y > 0$, i.e. if the horizontal line h_y moves upward, then the corresponding halfline r_y rotates in the positive direction around 0 through an angle Δy. If $0 < \Delta y < 2\pi$, then the open horizontal zone in the z-plane between the lines h_y and $h_{y+\Delta y}$ is mapped onto the angular region in the w-plane between the halflines r_y and $r_{y+\Delta y}$. If $\Delta y = 2\pi$, then the halflines r_y and $r_{y+\Delta y}$ coincide and then the open horizontal zone in the z-plane between the lines h_y and $h_{y+\Delta y}$ is mapped onto the whole w-plane without the halfline $r_y = r_{y+\Delta y}$ (and without 0). In this case, if the horizontal zone includes at least one of its two boundary lines, then its image is the whole w-plane (without 0). If $\Delta y > 2\pi$, then the horizontal zone in the z-plane between the lines h_y and $h_{y+\Delta y}$ is mapped onto the whole w-plane (without 0) “with repetitions”.

If the point $z = x + iy$ varies on the vertical line v_x in the z-plane which intersects the x-axis at the fixed point x, then $w = e^z = e^x(\cos y + i \sin y)$ varies on the circle $C_0(e^x)$, call it c_x, in the w-plane. Also, if z moves upward on the vertical line v_x, i.e. if y increases from $-\infty$ to $+\infty$, then $w = e^z$ covers the circle c_x infinitely many times in the positive direction. If y increases over an interval of length 2π, then $w = e^z$ describes the whole circle c_x once in the positive direction. If y increases over an interval of length $\Delta y < 2\pi$, then $w = e^z$ moves in the positive direction over an arc of central angle Δy. While, if $\Delta y > 2\pi$, then $w = e^z$ moves in the positive direction covering the whole circle c_x “with repetitions”. If x increases by $\Delta x > 0$, i.e. if the vertical line v_x moves to the right, then the circle c_x with radius e^x becomes the circle $c_{x+\Delta x}$ with radius $e^{x+\Delta x} = e^x e^{\Delta x}$. The vertical zone in the z-plane between the lines v_x and $v_{x+\Delta x}$ is mapped onto the ring in the w-plane between the circles c_x and $c_{x+\Delta x}$.

We may combine the above results. For instance, if we consider the rectangle

$$\Pi = \{x + iy \,|\, x_1 < x < x_2, y_1 < y < y_2\}$$

in the z-plane with sides parallel to the two main axes, then Π is the intersection of the horizontal zone between the lines h_{y_1} and h_{y_2} and the vertical zone between the lines v_{x_1} and v_{y_2}. If $y_2 - y_1 < 2\pi$, then Π is mapped onto the “circular rectangle”

$$R = \{re^{i\theta} \,|\, e^{x_1} < r < e^{x_2}, y_1 < \theta < y_2\},$$

in the w-plane, which is the intersection of the angular region between the halflines r_{y_1} and r_{y_2} and the open ring between the circles c_{x_1} and c_{x_2}. If $y_2 - y_1 = 2\pi$, then the “circular rectangle” R is the open ring between the circles c_{x_1} and c_{x_2} without its linear segment which belongs to the halfline $r_{y_1} = r_{y_2}$. Of course, in this case if Π in the z-plane includes at least one of its horizontal sides, then its image R in the w-plane is the whole open ring between the circles c_{x_1} and c_{x_2}. Finally, if we assume that $y_2 - y_1 > 2\pi$, then Π in the z-plane is mapped onto the whole ring R in the w-plane “with repetitions”.

Exercises.

5.2.1. Prove that $|e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|}$.

5.2.2. Let $z \to \infty$ on any halfline. Depending on the halfline, study the existence of the $\lim e^z$ in $\widehat{\mathbb{C}}$. Which characteristic of the halfline determines the existence and the value of the limit?

5.2.3. Find the images through the exponential function of: $\{x + iy \,|\, a < x < b, \theta < y < \theta + \pi\}$, $\{x + iy \,|\, a < x < b, \theta < y < \theta + 2\pi\}$, $\{x + iy \,|\, x < b, \theta < y < \theta + \pi\}$, $\{x + iy \,|\, x < b, \theta < y < \theta + 2\pi\}$, $\{x + iy \,|\, a < x, \theta < y < \theta + \pi\}$, $\{x + iy \,|\, a < x, \theta < y < \theta + 2\pi\}$.

5.2.4. Every horizontal and every vertical line in the z-plane are perpendicular. Also, every halfline with vertex 0 and every circle with center 0 in the w-plane are perpendicular. How do these facts relate to the conformality of the function $w = e^z$?

5.2.5. We define the functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

(i) Prove that these functions are extensions in $\mathbb{C}$ of the well known trigonometric functions in $\mathbb{R}$. Find their domains of definition and their periods. Prove that
· $\sin^2 z + \cos^2 z = 1$,
· $\sin(z + w) = \sin z \cos w + \cos z \sin w$, $\cos(z + w) = \cos z \cos w - \sin z \sin w$,
· $|\cos(x + iy)|^2 = \cos^2 x + \sinh^2 y$, $|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$.
(ii) Prove that these functions are holomorphic in their domains of definition and find their derivatives.
(iii) Study the function $w = \sin z$ in the vertical zone $\{x + iy \,|\, -\frac{\pi}{2} < x < \frac{\pi}{2}\}$ and the function $w = \cos z$ in the vertical zone $\{x + iy \,|\, 0 < x < \pi\}$. Examine the images through these functions of the various horizontal linear segments (of length π) and the various vertical lines inside these two vertical zones.

5.3 Branches of the logarithmic function.

In the last section we proved, for every $w \neq 0$, the equivalence

$$e^z = w \quad \Leftrightarrow \quad z \in \ln|w| + i \arg w. \tag{5.3}$$

Definition. *For every $w \neq 0$ we denote*

$$\log w = \ln|w| + i \arg w$$

and $\log w$ *is called* **logarithm** *of w.*
The elements of the set $\log w$ *are called* **values of the logarithm** *of w and the particular element*

$$\operatorname{Log} w = \ln|w| + i \operatorname{Arg} w$$

is called **principal logarithm** *of w.*

If $r = |w|$ and if θ is any of the values of the argument of w, i.e. if $w = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is any of the polar representations of w, then

$$\log w = \{\ln r + i(\theta + k2\pi) \,|\, k \in \mathbb{Z}\}.$$

Of course, (5.3) takes the form

$$e^z = w \quad \Leftrightarrow \quad z \in \log w,$$

and says that $\log w$ is the set of all solutions of $e^z = w$.

Example 5.3.1. (i) $\operatorname{Log} 1 = 0$ and $\log 1 = \{i2k\pi \mid k \in \mathbb{Z}\}$.
(ii) $\operatorname{Log}(-1) = i\pi$ and $\log(-1) = \{i(2k+1)\pi \mid k \in \mathbb{Z}\}$.
(iii) $\operatorname{Log} i = i\frac{\pi}{2}$ and $\log i = \{i(2k+\frac{1}{2})\pi \mid k \in \mathbb{Z}\}$.
(iv) $\operatorname{Log}(-3i) = \ln 3 - i\frac{\pi}{2}$ and $\log(-i) = \{\ln 3 + i(2k-\frac{1}{2})\pi \mid k \in \mathbb{Z}\}$.
(v) $\operatorname{Log}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$ and $\log(1+i) = \{\ln\sqrt{2} + i(2k+\frac{1}{4})\pi \mid k \in \mathbb{Z}\}$.
(vi) $\operatorname{Log}(1-i\sqrt{3}) = \ln 2 - i\frac{\pi}{3}$ and $\log(1-i\sqrt{3}) = \{\ln 2 + i(2k-\frac{1}{3})\pi \mid k \in \mathbb{Z}\}$.

For any fixed $w \neq 0$ there are infinitely many values z of $\log w$, and any two of them differ by an integral multiple of $i2\pi$. All values z of $\log w$ have the same real part $x = \ln|w|$ and hence they are on the same vertical line v_x with equation $x = \ln|w|$, and the vertical differences between them are the integral multiples of 2π. Therefore, every vertical segment of the line v_x, which has length 2π and includes only one of its endpoints, contains exactly one value z of $\log w$. Moreover, every horizontal zone, which has vertical width 2π and includes only one of its boundary lines (either the upper or the lower one), contains exactly one value z of $\log w$ for every $w \neq 0$. More precisely, if we consider any θ_0 and the horizontal zone

$$Z_{\theta_0} = \{x+iy \mid \theta_0 < y \leq \theta_0 + 2\pi\} \quad \text{or} \quad Z_{\theta_0} = \{x+iy \mid \theta_0 \leq y < \theta_0 + 2\pi\},$$

then this zone Z_{θ_0} contains exactly one value z of $\log w = \ln|w| + i\arg w$: the one with imaginary part y equal to the (unique) value θ of $\arg w$ which is such that

$$\theta_0 < \theta \leq \theta_0 + 2\pi \quad \text{or} \quad \theta_0 \leq \theta < \theta_0 + 2\pi,$$

respectively. Fro instance, if we consider the special zone determined by $\theta_0 = -\pi$ which contains its upper boundary line, i.e.

$$Z_{-\pi} = \{x+iy \mid -\pi < y \leq \pi\},$$

then, for every $w \neq 0$, the unique value of $\log w$ which is contained in this zone is the principal logarithm $z = \operatorname{Log} w$.

Proposition 5.9. *For all* $w_1, w_2 \neq 0$*:*

$$\log(w_1 w_2) = \log w_1 + \log w_2.$$

Proof. We have

$$\log(z_1 z_2) = \ln|z_1 z_2| + i\arg(z_1 z_2) = \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 = \log z_1 + \log z_2,$$

using the analogous property of the function ln in $(0, +\infty)$ and proposition 1.1. □

It is already clear that the exponential function $w = \exp z = e^z$ from $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$ is not one-to-one. In fact, it is *infinity-to-one* since there are infinitely many values of z corresponding to the same value of $w \neq 0$. Therefore, there is no inverse of the exponential function. If we want to produce some kind of inverse of the exponential function, we may take any w in the range $\mathbb{C} \setminus \{0\}$ of the function and select one value of z out of the infinitely many in $\mathbb{C}$ which satisfy the $e^z = w$. There are many instances of this method at a more elementary level. Let us consider for instance the function $y = x^2$ from $(-\infty, +\infty)$ onto $[0, +\infty)$, which is not one-to-one in $(-\infty, +\infty)$. We take any $y \in [0, +\infty)$ (the range of $y = x^2$) and find one x such that $x^2 = y$. There are exactly two such x: $x = \sqrt{y}$ and $x = -\sqrt{y}$. Therefore, one might say that we have only two choices for the inverse function: the choice $x = \sqrt{y}$ for every $y \in [0, +\infty)$ and the choice $x = -\sqrt{y}$ for every $y \in [0, +\infty)$. *But this is not correct.* We may choose $x = \sqrt{y}$ for some $y \in [0, +\infty)$ and $x = -\sqrt{y}$ for the remaining $y \in [0, +\infty)$, forming, for instance, the inverse function

$$x = \begin{cases} \sqrt{y}, & \text{if } 0 \leq y \leq 1 \\ -\sqrt{y}, & \text{if } 1 < y < +\infty \end{cases}$$

It is obvious that there are infinitely many such inverse functions, depending on the particular choice we make between $x = \sqrt{y}$ and $x = -\sqrt{y}$ for each value of y. Nevertheless, there is a criterion which reduces the number of our inverse functions to *exactly two*: the criterion of *continuity!* We observe that the last function, with the double formula, is not continuous. On the contrary, the function $x = \sqrt{y}$ for every $y \in [0, +\infty)$ and the function $x = -\sqrt{y}$ for every $y \in [0, +\infty)$ are both continuous. To prove that these are the only continuous inverse functions is a simple exercise in real analysis. Indeed, assume that there is some continuous inverse function $x = f(y)$ of $y = x^2$ defined in $[0, +\infty)$ (the range of $y = x^2$). I.e. $f : [0, +\infty) \to \mathbb{R}$ is continuous in $[0, +\infty)$ and $f(y)^2 = y$ for every $y \in [0, +\infty)$. Let there be $y_1, y_2 > 0$ with $y_1 \neq y_2$ such that $f(y_1) = \sqrt{y_1}$ and $f(y_2) = -\sqrt{y_2}$. Since f is continuous in the interval $[y_1, y_2]$ or $[y_2, y_1]$ and its values at the endpoints are opposite, there is some y in this interval so that: $f(y) = 0$. This is impossible, because $y > 0$ and either $f(y) = \sqrt{y} > 0$ or $f(y) = -\sqrt{y} < 0$. Therefore, there are no such $y_1, y_2 > 0$ and hence we have exactly two cases: either $f(y) = \sqrt{y}$ for every $y \geq 0$ or $f(y) = -\sqrt{y}$ for every $y \geq 0$. We may say that there are exactly two *continuous branches of the square root* in $[0, +\infty)$: the branch $x = \sqrt{y}$ and the branch $x = -\sqrt{y}$.

Now let us go back to the determination of possible inverses of the exponential function.

Definition. *Let $A \subseteq \mathbb{C} \setminus \{0\}$, $f : A \to \mathbb{C}$. We say that f is a* **continuous branch of $\log \mathbf{w}$** *in A if*
(i) f is continuous in A and
(ii) for every $w \in A$ we have $f(w) \in \log w$ or, equivalently, $e^{f(w)} = w$.

Proposition 5.10 gives many useful examples of continuous branches of the logarithm.

Proposition 5.10. *Let $\theta_0 \in \mathbb{R}$, We consider the set*

$$A_{\theta_0} = \{re^{i\theta} \,|\, 0 < r < +\infty, \theta_0 < \theta < \theta_0 + 2\pi\},$$

in the w-plane (i.e. $\mathbb{C}$ without the halfline with vertex 0 which forms angle θ_0 with the positive u-semiaxis) and the open horizontal zone

$$Z_{\theta_0} = \{x + iy \,|\, -\infty < x < +\infty, \theta_0 < y < \theta_0 + 2\pi\}$$

in the z-plane. We define the function

$$f : A_{\theta_0} \to Z_{\theta_0}$$

as follows: for every $w \in A_{\theta_0}$ we take $f(w)$ to be the unique value of $\log w$ in the zone Z_{θ_0}. It is clear that f satisies (ii) of the above definition for the set A_{θ_0}. Moreover, f is also continuous in A_{θ_0} and hence satisfies (i) of the above definition. Thus, f is a continuous branch of $\log w$ in A_{θ_0}.

Proof. Assume that f is not continuous at some w in A_{θ_0}. Then there is a sequence (w_n) in A_{θ_0} so that $w_n \to w$ and $f(w_n) \not\to f(w)$. This implies that there is $\delta > 0$ so that $|f(w_n) - f(w)| \geq \delta > 0$ for infinitely many n. These infinitely many n define a subsequence of (w_n). Now we ignore the rest of the sequence (w_n) and concentrate on the specific subsequence. For simplicity we rename the subsequence and call it (w_n) again. Therefore, we have a sequence (w_n) in A_{θ_0} such that

$$w_n \to w \qquad \text{and} \qquad |f(w_n) - f(w)| \geq \delta > 0 \quad \text{for every } n. \tag{5.4}$$

We set $z = f(w) \in Z_{\theta_0}$ and $z_n = f(w_n) \in Z_{\theta_0}$ for every n. Then $e^z = w$ and $e^{z_n} = w_n$ for every n and (5.4) becomes

$$e^{z_n} \to e^z \qquad \text{and} \qquad |z_n - z| \geq \delta > 0 \quad \text{for every } n. \tag{5.5}$$

The real parts of the z_n are equal to $\ln|w_n|$ and, since $\ln|w_n| \to \ln|w|$, the real parts of the z_n are bounded. Moreover, since $z_n \in Z_{\theta_0}$, the imaginary parts of the z_n are also bounded. Therefore, the sequence (z_n) is bounded and the Bolzano-Weierstrass theorem implies that there is a subsequence (z_{n_k}) so that

$$z_{n_k} \to z'. \tag{5.6}$$

for some z'. Since all z_{n_k} belong to Z_{θ_0}, we see that z' belongs to the closed zone

$$\overline{Z}_{\theta_0} = \{x + iy \,|\, -\infty < x < +\infty, \theta_0 \leq y \leq \theta_0 + 2\pi\}.$$

From (5.5) and (5.6) we get that $e^{z'} = e^z$ and $|z' - z| \geq \delta$. Therefore, z' and z differ by a non-zero integral multiple of $i2\pi$. But this is impossible, because z belongs to the open zone Z_{θ_0} and z' belongs to the closed zone $\overline{Z}_{\theta_0}$.
Thus f is continuous at every w in A_{θ_0}. □

Our study of the mapping properties of the exponential function in the previous section gives the following information about the mapping properties of the continuous branch $f : A_{\theta_0} \to Z_{\theta_0}$ of $\log w$, which is defined in proposition 5.10: f maps the halflines in A_{θ_0} with vertex 0 (without 0) onto the horizontal lines in Z_{θ_0} and the circles with center 0 (without their point on the halfline which is excluded from A_{θ_0}) onto the vertical segments of Z_{θ_0}.

Choosing any real θ_0, we have defined a continuous branch of $\log w$ in the subset A_{θ_0} of the w-plane, whose range is the zone Z_{θ_0} of the z-plane. If, instead of θ_0, we consider $\theta_0 + k2\pi$ with any $k \in \mathbb{Z}$, then the domain $A = A_{\theta_0+k2\pi}$ *remains the same* but the range, i.e. the zone $Z_{\theta_0+k2\pi}$, moves vertically by $k2\pi$. The various zones $Z_{\theta_0+k2\pi}$ are successive and cover the whole z-plane (except for their boundary lines with equations $y = \theta_0 + k2\pi$). We summarize:

If we exclude from the w-plane a halfline with vertex 0, then in the remaining open set A there are infinitely many continuous branches of $\log w$ *defined. Each of them maps A onto some open horizontal zone of the z-plane of width 2π. These various open zones, which correspond to the various continuous branches of* $\log w$ *(in the same set A), are mutually disjoint, successive and cover the z-plane (except for their boundary lines). Of course, if we change the original halfline which determines the set A, then the corresponding zones and the corresponding continuous branches of* $\log w$ *also change.*

Example 5.3.2. One particular example of a continuous branch of $\log w$ is defined when we choose $\theta_0 = -\pi$. Then the set

$$A_{-\pi} = \{re^{i\theta} \,|\, 0 < r < +\infty, -\pi < \theta < \pi\}$$

is the w-plane without the negative u-semiaxis (where $w = u + iv$) and the range of the branch is the zone

$$Z_{-\pi} = \{x + iy \,|\, -\infty < x < +\infty, -\pi < y < \pi\}.$$

It is obvious that this branch is the function which maps every $w \in A_{-\pi}$ onto the principal value $z = \mathrm{Log}\, w$ of $\log w$. I.e. we get the so-called **principal branch of log w**

$$\mathrm{Log} : A_{-\pi} \to Z_{-\pi}.$$

We must keep in mind that in the same set $A_{-\pi}$ of the w-plane, besides the principal branch, there are infinitely many other continuous branches of $\log w$ defined. Each of them maps $A_{-\pi}$ in a corresponding zone $Z_{-\pi+k2\pi}$, with $k \in \mathbb{Z}$, which is $Z_{-\pi}$ moved vertically by $k2\pi$. This branch results from the principal branch $z = \mathrm{Log}\, w$ by adding the constant $ik2\pi$ and its formula is

$$z = \mathrm{Log}\, w + i2k\pi.$$

We skip the proof of proposition 5.11, since it is a special case of proposition 5.14.

Proposition 5.11. *Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f : A \to \mathbb{C}$ be any continuous branch of* $\log w$ *in A. If w_0 is an interior point of A, then f is differentiable at w_0 and*

$$f'(w_0) = \frac{1}{w_0}.$$

Hence f is holomorphic in the interior of A.

Therefore, *every continuous branch of* $\log w$ *in an open set $A \subseteq \mathbb{C} \setminus \{0\}$ can be called* **holomorphic branch of log w** *in A.*

Example 5.3.3. We have defined infinitely many continuous branches of $\log w$ in the open set which results when we exclude any halfline with vertex 0 from the w-plane. All these branches are *holomorphic branches of* $\log w$. In particular the principal branch

$$\mathrm{Log} : A_{-\pi} \to Z_{-\pi}$$

is holomorphic in $A_{-\pi}$.

We skip the proof of proposition 5.12, since it is a special case of proposition 5.15.

Proposition 5.12. *Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f_1, f_2 : A \to \mathbb{C}$.*
(i) If f_1 is a continuous branch of $\log w$ *in A and $f_2(w) - f_1(w) = ik2\pi$ for every $w \in A$, where k is a fixed integer, then f_2 is also a continuous branch of* $\log w$ *in A.*
(ii) If, morever, A is connected and f_1, f_2 are continuous branches of $\log w$ *in A, then $f_2(w) - f_1(w) = ik2\pi$ for every $w \in A$, where k is a fixed integer. In particular, if $f_1(w_0) = f_2(w_0)$ for some $w_0 \in A$, then $f_1(w) = f_2(w)$ for every $w \in A$.*

Thus, *if we know one continuous branch of* $\log w$ *in the connected set A, then we find every other possible continuous branch of* $\log w$ *in A by adding to the known branch an arbitrary constant of the form $ik2\pi$ with $k \in \mathbb{Z}$.*

Example 5.3.4. Let $A = A_{-\pi}$ be the w-plane without the negative u-semiaxis (where $w = u+iv$). We want to find a continuous branch of $\log w$ in A having value $z = 0$ when $w = 1$.
We already know that the principal branch Log of the logarithm has value $z = \mathrm{Log}\, 1 = 0$ at $w = 1$.
Since A is connected, there is no other such continuous branch of $\log w$ in A.
Now, in the same set $A = A_{-\pi}$ we want to find a continuous branch of $\log w$ taking the value $z = i4\pi$ at $w = 1$.
Since A is connected the branch we are looking for has the form $z = \mathrm{Log}\, w + ik2\pi$ for some fixed integer k. We try $w = 1$ in this equality and get $k = 2$.

Example 5.3.5. Let $A = A_0 = \{re^{i\theta} \,|\, 0 < r < +\infty, 0 < \theta < 2\pi\}$ be the w-plane without the positive u-semiaxis (where $w = u+iv$). We want to find a continuous branch of $\log w$ in A taking the value $z = i(\frac{\pi}{2} + 4\pi)$ at $w = i$.
We consider the horizontal zones in the z-plane which correspond to the set A, i.e.

$$Z_{0+k2\pi} = \{x + iy \,|\, -\infty < x < +\infty, k2\pi < y < 2\pi + k2\pi\} \qquad \text{with } k \in \mathbb{Z}$$

and choose the one which contains the value $z = i(\frac{\pi}{2} + 4\pi)$. This zone corresponds to $k = 2$:

$$Z_{4\pi} = \{x + iy \,|\, -\infty < x < +\infty, 4\pi < y < 6\pi\}.$$

Then we consider the continuous branch f of $\log w$ which maps A onto $Z_{4\pi}$:

$$f(w) = \ln r + i\theta \qquad \text{for } w = re^{i\theta} \text{ and } r = |w| > 0,\ 4\pi < \theta < 6\pi,$$

where θ is the unique value of $\arg w$ which is contained in the interval $(4\pi, 6\pi)$.
Since A is connected, there is no other such continuous branch of $\log w$ in A.

Proposition 5.13. *There is no continuous branch of* $\log w$ *defined in any circle* $C_0(r)$ *and hence in any set* A *which contains such a circle.*

Proof. Assume that there is a continuous branch of $\log w$ in $C_0(r)$, i.e. $f : C_0(r) \to \mathbb{C}$ continuous in $C_0(r)$ and such that $e^{f(w)} = w$ for every $w \in C_0(r)$.
We consider the principal branch Log which is continuous in the w-plane without the negative u-semiaxis. Therefore, Log is continuous in $B = C_0(r) \setminus \{-r\}$.
Hence, both f and Log are continuous branches of $\log w$ in B. Since B is connected, there is a fixed integer k so that

$$\operatorname{Log} w = f(w) + ik2\pi \qquad \text{for every } \; w \in B. \tag{5.7}$$

We consider two sequences (w_n') and (w_n'') in B such that $w_n' \to -r$ on the upper semicircle of B and $w_n'' \to -r$ on the lower semicircle of B. The continuity of f in $C_0(r)$ implies $f(w_n') \to f(-r)$ and $f(w_n'') \to f(-r)$ and hence $f(w_n') - f(w_n'') \to 0$. From (5.7) we get $\operatorname{Log} w_n' - \operatorname{Log} w_n'' \to 0$. But $\operatorname{Log} w_n' \to \ln r + i\pi$ and $\operatorname{Log} w_n'' \to \ln r - i\pi$ and we arrive at a contradiction. $\square$

Example 5.3.6. There is no continuous branch of $\log w$ in any ring with center 0 neither in $\mathbb{C} \setminus \{0\}$

Now, we introduce a slight generalization of the notion of the branch of $\log w$, i.e. we define the notion of the branch of $\log g(w)$.

Definition. *Let* $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ *and* $g : A \to \mathbb{C} \setminus \{0\}$ *be continuous in* A. *We say that* f *is a* **continuous branch of log g(w)** *in* A *if*
(i) f *is continuous in* A *and*
(ii) for every $w \in A$ *we have* $f(w) \in \log g(w)$ *or, equivalently,* $e^{f(w)} = g(w)$.

We just state and prove the following results which are analogous to propositions 5.11 and 5.12.

Proposition 5.14. *Let* $A \subseteq \mathbb{C}$, $g : A \to \mathbb{C} \setminus \{0\}$ *be continuous in* A *and* $f : A \to \mathbb{C}$ *be any continuous branch of* $\log g(w)$ *in* A. *If* w_0 *is an interior point of* A *and* g *is differentiable at* w_0, *then* f *is differentiable at* w_0 *and*

$$f'(w_0) = \frac{g'(w_0)}{g(w_0)}.$$

Hence, if g *is holomorphic in the interior of* A, *then* f *is also holomorphic in the interior of* A.

Proof. We set $z_0 = f(w_0)$ and $z = f(w)$ for every $w \in A$. Then $e^{z_0} = g(w_0)$ and $e^z = g(w)$. Since f is continuous, $w \to w_0$ implies $z \to z_0$. Therefore, using the derivative of the exponential function at z_0, we see that

$$\frac{f(w) - f(w_0)}{w - w_0} = \frac{z - z_0}{e^z - e^{z_0}} \, \frac{g(w) - g(w_0)}{w - w_0} \to \frac{g'(w_0)}{e^{z_0}} = \frac{g'(w_0)}{g(w_0)} \qquad \text{when } w \to w_0.$$

Thus f is differentiable at w_0 and $f'(w_0) = \frac{g'(w_0)}{g(w_0)}$. $\square$

Proposition 5.15. *Let* $A \subseteq \mathbb{C}$, $g : A \to \mathbb{C} \setminus \{0\}$ *be continuous in* A *and* $f_1, f_2 : A \to \mathbb{C}$.
(i) If f_1 *is a continuous branch of* $\log g(w)$ *in* A *and* $f_2(w) - f_1(w) = ik2\pi$ *for every* $w \in A$, *where* k *is a fixed integer, then* f_2 *is also a continuous branch of* $\log g(w)$ *in* A.
(ii) If, morever, A *is connected and* f_1, f_2 *are continuous branches of* $\log g(w)$ *in* A, *then* $f_2(w) - f_1(w) = ik2\pi$ *for every* $w \in A$, *where* k *is a fixed integer. In particular, if* $f_1(w_0) = f_2(w_0)$ *for some* $w_0 \in A$, *then* $f_1(w) = f_2(w)$ *for every* $w \in A$.

Proof. (i) The continuity of f_1 in A implies the continuity of f_2 in A. We also have $e^{f_1(w)} = g(w)$ for every $w \in A$ and hence $e^{f_2(w)} = e^{f_1(w)+ik2\pi} = e^{f_1(w)}e^{ik2\pi} = g(w)1 = g(w)$ for every $w \in A$. Therefore, f_2 is a continuous branch of $\log g(w)$ in A.
(ii) We consider the function $k : A \to \mathbb{C}$ defined by

$$k(w) = \frac{1}{i2\pi}(f_2(w) - f_1(w)) \qquad \text{for every } w \in A.$$

Since for every $w \in A$ both $f_2(w)$ and $f_1(w)$ are values of $\log g(w)$, we have that $k(w)$ is an integer. I.e. $k : A \to \mathbb{Z}$. Also, since both f_1, f_2 are continuous in A, k is continuous in A. Now, k is a continuous real function in the connected set A, and hence it has the intermediate value property. But since its only values are integers, it is constant in A. Therefore, there is a fixed integer k so that $\frac{1}{i2\pi}(f_2(w) - f_1(w)) = k$ or, equivalently, $f_2(w) - f_1(w) = ik2\pi$ for every $w \in A$.
If $f_2(w_0) = f_1(w_0)$ for some $w_0 \in A$, then the integer k is 0 and we get that $f_2(w) = f_1(w)$ for every $w \in A$. □

Exercises.

5.3.1. Prove that for every $z \neq 0$ we have $\exp(\log z) = \{z\}$ and $\log(\exp z) = \{z + k2\pi i \,|\, k \in \mathbb{Z}\}$.

5.3.2. If A is any of the sets $\{w \,|\, r_1 \le |w| \le r_2\} \setminus [-r_2, -r_1]$, $\{w \,|\, 0 < |w| \le r_2\} \setminus [-r_2, 0)$, $\{w \,|\, r_1 \le |w| < +\infty\} \setminus (-\infty, -r_1]$, find $\mathrm{Log}(A)$.

5.3.3. Work on the following in both cases: $\theta_0 = -\pi$ and $\theta_0 = 0$.
Consider A_{θ_0}, i.e. the w-plane without the halfline with vertex 0 which forms angle θ_0 with the positive u-semiaxis. Consider also θ_1, θ_2 with $\theta_0 < \theta_1 < \theta_2 < \theta_0 + 2\pi$ as well as r_1, r_2 with $0 < r_1 < r_2 < +\infty$. Draw the set $P = \{w = re^{i\theta} \,|\, r_1 < r < r_2, \theta_1 < \theta < \theta_2\}$ and its images through the various continuous branches of $\log w$ in A_{θ_0}.

5.3.4. Let $P = \{re^{i\theta} \,|\, 1 < r < 2, -\frac{3\pi}{4} < \theta < \frac{3\pi}{4}\}$, $Q = \{w = re^{i\theta} \,|\, 1 < r < 2, \frac{\pi}{4} < \theta < \frac{7\pi}{4}\}$. We know that there is a continuous branch f of $\log w$ in P and a continuous branch g of $\log w$ in Q. Is it possible for f and g to coincide in $P \cap Q$?

5.3.5. Look back at exercise 1.2.1 and prove that

$$\mathrm{Log}(z_1 z_2) = \begin{cases} \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2, & \text{if } -\pi < \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \le \pi \\ \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 + 2\pi i, & \text{if } -2\pi < \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \le -\pi \\ \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2 - 2\pi i, & \text{if } \pi < \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \le 2\pi \end{cases}$$

Are there any other possible values of $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$?

5.3.6. Define $w^a = e^{a \,\mathrm{Log}\, w}$ for every $w \in D_1(1)$, and prove that $\lim_{x \to +\infty} \left(1 + \frac{z}{x}\right)^x = e^z$ for every z.

5.3.7. Let $A \subseteq \mathbb{C} \setminus \{0\}$. If A is connected and if f_1, f_2 are two different continuous branches of $\log w$ in A, prove that $f_1(A) \cap f_2(A) = \emptyset$. (Observe how this result is confirmed by the special case of A being $\mathbb{C}$ without a halfline with vertex 0 in which case the various continuous branches of $\log w$ in A map A onto disjoint horizontal zones.)

5.3.8. Let $A \subseteq \mathbb{C}$ and $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A. If there is a continuous branch $f : g(A) \to \mathbb{C}$ of the logarithm in $g(A)$, prove that $f \circ g : A \to \mathbb{C}$ is a continuous branch of $\log g(w)$ in A.

5.3.9. Let $a < b$. Discuss the geometric meaning of the number $\mathrm{Log}\, \frac{z-b}{z-a}$ for every z with $\mathrm{Im}\, z > 0$.

5.4 Powers and branches of roots.

If $n \in \mathbb{N}$, $n \geq 2$, the function

$$w = z^n$$

is holomorphic in the z-plane $\mathbb{C}$ and we shall examine some mapping properties of this function.

We work with polar representations:

$$z = re^{i\theta}, \qquad w = r^n e^{in\theta}.$$

If $\theta \in \mathbb{R}$ is constant and r varies in $(0, +\infty)$, i.e. if z moves on the halfline r_θ in the z-plane with vertex 0 (without 0) which forms angle θ with the positive x-semiaxis, then $w = z^n$ moves on the halfline r_ϕ in the w-plane with vertex 0 (without 0) which forms angle $\phi = n\theta$ with the positive u-semiaxis. Also, if z moves on the halfline r_θ from 0 to ∞, then $w = z^n$ moves on the halfline r_ϕ from 0 to ∞. If θ increases by $\Delta\theta > 0$, i.e. if the halfline r_θ turns in the positive direction through an angle $\Delta\theta$, then the corresponding halfline r_ϕ turns in the positive direction through an angle $\Delta\phi = n\Delta\theta$. If $0 < \Delta\theta < \frac{2\pi}{n}$, then the angular region in the z-plane between the halflines r_θ and $r_{\theta+\Delta\theta}$ is mapped onto the angular region in the w-plane between the halflines r_ϕ and $r_{\phi+\Delta\phi}$. If $\Delta\theta = \frac{2\pi}{n}$, then the halflines r_ϕ and $r_{\phi+\Delta\phi}$ coincide and then the angular region in the z-plane between the halflines r_θ and $r_{\theta+\Delta\theta}$ is mapped in the whole w-plane without the halfline $r_\phi = r_{\phi+\Delta\phi}$. In this case, if the original angular region includes at least one of its boundary halflines, then its image is the whole w-plane (without 0). If $\Delta\theta > \frac{2\pi}{n}$, then the angular region in the z-plane between the halflines r_θ and $r_{\theta+\Delta\theta}$ is mapped onto the whole w-plane (without 0) "with repetitions".

If $r \in (0, +\infty)$ is constant and θ varies in $\mathbb{R}$, i.e. if the point z moves on the circle $C_0(r)$ in the z-plane, then $w = z^n$ moves on the circle $C_0(r^n)$ in the w-plane. Also, if z covers $C_0(r)$ once in the positive direction, i.e. if θ increases in an interval of length 2π, then $w = z^n$ covers $C_0(r^n)$ n times in the positive direction. If θ increases in an interval of length $\frac{2\pi}{n}$, then $w = z^n$ covers $C(0; r^n)$ once in the positive direction. If z moves on $C_0(r)$ in the positive direction over an arc with central angle $\Delta\theta < \frac{2\pi}{n}$, then $w = z^n$ moves on $C_0(r^n)$ in the positive direction over an arc with central angle $n\Delta\theta$. If $\Delta\theta > \frac{2\pi}{n}$, then $w = z^n$ covers $C(0; r^n)$ in the positive direction "with repetition". If r increases, i.e. if the circle $C(0; r)$ expands, then the corresponding circle $C(0; r^n)$ also expands. The ring in the z-plane between the circles $C(0; r_1)$ and $C(0; r_2)$, where $0 < r_1 < r_2$, is mapped onto the ring in the w-plane between the circles $C(0; {r_1}^n)$ and $C(0; {r_2}^n)$.

In the proof of the following propostion as well as in the whole course, we shall use the symbol $\sqrt[n]{x}$ only to denote the unique nonnegative n-th root of a nonnegative real number x.

Proposition 5.16. *If $w \neq 0$ and $n \in \mathbb{N}$, $n \geq 2$, the equation $z^n = w$ has exactly n solutions.*

Proof. We consider a polar representation $w = Re^{i\Theta}$ of w and a polar representation $z = re^{i\theta}$ of the unknown z. Then the equation $z^n = w$ takes the equivalent form $r^n e^{in\theta} = Re^{i\Theta}$ and this is equivalent to $r^n = R$ and $n\theta = \Theta + k2\pi$, $k \in \mathbb{Z}$. Solving for r and θ, we find the solutions of the equation $z^n = w$:

$$z_k = \sqrt[n]{R}\, e^{i(\frac{\Theta}{n} + k\frac{2\pi}{n})}, \quad k \in \mathbb{Z}.$$

It is trivial to see that two solutions z_k are the same if and only if the corresponding values of k differ by a multiple of n. Hence, the n numbers

$$z_k = \sqrt[n]{R}\, e^{i(\frac{\Theta}{n} + k\frac{2\pi}{n})}, \quad k = 0, 1, \ldots, n-1,$$

are the distinct solutions of $z^n = w$. $\square$

We easily see that the solutions of $z^n = w$ are the vertices of a regular n-gon inscribed in the circle $C_0(\sqrt[n]{R})$, where $R = |w|$.

Definition. *Let $w \neq 0$. We call* **n-th root** *of w the set $\{z_0, z_1, \dots, z_{n-1}\}$, where $z_0, z_1, \dots, z_{n-1}$ are the n solutions of the equation $z^n = w$. We denote this set by*

$$w^{\frac{1}{n}} = \{z_0, z_1, \dots, z_{n-1}\}.$$

If $w = 0$, we define

$$0^{\frac{1}{n}} = \{0\},$$

since 0 is the only solution of $z^n = 0$.
Every element of the n-root of w is called **a value of the n-root** *of w.*

Note that, as in the case of the argument, the n-th root of a complex number is a set.
The following equivalence is clear:

$$z^n = w \quad \Leftrightarrow \quad z \in w^{\frac{1}{n}}.$$

Definition. *The n-th root of 1, i.e. the set of solutions of $z^n = 1$, is called* **n-th root of unity** *and each of its elements is called* **a value of the n-th root of unity**.

Since $1 = 1e^{i0}$ is a polar representation of 1, the values of the n-th root of unity are $z_k = e^{i(\frac{0}{n}+k\frac{2\pi}{n})} = e^{ik\frac{2\pi}{n}}$, $k = 0, 1, \dots, n-1$. Obviously, one of them is $z_0 = 1$ and, if we denote $z_1 = e^{i\frac{2\pi}{n}}$ by the symbol ω_n,

$$\omega_n = e^{i\frac{2\pi}{n}},$$

we find that the values of the n-th root of unity are

$$1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}.$$

We saw that $w^{\frac{1}{n}}$ has exactly n values which are on the vertices of a regular n-gon inscribed in the circle $C(0; \sqrt[n]{|w|})$ of the z-plane. Therefore, every arc of this circle with central angle $\frac{2\pi}{n}$, which includes only one of its endpoints, contains exactly one of the values of $w^{\frac{1}{n}}$. Thus, every angular set in the z-plane with vertex 0 and angle $\frac{2\pi}{n}$, which includes only one of its boundary halflines, contains, for every $w \neq 0$, exactly one value of $w^{\frac{1}{n}}$. In particular, if we consider any θ_0 and the angular set

$$A_{\theta_0} = \left\{re^{i\theta} \,\middle|\, r > 0, \theta_0 < \theta \leq \theta_0 + \frac{2\pi}{n}\right\} \quad \text{or} \quad A_{\theta_0} = \left\{re^{i\theta} \,\middle|\, r > 0, \theta_0 \leq \theta < \theta_0 + \frac{2\pi}{n}\right\},$$

then A_{θ_0} contains exactly one value of $w^{\frac{1}{n}}$.
Clearly, the function $w = z^n$ from $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$ is n-to-one and has no inverse.

Definition. *Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f : A \to \mathbb{C} \setminus \{0\}$. We say that f is a* **continuous branch of $\mathbf{w^{\frac{1}{n}}}$** *in A if*
(i) f is continuous in A and
(ii) for every $w \in A$ we have $f(w) \in w^{\frac{1}{n}}$ or, equivalently, $f(w)^n = w$.

Proposition 5.17 gives many examples of continuous branches of $w^{\frac{1}{n}}$.

Proposition 5.17. *Let $\phi_0 \in \mathbb{R}$. We consider the set*

$$A_{\phi_0} = \{se^{i\phi} \mid s > 0, \phi_0 < \phi < \phi_0 + 2\pi\},$$

in the w-plane (i.e. $\mathbb{C}$ without the halfline with vertex 0 which forms angle ϕ_0 with the positive u-semiaxis) and the angular region

$$B_{\phi_0/n} = \left\{re^{i\theta} \,\middle|\, r > 0, \frac{\phi_0}{n} < \theta < \frac{\phi_0}{n} + \frac{2\pi}{n}\right\}$$

in the z-plane. We define the function

$$f : A_{\phi_0} \to B_{\phi_0/n}$$

as follows: for every $w \in A_{\phi_0}$ we take $f(w)$ to be the unique value of $w^{\frac{1}{n}}$ in the angular region $B_{\phi_0/n}$. It is clear that f satisfies (ii) of the above definition for the set A_{ϕ_0}. Moreover, f is also continuous in A_{ϕ_0} and hence satisfies (i) of the above definition. Therefore, f is a continuous branch of $w^{\frac{1}{n}}$ in A_{ϕ_0}.

Proof. Assume that f is not continuous at some w in A_{ϕ_0}. Then there is a sequence (w_k) in A_{ϕ_0} so that $w_k \to w$ and $f(w_k) \not\to f(w)$. Then there is $\delta > 0$ so that $|f(w_k) - f(w)| \geq \delta > 0$ for infinitely many k. These infinitely many k define a subsequence of (w_k). Now we ignore the rest of the sequence (w_k) and concentrate on the specific subsequence. For simplicity we rename the subsequence and call it (w_k) again. Therefore, we have a sequence (w_k) in A_{ϕ_0} such that

$$w_k \to w \qquad \text{and} \qquad |f(w_k) - f(w)| \geq \delta > 0 \quad \text{for every } k. \tag{5.8}$$

We set $z = f(w) \in B_{\phi_0/n}$ and $z_k = f(w_k) \in B_{\phi_0/n}$ for every k. Then $z^n = w$ and $z_k^n = w_k$ for every k and (5.8) becomes

$$z_k^n \to z^n \qquad \text{and} \qquad |z_k - z| \geq \delta > 0 \quad \text{for every } k. \tag{5.9}$$

Since $|z_k|^n \to |z|^n$ and hence $|z_k| \to |z|$, we get that the sequence (z_k) is bounded and the Bolzano-Weierstrass theorem implies that there is a subsequence (z_{k_m}) so that

$$z_{k_m} \to z' \tag{5.10}$$

for some z'. Since all z_{k_m} belong to $B_{\phi_0/n}$, we have that z' belongs to the closed angular region

$$\overline{B}_{\phi_0/n} = \left\{ z = re^{i\theta} \,\middle|\, r \geq 0, \frac{\phi_0}{n} \leq \theta \leq \frac{\phi_0}{n} + \frac{2\pi}{n} \right\}$$

From (5.9) and (5.10) we get $z'^n = z^n$ and $|z' - z| \geq \delta$. This is impossible, because z belongs to $B_{\phi_0/n}$ and z' belongs to $\overline{B}_{\phi_0/n}$.
Thus f is continuous at every w in A_{ϕ_0}. □

From the mapping properties of the function $w = z^n$ we get the following for the mapping properties of the continuous branch $f : A_{\phi_0} \to B_{\phi_0/n}$ of $w^{\frac{1}{n}}$ defined in proposition 5.17. The function f maps the halflines in A_{ϕ_0} with vertex 0 (without 0) onto the halflines in $B_{\phi_0/n}$ with vertex 0 (without 0) and the circular arcs in A_{ϕ_0} with center 0 onto the circular arcs in $B_{\phi_0/n}$ with center 0.

Choosing any real ϕ_0, we have defined a continuous branch of $w^{\frac{1}{n}}$ in the subset A_{ϕ_0} of the w-plane, whose range is the angular region $B_{\phi_0/n}$ of the z-plane. If, instead of ϕ_0, we consider $\phi_0 + k2\pi$ with any $k = 0, 1, \ldots, n-1$, then the set $A = A_{\phi_0 + k2\pi}$ *remains the same* but the range, i.e. the angular region $B_{(\phi_0+k2\pi)/n}$ rotates through the angle $k\frac{2\pi}{n}$. The n angular regions $B_{(\phi_0+k2\pi)/n}$ with $k = 0, 1, \ldots, n-1$ are successive and cover the z-plane (except for their n boundary halflines with vertex 0). We summarize:

If we exclude from the w-plane any halfline with vertex 0, then in the remaining open set A there are n continuous branches of $w^{\frac{1}{n}}$ defined. Each of them maps A onto some open angular region of the z-plane with vertex 0 and angle $\frac{2\pi}{n}$. These various angular regions, which correspond to the various continuous branches of $w^{\frac{1}{n}}$ (in the same set A), are mutually disjoint, successive and cover the z-plane (except for their boundary halflines). Of course, if we change the original halfline which determines the set A, then the corresponding angular regions and the corresponding branches of $w^{\frac{1}{n}}$ also change.

Example 5.4.1. We get a concrete example of a continuous branch of $w^{\frac{1}{n}}$ when we consider $\phi_0 = -\pi$. Then the set

$$A_{-\pi} = \{se^{i\phi} \,|\, s > 0, -\pi < \phi < \pi\}$$

is the w-plane without the negative u-semiaxis (where $w = u + iv$) and the range of the continuous branch of $w^{\frac{1}{n}}$ is the angular region

$$B_{-\pi/n} = \Big\{re^{i\theta} \,\Big|\, r > 0, -\frac{\pi}{n} < \theta < \frac{\pi}{n}\Big\}.$$

This branch is given by

$$z = \sqrt[n]{s}\, e^{i\frac{\phi}{n}} \qquad \text{for } w = se^{i\phi} \text{ with } -\pi < \phi < \pi.$$

Clearly,

$$z = \sqrt[n]{|w|}\, e^{i\frac{\operatorname{Arg} w}{n}}.$$

On the same set $A_{-\pi}$ of the w-plane, besides the above continuous branch of $w^{\frac{1}{n}}$, we may define n continuous branches of $w^{\frac{1}{n}}$. Each of them maps $A_{-\pi}$ onto a corresponding angular region $B_{(-\pi+k2\pi)/n}$ with $k = 0, 1, \ldots, n-1$, which results by rotating $B_{-\pi/n}$ in the positive direction through the angle $k\frac{2\pi}{n}$. This branch results from the original branch by multiplication by the constant $e^{ik\frac{2\pi}{n}}$ and it is given by

$$z = \sqrt[n]{s}\, e^{i(\frac{\phi}{n}+k\frac{2\pi}{n})} \qquad \text{for } w = se^{i\phi} \text{ with } -\pi < \phi < \pi.$$

We skip the proof of proposition 5.18, since it is a special case of proposition 5.20.

Proposition 5.18. *Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f : A \to \mathbb{C} \setminus \{0\}$ be a continuous branch of $w^{\frac{1}{n}}$ in A. If w_0 is an interior point of A, then f is differentiable at w_0 and*

$$f'(w_0) = \frac{f(w_0)}{nw_0}.$$

Hence f is holomorphic in the interior of A.

Therefore, *every continuous branch of $w^{\frac{1}{n}}$ in an open set $A \subseteq \mathbb{C} \setminus \{0\}$ can be called* **holomorphic branch of $\mathbf{w}^{\frac{1}{n}}$** *in A.*

Example 5.4.2. We have defined n distinct continuous branches of $w^{\frac{1}{n}}$ in the open set A which results when we exclude any halfline with vertex 0 from the w-plane. All these branches are holomorphic branches of $w^{\frac{1}{n}}$ in A.

We skip the proof of proposition 5.19, since it is a special case of proposition 5.21.

Proposition 5.19. *Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f_1, f_2 : A \to \mathbb{C} \setminus \{0\}$. Let also ω_n be the principal n-th root of unity.*
(i) If f_1 is a continuous branch of $w^{\frac{1}{n}}$ in A and $\frac{f_2(w)}{f_1(w)} = \omega_n^k$ for every $w \in A$, where $k = 0, 1, \ldots, n-1$ is fixed, then f_2 is also a continuous branch of $w^{\frac{1}{n}}$ in A.
(ii) If, moreover, A is connected and f_1, f_2 are continuous branches of $w^{\frac{1}{n}}$ in A, then $\frac{f_2(w)}{f_1(w)} = \omega_n^k$ for every $w \in A$, where $k = 0, 1, \ldots, n-1$ is fixed. In particular, if $f_1(w_0) = f_2(w_0)$ for some $w_0 \in A$, then $f_1(w) = f_2(w)$ for every $w \in A$.

Thus, *if we know one continuous branch of $w^{\frac{1}{n}}$ in the connected set A, then we can find every other possible continuous branch of $w^{\frac{1}{n}}$ in A by multiplying the known branch with any constant n-th root of unity.*

Example 5.4.3. Let $A = A_{-\pi} = \{se^{i\phi} \,|\, s > 0, -\pi < \phi < \pi\}$ be the w-plane without the negative u-semiaxis (where $w = u + iv$). We want to find a continuous branch of the square root $w^{\frac{1}{2}}$ in A taking the value $z = 1$ at $w = 1$.
From the example 5.4.1 we already know the continuous branch of the square root which maps A onto the angular region

$$B_{-\pi/2} = \left\{ re^{i\theta} \,\middle|\, r > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2} \right\},$$

i.e. onto the right halfplane of the z-plane, which is given by

$$z = \sqrt{s}\, e^{i\frac{\phi}{2}} \qquad \text{for } w = se^{i\phi} \text{ with } -\pi < \phi < \pi.$$

Since A is connected, there is no other continuous branch of the square root in A taking the value $z = 1$ at $w = 1$.

Example 5.4.4. Let $A = A_{-\pi} = \{se^{i\phi} \,|\, s > 0, -\pi < \phi < \pi\}$ again. Now we want to find a continuous branch of the square root $w^{\frac{1}{2}}$ in A taking the value $z = -1$ at $w = 1$.
In the previous example we found one continuous branch of the square root in A. Since A is connected, there are exactly two continuous branches of the square root in A. We consider the principal square root of 1, i.e. $\omega_2 = e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$. (Trivial: the square roots of 1 are the solutions of $z^2 = 1$, i.e. the numbers $1, -1$.) Then the second continuous branch of the square root in A is given by

$$z = \sqrt{s}\, e^{i\frac{\phi}{2}} \omega_2 = -\sqrt{s}\, e^{i\frac{\phi}{2}} \qquad \text{for } w = se^{i\phi} \text{ with } -\pi < \phi < \pi,$$

i.e. the opposite of the previous branch. This branch maps A onto the angular region

$$B_{(-\pi+2\pi)/2} = B_{\pi/2} = \left\{ re^{i\theta} \,\middle|\, r > 0, \frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\},$$

i.e. onto the left halfplane of the z-plane.

Now, we introduce a slight generalization of the notion of continuous branch of $w^{\frac{1}{n}}$, i.e. we define the notion of continuous branch of $g(w)^{\frac{1}{n}}$.

Definition. *Let $A \subseteq \mathbb{C}$, $f : A \to \mathbb{C}$ and $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A. We say that f is a* **continuous branch of g(w)$^{\frac{1}{n}}$** *in A if*
(i) f is continuous in A and
(ii) for every $w \in A$ we have $f(w) \in g(w)^{\frac{1}{n}}$ or, equivalently, $f(w)^n = g(w)$.

Proposition 5.20. *Let $A \subseteq \mathbb{C}$, $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A and $f : A \to \mathbb{C}$ be any continuous branch of $g(w)^{\frac{1}{n}}$ in A. If w_0 is an interior point of A and g is differentiable at w_0, then f is differentiable at w_0 and*

$$f'(w_0) = \frac{g'(w_0) f(w_0)}{n g(w_0)}.$$

Hence, if g is holomorphic in the interior of A, then f is also holomorphic in the interior of A.

Proof. We set $z_0 = f(w_0)$ and $z = f(w)$ for every $w \in A$. Then $z_0^n = g(w_0)$ and $z^n = g(w)$. Since f is continuous, $w \to w_0$ implies $z \to z_0$. Therefore, using the derivative of the exponential function at z_0, we see that

$$\frac{f(w) - f(w_0)}{w - w_0} = \frac{z - z_0}{z^n - z_0^n} \frac{g(w) - g(w_0)}{w - w_0} \to \frac{g'(w_0)}{n z_0^{n-1}} = \frac{g'(w_0) f(w_0)}{n g(w_0)} \qquad \text{when } w \to w_0.$$

Thus f is differentiable at w_0 and $f'(w_0) = \frac{g'(w_0) f(w_0)}{n g(w_0)}$. □

Proposition 5.21. *Let $A \subseteq \mathbb{C}$ $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A and $f_1, f_2 : A \to \mathbb{C} \setminus \{0\}$. Let also ω_n be the principal n-th root of unity.*
(i) If f_1 is a continuous branch of $g(w)^{\frac{1}{n}}$ in A and $\frac{f_2(w)}{f_1(w)} = \omega_n^k$ for every $w \in A$, where $k = 0, 1, \ldots, n-1$ is fixed, then f_2 is also a continuous branch of $g(w)^{\frac{1}{n}}$ in A.
(ii) If, moreover, A is connected and f_1, f_2 are continuous branches of $g(w)^{\frac{1}{n}}$ in A, then $\frac{f_2(w)}{f_1(w)} = \omega_n^k$ for every $w \in A$, where $k = 0, 1, \ldots, n-1$ is fixed. In particular, if $f_1(w_0) = f_2(w_0)$ for some $w_0 \in A$, then $f_1(w) = f_2(w)$ for every $w \in A$.

Proof. (i) The continuity of f_1 in A implies the continuity of f_2 in A. We also have $f_1(w)^n = g(w)$ for every $w \in A$ and hence $f_2(w)^n = f_1(w)^n(\omega_n^k)^n = g(w)(\omega_n^n)^k = g(w)$ for every $w \in A$. Thus, f_2 is a continuous branch of $g(w)^{\frac{1}{n}}$ in A.
(ii) Since for every $w \in A$ the two numbers $f_2(w), f_1(w)$ are values of $g(w)^{\frac{1}{n}}$, we have that

$$\Big(\frac{f_2(w)}{f_1(w)}\Big)^n = \frac{f_2(w)^n}{f_1(w)^n} = \frac{g(w)}{g(w)} = 1.$$

Hence for every $w \in A$ the number $\frac{f_2(w)}{f_1(w)}$ is an n-th root of unity and $\frac{f_2}{f_1} : A \to \{1, \omega_n, \ldots, \omega_n^{n-1}\}$. Now, the function $\frac{f_2}{f_1}$ is continuous in A and A is connected, hence the set $\frac{f_2}{f_1}(A)$ is also connected. Since $\frac{f_2}{f_1}(A) \subseteq \{1, \omega_n, \ldots, \omega_n^{n-1}\}$, the set $\frac{f_2}{f_1}(A)$ contains only one point. I.e. $\frac{f_2}{f_1}$ is constant in A and hence $\frac{f_2(w)}{f_1(w)} = \omega_n^k$ in A, where $k = 0, 1, \ldots, n-1$ is fixed.
In case $f_2(w_0) = f_1(w_0)$, then the integer k is 1 and we get $f_2(w) = f_1(w)$ for every $w \in A$. $\square$

Exercises.

5.4.1. Find $(-1)^{\frac{1}{2}}, (-1)^{\frac{1}{3}}, (-1)^{\frac{1}{4}}, i^{\frac{1}{2}}, i^{\frac{1}{3}}, i^{\frac{1}{4}}, (\frac{1-i\sqrt{3}}{2})^{\frac{1}{2}}, (\frac{1-i\sqrt{3}}{2})^{\frac{1}{3}}, (\frac{1-i\sqrt{3}}{2})^{\frac{1}{4}}$.

5.4.2. (i) Find the values of $\log i^2$ and $2\log i$ and observe that $\log i^2 \neq 2\log i$.
(ii) Find the values of $\log i^{\frac{1}{2}}$ and $\frac{1}{2}\log i$ and observe that $\log i^{\frac{1}{2}} = \frac{1}{2}\log i$. Generalizing, prove that for every $w \neq 0$ and every $n \in \mathbb{N}$ we have $\log w^{\frac{1}{n}} = \frac{1}{n}\log w$.

5.4.3. Let $w \neq 0$ and z be any of the values of $w^{\frac{1}{n}}$. Prove that $w^{\frac{1}{n}} = \{z, z\omega_n, z\omega_n^2, \ldots, z\omega_n^{n-1}\}$ or, equivalently, $w^{\frac{1}{n}} = z1^{\frac{1}{n}}$.

5.4.4. The set $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a group under multiplication of complex numbers.
(i) Prove that, if $n \in \mathbb{N}$, $n \geq 2$, then the n-th root of unity, $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$, is a subgroup of $\mathbb{C}^*$.
(ii) Let $z = \omega_n^k$ be any of the values of the n-th root of unity and $\langle z\rangle = \{z^m \,|\, m \in \mathbb{Z}\}$ be the group generated by z. Prove that z is a generator of $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ or, equivalently, $\langle z\rangle = \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ if and only if $\gcd\{k, n\} = 1$.
(iii) Prove that $\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ has no subgroups other than $\{1\}$ and itself if and only if n is a prime number.

5.4.5. Look at exercise 4.3.2. Consider the curves on the z-plane with equations $x^2 - y^2 = \alpha$ and $2xy = \beta$. If the two curves intersect at a point (x_0, y_0), find in two ways their angle at this point.

5.4.6. Prove that there is no continuous branch of $w^{\frac{1}{n}}$ in any circle $C_0(r)$ and hence in any set A which contains such a circle.

5.4.7. Let $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ be given by

$$f(u,v) = \begin{cases} \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} + i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}, & \text{if } u \in \mathbb{R}, v > 0 \\ \sqrt{u}, & \text{if } u > 0, v = 0 \\ \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} - i\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}, & \text{if } u \in \mathbb{R}, v < 0 \end{cases}$$

where $w = u + iv = (u, v)$.
(i) prove that $f(w)^2 = w$ for every $w \in \mathbb{C} \setminus (-\infty, 0]$ as well as that f coincides with the continuous branch of the square root which we saw in the example 5.4.3.
(ii) Prove that f is one-to-one and onto $\{z \mid \operatorname{Re} z > 0\}$.
(iii) Prove that f is continuous in $\mathbb{C} \setminus (-\infty, 0]$, using either the formula of f or sequences and the identity in (i).
(iv) Prove that f is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$.

5.4.8. Find the continuous branches of the square root in $\mathbb{C} \setminus [0, +\infty)$.

5.4.9. Find the continuous branches of the cube root in $\mathbb{C} \setminus (-\infty, 0]$.

5.4.10. Let $g : A \to \mathbb{C} \setminus \{0\}$ be continuous, let $f : A \to \mathbb{C}$ be a continuous branch of $\log g$ in A and let $n \in \mathbb{N}$, $n \geq 2$. Prove that $h : A \to \mathbb{C}$ with $h = e^{\frac{1}{n} f}$ is a continuous branch of $g^{\frac{1}{n}}$ in A.

5.4.11. (i) Considering a continuous branch of $(w+1)^{\frac{1}{2}}$ in $\mathbb{C} \setminus (-\infty, -1]$ and a continuous branch of $(w-1)^{\frac{1}{2}}$ in $\mathbb{C} \setminus [1, +\infty)$, prove that there is a continuous branch of $(w^2 - 1)^{\frac{1}{2}}$ in $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$, i.e. $f : \Omega \to \mathbb{C}$ continuous in Ω so that $f(w)^2 = w^2 - 1$ for every $w \in \Omega$.
(ii) Considering a continuous branch of $(w+1)^{\frac{1}{2}}$ in $\mathbb{C} \setminus (-\infty, -1]$ and a continuous branch of $(w-1)^{\frac{1}{2}}$ in $\mathbb{C} \setminus (-\infty, 1]$, prove that there is a continuous branch of $(w^2-1)^{\frac{1}{2}}$ in $\Omega' = \mathbb{C} \setminus [-1, 1]$, i.e. $f : \Omega' \to \mathbb{C}$ continuous in Ω' so that $f(w)^2 = w^2 - 1$ for every $w \in \Omega'$. *This is more difficult than (i).*
(iii) What is the possible relation between two continuous branches of $(w^2 - 1)^{\frac{1}{2}}$ in the same set, either Ω or Ω'?
(iv) Prove that there is no continuous branch of $(w^2 - 1)^{\frac{1}{2}}$ in any circle which surrounds one of the points ± 1 but not the other.
(v) Prove that the continuous branches of $(w^2 - 1)^{\frac{1}{2}}$ in Ω and in Ω' are holomorphic.

5.4.12. Prove that we can define a holomorphic branch f of $(1-w)^{\frac{1}{2}} + (1+w)^{\frac{1}{2}}$ in the region A which results when we exclude from $\mathbb{C}$ two non-intersecting halflines, one with vertex $+1$ and another with vertex -1. Prove that every such f satisfies $f(w)^4 - 4f(w)^2 + 4w^2 = 0$ for every $w \in A$. How many such branches f exist in A?

5.4.13. (i) Prove that $w^a \in \exp(a \log w)$ for every $w \neq 0$ and every $a \in \mathbb{Z}$.
(ii) Generalizing (i), we define $w^a = e^{a \log w} = \{e^{az} \mid z \in \log w\}$ for every $w \neq 0$ and every $a \notin \mathbb{Z}$. (Thus, w^a is a set.)
(iii) Find $(\frac{1-i\sqrt{3}}{2})^{\frac{1}{2}}$, $i^{\frac{1}{4}}$ and draw 2^i, $i^{\sqrt{2}}$.
(iv) Prove that $w^{a+b} \subseteq w^a w^b$, $w^{a-b} \subseteq \frac{w^a}{w^b}$ and $w^{ab} \subseteq (w^a)^b$.
(v) Let $A \subseteq \mathbb{C} \setminus \{0\}$ and $f : A \to \mathbb{C}$ be a continuous branch of $\log w$. Then $g : A \to \mathbb{C}$ with $g(w) = e^{af(w)}$ for every $w \in A$ is called a **continuous branch of $\mathbf{w^a}$** in A. Prove that g is differentiable at every interior point w_0 of A and $g'(w_0) = \frac{ag(w_0)}{w_0}$. Therefore, if we can define a holomorphic branch of $\log w$ in the open set $A \subseteq \mathbb{C} \setminus \{0\}$, then we can also define a holomorphic branch of w^a in A.

5.4.14. Prove that there is a unique holomorphic branch f of $(1-w)^i = e^{i \log(1-w)}$ in $D_0(1)$ so that $f(0) = 1$. Then prove that there are $c_1, c_2 > 0$ so that $c_1 < |f(w)| < c_2$ for every $w \in D_0(1)$. Find the best such c_1, c_2.

5.4.15. Look at exercise 5.2.5. We define

$$\arccos w = \{z \mid \cos z = w\}, \quad \arcsin w = \{z \mid \sin z = w\}, \quad \arctan w = \{z \mid \tan z = w\}.$$

(i) Prove that $\arccos w$ and $\arcsin w$ are non-empty sets for every w and that $\arctan w$ is non-empty for every $w \neq \pm i$ and empty for $w = \pm i$.

(ii) Express arccos, arcsin and arctan in terms of log.
(iii) It should be clear from exercise 5.2.5 that $w = \sin z$ is one-to-one from $\{x+iy \,|\, -\frac{\pi}{2} < x < \frac{\pi}{2}\}$ onto $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$. Prove that the inverse function g_0 is a continuous branch of $\arcsin w$ in Ω, i.e. g_0 is continuous in Ω and $\sin g_0(w) = w$ for every $w \in \Omega$. Describe all continuous branches g of $\arcsin w$ in Ω and prove that they are holomorphic in Ω with $g'(w) = 1/(1-w^2)^{\frac{1}{2}}$ for every $w \in \Omega$, where at the denominator appears a specific continuous branch of $(1-w^2)^{\frac{1}{2}}$ in Ω.
(iv) From exercise 5.2.5 again, it is clear that $w = \cos z$ is one-to-one from $\{x+iy \,|\, 0 < x < \pi\}$ onto $\Omega = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$. Prove that the inverse function h_0 is a continuous branch of $\arccos w$ in Ω, i.e. h_0 is continuous in Ω and $\cos h_0(w) = w$ for every $w \in \Omega$. Describe all continuous branches h of $\arccos w$ in Ω and prove that they are holomorphic in Ω with $h'(w) = -1/(1-w^2)^{\frac{1}{2}}$ for every $w \in \Omega$, where at the denominator appears a specific continuous branch of $(1-w^2)^{\frac{1}{2}}$ in Ω.
(v) Prove that the function $w = \tan z$ is one-to-one from $\{x+iy \,|\, -\frac{\pi}{2} < x < \frac{\pi}{2}\}$ onto $U = \mathbb{C} \setminus \{iv \,|\, v \leq -1 \text{ or } 1 \leq v\}$. Prove that the inverse function k_0 is a continuous branch of $\arctan w$ in U, i.e. k_0 is continuous in U and $\tan k_0(w) = w$ for every $w \in U$. Describe all continuous branches k of $\arctan w$ in U and prove that they are holomorphic in U with $k'(w) = \frac{1}{1+w^2}$ for every $w \in U$.

5.4.16. Considering appropriate continuous branches of $w^{\frac{1}{2}}$, evaluate $\int_\gamma \frac{1}{w^{1/2}}\, dw$ for both curves $\gamma_1(t) = e^{it}$, $t \in [0, \pi]$, and $\gamma_2(t) = e^{-it}$, $t \in [0, \pi]$.

5.5 Functions defined by curvilinear integrals.

5.5.1 Indefinite integrals.

Definition. *Let $\Omega \subseteq \mathbb{C}$ be a region and $f, F : \Omega \to \mathbb{C}$. We say that F is a primitive of f in Ω if $F'(z) = f(z)$ for every $z \in \Omega$.*

Proposition 5.22. *Let $\Omega \subseteq \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$ be continuous. The following are equivalent.*
(i) For every closed curve γ in Ω we have $\oint_\gamma f(z)\, dz = 0$.
(ii) If γ_1, γ_2 are any two curves in Ω with the same endpoints, then $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$.
(iii) There is a primitive of f in Ω.

Proof. (iii) ⇒ (i) Let $F : \Omega \to \mathbb{C}$ be any primitive of f in Ω. We take an arbitrary curve $\gamma : [a, b] \to \Omega$ with $\gamma(a) = \gamma(b)$. Then

$$\oint_\gamma f(z)\, dz = \oint_\gamma F'(z)\, dz = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_a^b (F \circ \gamma)'(t)\, dt$$
$$= (F \circ \gamma)(b) - (F \circ \gamma)(a) = F(\gamma(b)) - F(\gamma(a)) = 0.$$

(i) ⇒ (ii) Assume that the curves γ_1, γ_2 in Ω have the same endpoints. Then the curve $\gamma = \gamma_1 \dot{+} (\neg\, \gamma_2)$ is a closed curve in Ω and then

$$\int_{\gamma_1} f(z)\, dz - \int_{\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\neg\, \gamma_2} f(z)\, dz = \oint_\gamma f(z)\, dz = 0.$$

Therefore, $\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$.
(ii) ⇒ (iii) We consider an arbitrary fixed $z_0 \in \Omega$. Then for every $z \in \Omega$ there is at least one curve γ in Ω with initial point z_0 and final point z. We define the function $F : \Omega \to \mathbb{C}$ by

$$F(z) = \int_\gamma f(\zeta)\, d\zeta. \tag{5.11}$$

This formula defines $F(z)$ uniquely, since the value of the curvilinear integral depends only on the point z and not on the particular curve γ which we use to join z_0 to z.
Now we shall prove that F is a primitive of f in Ω. We take an arbitrary $z \in \Omega$ and a disc $D_z(r) \subseteq \Omega$. We also take a curve γ in Ω with initial point z_0 and final point z. Then the value of $F(z)$ is given by (5.11). Now we consider any $w \in D_z(r)$ and the curve $\gamma \dot{+} [z,w]$. This curve is in Ω and has initial point z_0 and final point w. Therefore,

$$F(w) = \int_{\gamma \dot{+} [z,w]} f(\zeta)\, d\zeta = \int_\gamma f(\zeta)\, d\zeta + \int_{[z,w]} f(\zeta)\, d\zeta. \tag{5.12}$$

From (5.11) and (5.12) we get

$$F(w) - F(z) - f(z)(w-z) = \int_{[z,w]} f(\zeta)\, d\zeta - f(z) \int_{[z,w]} d\zeta = \int_{[z,w]} (f(\zeta) - f(z))\, d\zeta. \tag{5.13}$$

Now, since f is continuous, for every $\epsilon > 0$ there is $\delta > 0$ so that $|f(z') - f(z)| < \epsilon$ for every $z' \in \Omega$ with $|z' - z| < \delta$. Taking $w \in D_z(r)$ with $|w - z| < \delta$ we automatically have $|\zeta - z| < \delta$ for every $\zeta \in [z,w]$ and hence (5.13) implies

$$|F(w) - F(z) - f(z)(w-z)| = \Big| \int_{[z,w]} (f(\zeta) - f(z))\, d\zeta \Big| \leq \epsilon |w - z|.$$

Therefore, $\left| \frac{F(w)-F(z)}{w-z} - f(z) \right| < \epsilon$ for every w with $|w - z| < \delta$ and hence $F'(z) = f(z)$. $\square$

Definition. *Let $\Omega \subseteq \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$ be continuous. If either one of the equivalent conditions (i), (ii) of proposition 5.22 is satisfied, then as we saw in the proof of (ii)$\Rightarrow$(iii) of proposition 5.22, we may define a function $F : \Omega \to \mathbb{C}$ by choosing a fixed point $z_0 \in \Omega$ and setting $F(z) = \int_\gamma f(\zeta)\, d\zeta$ for every $z \in \Omega$, where γ is an arbitrary curve in Ω with initial point z_0 and final point z.*
Now, any function $F : \Omega \to \mathbb{C}$ of the form

$$F(z) = \int_\gamma f(\zeta)\, d\zeta + c \qquad \textit{for every } z \in \Omega,$$

where γ is an arbitrary curve in Ω with fixed (but otherwise arbitrary) initial point z_0 and final point z and where c is an arbitrary constant, is called **indefinite integral** *of f in Ω.*

The crucial condition for the existence of an indefinite integral is (ii) (or its equivalent (i)) of proposition 5.22. As soon as this is satisfied, then by changing the *base point* $z_0 \in \Omega$ or the constant c we get different indefinite integrals F.

In the proof of proposition 5.22 we saw that every indefinite integral of f is a primitive of f. The converse is also true. Indeed, let F be any primitive of f in the region Ω, i.e. let $F'(z) = f(z)$ for every $z \in \Omega$. Proposition 5.22 implies that condition (ii) is satisfied and, if we take any curve $\gamma : [a,b] \to \Omega$ with initial point a fixed $z_0 \in \Omega$ and final point $z \in \Omega$, then

$$\begin{aligned} \int_\gamma f(\zeta)\, d\zeta = \int_\gamma F'(\zeta)\, d\zeta = \int_a^b F'(\gamma(t))\gamma'(t)\, dt = \int_a^b (F \circ \gamma)'(t)\, dt \\ = (F \circ \gamma)(b) - (F \circ \gamma)(a) = F(z) - F(z_0). \end{aligned} \tag{5.14}$$

Therefore, F has the form

$$F(z) = \int_\gamma f(\zeta)\, d\zeta + F(z_0)$$

and hence it is an indefinite integral of f in Ω.

We summarize. *Let $f : \Omega \to \mathbb{C}$ be continuous in the region Ω. Then the notion of primitive of f in Ω coincides with the notion of indefinite integral of f in Ω. Moreover, the existence of a* *primitive or, equivalently, of an indefinite integral of f in Ω is equivalent to the validity of condition (ii) or (i) of proposition 5.22.*

Regarding the number of possible primitives of f in Ω we may easily see that, if there is at least one primitive F of f in Ω, then all others are of the form $F + c$ for an arbitrary constant c. Indeed, it is obvious that $F + c$ is a primitive of f in Ω. Conversely, if G is a primitive of f in Ω, then we have $(G - F)'(z) = G'(z) - F'(z) = f(z) - f(z) = 0$ for every $z \in \Omega$. Now, theorem 4.3 implies that $G - F$ is a constant in Ω.

Since it is useful for calculations of curvilinear integrals, we state relation (5.14) as a separate proposition.

Proposition 5.23. *Let $f, F : \Omega \to \mathbb{C}$ and let F be a primitive of the continuous f in the region Ω. Then for every curve γ in Ω with initial endpoint z_1 and final endpoint z_2 we have*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Example 5.5.1. Every polynomial function $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ has the primitive $a_0 z + \frac{a_1}{2} z^2 + \cdots + \frac{a_n}{n+1} z^{n+1}$ in $\mathbb{C}$. Therefore, we have

$$\oint_\gamma p(z)\,dz = \oint_\gamma (a_0 + a_1 z + \cdots + a_n z^n)\,dz = 0$$

for every closed curve γ.
In particular, if $n \in \mathbb{Z}$, $n \geq 0$, we have $\oint_\gamma (z - z_0)^n\,dz = 0$ for every closed curve γ. A very special case of this, with the circle $C_{z_0}(r)$, we saw in examples 3.2.8 and 5.2.2.

Example 5.5.2. The exponential function e^z has the primitive e^z in $\mathbb{C}$. Hence

$$\oint_\gamma e^z\,dz = 0$$

for every closed curve γ.

Example 5.5.3. Let $z_0 \in \mathbb{C}$ and $n \in \mathbb{N}$, $n \geq 2$. Then the function $\frac{1}{(z-z_0)^n}$ has the primitive $-\frac{1}{(n-1)(z-z_0)^{n-1}}$ in $\mathbb{C} \setminus \{z_0\}$. Therefore,

$$\oint_\gamma \frac{1}{(z - z_0)^n}\,dz = 0, \qquad n \in \mathbb{N}, n \geq 2,$$

for every closed curve γ in $\mathbb{C} \setminus \{z_0\}$.
A very special case of this, with the circle $C_{z_0}(r)$, we saw in examples 3.2.8 and 5.2.2.

Example 5.5.4. The function $\frac{1}{z-z_0}$ (the case $n = 1$ of the previous example) has no primitive in $\mathbb{C} \setminus \{z_0\}$ or even in any open ring $D_{z_0}(r_1, r_2) = \{z \,|\, r_1 < |z - z_0| < r_2\}$.
Indeed, if $\frac{1}{z-z_0}$ had a primitive in $D_{z_0}(r_1, r_2)$, then we would have $\oint_\gamma \frac{1}{z-z_0}\,dz = 0$ for every closed curve γ in $D_{z_0}(r_1, r_2)$. Now, if we take a radius r so that $r_1 < r < r_2$ and the curve $\gamma : [0, 2\pi] \to D_{z_0}(r_1, r_2)$ with parametric equation $\gamma(t) = z_0 + re^{it}$, then we have

$$\oint_\gamma \frac{1}{z - z_0}\,dz = \oint_{C_{z_0}(r)} \frac{1}{z - z_0}\,dz = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it}\,dt = 2\pi i \neq 0.$$

In fact, we did exactly the same calculation in example 5.2.2.

The following result is important.

Proposition 5.24. *Let $g : \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic in the region Ω and let g' be continuous in Ω. Then a holomorphic branch of $\log g$ exists in Ω if and only if $\oint_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ for every closed curve γ in Ω.*

Proof. Assume that there is a holomorphic branch of $\log g$ in Ω, i.e. there is $F : \Omega \to \mathbb{C}$ holomorphic in Ω so that $e^{F(z)} = g(z)$ for every $z \in \Omega$. Then $F'(z)e^{F(z)} = g'(z)$ for every $z \in \Omega$ and hence $F'(z) = \frac{g'(z)}{g(z)}$ for every $z \in \Omega$. Therefore, $F(z)$ is a primitive of $\frac{g'(z)}{g(z)}$ in Ω and thus, $\oint_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ for every closed curve γ in Ω.
Conversely, assume $\oint_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ for every closed curve γ in Ω. Then $\frac{g'(z)}{g(z)}$ has a primitive in Ω, i.e. there is $F : \Omega \to \mathbb{C}$ so that $F'(z) = \frac{g'(z)}{g(z)}$ for every $z \in \Omega$. Now,

$$\frac{d}{dz}\big(g(z)e^{-F(z)}\big) = g'(z)e^{-F(z)} - g(z)F'(z)e^{-F(z)} = 0$$

for every $z \in \Omega$. This implies that, for some constant c, we have $g(z)e^{-F(z)} = c$ for every $z \in \Omega$. Since $c \neq 0$, there is a constant d so that $e^d = c$ and we finally get that

$$e^{F(z)+d} = g(z) \qquad \text{for every } z \in \Omega.$$

Now the function $F + d$ is a holomorphic branch of $\log g$ in Ω. □

In the next chapter we shall prove that for every holomorphic g the derivative g' is automatically continuous. Therefore, *a posteriori* the assumption in proposition 5.24 that g' is continuous is unnecessary.

Example 5.5.5. If the region $\Omega \subseteq \mathbb{C} \setminus \{z_0\}$ contains a circle $C_{z_0}(r)$, then there is no holomorphic branch of $\log(z - z_0)$ in Ω. In fact, example 5.5.4 shows that $\oint_{C_{z_0}(r)} \frac{1}{z-z_0}\, dz \neq 0$.

Example 5.5.6. Let $g : \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic in the region Ω and let g' be continuous in Ω and suppose that there is a halfline with vertex 0 so that $g(\Omega) \subseteq \mathbb{C} \setminus l$.
We know that a holomorphic branch of $\log w$ exists in $\mathbb{C} \setminus l$. If $f : \mathbb{C} \setminus l \to \mathbb{C}$ is such a branch, then $e^{f(w)} = w$ for every $\in \mathbb{C} \setminus l$. This implies that $e^{f(g(z))} = g(z)$ for every $z \in \Omega$ and this means that the function $f \circ g : \Omega \to \mathbb{C}$ is a holomorphic branch of $\log g$ in Ω. From proposition 5.24 we also get that $\oint_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ for every closed curve γ in Ω.

5.5.2 Integrals with parameter.

Lemma 5.1. *Let $n \in \mathbb{N}$ and γ be any curve. If $\phi : \gamma^* \to \mathbb{C}$ is continuous in the trajectory γ^*, we define the function $f : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ by*

$$f(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}\, d\zeta \qquad \textit{for every } z \notin \gamma^*.$$

Then f is holomorphic in the open set $\mathbb{C} \setminus \gamma^*$ and

$$f'(z) = n \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \textit{for every } z \notin \gamma^*.$$

Proof. We take any $z \in \mathbb{C} \setminus \gamma^*$. Since $\mathbb{C} \setminus \gamma^*$ is open, there is $\delta > 0$ so that $D_z(\delta) \subseteq \mathbb{C} \setminus \gamma^*$. We consider the smaller circle $D_z(\frac{\delta}{2})$ and we have

$$|\zeta - w| \geq \frac{\delta}{2} \qquad \text{for every } \zeta \in \gamma^* \text{ and every } w \in D_z\Big(\frac{\delta}{2}\Big). \tag{5.15}$$

Now for every $w \in D_z(\frac{\delta}{2})$ we get

$$f(w) - f(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - w)^n}\,d\zeta - \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}\,d\zeta = \int_\gamma \Big(\frac{1}{(\zeta - w)^n} - \frac{1}{(\zeta - z)^n}\Big)\,\phi(\zeta)\,d\zeta,$$

hence

$$\frac{f(w) - f(z)}{w - z} - n\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta = \int_\gamma \Big(\frac{\frac{1}{(\zeta-w)^n} - \frac{1}{(\zeta-z)^n}}{w - z} - \frac{n}{(\zeta - z)^{n+1}}\Big)\,\phi(\zeta)\,d\zeta. \tag{5.16}$$

To simplify the notation, we temporarily set $A = \zeta - w$ and $B = \zeta - z$, and, to estimate the parenthesis in (5.16), we use the algebraic identity

$$\frac{\frac{1}{A^n} - \frac{1}{B^n}}{B - A} - \frac{n}{B^{n+1}} = (B - A)\Big(\frac{1}{A^n B^2} + \frac{2}{A^{n-1}B^3} + \cdots + \frac{n-1}{A^2 B^n} + \frac{n}{AB^{n+1}}\Big).$$

From (5.15) we have that $|A| \geq \frac{\delta}{2}$ and $|B| \geq \frac{\delta}{2}$ for every $\zeta \in \gamma^*$ and $w \in D_z(\frac{\delta}{2})$ and hence

$$\begin{aligned}\Big|\frac{\frac{1}{A^n} - \frac{1}{B^n}}{B - A} - \frac{n}{B^{n+1}}\Big| &\leq |B - A|\Big(\frac{1}{|A|^n|B|^2} + \cdots + \frac{n}{|A||B|^{n+1}}\Big)\\ &\leq |w - z|\frac{1 + 2 + \cdots + (n-1) + n}{(\frac{\delta}{2})^{n+2}} \leq |w - z|\,\frac{n^2 2^{n+2}}{\delta^{n+2}}.\end{aligned} \tag{5.17}$$

Now, γ^* is compact and ϕ is continuous in γ^* and hence there is $M \geq 0$ so that $|\phi(\zeta)| \leq M$ for every $\zeta \in \gamma^*$. Therefore, (5.16) and (5.17) imply

$$\Big|\frac{f(w) - f(z)}{w - z} - n\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta\Big| \leq |w - z|\,\frac{n^2 2^{n+2}}{\delta^{n+2}}\,M\,l(\gamma)$$

for every $w \in D_z(\frac{\delta}{2})$. Thus, $\lim_{w\to z}\frac{f(w)-f(z)}{w-z} = n\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta$ and f is differentiable at z with $f'(z) = n\int_\gamma \frac{\phi(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta$. □

Observe that lemma 5.1 justifies the change of order of the operations of integration and differentiation with respect to the parameter z:

$$f'(z) = \frac{d}{dz}f(z) = \frac{d}{dz}\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^n}\,d\zeta = \int_\gamma \frac{d}{dz}\Big(\frac{\phi(\zeta)}{(\zeta - z)^n}\Big)\,d\zeta = n\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Proposition 5.25. *Let γ be any curve and $\phi : \gamma^* \to \mathbb{C}$ be continuous in the trajectory γ^*. Then the function $f : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$ defined by*

$$f(z) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z}\,d\zeta \qquad \textit{for every } z \notin \gamma^*$$

is infinitely many times differentiable in the open set $\mathbb{C} \setminus \gamma^*$ and

$$f^{(n)}(z) = n!\int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \qquad \textit{for every } z \notin \gamma^*.$$

Proof. Successive application of lemma 5.1. □

Exercises.

5.5.1. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic in Ω and let f', g' be continuous in Ω.
(i) If $|f(z) - 1| < 1$ for every $z \in \Omega$, prove that $\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 0$ for every closed curve γ in Ω.
(ii) If $|f(z) - g(z)| < |g(z)|$ for every $z \in \Omega$, prove that $\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \oint_\gamma \frac{g'(z)}{g(z)}\,dz$ for every closed curve γ in Ω.

5.5.2. Let γ be any curve and $\phi : \gamma^* \to \mathbb{C}$ be continuous in the trajectory γ^*. We know that the function $f(z) = \int_\gamma \frac{\phi(\zeta)}{\zeta - z}\, d\zeta$ is holomorphic in $\mathbb{C} \setminus \gamma^*$. Prove that f is holomorphic at ∞.

5.5.3. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous in $\mathbb{R}$ and let $\int_{-\infty}^{+\infty} \frac{|f(t)|}{1+|t|}\, dt < +\infty$. Prove that the function $F(z) = \int_{-\infty}^{+\infty} \frac{f(t)}{t-z}\, dt$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$.

5.5.4. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous in $\mathbb{R}$ and $\int_{-\infty}^{+\infty} |f(t)| e^{M|t|}\, dt < +\infty$ for every $M > 0$. Prove that the functions $F(z) = \int_{-\infty}^{+\infty} f(t) e^{tz}\, dt$, $G(z) = \int_{-\infty}^{+\infty} f(t) \cos tz\, dt$ and $H(z) = \int_{-\infty}^{+\infty} f(t) \sin tz\, dt$ are holomorphic in $\mathbb{C}$.

5.5.5. Find the domains of holomorphy of the functions $f(z) = \int_0^1 \frac{1}{1+tz}\, dt$, $g(z) = \int_{-1}^1 \frac{e^{tz}}{1+t^2}\, dt$ and $h(z) = \int_0^{+\infty} \frac{e^{tz}}{1+t^2}\, dt$, $k(z) = \int_0^{+\infty} e^{-tz^2}\, dt$.

5.6 Functions defined by power series.

Definition. *Every series of the form*

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

is called **power series** *with* **center** z_0 *and* **coefficients** a_n*. The* $R \in [0, +\infty]$ *defined by*

$$R = \frac{1}{\overline{\lim}\, \sqrt[n]{|a_n|}}$$

is called **radius of convergence** *of the power series. (Of course we understand that* $R = 0$ *if* $\overline{\lim}\, \sqrt[n]{|a_n|} = +\infty$ *and* $R = +\infty$ *if* $\overline{\lim}\, \sqrt[n]{|a_n|} = 0$*.)*

Proposition 5.26. *Let* $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ *be a power series with radius of convergence* R*.*
If $R = 0$*, then the series converges only at* z_0*. If* $R > 0$*, then*
(i) The power series converges absolutely at every $z \in D_{z_0}(R)$*.*
(ii) The power series diverges at every $z \notin \overline{D}_{z_0}(R)$*.*
(iii) The power series converges uniformly in every closed disc $\overline{D}_{z_0}(r)$ *with* $r < R$*.*
(iv) The sum $s(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$*, which is defined as a function* $s : D_{z_0}(R) \to \mathbb{C}$*, is holomorphic in* $D_{z_0}(R)$*. The derivative of* s *in* $D_{z_0}(R)$ *is the sum* $t(z) = \sum_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}$ *of the power series which results from* $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ *by formal termwise differentiation.*

Proof. If $z = z_0$, then the power series consists only of its constant term a_0 and hence converges. If $z \neq z_0$, then by the definition of R we get

$$\overline{\lim}\, \sqrt[n]{|a_n (z - z_0)^n|} = \overline{\lim}\, \sqrt[n]{|a_n|}\, |z - z_0| = \frac{|z - z_0|}{R}.$$

The root test of Cauchy for general series implies that the power series converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0 > R$ and this is the content of (i) and (ii).
(iii) Let $0 < r < R$. We consider any R' so that $r < R' < R$. Then $\overline{\lim}\, \sqrt[n]{|a_n|} < \frac{1}{R'}$ and hence there is n_0 so that $\sqrt[n]{|a_n|} \leq \frac{1}{R'}$ for every $n \geq n_0$. Then for every $z \in \overline{D}_{z_0}(r)$ we have

$$|a_n (z - z_0)^n| = |a_n|\, |z - z_0|^n \leq \Big(\frac{r}{R'}\Big)^n \qquad \text{for every } n \geq n_0.$$

Since $\frac{r}{R'} < 1$, we have $\sum_{n=0}^{+\infty} (\frac{r}{R'})^n < +\infty$ and the test of Weierstrass implies that the power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ converges uniformly in $\overline{D}_{z_0}(r)$.
(iv) Besides $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$, we also consider the power series $\sum_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}$. The

second power series results from the first by formal termwise differentiation. We shall prove that the second series converges at every $z \in D_{z_0}(R)$ and that its sum is the derivative of the sum s of the first series at every $z \in D_{z_0}(R)$.
Since $\sqrt[n]{n} \to 1$, we have

$$\overline{\lim}\ \sqrt[n]{|na_n|} = \overline{\lim}\ \sqrt[n]{n}\ \sqrt[n]{|a_n|} = \overline{\lim}\ \sqrt[n]{|a_n|}$$

and the radius of convergence of the series $\sum_{n=1}^{+\infty} na_n(z-z_0)^n$ is also R. Thus, $\sum_{n=1}^{+\infty} na_n(z-z_0)^n$ and hence $\sum_{n=1}^{+\infty} na_n(z-z_0)^{n-1}$ converges at every $z \in D_{z_0}(R)$.
We define $t(z) = \sum_{n=1}^{+\infty} na_n(z-z_0)^{n-1}$ at every $z \in D_{z_0}(R)$.
Now at every $z, w \in D_{z_0}(R)$ we have $s(w) - s(z) = \sum_{n=0}^{+\infty} a_n((w-z_0)^n - (z-z_0)^n)$. For simplicity, we shall set temporarily $B = w - z_0$ and $A = z - z_0$ and then we have

$$s(w) - s(z) = (w-z)\sum_{n=1}^{+\infty} a_n(B^{n-1} + B^{n-2}A + \cdots + BA^{n-2} + A^{n-1})$$

and hence

$$\begin{aligned}\frac{s(w)-s(z)}{w-z} - t(z) &= \sum_{n=2}^{+\infty} a_n(B^{n-1} + B^{n-2}A + \cdots + BA^{n-2} + A^{n-1} - nA^{n-1})\\ &= (w-z)\sum_{n=2}^{+\infty} a_n(B^{n-2} + 2B^{n-3}A + \cdots + (n-2)BA^{n-3} + (n-1)A^{n-2}).\end{aligned} \tag{5.18}$$

Now we fix $z \in D_{z_0}(R)$ and take $\delta = \frac{R-|z-z_0|}{2} > 0$. We also set $R_1 = |z-z_0| + \delta = R - \delta$. If $w \in D_z(\delta)$, then $|B| \leq R_1$ and $|A| \leq R_1$ and (5.18) implies

$$\Big|\frac{s(w)-s(z)}{w-z} - t(z)\Big| \leq |w-z| \sum_{n=2}^{+\infty} n^2|a_n|R_1^{n-2}.$$

Since $\overline{\lim}\ \sqrt[n]{|n^2 a_n R_1^n|} = \frac{R_1}{R} < 1$, the last sum is a finite number independent of $w \in D_z(\delta)$. Therefore, $\lim_{w\to z} \frac{s(w)-s(z)}{w-z} = t(z)$ and s is differentiable at z with $s'(z) = t(z)$. $\square$

If R is the radius of convergence of $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$, then the open disc $D_{z_0}(R)$ is called **disc of convergence** of the power series.

If $R = 0$ then the disc of convergence is empty and the power series converges only at the center z_0. If $R = +\infty$ then the disc of convergence is the whole z-plane. In this case the power series converges (absolutely) at every z. If $0 < R < +\infty$, then the power series converges (absolulety) at every $z \in D_{z_0}(R)$ and diverges at every z outside the closed disc $\overline{D}_{z_0}(R)$. There is no general result for the convergence at the points $z \in C_{z_0}(R)$: the series may converge at some points of the circle and diverge at its remaining points. If $0 < R \leq +\infty$, the power series converges uniformly in every smaller closed disc $\overline{D}_{z_0}(r)$ with $r < R$ and its sum $s(z)$ is a holomorphic function in $D_{z_0}(R)$. In fact the derivative of the sum $s(z)$ of the power series is the function $t(z)$ which is the sum of the power series we get by formal termwise differentiation of the original power series. We saw that the differentiated power series has the same disc of convergence as the original series. Therefore, we may repeat our arguments: the function $t(z)$ is holomorphic in $D_{z_0}(R)$ and its derivative, i.e. the second derivative of $s(z)$, is the sum of the power series which we get by a second formal termwise differentiation of the original power series. We conclude that *the function $s(z)$ is infinitely many times differentiable in the disc of convergence $D_{z_0}(R)$ and*

$$s^{(k)}(z) = \sum_{n=k}^{+\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k} \qquad \text{for every } z \in D_{z_0}(R).$$

Example 5.6.1. For the power series $\sum_{n=1}^{+\infty}\frac{z^n}{n}$ we get $\overline{\lim}\sqrt[n]{|\frac{1}{n}|}=\lim\frac{1}{\sqrt[n]{n}}=1$, and hence $R=1$. The disc of convergence is $\mathbb{D}$. If s is the function defined by the power series in $\mathbb{D}$, then $s'(z)=\sum_{n=1}^{+\infty}z^{n-1}=\frac{1}{1-z}$ for every $z\in\mathbb{D}$. We observe that $-\operatorname{Log}(1-z)$ is defined and is holomorphic in $\mathbb{D}$. Its derivative is $\frac{1}{1-z}$ and its value at 0 is 0. Since the functions $s(z)$ and $-\operatorname{Log}(1-z)$ have the same derivative in the region $\mathbb{D}$ and the same value at 0, we conclude that

$$\sum_{n=1}^{+\infty}\frac{z^n}{n}=-\operatorname{Log}(1-z)\qquad\text{for every } z\in\mathbb{D}.$$

We shall come back to this identity later, when we study the Taylor series of the function $-\operatorname{Log}(1-z)$ in $\mathbb{D}$.

Example 5.6.2. For $\sum_{n=1}^{+\infty}\frac{z^n}{n^2}$ we get $\overline{\lim}\sqrt[n]{|\frac{1}{n^2}|}=\lim\frac{1}{(\sqrt[n]{n})^2}=1$, and hence $R=1$. The disc of convergence is $\mathbb{D}$.

Example 5.6.3. For $\sum_{n=0}^{+\infty}\frac{z^n}{n!}$ we have $\overline{\lim}\sqrt[n]{|\frac{1}{n!}|}=\lim\frac{1}{\sqrt[n]{n!}}=0$, and hence $R=+\infty$. The disc of convergence is $\mathbb{C}$. If s is the function defined by the power series in $\mathbb{C}$, then $s'(z)=\sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}=\sum_{n=0}^{+\infty}\frac{z^n}{n!}=s(z)$ for every $z\in\mathbb{C}$. Now we have that $\frac{d}{dz}(e^{-z}s(z))=-e^{-z}s(z)+e^{-z}s'(z)=0$ for every $z\in\mathbb{C}$. Since the value of $e^{-z}s(z)$ at 0 is 1, we find that $e^{-z}s(z)=1$ for every $z\in\mathbb{C}$ and thus

$$\sum_{n=0}^{+\infty}\frac{z^n}{n!}=e^z\qquad\text{for every } z.$$

We shall reprove this identity later, when we study the Taylor series of the function e^z.

Example 5.6.4. For $\sum_{n=1}^{+\infty}n!z^n$ we have $\overline{\lim}\sqrt[n]{|n!|}=\lim\sqrt[n]{n!}=+\infty$, and hence $R=0$. The power series converges only at 0.

Definition. *Every series of the form*

$$\sum_{-\infty}^{n=-1}a_n(z-z_0)^n=\cdots+\frac{a_{-3}}{(z-z_0)^3}+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}$$

is called **power series of second type** *with* **center** z_0 *and* **coefficients** a_n. *The* $R\in[0,+\infty]$ *defined by*

$$R=\overline{\lim}\sqrt[m]{|a_{-m}|}$$

is called **radius of convergence** *of the power series.*

The usual power series of the form $\sum_{n=0}^{+\infty}a_n(z-z_0)^n$ are also called *power series of first type*, to distinguish them from the power series of second type.

We observe that a power series of second type has no meaning at z_0, in the same way that any power series of first type (with $a_n\neq 0$ for at least one $n\geq 1$) has no meaning at ∞. On the other hand, if $z=\infty$, then a power series of second type becomes $\sum_{-\infty}^{n=-1}0=0$ and hence converges with sum 0.

From now on in these notes we shall use the notations

$$D_{z_0}(R,+\infty)=\{z\,|\,R<|z-z_0|\},\qquad \overline{D}_{z_0}(R,+\infty)=\{z\,|\,R\leq|z-z_0|\}$$

for the open and the closed unbounded ring with center z_0 and internal radius R. We also use

$$D_{z_0}(R_1,R_2)=\{z\,|\,R_1<|z-z_0|<R_2\},\qquad \overline{D}_{z_0}(R_1,R_2)=\{z\,|\,R_1\leq|z-z_0|\leq R_2\}$$

to denote the open and the closed bounded ring with center z_0, internal radius R_1 and external radius R_2.

Proposition 5.27. *Let $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$ be a power series of second type with radius of convergence R.*
If $R=+\infty$, then the series converges only at ∞. If $R<+\infty$, then
(i) The power series converges absolutely at every $z \in D_{z_0}(R,+\infty)$.
(ii) The power series diverges at every $z \notin \overline{D}_{z_0}(R,+\infty)$.
(iii) The power series converges uniformly in every $\overline{D}_{z_0}(r,+\infty)$ with $r>R$.
(iv) The sum $s(z)=\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$, defined as a function $s: D_{z_0}(R,+\infty)\to\mathbb{C}$, is holomorphic in $D_{z_0}(R,+\infty)$. The derivative of s in $D_{z_0}(R,+\infty)$ is $t(z)=\sum_{-\infty}^{n=-1} na_n(z-z_0)^{n-1}$, i.e. the sum of the power series which results from $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$ by formal termwise differentiation. Finally, the function s is also differentiable at ∞.

Proof. The easiest way is to reduce a power series of second type to a power series of first type with the simple change of variable
$$w=\frac{1}{z-z_0}.$$
Then the power series $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$ takes the form
$$\sum_{-\infty}^{n=-1} a_n w^{-n}=\sum_{m=1}^{+\infty} a_{-m}w^m$$
of a power series of first type with center 0. We also observe that z varies in the unbounded ring $D_{z_0}(R,+\infty)$ if and only if w varies in the punctured disc $D_0(\frac{1}{R})\setminus\{0\}$. Also, z varies in the unbounded ring $\overline{D}_{z_0}(r,+\infty)$ if and only if w varies in the punctured disc $\overline{D}_0(\frac{1}{r})\setminus\{0\}$. Now we can use everything we know about the series $\sum_{m=1}^{+\infty} a_{-m}w^m$ from proposition 5.26 to get the corresponding results about the series $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$. For example, the differentiability of $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$ results from the differentiability of $\sum_{m=1}^{+\infty} a_{-m}w^m$ and the differentiability of the function $w=\frac{1}{z-z_0}$. We leave all the details to the reader. We shall only say a few things about the differentiability of $s(z)=\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$ at ∞, using again the transformed power series $s_*(w)=\sum_{m=1}^{+\infty} a_{-m}w^m$. Since $s(\infty)=0$ and $s_*(0)=0$, we have
$$\lim_{z\to\infty} z(s(z)-s(\infty))=\lim_{z\to\infty} zs(z)=\lim_{w\to 0}\frac{1+z_0w}{w}s_*(w)=\lim_{w\to 0}\frac{s_*(w)}{w}=s_*'(0)=a_{-1}.$$
Therefore, s is differentiable at ∞ □

If R is the radius of convergence of $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$, then the open ring $D_{z_0}(R,+\infty)$ is called **ring of convergence** of the power series.

If $R=+\infty$ then the ring of convergence is empty and the power series converges only at ∞. If $R=0$ then the disc of convergence is the whole z-plane without z_0. In this case the power series converges (absolutely) at every $z\neq z_0$. If $0<R<+\infty$, then the power series converges (absolutely) at every $z\in D_{z_0}(R,+\infty)$ and diverges at every z inside the open disc $D_{z_0}(R)$. There is no general result for the convergence at the points $z\in C_{z_0}(R)$: the series may converge at some points of the circle and diverge at its remaining points. If $0\leq R<+\infty$, the power series converges uniformly in every smaller closed ring $\overline{D}_{z_0}(r,+\infty)$ with $r>R$ and its sum $s(z)$ is a holomorphic function in $D_{z_0}(R,+\infty)$. In fact the derivative of the sum $s(z)$ of the power series is the function $t(z)$ which is the sum of the power series we get by formal termwise differentiation of the original power series. The differentiated power series has the same ring of convergence as the original series. Therefore, we may repeat our arguments: the function $t(z)$ is holomorphic in $D_{z_0}(R,+\infty)$ and its derivative, i.e. the second derivative of $s(z)$, is the sum of the power series which we get by a second formal termwise differentiation of the original power series. We conclude that *the function $s(z)$ is infinitely many times differentiable in the ring of* *convergence* $D_{z_0}(R,+\infty)$ *and*

$$s^{(k)}(z)=\sum_{-\infty}^{n=-1} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k} \qquad \text{for every } z\in D_{z_0}(R,+\infty).$$

Example 5.6.5. $\sum_{-\infty}^{n=-1}\frac{z^n}{-n}=\sum_{m=1}^{+\infty}\frac{1}{mz^m}$ has ring of convergence $D_0(1,+\infty)$.

Example 5.6.6. $\sum_{-\infty}^{n=-1}\frac{z^n}{n^2}=\sum_{m=1}^{+\infty}\frac{1}{m^2z^m}$ has ring of convergence $D_0(1,+\infty)$.

Example 5.6.7. $\sum_{-\infty}^{n=-1}\frac{z^n}{(-n)!}=\sum_{m=1}^{+\infty}\frac{1}{m!z^m}$ has ring of convergence $D_0(0,+\infty)=\mathbb{C}\setminus\{0\}$.

Example 5.6.8. $\sum_{-\infty}^{n=-1}(-n)!z^n=\sum_{m=1}^{+\infty}\frac{m!}{z^m}$ has empty ring of convergence and converges only at ∞.

Definition. *We consider a series of the form*

$$\sum_{-\infty}^{+\infty}a_n(z-z_0)^n=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

which consists of a power series of first type and a power series of second type. We assume that $a_n\neq 0$ *for at least one* $n<0$ *and for at least one* $n>0$. *Then the original series is called* **power series of third type** *with* **center** z_0 *and* **coefficients** a_n. *The radius of convergence* R_1 *of* $\sum_{-\infty}^{n=-1}a_n(z-z_0)^n$ *and the radius of convergence* R_2 *of* $\sum_{n=0}^{+\infty}a_n(z-z_0)^n$ *are called* **radii of convergence** *of our power series. We say that* $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ *converges at z if both* $\sum_{-\infty}^{n=-1}a_n(z-z_0)^n$ *and* $\sum_{n=0}^{+\infty}a_n(z-z_0)^n$ *converge at z, and we say that* $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ *diverges at z in all other cases.*

A power series of third type with center z_0 has no meaning at the points z_0 and ∞.

A power series of third type is a combination of a power series of first type and a power series of second type. Therefore, we expect that the properties of a power series of this new type are a combination of properties of power series of the two previous types. Indeed, the next result is a direct combination of propositions 5.26 and 5.27 and we omit the proof.

Proposition 5.28. *Let* $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ *be a power series of third type with radii of convergence* R_1,R_2.
If $R_2\leq R_1$, *then the series diverges at every z, except in the case* $0<R_1=R_2=R<+\infty$ *and then it may converge only at some* $z\in C_{z_0}(R)$. *If* $R_1<R_2$, *then*
(i) The power series converges absolutely at every $z\in D_{z_0}(R_1,R_2)$.
(ii) The power series diverges at every $z\notin\overline{D}_{z_0}(R_1,R_2)$.
(iii) The power series converges uniformly in every $\overline{D}_{z_0}(r_1,r_2)$ *with* $R_1<r_1<r_2<R_2$.
(iv) The sum $s(z)=\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$, *defined as a function* $s:D_{z_0}(R_1,R_2)\to\mathbb{C}$, *is holomorphic in* $D_{z_0}(R_1,R_2)$. *The derivative of s in* $D_{z_0}(R_1,R_2)$ *is* $t(z)=\sum_{-\infty}^{+\infty}na_n(z-z_0)^{n-1}$, *i.e. the sum of the power series which results from* $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ *by formal termwise differentiation.*

If $R_1<R_2$, then $D_{z_0}(R_1,R_2)$ is called **ring of convergence** of $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$.

Example 5.6.9. We consider $\sum_{-\infty}^{n=-1}\frac{2^n}{-n}z^n+1+\sum_{n=1}^{+\infty}\frac{1}{n^2}z^n$.
Then $\sum_{-\infty}^{n=-1}\frac{2^n}{-n}z^n$ has radius of convergence $\frac{1}{2}$ and $1+\sum_{n=1}^{+\infty}\frac{1}{n^2}z^n$ has radius of convergence 1. Therefore, $D_0(\frac{1}{2},1)$ is the ring of convergence of $\sum_{-\infty}^{n=-1}\frac{2^n}{-n}z^n+1+\sum_{n=1}^{+\infty}\frac{1}{n^2}z^n$.

Exercises.

5.6.1. Find the discs of convergence of $\sum_{n=1}^{+\infty}a_nz^n$ when $a_n=n^{13}$, $a_n=\frac{1}{n^5}$, $a_n=\frac{1}{n^n}$, $a_n=n^{\ln n}$, $a_n=(\ln n)^n$, $a_n=\frac{n!}{n^n}$, $a_n=\frac{(n!)^2}{n^n}$, $a_n=\frac{(n!)^2}{(2n)!}$.

5.6.2. Find the rings of convergence of $\sum_{-\infty}^{n=-1} a_n z^n$ when $a_n = n^3$, $a_n = \frac{1}{n^2}$, $a_n = \frac{1}{2^n}$, $a_n = 3^n$, $a_n = \frac{1}{(-n)!n^n}$.

5.6.3. Find the ring of convergence and the sum of $\sum_{-\infty}^{n=-1}(-1)^n z^n + \sum_{n=1}^{+\infty}(\frac{1}{2i})^{n+1} z^n$.

5.6.4. (i) Write $\frac{1}{z-1}$ as a power series with disc of convergence $D_0(1)$ and as power series with ring of convergence $D_0(1,+\infty)$.
(ii) Write $\frac{1}{(z-3)(z-4)}$ as a power series with disc of convergence $D_0(3)$, as a power series with ring of convergence $D_0(3,4)$ and as a power series with ring of convergence $D_0(4,+\infty)$.

5.6.5. If $m \in \mathbb{N}$, using the geometric power series $\sum_{n=0}^{+\infty} z^n$, write $\frac{1}{(1-z)^m}$ as a power series $\sum_{n=0}^{+\infty} a_n z^n$, and determine its disc of convergence.

5.6.6. Find the radius of convergence of $1 + \sum_{n=1}^{+\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{1\cdot 2\cdots n\cdot c(c+1)\cdots(c+n-1)} z^n$, where $c \neq 0, -1, -2, \ldots$. This power series is called **hypergeometric series** with parameters a, b, c. Prove that the function $w = F(z; a, b, c)$, which is defined by the hypergeometric series in its disc of convergence, is a solution of the differential equation $z(1-z)w'' + (c-(a+b+1)z)w' - abw = 0$.

5.6.7. (i) Prove that, if two power series of the type $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ with positive radii of convergence define the same function in the intersection of their discs of convergence, then the two series coincide, i.e. they have the same coefficients a_n.
(ii) Prove a result analogous to (i) for two power series of the type $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$.
(iii) Can you prove *now* the analogous result for two power series of the type $\sum_{-\infty}^{+\infty} a_n(z-z_0)^n$?

5.6.8. Let $0 < R < +\infty$.
(i) If $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ converges absolutely for some $z \in C_{z_0}(R)$, prove that it converges absolutely for every $z \in \overline{D}_{z_0}(R)$.
(ii) If $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ converges for some $z \in C_{z_0}(R)$, prove that it converges absolutely for every $z \in D_{z_0}(R)$.

5.6.9. Let R', R'' and R be the radii of convergence of $\sum_{n=0}^{+\infty} a_n{}'(z-z_0)^n$, $\sum_{n=0}^{+\infty} a_n{}''(z-z_0)^n$ and $\sum_{n=0}^{+\infty}(a_n{}' + a_n{}'')(z-z_0)^n$, respectively. If $R' \neq R''$, prove that $R = \min\{R', R''\}$. If $R' = R''$, prove that $R \geq R' = R''$.

5.6.10. Let R be the radius of convergence of $\sum_{n=1}^{+\infty} a_n(z-z_0)^n$. If $0 < R < +\infty$, find the radii of convergence of $\sum_{n=1}^{+\infty} n^k a_n(z-z_0)^n$, $\sum_{n=1}^{+\infty} n! a_n(z-z_0)^n$, $\sum_{n=1}^{+\infty} \frac{a_n}{n!}(z-z_0)^n$.

5.6.11. Let $k \in \mathbb{N}$, $k \geq 2$. Find the z at which $\sum_{n=1}^{+\infty} \frac{z^{kn}}{n}$ converges.

5.6.12. Find the z at which $\sum_{n=1}^{+\infty} z^{n!}$ converges.

5.6.13. Let $0 < b < 1$. Find the ring of convergence of $\sum_{n=-\infty}^{+\infty} b^{n^2} z^n$.

5.6.14. If $\sum_{n=0}^{+\infty} a_n(z-z_0)^n = s(z)$ for every $z \in D_{z_0}(R)$ and $|a_1| \geq \sum_{n=2}^{+\infty} n|a_n|R^{n-1}$, prove that s is one-to-one in $D_{z_0}(R)$.

Chapter 6

Local behaviour and basic properties of holomorphic functions.

6.1 The theorem of Cauchy for triangles.

Let Δ be a closed triangular region. When we write

$$\oint_{\partial\Delta} f(z)\, dz$$

we mean the curvilinear integral over a curve γ with trajectory $\gamma^* = \partial\Delta$ which describes the triangle $\partial\Delta$, the boundary of Δ, once and in the positive direction. For instance, if z_1, z_2, z_2 are the vertices of the triangle in the order which agrees with the positive direction of $\partial\Delta$, then a valid curve is $\gamma = [z_1, z_2] \dot{+} [z_2, z_3] \dot{+} [z_3, z_1]$. Hence,

$$\oint_{\partial\Delta} f(z)\, dz = \int_{[z_1,z_2]} f(z)\, dz + \int_{[z_2,z_3]} f(z)\, dz + \int_{[z_3,z_1]} f(z)\, dz.$$

Of course there are analogous statements for integrals

$$\oint_{\partial R} f(z)\, dz,$$

when R is a closed rectangular region or, more generally, a closed convex polygonal region.

The theorem of Cauchy-Goursat. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in an open set Ω which contains the closed triangular region Δ. Then*

$$\oint_{\partial\Delta} f(z)\, dz = 0.$$

Proof. We write $I = \oint_{\partial\Delta} f(z)\, dz$, and we have to show that $I = 0$.
Let $\Delta = \Delta(z_1, z_2, z_3)$ be the given closed triangular region with vertices z_1, z_2, z_3 written in the order which agrees with the positive direction of $\partial\Delta$. We take the points w_3, w_1, w_2, which are the midpoints of the linear segments $[z_1, z_2]$, $[z_2, z_3]$, $[z_3, z_1]$, respectively. Then the closed triangular region $\Delta(z_1, z_2, z_3)$ splits into the four closed triangular regions $\Delta^{(1)} = \Delta(z_1, w_3, w_2)$, $\Delta^{(2)} = \Delta(w_3, z_2, w_1)$, $\Delta^{(3)} = \Delta(w_1, z_3, w_2)$ and $\Delta^{(4)} = \Delta(w_3, w_1, w_2)$ and we define the corresponding curvilinear integrals: $I^{(1)} = \oint_{\partial\Delta^{(1)}} f(z)\, dz$, $I^{(2)} = \oint_{\partial\Delta^{(2)}} f(z)\, dz$, $I^{(3)} = \oint_{\partial\Delta^{(3)}} f(z)\, dz$ and $I^{(4)} = \oint_{\partial\Delta^{(4)}} f(z)\, dz$. Now, we analyse each of the four integrals into three integrals over the three linear segments of the corresponding triangle, and then we add the resulting twelve integrals and observe the cancellations which occur between integrals over pairs of linear segments with

opposite directions. We end up with six integrals over six successive linear segments which add up to give the three linear segments of the original triangle $\partial\Delta$. The result is

$$I = I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)}.$$

This implies $|I| \leq |I^{(1)}| + |I^{(2)}| + |I^{(3)}| + |I^{(4)}|$ and hence $|I^{(j)}| \geq \frac{1}{4}\,|I|$ for at least one j. Now we take the corresponding closed triangular region $\Delta^{(j)}$ and, for simplicity, we denote it Δ_1. We also denote I_1 the corresponding integral $I^{(j)}$. We have proved that there is a closed triangular region Δ_1 contained in the original Δ such that, if $I = \oint_{\partial\Delta} f(z)\,dz$ and $I_1 = \oint_{\partial\Delta_1} f(z)\,dz$, then $|I_1| \geq \frac{1}{4}|I|$. We also observe that $\operatorname{diam}\Delta_1 = \frac{1}{2}\operatorname{diam}\Delta$. We may continue inductively and produce a sequence of closed triangular regions Δ_n and the corresponding sequence of curvilinear integrals $I_n = \oint_{\partial\Delta_n} f(z)\,dz$ so that:
(i) $\Delta \supseteq \Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1} \supseteq \cdots$,
(ii) $|I_n| \geq \frac{1}{4^n}|I|$,
(iii) $\operatorname{diam}\Delta_n = \frac{1}{2^n}\operatorname{diam}\Delta$.
Now, (i), (iii) and proposition 1.16 imply that there is a (unique) point z contained in all Δ_n. In particular, $z \in \Delta$ and hence f is differentiable at z. If we take an arbitrary $\epsilon > 0$, then there is $\delta > 0$ so that $|\frac{f(\zeta)-f(z)}{\zeta-z} - f'(z)| < \epsilon$ for every ζ with $0 < |\zeta - z| < \delta$. Thus,

$$|f(\zeta) - f(z) - f'(z)(\zeta - z)| \leq \epsilon|\zeta - z| \qquad \text{for every } \zeta \text{ with } |\zeta - z| < \delta. \tag{6.1}$$

Because of (iii), there is some large n so that $\operatorname{diam}\Delta_n < \delta$. Since $z \in \Delta_n$ and $\operatorname{diam}\Delta_n < \delta$, we get $|\zeta - z| \leq \operatorname{diam}\Delta_n < \delta$ for every $\zeta \in \partial\Delta_n \subseteq \Delta_n$ and now (6.1) and (iii) imply

$$|f(\zeta) - f(z) - f'(z)(\zeta - z)| \leq \epsilon|\zeta - z| \leq \epsilon \operatorname{diam}\Delta_n = \frac{\epsilon}{2^n}\operatorname{diam}\Delta \qquad \text{for every } \zeta \in \partial\Delta_n.$$

Therefore,

$$\Big|\oint_{\partial\Delta_n} (f(\zeta) - f(z) - f'(z)(\zeta - z))\,d\zeta\Big| \leq \frac{\epsilon}{2^n}\operatorname{diam}\Delta\, l(\partial\Delta_n) \leq \frac{3\epsilon}{4^n}(\operatorname{diam}\Delta)^2. \tag{6.2}$$

Since $f(z) + f'(z)(\zeta - z)$ is a polynomial function of ζ, example 5.5.1 shows that

$$\oint_{\partial\Delta_n} (f(z) + f'(z)(\zeta - z))\,d\zeta = 0$$

and (6.2) becomes $|I_n| = |\oint_{\partial\Delta_n} f(\zeta)\,d\zeta| \leq \frac{3\epsilon}{4^n}(\operatorname{diam}\Delta)^2$. Finally, (ii) implies $|I| \leq 3\epsilon(\operatorname{diam}\Delta)^2$ and since $\epsilon > 0$ is arbitrary, we conclude that $I = 0$. ☐

6.2 Primitives and the theorem of Cauchy in convex regions.

Proposition 6.1. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the convex region Ω. Then f has a primitive in Ω.*

Proof. We fix $z_0 \in \Omega$. Then for every $z \in \Omega$ the linear segment $[z_0, z]$ is contained in Ω and we define $F : \Omega \to \mathbb{C}$ by

$$F(z) = \int_{[z_0,z]} f(\zeta)\,d\zeta.$$

We shall prove that F is a primitive of f in Ω. We take arbitrary $z, w \in \Omega$ and consider the closed triangular region Δ with vertices z_0, z, w. Since Ω is convex, Δ is contained in Ω and the Cauchy-Goursat theorem implies $\oint_{\partial\Delta} f(z)\,dz = 0$, i.e.

$$\int_{[z_0,z]} f(\zeta)\,d\zeta + \int_{[z,w]} f(\zeta)\,d\zeta + \int_{[w,z_0]} f(\zeta)\,d\zeta = 0.$$

Therefore $F(w) - F(z) = \int_{[z,w]} f(\zeta)\, d\zeta$ and hence

$$F(w) - F(z) - f(z)(w-z) = \int_{[z,w]} f(\zeta)\, d\zeta - f(z) \int_{[z,w]} d\zeta = \int_{[z,w]} (f(\zeta) - f(z))\, d\zeta. \quad (6.3)$$

Since f is continuous, for every $\epsilon > 0$ there is $\delta > 0$ so that $|f(z') - f(z)| < \epsilon$ for every $z' \in \Omega$ with $|z' - z| < \delta$. Taking $w \in \Omega$ with $|w - z| < \delta$ we automatically have $|\zeta - z| < \delta$ for every $\zeta \in [z, w]$ and (6.3) implies

$$|F(w) - F(z) - f(z)(w-z)| = \Big| \int_{[z,w]} (f(\zeta) - f(z))\, d\zeta \Big| \leq \epsilon |w - z|.$$

Therefore, $\big| \frac{F(w)-F(z)}{w-z} - f(z) \big| < \epsilon$ for every w with $0 < |w-z| < \delta$ and hence $F'(z) = f(z)$. □

The theorem of Cauchy in convex regions. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the convex region Ω. Then*

$$\oint_\gamma f(z)\, dz = 0$$

for every closed curve γ in Ω.

Proof. Direct from propositions 5.22 and 6.1. □

Now we shall decribe a very useful technique to handle curvilinear integrals of holomorphic functions. Every closed curve γ we shall refer to will be *visually simple*, for instance a circle or a triangle or a rectangle, and we shall be able to distinguish between the points *inside* γ and the points *outside* γ. We assume that γ surrounds every point inside it once and in the positive direction and that it does not surround the points outside it. The points inside γ form the *region inside* γ and the points outside γ form the *region outside* γ. Then γ^* is the common boundary of the region inside γ and the region outside γ. We shall concentrate on two characteristic cases.

First case. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω and let γ be a closed curve in Ω. We want to evaluate $\oint_\gamma f(z)\, dz$.

If Ω is convex, then $\oint_\gamma f(z)\, dz = 0$. So let us assume that Ω is not convex. To continue, we assume that the region inside γ, call it D, is contained in Ω, and hence f is holomorphic in D as well as in $\partial D = \gamma^*$. Now our technique is the following. We split D into specific disjoint open sets $E_1, \dots, E_m$ so that their boundaries $\partial E_1, \dots, \partial E_m$ are trajectories of closed curves $\sigma_1, \dots, \sigma_m$, so that $\overline{D} = \overline{E}_1 \cup \cdots \cup \overline{E}_m$ and, finally, so that, when we analyse in an appropriate way each of $\sigma_1, \dots, \sigma_m$ in successive subcurves and drop those subcurves which appear as pairs of opposite curves, the remaining subcurves can be summed up to give the original curve γ. The result is:

$$\oint_\gamma f(z)\, dz = \oint_{\sigma_1} f(z)\, dz + \cdots + \oint_{\sigma_m} f(z)\, dz.$$

In fact we applied this technique in the proof of the theorem of Cauchy-Goursat.

Now, if the various $E_1, \dots, E_m$ can be chosen so that each $\overline{E}_1, \dots, \overline{E}_m$ is contained in a corresponding *convex* open subset of Ω, then we conclude that

$$\oint_\gamma f(z)\, dz = \oint_{\sigma_1} f(z)\, dz + \cdots + \oint_{\sigma_m} f(z)\, dz = 0 + \cdots + 0 = 0.$$

Second case. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω and let $\gamma, \gamma_1, \dots, \gamma_n$ be $n+1$ closed curves in Ω. We want to relate the integrals $\oint_\gamma f(z)\, dz, \oint_{\gamma_1} f(z)\, dz, \dots, \oint_{\gamma_n} f(z)\, dz$.

We assume that the regions inside $\gamma_1, \dots, \gamma_n$ are disjoint and that they are all contained in the region inside γ. Let us call D the *intermediate* region, i.e. the set consisting of the points which are inside γ and outside every $\gamma, \dots, \gamma_n$, i.e. the intersection of the region inside γ and the regions ouside $\gamma_1, \ldots, \gamma_n$. We further assume that D is a subset of Ω, and hence f is holomorphic in D as well as in $\partial D = \gamma^* \cup \gamma_1^* \cup \cdots \cup \gamma_n^*$. Now, here is the technique. We split D into specific disjoint open sets $E_1, \ldots, E_m$ so that their boundaries $\partial E_1, \ldots, \partial E_m$ are trajectories of closed curves $\sigma_1, \ldots, \sigma_m$, so that $\overline{E} = \overline{E}_1 \cup \cdots \cup \overline{E}_m$ and, finally, so that, when we analyse in an appropriate way each of $\sigma_1, \ldots, \sigma_m$ in successive subcurves and drop those subcurves which appear as pairs of opposite curves, the remaining subcurves can be summed up to give γ as well as the opposites of $\gamma_1, \ldots, \gamma_n$. The result is:

$$\oint_\gamma f(z)\,dz - \oint_{\gamma_1} f(z)\,dz - \cdots - \oint_{\gamma_n} f(z)\,dz = \oint_{\sigma_1} f(z)\,dz + \cdots + \oint_{\sigma_m} f(z)\,dz.$$

If the various $E_1, \ldots, E_m$ can be chosen so that each $\overline{E}_1, \ldots, \overline{E}_m$ is contained in a corresponding *convex* open subset of Ω, then $\oint_\gamma f(z)\,dz - \oint_{\gamma_1} f(z)\,dz - \cdots - \oint_{\gamma_n} f(z)\,dz = 0 + \cdots + 0 = 0$ and hence

$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_n} f(z)\,dz.$$

Corollary 6.1. *Let $C, C_1, \ldots, C_n$ be $n+1$ circles and let $D, D_1, \ldots, D_n$ be the corresponding open discs. Assume that $D_1, \ldots, D_n$ are disjoint and that they are all contained in D. Consider also the closed region $M = \overline{D} \setminus (D_1 \cup \cdots \cup D_n)$. If $f : \Omega \to \mathbb{C}$ is holomorphic in an open set Ω which contains M, then*

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz.$$

Instead of circles we may consider rectangles or triangles or any combination of the three shapes.

Proof. Clear after the previous discussion. □

Exercises.

6.2.1. Let γ_R be the closed curve which is the sum of the linear segment $[0, R]$, the arc of the circle $C_0(R)$ from R to $Re^{i\frac{\pi}{4}}$ in the positive direction and the linear segment $[Re^{i\frac{\pi}{4}}, 0]$. Also, let σ_R be the curve wich describes only the above arc from R to $Re^{i\frac{\pi}{4}}$.
(i) Prove that $\int_{\sigma_R} e^{-z^2}\,dz \to 0$ when $R \to +\infty$.
(ii) Using γ_R appropriately together with the formula $\int_0^{+\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, prove the formulas for the so-called Fresnel integrals: $\int_0^{+\infty} \sin t^2\,dt = \int_0^{+\infty} \cos t^2\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$.

6.2.2. Let $y, R > 0$ and $\gamma_{R,y}$ be the closed curve which is the sum of the linear segments $[-R, R]$, $[R, R+iy]$, $[R+iy, -R+iy]$ and $[-R+iy, -R]$.
(i) If $y > 0$ is constant, prove that $\int_{[R,R+iy]} e^{-z^2}\,dz \to 0$ and $\int_{[-R+iy,-R]} e^{-z^2}\,dz \to 0$ when $R \to +\infty$.
(ii) Using $\gamma_{R,y}$ appropriately, prove that $\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\,dx$ does not depend on $y \in [0, +\infty)$.
(iii) Using the formula $\int_0^{+\infty} e^{-x^2}\,dt = \frac{\sqrt{\pi}}{2}$, prove that $\int_{-\infty}^{+\infty} e^{-x^2}\cos(2xy)\,dx = \sqrt{\pi}e^{-y^2}$ for every $y \geq 0$ (and hence for every $y \leq 0$ also). This identity is very important for harmonic analysis.

6.3 Cauchy's formulas for circles and infinite differentiability.

Cauchy's formula for circles. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in an open set Ω containing the closed disc $\overline{D}_{z_0}(R)$. Then*

$$f(z) = \frac{1}{2\pi i}\oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad \textit{for every } z \in D_{z_0}(R).$$

Proof. Let $z \in D_{z_0}(R)$. We consider any open disc $D_z(r)$ with $r < R - |z - z_0|$. Then $\overline{D}_z(r) \subseteq D_{z_0}(R)$ and the function $\frac{f(\zeta)}{\zeta - z}$ is holomorphic in the open set $\Omega \setminus \{z\}$ which contains the closed region between the circles $C_z(r)$ and $C_{z_0}(R)$. Corollary 6.1 implies

$$\oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{6.4}$$

Now, we have $\oint_{C_z(r)} \frac{1}{\zeta - z}\, d\zeta = \int_0^{2\pi} \frac{1}{re^{it}}\, ire^{it}\, dt = 2\pi i$ and hence

$$\oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i f(z) = \oint_{C_z(r)} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta. \tag{6.5}$$

We take $\epsilon > 0$. Since f is continuous at z, there is $\delta > 0$ so that $|f(\zeta) - f(z)| < \epsilon$ for every $\zeta \in \Omega$ with $|\zeta - z| < \delta$. Therefore, if $r < \delta$, (6.5) implies

$$\Big| \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - 2\pi i f(z) \Big| \le \frac{\epsilon}{r}\, 2\pi r = 2\pi\epsilon.$$

Since ϵ is arbitrary, we conclude that $\lim_{r\to 0} \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z)$. Now, letting $r \to 0$ in (6.4), we get $\oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z)$. □

A particular instance of the formula of Cauchy is when we take $z = z_0$, the center of the circle $C_{z_0}(R)$. Using the parametric equation $\zeta = z_0 + Re^{it}$, $t \in [0, 2\pi]$, we get

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

and this is called **mean value property** of the holomorphic function f.

Cauchy's formula for derivatives and circles. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in an open set Ω containing the closed disc $\overline{D}_{z_0}(R)$. Then f is infinitely many times differentiable at every $z \in D_{z_0}(R)$ and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C_{z_0}(R)} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta \qquad \text{for every } z \in D_{z_0}(R) \text{ and every } n \in \mathbb{N}.$$

Proof. Proposition 5.25 says that $\frac{1}{2\pi i} \oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ is an infinitely many times differentiable function of z in the disc $D_{z_0}(R)$. On the other hand, Cauchy's formula says that this function coincides with the function $f(z)$ in the same disc. Therefore $f(z)$ is infinitely many times differentiable in $D_{z_0}(R)$. Moreover, the derivatives of $f(z)$ are the same as the derivatives of $\frac{1}{2\pi i} \oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ and these are given by the formulas in proposition 5.25. □

Example 6.3.1. Let $n \in \mathbb{N}$. Then

$$\oint_{C_{z_0}(R)} \frac{1}{(\zeta - z)^n}\, d\zeta = 0, \qquad \text{for every } z \notin \overline{D}_{z_0}(R).$$

To see this we observe that the circle $C_{z_0}(R)$ is contained in a slightly larger open disc $D_{z_0}(R')$ which does not contain z: it is enough to take $R < R' < |z - z_0|$. Then the disc $D_{z_0}(R')$ is a convex region and $\frac{1}{(\zeta - z)^n}$ is a holomorphic function of ζ in $D_{z_0}(R')$. Now the result is an application of the theorem of Cauchy in convex regions.
On the other hand, we have

$$\oint_{C_{z_0}(R)} \frac{1}{(\zeta - z)^n}\, d\zeta = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0, & \text{if } n \ge 2 \end{cases} \qquad \text{for every } z \in D_{z_0}(R).$$

This is a simple application of Cauchy's formula (for a function and its derivatives) to the constant function 1. The special case $z = z_0$ we have already seen in examples 3.2.8 and 5.2.2.

Theorem 6.1. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω. Then f is infinitely many times differentiable in Ω.*

Proof. Let $z_0 \in \Omega$. We take a closed disc $\overline{D}_{z_0}(R) \subseteq \Omega$ and then f is infinitely many times differentiable in $D_{z_0}(R)$ and hence at z_0. $\square$

It is time to recall the remark after proposition 5.24. The assumption of continuity of the derivative in proposition 5.24 is superfluous. The same we may say for the hypothesis in example 5.5.6 and in exercise 5.5.1.

Cauchy's estimates. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in an open set containing the closed disc $\overline{D}_{z_0}(R)$. If $|f(\zeta)| \le M$ for every $\zeta \in C_{z_0}(R)$, then*

$$|f^{(n)}(z_0)| \le \frac{n!M}{R^n} \qquad \text{for every } n \in \mathbb{N}.$$

Proof. Direct application of Cauchy's formulas. $\square$

Exercises.

6.3.1. Evaluate $\oint_{C_0(r)} \frac{z^2+1}{z(z^2+4)}\,dz$ for $0 < r < 2$ and for $2 < r < +\infty$.

6.3.2. If $n \in \mathbb{N}$, evaluate $\oint_{C_0(1)} \frac{e^z}{z^n}\,dz$ and $\int_0^{2\pi} e^{\cos\theta}\sin(n\theta-\sin\theta)\,d\theta$, $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta-\sin\theta)\,d\theta$.

6.3.3. If $n \in \mathbb{N}$, evaluate $\oint_{C_0(1)} \frac{e^{iz}}{z^n}\,dz$, $\oint_{C_0(1)} \frac{\sin z}{z^n}\,dz$, $\oint_{C_0(1)} \frac{e^z-e^{-z}}{z^n}\,dz$, $\oint_{C_1(\frac{1}{2})} \frac{\operatorname{Log} z}{(z-1)^n}\,dz$.

6.3.4. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in $\mathbb{C}$ and assume that there are A, M and $n \in \mathbb{N}$ so that $|f(z)| \le A + M|z|^n$ for every z. Prove that $f^{(n+1)}(z) = 0$ for every z and that f is a polynomial function of degree $\le n$.

6.3.5. Let $f : \overline{D}_{z_0}(R) \to \mathbb{C}$ be continuous in $\overline{D}_{z_0}(R)$ and holomorphic in $D_{z_0}(R)$. Prove that $f(z) = \frac{1}{2\pi i}\oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ for every $z \in D_{z_0}(R)$.

6.3.6. Let $f : \Omega \to \mathbb{C}$ be holomorphic in an open set containing the closed disc $\overline{D}_{z_0}(R)$ and let $0 < r < R$. If $|f(z)| \le M$ for every $z \in C_{z_0}(R)$, find an upper bound for $|f^{(n)}|$ in $\overline{D}_{z_0}(r)$, which depends only on n, r, R, M and not on f or z_0.

6.3.7. Let $f : D_{z_0}(R) \to \mathbb{C}$ be holomorphic in $D_{z_0}(R)$. If $|f(z)| \le \frac{1}{R-|z-z_0|}$ for every $z \in D_{z_0}(R)$, find the smallest possible upper bound for $|f^{(n)}(z_0)|$, which depends only on n, R and not on f or z_0.

6.3.8. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and bounded in $\mathbb{D}$. Prove that $f(w) = \frac{1}{\pi}\iint_{\mathbb{D}} \frac{f(z)}{(1-\bar{z}w)^2}\,dxdy$ for every $w \in \mathbb{D}$.

6.4 Morera's theorem.

Theorem 6.1 and proposition 5.22 imply the following corollary. *Let $f : \Omega \to \mathbb{C}$ be continuous in the region Ω. If $\oint_\gamma f(z)\,dz = 0$ for every closed curve γ in Ω, then f is holomorphic in Ω.* Indeed, since $\oint_\gamma f(z)\,dz = 0$ for every closed curve γ in Ω, we get that f has a primitive, say F, in Ω. This means that $F' = f$ in Ω and hence F is holomorphic in Ω. Therefore, F is infinitely many times differentiable in Ω and then f is also infinitely many times differentiable in Ω. In particular, f is holomorphic in Ω.

The next theorem proves the same result with weaker assumptions.

Morera's theorem. *Let $f : \Omega \to \mathbb{C}$ be continuous in the open set Ω. If $\oint_{\partial\Delta} f(z)\,dz = 0$ for every closed triangular region Δ in Ω, then f is holomorphic in Ω.*

Proof. Let $z_0 \in \Omega$. We consider a disc $D_{z_0}(R) \subseteq \Omega$. This disc is a convex set and we have that $\oint_{\partial\Delta} f(z)\,dz = 0$ for every closed triangular region Δ in $D_{z_0}(R)$. Then the *proof* of proposition 6.1 applies and we get that f has a primitive, say F, in $D_{z_0}(R)$. This means that $F' = f$ in $D_{z_0}(R)$ and hence F is holomorphic in $D_{z_0}(R)$. Therefore, F is infinitely many times differentiable in $D_{z_0}(R)$ and f is also infinitely many times differentiable in $D_{z_0}(R)$. In particular, f is holomorphic in $D_{z_0}(R)$ and hence at z_0. $\square$

Exercises.

6.4.1. Let $f : \Omega \to \mathbb{C}$ and l be a line. If f is continuous in the open set Ω and holomorphic in $\Omega \setminus l$, prove that f is holomorphic in Ω.

6.5 Liouville's theorem. The fundamental theorem of algebra.

Liouville's theorem. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded in $\mathbb{C}$, then f is constant in $\mathbb{C}$.*

Proof. There is $M \geq 0$ so that $|f(z)| \leq M$ for every z. We take any z_0 and apply Cauchy's estimate for $n = 1$ with an arbitrary circle $C_{z_0}(R)$ and we find that $|f'(z_0)| \leq \frac{M}{R}$. Letting $R \to +\infty$, we get $f'(z_0) = 0$. Since z_0 is arbitrary, we conclude that f is constant. $\square$

Fundamental theorem of algebra. *Every polynomial of degree ≥ 1 has at least one root in $\mathbb{C}$.*

Proof. Let p be a polynomial of degree ≥ 1 and assume that p has no root in $\mathbb{C}$.
We consider the function $f = \frac{1}{p}$, which is holomorphic in $\mathbb{C}$, and we see easily that it is also bounded in $\mathbb{C}$. Indeed, since $\lim_{z\to\infty} p(z) = \infty$, we have $\lim_{z\to\infty} f(z) = 0$, and hence there is $R > 0$ so that $|f(z)| \leq 1$ for every z with $|z| > R$. Since $|f|$ is continuous in the compact disc $\overline{D}_0(R)$, there is $M' \geq 0$ so that $|f(z)| \leq M'$ for every z with $|z| \leq R$. Taking $M = \max\{M', 1\}$, we have that $|f(z)| \leq M$ for every z and hence f is bounded.
Liouville's theorem implies that f and hence p is constant and we arrive at a contradiction. $\square$

Having proved that a polynomial p has a root z_1, we may prove in a purely algebraic way that $z - z_1$ is a factor of p, i.e. there is a polynomial p_1 so that $p(z) = (z - z_1)p_1(z)$ for every z. If p_1 is of degree ≥ 1, then it has a root z_2 and, as before, there is a polynomial p_2 so that $p_1(z) = (z-z_2)p_2(z)$ and hence $p(z) = (z-z_1)(z-z_2)p_2(z)$ for every z. Continuing inductively, we conclude that, if $n \geq 1$ is the degree of p, there are $z_1, \ldots, z_n$ so that

$$p(z) = c(z - z_1)\cdots(z - z_n) \qquad \text{for every } z$$

where c is a constant. It is clear that c is the coefficient of the monomial of highest degree of p. We have proved that *every polynomial p of degree $n \geq 1$ has exactly n roots in $\mathbb{C}$.*

Exercises.

6.5.1. We say that z, w are symmetric with respect to $\mathbb{T}$ if either $z = 0, w = \infty$ or $z = \infty, w = 0$ or $z, w \in \mathbb{C}$, $z = \frac{1}{\overline{w}}$.
Let $r = \frac{p}{q}$ be a non-constant rational function so that the polynomials p, q have no common root and so that $|r(z)| = 1$ for every $z \in \mathbb{T}$. Prove that, if $a \in \mathbb{C} \setminus \{0\}$ is a root of p of multiplicity k, then $b = \frac{1}{\overline{a}}$ is a root of q of multiplicity k and conversely. I.e. the roots of p and the roots of q form pairs of points symmetric with respect to $\mathbb{T}$. (In particular, p and q have the same degree.)

6.5.2. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic in $\mathbb{C}$ and $\operatorname{Re} f$ is bounded in $\mathbb{C}$, prove that f is constant in $\mathbb{C}$.

6.6 Maximum principle.

Maximum principle. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω, $M = \sup\{|f(z)| \,|\, z \in \Omega\}$. If there is $z_0 \in \Omega$ so that $|f(z_0)| = M$, then f is constant in Ω.*

Proof. We take any $z \in \Omega$ for which $|f(z)| = M$. We consider an open disc $D_z(R) \subseteq \Omega$ and any r with $0 < r < R$. We apply Cauchy's formula and we get

$$f(z) = \frac{1}{2\pi i} \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{it})}{re^{it}}\, ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt.$$

Since $|f(z + re^{it})| \leq M$ for every $t \in [0, 2\pi]$, we have

$$M = |f(z)| = \Big|\frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it})\, dt\Big| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{it})|\, dt \leq M. \tag{6.6}$$

Hence, $\frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{it})|\, dt = M$ and, since $|f(z + re^{it})|$ is a continuous function of t, we get $|f(z + re^{it})| = M$ for every $t \in [0, 2\pi]$. Now, r is arbitrary in the interval $(0, R)$ and we find that $|f(z + re^{it})| = M$ for every $t \in [0, 2\pi]$ and every $r \in (0, R)$. So we get

$$|f(w)| = M \qquad \text{for every } w \in D_z(R).$$

We proved that, if $|f(z)| = M$ for a $z \in \Omega$, then this equality holds in a neighborhood of z. Now we define

$$B = \{z \in \Omega \,|\, |f(z)| = M\}, \qquad C = \{z \in \Omega \,|\, |f(z)| < M\}$$

and it is clear that $B \cup C = \Omega$ and $B \cap C = \emptyset$.
If $z \in B$, then $|f(z)| = M$ and hence the same is true at every point in a neighborhood of z. Therefore some neighborhood of z contains no point of C and, thus, z is not a limit point of C. Moreover, if C contains a limit point z of B, then $|f(z)| < M$ and there is a sequence (z_n) in B so that $z_n \to z$. Then $|f(z_n)| = M$ for every n and by the continuity of f we have $|f(z)| = M$ which is wrong. Therefore C contains no limit point of B.
If both B and C are non-empty, then they form a decomposition of Ω. But Ω is connected and, since $z_0 \in B$, we get that $C = \emptyset$. Therefore,

$$|f(z)| = M \qquad \text{for every } z \in \Omega. \tag{6.7}$$

Now we shall prove that f is constant in Ω. If $M = 0$, then clearly $f = 0$ in Ω. Let us assume that $M > 0$. If u and v are the real and the imaginary part of f, then (6.7) says that $u^2 + v^2 = M^2$ in Ω and hence $u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ and $u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ in Ω. Using the C-R equations, we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0, \quad v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} = 0 \qquad \text{in } \Omega.$$

Viewing this as a system with unknowns $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$, we see that its determinant is $u^2 + v^2 = M^2 > 0$, and we find that $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial x} = 0$ in Ω. Therefore, $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$ in Ω and hence f is constant in the region Ω. $\square$

Maximum principle. *Let $f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the bounded region Ω and continuous in $\overline{\Omega}$. Then either f is constant in $\overline{\Omega}$ or $|f|$ has a maximum value, say M, attained at a point of $\partial\Omega$ and $|f(z)| < M$ for every $z \in \Omega$. In every case, $|f|$ has a maximum value which is attained at a point of $\partial\Omega$.*

Proof. If f is constant in $\overline{\Omega}$, then $|f|$ is also constant, say M, in $\overline{\Omega}$. Then, obviously, M is the maximum value of $|f|$ and it is attained (everywhere and hence) at every point of $\partial\Omega$.
Now we assume that f is not constant in $\overline{\Omega}$. This implies that f is not constant in Ω either. (If $f = c$ in Ω, then for every $z \in \overline{\Omega}$ there is a sequence (z_n) in Ω so that $z_n \to z$ and then, by continuity, we get $c = f(z_n) \to f(z)$ and hence $f(z) = c$ for every $z \in \overline{\Omega}$.)
Now, $|f|$ is continuous in the compact set $\overline{\Omega}$ and hence attains its maximum value, say M, at some point $z_0 \in \overline{\Omega}$. I.e. we have $|f(z_0)| = M$ and $|f(z)| \leq M$ for every $z \in \overline{\Omega}$.
If any such z_0 belongs to Ω, then the previous maximum principle implies that f is constant in Ω and we arrive at a contradiction. We conclude that $z_0 \in \partial\Omega$ and $|f(z)| < M$ for every $z \in \Omega$. $\square$

Exercise 6.6.3 refers to the case of an unbounded region Ω.

Exercises.

6.6.1. Prove the **minimum principle**. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and let $m = \inf\{|f(z)| \,|\, z \in \Omega\}$. If there is $z_0 \in \Omega$ so that $|f(z_0)| = m$, then either $m = 0$ (and hence $f(z_0) = 0$) or $m > 0$ and then f is constant in Ω.

6.6.2. Let f be holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, let $|f(z)| > 1$ for every $z \in \mathbb{T}$ and $f(0) = 1$. Does f have a root in $\mathbb{D}$?

6.6.3. State and prove the second maximum principle in the case of an *unbounded* region Ω. In this case we must include the point ∞ in $\overline{\Omega}$.

6.6.4. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the bounded region Ω and $\lim_{z\to\zeta} f(z) = 0$ for every $\zeta \in \partial\Omega$. Prove that f is constant 0 in Ω.
In the case of an *unbounded* region Ω, we must include the point ∞ in $\partial\Omega$.

6.6.5. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and $K = \sup\{\mathrm{Re}\, f(z) \,|\, z \in \Omega\}$. If there is $z_0 \in \Omega$ so that $\mathrm{Re}\, f(z_0) = K$, prove that f is constant in Ω.

6.6.6. Prove the fundamental theorem of algebra using the maximum principle.

6.6.7. Let $f_n, f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the bounded region Ω and continuous in $\overline{\Omega}$. If $f_n \to f$ uniformly in $\partial\Omega$, prove that $f_n \to f$ uniformly in $\overline{\Omega}$.
In the case of an *unbounded* region Ω, we must include the point ∞ in $\partial\Omega$.

6.6.8. Let R be a square region with center z_0. Let $f : \overline{R} \to \mathbb{C}$ be holomorphic in R and continuous in $\overline{R}$. If $|f(z)| \leq m$ for every z in one of the four sides of R and $|f(z)| \leq M$ for every z in the other three sides of R, prove that $|f(z_0)| \leq \sqrt[4]{mM^3}$.

6.6.9. Let $\Omega = \{x + iy \,|\, -\frac{\pi}{2} < y < \frac{\pi}{2}\}$ and $f(z) = e^{e^z}$. Then f is holomorphic in Ω and continuous in $\overline{\Omega} = \{x + iy \,|\, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\}$. Prove that $|f(x - i\frac{\pi}{2})| = |f(x + i\frac{\pi}{2})| = 1$ for every $x \in \mathbb{R}$ and that $\lim_{x\to+\infty} f(x) = +\infty$. Does this contradict the maximum principle?

6.6.10. Let $f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the bounded region Ω and continuous in $\overline{\Omega}$. If $|f|$ is constant in $\partial\Omega$, prove that either f has at least one root in Ω or f is constant in Ω.

6.6.11. Let $f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the bounded region Ω and continuous in $\overline{\Omega}$. If $\mathrm{Re}\, f = 0$ in $\partial\Omega$, prove that f is constant in Ω.

6.6.12. (i) Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant in the region Ω. For every $\mu > 0$ prove that $\overline{\{z \in \Omega \,|\, |f(z)| < \mu\}} \cap \Omega = \{z \in \Omega \,|\, |f(z)| \leq \mu\}$.
(ii) Let p be a polynomial of degree $n \geq 1$. Prove that for every $\mu > 0$ the set $\{z \,|\, |p(z)| < \mu\}$ has at most n connected components and each of them contains at least one root of p. How do these connected components behave when $\mu \to 0+$ and when $\mu \to +\infty$?

6.6.13. Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant in the bounded region Ω. If we assume that $\limsup_{\Omega \ni z \to \zeta} |f(z)| \leq M$ for every $\zeta \in \partial\Omega$, prove that $|f(z)| < M$ for every $z \in \Omega$.
In the case of an *unbounded* region Ω, we must include the point ∞ in $\partial\Omega$.

6.6.14. Let $f : \overline{\Omega} \to \mathbb{C}$ be holomorphic in the bounded region Ω and continuous in $\overline{\Omega}$. If U is an open set so that $\overline{U} \subseteq \Omega$, prove that $\sup\{|f(z)| \,|\, z \in \partial U\} \leq \sup\{|f(z)| \,|\, z \in \partial\Omega\}$. If equality holds, prove that f is constant in $\overline{\Omega}$.

6.6.15. Let $f : D_0(R_1, R_2) \to \mathbb{C}$ be holomorphic in $D_0(R_1, R_2)$ and $a \in \mathbb{R}$. Prove that $|z|^a|f(z)|$ has no maximum value in $D_0(R_1, R_2)$, except if $a \in \mathbb{Z}$ and there is c so that $f(z) = cz^{-a}$ for every $z \in D_0(R_1, R_2)$.

6.6.16. The three circles theorem of Hadamard. Let $f : D_{z_0}(R_1, R_2) \to \mathbb{C}$ be holomorphic in $D_{z_0}(R_1, R_2)$ and let $M(r) = \max\{|f(z)| \,|\, z \in C_{z_0}(r)\}$ for $R_1 < r < R_2$. Prove that $\ln M(r)$ is a convex function of $\ln r$ in (R_1, R_2). I.e. prove that, if $R_1 < r_1 < r < r_2 < R_2$ and $\ln r = (1-t)\ln r_1 + t \ln r_2$ for $0 < t < 1$, then $\ln M(r) \leq (1-t)\ln M(r_1) + t \ln M(r_2)$.

6.6.17. The three lines theorem. Let $f : K \to \mathbb{C}$ be holomorphic and bounded in the vertical zone $K = \{x + iy \,|\, X_1 < x < X_2\}$ and let $M(x) = \sup\{|f(x+iy)| \,|\, -\infty < y < +\infty\}$ for $X_1 < x < X_2$. Prove that $\ln M(x)$ is a convex function of x in (X_1, X_2). I.e. prove that, if $X_1 < x_1 < x < x_2 < X_2$ and $x = (1-t)x_1 + tx_2$ for $0 < t < 1$, then $\ln M(x) \leq (1-t)\ln M(x_1) + t \ln M(x_2)$.

6.6.18. The Phragmén-Lindelöf theorem. Let $f : \Omega \to \mathbb{C}$ be holomorphic in the bounded region Ω, let $\phi : \Omega \to \mathbb{C}$ be holomorphic and bounded in Ω and let ϕ have no root in Ω. Let also $A \cap B = \emptyset$ and $A \cup B = \partial\Omega$. If
(i) $\limsup_{\Omega \ni z \to \zeta} |f(z)| \leq M$ for every $\zeta \in A$ and
(ii) $\limsup_{\Omega \ni z \to \zeta} |f(z)||\phi(z)|^\epsilon \leq M$ for every $\zeta \in B$ and every $\epsilon > 0$,
then prove that $|f(z)| \leq M$ for every $z \in \Omega$.
If, moreover, f is non-constant in Ω, prove that $|f(z)| < M$ for every $z \in \Omega$.

6.7 Taylor series and Laurent series.

Proposition 6.2. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω, $z_0 \in \Omega$ and let $D_{z_0}(R)$ be the largest disc with center z_0 which is contained in Ω. Then there is a unique power series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ so that*

$$f(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n \qquad \text{for every } z \in D_{z_0}(R).$$

The coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \qquad \text{for } 0 < r < R.$$

Proof. We take $z \in D_{z_0}(R)$, and then $|z - z_0| < R$. If $|z - z_0| < r < R$, then $z \in D_{z_0}(r)$ and, according to the formula of Cauchy, we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_{z_0}(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{6.8}$$

Now for every $\zeta \in C_{z_0}(r)$ we have $|\frac{z-z_0}{\zeta-z_0}| = \frac{|z-z_0|}{r} < 1$ and hence

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \frac{1}{\zeta - z_0}\sum_{n=0}^{+\infty}\Big(\frac{z-z_0}{\zeta-z_0}\Big)^n.$$

Therefore, (6.8) becomes

$$f(z) = \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{+\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n \, d\zeta.$$

The test of Weierstrass implies that $\sum_{n=0}^{+\infty} \big(\frac{z-z_0}{\zeta-z_0}\big)^n$ converges, as a series of functions of ζ, uniformly in $C_{z_0}(r)$. Indeed, $|\frac{z-z_0}{\zeta-z_0}|^n = (\frac{|z-z_0|}{r})^n$ for every $\zeta \in C_{z_0}(r)$ and $\sum_{n=0}^{+\infty}(\frac{|z-z_0|}{r})^n < +\infty$. Hence,

$$f(z) = \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \, (z - z_0)^n. \tag{6.9}$$

Now, we observe that the radius r has been chosen to satisfy the inequality $|z - z_0| < r < R$ and hence the integrals $\frac{1}{2\pi i} \int_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} \, d\zeta$ depend *a priori* on z. But there are two reasons that these integrals actually do not depend on the value of r in the interval $(0, R)$ and hence on z. The first reason is that from the formulas of Cauchy for the derivatives we get

$$\frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = \frac{f^{(n)}(z_0)}{n!}.$$

The second reason is that $\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}$ is, as a function of ζ, holomorphic in $D_{z_0}(R) \setminus \{z_0\}$, and because of corollary 6.1, we have

$$\frac{1}{2\pi i} \oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta = \frac{1}{2\pi i} \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \qquad \text{when } 0 < r_1 < r_2 < R.$$

We conclude from (6.9) that

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \text{for every } z \in D_{z_0}(R), \tag{6.10}$$

where $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} \, d\zeta$ for $0 < r < R$.
Regarding uniqueness, assume that $f(z) = \sum_{n=0}^{+\infty} b_n (z - z_0)^n$ for every $z \in D_{z_0}(R)$. Then, if $0 < r < R$, the series $\sum_{n=0}^{+\infty} b_n (z - z_0)^n$ converges uniformly in $C_{z_0}(r)$ and we get

$$\begin{aligned} 2\pi i a_k &= \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} \, d\zeta = \oint_{C_{z_0}(r)} \frac{1}{(\zeta - z_0)^{k+1}} \sum_{n=0}^{+\infty} b_n (\zeta - z_0)^n \, d\zeta \\ &= \sum_{n=0}^{+\infty} b_n \oint_{C_{z_0}(r)} (\zeta - z_0)^{n-k-1} \, d\zeta = 2\pi i b_k. \end{aligned}$$

The last equality uses the calculation in example 5.2.2.
Finally, we get that $b_k = a_k$ for every k and we conclude that the power series which satisfies (6.10) is unique. ☐

Definition. *The power series* $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ *with* $a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} \, d\zeta$ *for* $0 < r < R$ *is called* **Taylor series** *of* f *in the disc* $D_{z_0}(R)$*, the largest open disc with center* z_0 *which is contained in the domain of holomorphy of* f*.*

Example 6.7.1. The function $f(z) = \frac{1}{1-z}$ is holomorphic in $\mathbb{C} \setminus \{1\}$ and the largest open disc with center 0 which is contained in $\mathbb{C} \setminus \{1\}$ is $D_0(1)$. To find the Taylor series of f in $D_0(1)$ we calculate the derivatives $f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$ for every $n \geq 0$. Thus, $a_n = \frac{f^{(n)}(0)}{n!} = 1$ for every $n \geq 0$ and the Taylor series of f is $\sum_{n=0}^{+\infty} z^n$. I.e. $\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$ for every $z \in D_0(1)$. Of course, this is already known.

Example 6.7.2. The function $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$ is holomorphic in the open set $\mathbb{C} \setminus \{i, -i\}$ and the largest open disc with center 0 which is contained in $\mathbb{C} \setminus \{i, -i\}$ is $D_0(1)$. To find the Taylor series of f in $D_0(1)$ we calculate the derivatives of f. We write $f(z) = -\frac{1}{2i}(\frac{1}{i-z} + \frac{1}{i+z})$ and get $f^{(n)}(z) = -\frac{1}{2i}(\frac{n!}{(i-z)^{n+1}} + (-1)^n \frac{n!}{(i+z)^{n+1}})$ for every $n \geq 0$. Hence $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1+(-1)^n}{2i^n}$ for every $n \geq 0$. If n is odd, then $a_n = 0$. If n is even, then $a_n = \frac{1}{i^n} = (-1)^{\frac{n}{2}}$ and the Taylor series of f is $\sum_{k=0}^{+\infty} (-1)^k z^{2k}$. I.e. $\frac{1}{1+z^2} = \sum_{k=0}^{+\infty} (-1)^k z^{2k}$ for every $z \in D_0(1)$.
We may find the same formula if we use the Taylor series of $\frac{1}{1-z}$, i.e. $\frac{1}{1-z} = \sum_{n=0}^{+\infty} z^n$. We replace z with $-z^2$ and find $\frac{1}{1+z^2} = \sum_{n=0}^{+\infty} (-1)^n z^{2n}$. From the moment that we have found *some* power series which coincides with our function in $D_0(1)$, then, because of uniqueness, *this* is the Taylor series of our function.

Example 6.7.3. The exponential function $f(z) = e^z$ is holomorphic in $\mathbb{C}$ and the largest open disc with center 0 which is contained in $\mathbb{C}$ is $D_0(+\infty) = \mathbb{C}$. The derivatives of f are $f^{(n)}(z) = e^z$ for every $n \geq 0$ and the coefficients of the Taylor series of f are $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$ for every $n \geq 0$. Thus, the Taylor series of f is $\sum_{n=0}^{+\infty} \frac{1}{n!} z^n$ and we have

$$e^z = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n \qquad \text{for every } z.$$

Example 6.7.4. The function $f(z) = \cos z$, defined in exercise 5.2.5, is holomorphic in $\mathbb{C}$ and the largest open disc with center 0 which is contained in $\mathbb{C}$ is $D_0(+\infty) = \mathbb{C}$. The derivatives of f are $f^{(n)}(z) = (-1)^{\frac{n}{2}} \cos z$ for even n and $f^{(n)}(z) = (-1)^{\frac{n+1}{2}} \sin z$ for odd n. Therefore, the coefficients of the Taylor series are $a_n = \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{\frac{n}{2}}}{n!}$ for even n and $a_n = \frac{f^{(n)}(0)}{n!} = 0$ for odd n. Thus, the Taylor series of f is $\sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ and we have

$$\cos z = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} z^{2k} \qquad \text{for every } z.$$

In the same manner we can prove that

$$\sin z = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{(2k-1)!} z^{2k-1} \qquad \text{for every } z.$$

Another way to find the Taylor series of $\cos z$ and $\sin z$ is through the definitions of the two functions and the Taylor series of e^z. For instance:

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{n!} (iz)^n + \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1}{n!} (-iz)^n = \sum_{n=0}^{+\infty} \frac{i^n (1 + (-1)^n)}{2\, n!} z^n \\ &= \sum_{k=0}^{+\infty} \frac{i^{2k}}{(2k)!} z^{2k} = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} z^{2k}. \end{aligned}$$

The power series we found coincides with the function $\cos z$ in the largest open disc with center 0 which is contained in the domain of holomorphy of $\cos z$ and, because of uniqueness, this is the Taylor series of $\cos z$.

Example 6.7.5. The function $f(z) = -\operatorname{Log}(1 - z)$ is defined and holomorphic in $\mathbb{C} \setminus [1, +\infty)$. The largest disc with center 0 in $\mathbb{C} \setminus [1, +\infty)$ is $\mathbb{D}$. The derivatives of f are $f^{(n)}(z) = \frac{(n-1)!}{(1-z)^n}$ for every $n \geq 1$. Thus, $a_0 = 0$ and $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n}$ for every $n \geq 1$ and the Taylor series of f is $\sum_{n=1}^{+\infty} \frac{z^n}{n}$. I.e.

$$-\operatorname{Log}(1 - z) = \sum_{n=1}^{+\infty} \frac{z^n}{n} \qquad \text{for every } z \in \mathbb{D}.$$

Proposition 6.3. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω and let $D_{z_0}(R_1, R_2)$ be a largest open ring with center z_0 which is contained in Ω. Then there is a unique power series $\sum_{-\infty}^{+\infty} a_n(z-z_0)^n$ so that*

$$f(z) = \sum_{-\infty}^{+\infty} a_n(z-z_0)^n \qquad \text{for every } z \in D_{z_0}(R_1, R_2).$$

The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \qquad \text{for } R_1 < r < R_2.$$

Proof. We take $z \in D_{z_0}(R_1, R_2)$, and then $R_1 < |z - z_0| < R_2$. We choose any r_1, r_2 so that $R_1 < r_1 < |z - z_0| < r_2 < R_2$. Then $z \in D_{z_0}(r_1, r_2)$ and

$$f(z) = \frac{1}{2\pi i} \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{6.11}$$

To prove (6.11), we consider an open disc $D_z(r)$ with $r < \min\{r_2 - |z - z_0|, |z - z_0| - r_1\}$. Then $\overline{D}_z(r) \subseteq D_{z_0}(r_1, r_2)$ and we apply corollary 6.1 to $\frac{f(\zeta)}{\zeta - z}$, which is a holomorphic function of ζ in $D_{z_0}(R_1, R_2) \setminus \{z\}$. We get

$$\oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{\zeta - z}\, dz - \oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{\zeta - z}\, dz = \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Now as in the proof of Cauchy's formula for circles, we have $\lim_{r\to 0} \oint_{C_z(r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = 2\pi i f(z)$ and the proof of (6.11) is complete.
For every $\zeta \in C_{z_0}(r_2)$ we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{+\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n,$$

because $|\frac{z - z_0}{\zeta - z_0}| = \frac{|z - z_0|}{r_2} < 1$. Similarly, for every $\zeta \in C_{z_0}(r_1)$ we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\frac{1}{z - z_0} \sum_{n=0}^{+\infty} \Big(\frac{\zeta - z_0}{z - z_0}\Big)^n$$

because $|\frac{\zeta - z_0}{z - z_0}| = \frac{r_1}{|z - z_0|} < 1$. Hence (6.11) becomes

$$f(z) = \frac{1}{2\pi i} \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{+\infty} \Big(\frac{z - z_0}{\zeta - z_0}\Big)^n d\zeta + \frac{1}{2\pi i} \oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{z - z_0} \sum_{n=0}^{+\infty} \Big(\frac{\zeta - z_0}{z - z_0}\Big)^n d\zeta.$$

Because of uniform convergence of the series inside these two integrals, we get

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\, (z - z_0)^n \\ &\quad + \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \oint_{C_{z_0}(r_1)} f(\zeta)(\zeta - z_0)^n\, d\zeta\, \frac{1}{(z - z_0)^{n+1}}. \end{aligned}$$

In the last series we change $n + 1$ to $-n$ and get

$$\begin{aligned} f(z) &= \sum_{n=0}^{+\infty} \frac{1}{2\pi i} \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\, (z - z_0)^n \\ &\quad + \sum_{-\infty}^{n=-1} \frac{1}{2\pi i} \oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\, (z - z_0)^n. \end{aligned} \tag{6.12}$$

Now, $\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}$ is, as a function of ζ, holomorphic in $D_{z_0}(R_1,R_2)$, and another application of corollary 6.1 implies that

$$\oint_{C_{z_0}(r_1)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta = \oint_{C_{z_0}(r_2)} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta \qquad \text{for } R_1<r_1<r_2<R_2.$$

Therefore the coefficients of both series in (6.12) do not depend on the values of r_1, r_2, and we replace both radii with any r with $R_1<r<R_2$. We proved that

$$f(z)=\sum_{-\infty}^{+\infty} a_n(z-z_0)^n \qquad \text{for every } z\in D_{z_0}(R_1,R_2), \tag{6.13}$$

where $a_n=\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$ for $R_1<r<R_2$.

Regarding uniqueness, assume that $f(z)=\sum_{-\infty}^{+\infty} b_n(z-z_0)^n$ for every $z\in D_{z_0}(R_1,R_2)$. We take any r with $R_1<r<R_2$, and then $\sum_{-\infty}^{+\infty} b_n(z-z_0)^n$ converges uniformly in $C_{z_0}(r)$ Then

$$\begin{aligned} 2\pi i a_k &= \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta = \oint_{C_{z_0}(r)} \frac{1}{(\zeta-z_0)^{k+1}} \sum_{-\infty}^{+\infty} b_n(\zeta-z_0)^n\,d\zeta \\ &= \sum_{-\infty}^{+\infty} b_n \oint_{C_{z_0}(r)} (\zeta-z_0)^{n-k-1}\,d\zeta = 2\pi i b_k \end{aligned}$$

and the power series $\sum_{-\infty}^{+\infty} b_n(z-z_0)^n$ satisfying (6.13) is unique. □

Definition. *The power series* $\sum_{-\infty}^{+\infty} a_n(z-z_0)^n$ *with* $a_n=\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta$ *for* $R_1<r<R_2$ *is called* **Laurent series** *of f in the ring $D_{z_0}(R_1,R_2)$, a largest open ring with center z_0 which is contained in the domain of holomorphy of f.*

Example 6.7.6. The function $f(z)=\frac{1}{z}$ is holomorphic in $\mathbb{C}\setminus\{0\}$. The ring $D_0(0,+\infty)=\mathbb{C}\setminus\{0\}$ is the largest open ring with center 0 which is contained in $\mathbb{C}\setminus\{0\}$. To find the Laurent series of f in $D_0(0,+\infty)$ we evaluate the coefficients a_n. We take any r with $0<r<+\infty$, and then

$$a_n=\frac{1}{2\pi i}\oint_{C_0(r)}\frac{1/\zeta}{\zeta^{n+1}}\,d\zeta=\frac{1}{2\pi i}\oint_{C_0(r)}\frac{1}{\zeta^{n+2}}\,d\zeta \qquad \text{for every } n.$$

If $n\neq -1$, then $a_n=0$ and, if $n=-1$, then $a_{-1}=1$. Therefore, the Laurent series of f in $D_0(0,+\infty)$ is $\sum_{-\infty}^{+\infty} a_n z^n=z^{-1}$ and hence we have the obvious identity $\frac{1}{z}=z^{-1}$ for every $z\in D_0(0,+\infty)$.

In the following examples we shall use the uniqueness of the Laurent series to find the Laurent series of certain functions without evaluating integrals: we find in an indirect way a power series which coincides with the function in a specific ring and then, because of uniqueness, this is the Laurent series of the function in the ring.

Example 6.7.7. The function $f(z)=\frac{1}{1-z}$ is holomorphic in the open set $\mathbb{C}\setminus\{1\}$. We have seen that the largest open disc with center 0 which is contained in $\mathbb{C}\setminus\{1\}$ is $D_0(1)$ and that the Taylor series of f in this disc is $\sum_{n=0} z^n$.

Another largest open ring with center 0 which is contained in $\mathbb{C}\setminus\{1\}$ is $D_0(1,+\infty)$. To find the Laurent series of f in this ring, we may evaluate the coefficients a_n using their formulas with the integrals. But we can do something simpler. If $z\in D_0(1,+\infty)$, then $|\frac{1}{z}|<1$ and hence

$$\frac{1}{1-z}=-\frac{1}{z}\frac{1}{1-\frac{1}{z}}=-\frac{1}{z}\sum_{n=0}^{+\infty}\Big(\frac{1}{z}\Big)^n=-\sum_{-\infty}^{n=-1} z^n.$$

Because of uniqueness, the Laurent series of f in $D_0(1,+\infty)$ is $-\sum_{-\infty}^{n=-1} z^n$.

Example 6.7.8. The function $f(z)=\frac{1}{(z-1)(z-2)}$ is holomorphic in $\mathbb{C}\setminus\{1,2\}$. There is a largest open disc and two largest open rings with center 0 which are contained in $\mathbb{C}\setminus\{1,2\}$: the disc $D_0(1)$ and the rings $D_0(1,2)$ and $D_0(2,+\infty)$. To find the corresponding Taylor and Laurent series we write f as a sum of simple fractions: $f(z)=\frac{1}{z-2}-\frac{1}{z-1}$.
If $z\in D_0(1)$, then $|z|<1$ and $|\frac{z}{2}|<1$, and hence

$$f(z)=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}+\frac{1}{1-z}=-\frac{1}{2}\sum_{n=0}^{+\infty}\Big(\frac{z}{2}\Big)^n+\sum_{n=0}^{+\infty}z^n=\sum_{n=0}^{+\infty}\Big(1-\frac{1}{2^{n+1}}\Big)z^n.$$

Therefore, the Taylor series of f in $D_0(1)$ is $\sum_{n=0}^{+\infty}(1-\frac{1}{2^{n+1}})z^n$.
If $z\in D_0(1,2)$, then $|\frac{1}{z}|<1$ and $|\frac{z}{2}|<1$, and hence

$$f(z)=-\frac{1}{2}\frac{1}{1-\frac{z}{2}}-\frac{1}{z}\frac{1}{1-\frac{1}{z}}=-\frac{1}{2}\sum_{n=0}^{+\infty}\Big(\frac{z}{2}\Big)^n-\frac{1}{z}\sum_{n=0}^{+\infty}\Big(\frac{1}{z}\Big)^n=-\sum_{-\infty}^{n=-1}z^n-\sum_{n=0}^{+\infty}\frac{1}{2^{n+1}}\,z^n.$$

Therefore, the Laurent series of f in $D_0(1,2)$ is $-\sum_{-\infty}^{n=-1}z^n-\sum_{n=0}^{+\infty}\frac{1}{2^{n+1}}\,z^n$.
If $z\in D_0(2,+\infty)$, then $|\frac{1}{z}|<1$ and $|\frac{2}{z}|<1$, and hence

$$f(z)=\frac{1}{z}\frac{1}{1-\frac{2}{z}}-\frac{1}{z}\frac{1}{1-\frac{1}{z}}=\frac{1}{z}\sum_{n=0}^{+\infty}\Big(\frac{2}{z}\Big)^n-\frac{1}{z}\sum_{n=1}^{+\infty}\Big(\frac{1}{z}\Big)^n=\sum_{-\infty}^{n=-2}\Big(\frac{1}{2^{n+1}}-1\Big)z^n.$$

Therefore, the Laurent series of f in $D_0(2,+\infty)$ is $\sum_{-\infty}^{n=-2}(\frac{1}{2^{n+1}}-1)z^n$.

Example 6.7.9. The function $f(z)=e^{\frac{1}{z}}$ is holomorphic in $\mathbb{C}\setminus\{0\}$. Then $D_0(0,+\infty)=\mathbb{C}\setminus\{0\}$ is the only largest open ring with center 0 which is contained in $\mathbb{C}\setminus\{0\}$. We find the Laurent series of f in $D_0(0,+\infty)$ using the Taylor series of e^z in $\mathbb{C}$. In the identity $e^z=\sum_{n=0}^{+\infty}\frac{1}{n!}z^n$ we replace z with $\frac{1}{z}$ and we find

$$e^{\frac{1}{z}}=\sum_{-\infty}^{n=-1}\frac{1}{(-n)!}z^n+1\qquad\text{for every } z\neq 0.$$

Therefore, the Laurent series of f in $D_0(0,+\infty)$ is $\sum_{-\infty}^{n=-1}\frac{1}{(-n)!}z^n+1$.

Exercises.

6.7.1. Let $0<|a|<|b|$. Find the three Laurent series with center 0, the two Laurent series with center a and the two Laurent series with center b of the function $\frac{z}{(z-a)(z-b)}$.

6.7.2. Find the Taylor series of $\frac{1}{1+z^2}$ with center any $a\in\mathbb{R}$.

6.7.3. Find the Taylor series with center 1 of the holomorphic branch of $z^{\frac{1}{2}}$ with value 1 at 1.

6.7.4. Let f be holomorphic in $D_{z_0}(R)$ and let $\sum_{n=0}^{+\infty}a_n(z-z_0)^n$ be the Taylor series of f.
(i) Prove that, if $0\leq r<R$, then $\frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{it})|^2\,dt=\sum_{n=0}^{+\infty}|a_n|^2r^{2n}$.
(ii) If $|f(z)|\leq M$ for every $z\in D_{z_0}(R)$, prove that $\sum_{n=0}^{+\infty}|a_n|^2R^{2n}\leq M^2$.
(iii) If g is also holomorphic in $D_{z_0}(R)$ with Taylor series $\sum_{n=0}^{+\infty}b_n(z-z_0)^n$, prove that, if $0\leq r<R$, then $\frac{1}{2\pi}\int_0^{2\pi}f(z_0+re^{it})\,\overline{g(z_0+re^{it})}\,dt=\sum_{n=0}^{+\infty}a_n\overline{b_n}r^{2n}$.

6.7.5. Let f be holomorphic in $D_{z_0}(R_1,R_2)$. Prove that there are functions f_1 and f_2 so that f_2 is holomorphic in $D_{z_0}(R_2)$ and f_1 is holomorphic in $D_{z_0}(R_1,+\infty)$ and so that $f(z)=f_1(z)+f_2(z)$ for every $z\in D_{z_0}(R_1,R_2)$. Prove that, if f is bounded in $D_{z_0}(R_1,R_2)$, then f_1,f_2 are bounded in $D_{z_0}(R_1,R_2)$.

6.7.6. Let $\Omega \subseteq \mathbb{C}$ be an open set and let $D_0(R,+\infty) = \{z \,|\, |z| > R\}$ be the largest ring of this kind which is contained in Ω (i.e. R is the smallest possible). Let $f : \Omega \cup \{\infty\} \to \mathbb{C}$ be holomorphic in Ω. Prove that f is holomorphic also at ∞ if and only if the Laurent series of f in $D_0(R,+\infty)$ is of the form $\sum_{-\infty}^{n=-1} a_n z^n + a_0$. Observe that $f(\infty) = a_0$.

6.7.7. Prove that $\frac{1}{\cos z} = 1 + \sum_{k=1}^{+\infty} \frac{E_{2k}}{(2k)!} z^{2k}$ for $|z| < \frac{\pi}{2}$, where the numbers E_{2k} satisfy the recursive relations $E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} - \cdots + (-1)^{n-1}\binom{2n}{2} E_2 + (-1)^n = 0$. Evaluate E_2, E_4, E_6, E_8. The numbers E_{2k} are called Euler constants.

6.7.8. Let $\Omega = \{x + iy \,|\, A < y < B\}$, let $f : \Omega \to \mathbb{C}$ be holomorphic in the horizontal zone Ω and let f be periodic with period 1, i.e. $f(z+1) = f(z)$ for every $z \in \Omega$.
(i) Prove that there are c_n, $n \in \mathbb{Z}$, so that $f(z) = \sum_{-\infty}^{+\infty} c_n e^{2\pi i n z}$ for every $z \in \Omega$ and find formulas for the coefficients c_n.
(ii) Prove that the series in (i) converges uniformly in every smaller zone $\{x + iy \,|\, a < y < b\}$ with $A < a < b < B$.

6.7.9. (i) Prove that $e^{\frac{w}{2}(z-\frac{1}{z})} = b_0(w) + \sum_{n=1}^{+\infty} b_n(w)(z^n + \frac{(-1)^n}{z^n})$ for every $z \neq 0$, where $b_n(w) = \frac{1}{\pi}\int_0^\pi \cos(nt - w\sin t)\,dt$ for $n \in \mathbb{N}_0$.
(ii) If $m, n \in \mathbb{N}_0$, prove that $\frac{1}{2\pi i}\int_{C_0(1)} \frac{(z^2 \pm 1)^m}{z^{m+n+1}}\,dz = \begin{cases} \frac{(\pm 1)^p (n+2p)!}{p!(n+p)!} & \text{if } m = n+2p,\ p \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$
(iii) The function $b_n(w)$ is called Bessel function of the first kind. Find the Taylor series of $b_n(w)$ with center 0.

6.7.10. Let I be an open interval in $\mathbb{R}$. The function $f : I \to \mathbb{C}$ is called **real analytic** in I if for every $t_0 \in I$ there are $\epsilon > 0$ and $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, so that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq I$ and $f(t) = \sum_{n=0}^{+\infty} a_n (t - t_0)^n$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$.
Prove that, if f is real analytic in I, then there is an open set $\Omega \subseteq \mathbb{C}$ so that $I \subseteq \Omega$ and so that f can be extended as a function $f : \Omega \to \mathbb{C}$ holomorphic in Ω.

6.8 Roots and the principle of identity.

Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω and take $z_0 \in \Omega$. We consider the largest open disc $D_{z_0}(R)$ which is contained in Ω and the Taylor series of f in this disc. Then

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad \text{for every } z \in D_{z_0}(R).$$

We assume that z_0 is a root of f or, equivalently, that $a_0 = 0$ and we distinguish between two cases.

First case: $a_n = 0$ *for every* n.
Then, obviously, $f(z) = 0$ for every $z \in D_{z_0}(R)$, i.e. f is identically 0 in $D_{z_0}(R)$. Because of the formulas for a_n, the condition $a_n = 0$ for every n is equivalent to $f^{(n)}(z_0) = 0$ for every n.

Second case: $a_n \neq 0$ *for at least one* n.
We consider the smallest $n \geq 1$ with $a_n \neq 0$ and let this be N. I.e. $a_0 = a_1 = \ldots = a_{N-1} = 0$ and $a_N \neq 0$. This is equivalent to $f(z_0) = f^{(1)}(z_0) = \ldots = f^{(N-1)}(z_0) = 0$ and $f^{(N)}(z_0) \neq 0$. Then we have

$$f(z) = (z - z_0)^N \sum_{n=N}^{+\infty} a_n (z - z_0)^{n-N} = (z - z_0)^N \sum_{n=0}^{+\infty} a_{N+n}(z - z_0)^n \quad \text{for every } z \in D_{z_0}(R).$$

The power series $\sum_{n=0}^{+\infty} a_{N+n}(z - z_0)^n = a_N + a_{N+1}(z - z_0) + a_{N+2}(z - z_0)^2 + \cdots$ converges in the disc $D_{z_0}(R)$ and defines a holomorphic function $g : D_{z_0}(R) \to \mathbb{C}$. Then

$$f(z) = (z - z_0)^N g(z) \qquad \text{for every } z \in D_{z_0}(R),$$

and thus $g(z) = \frac{f(z)}{(z-z_0)^N}$ for every $z \in D_{z_0}(R) \setminus \{z_0\}$. We observe that $\frac{f(z)}{(z-z_0)^N}$ is a holomorphic function in $\Omega \setminus \{z_0\}$ and not only in $D_{z_0}(R) \setminus \{z_0\}$. Therefore, we may consider g as defined in $\Omega \setminus \{z_0\}$ with the same formula: $g(z) = \frac{f(z)}{(z-z_0)^N}$. We also recall that g is defined, through its power series, at z_0 and it is holomorphic in $D_{z_0}(R) \subseteq \Omega$. In fact its value at z_0 is $g(z_0) = a_N = \frac{f^{(N)}(z_0)}{N!}$. Thus, the formula of $g : \Omega \to \mathbb{C}$, as a function holomorphic in Ω, can be written:

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^N}, & \text{if } z \in \Omega \setminus \{z_0\} \\ a_N = \frac{f^{(N)}(z_0)}{N!}, & \text{if } z = z_0 \end{cases} \tag{6.14}$$

Since $g(z_0) = a_N \neq 0$ and since g is continuous at z_0, there is r with $0 < r \leq R$ so that $g(z) \neq 0$ for every $z \in D_{z_0}(r)$, and (6.14) implies

$$f(z) \neq 0 \qquad \text{for every } z \in D_{z_0}(r) \setminus \{z_0\}.$$

Definition. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω and let $z_0 \in \Omega$ with $f(z_0) = 0$. Also, let $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ be the Taylor series of f at z_0.*
If $a_n = 0$ for every n, then we say that z_0 is a root of f of multiplicity $+\infty$.
If $a_0 = a_1 = \ldots = a_{N-1} = 0$ and $a_N \neq 0$, then we say that z_0 is a root of f of multiplicity N.
In case $f(z_0) = a_0 \neq 0$ we say that z_0 is a root of f of multiplicity 0.

We saw that, if z_0 is a root of f of infinite multiplicity, then f is identically 0 in the largest disc with center z_0 which is contained in the domain of holomorphy of f. If z_0 is a root of f of finite multiplicity, then there is some disc $D_{z_0}(r)$ which contains no other root of f besides z_0 and hence we say that the root z_0 is **isolated**. Moreover, if the multiplicity of z_0 is N, then the function $g(z) = \frac{f(z)}{(z-z_0)^N}$, which is holomorphic in $\Omega \setminus \{z_0\}$, can be defined at z_0 as $g(z_0) = a_N = \frac{f^{(N)}(z_0)}{N!}$ and then it is holomorphic in Ω. In other words, *we can factorize $(z-z_0)^N$ from $f(z)$, i.e. we can write $f(z) = (z-z_0)^N g(z)$ with a function g holomorphic in Ω.* This is a striking generalization of the analogous factorization for polynomials: is z_0 is a root of the polynomial $p(z)$ of multiplicity N, then we write $p(z) = (z-z_0)^N q(z)$, where $q(z)$ is another polynomial.

Example 6.8.1. The function $e^{z^3} - 1$ is holomorphic in $\mathbb{C}$ and its Taylor series with center 0 is $\sum_{n=1}^{+\infty} \frac{1}{n!} z^{3n}$. Therefore, $e^{z^3} - 1 = z^3 \sum_{n=1}^{+\infty} \frac{1}{n!} z^{3(n-1)} = z^3 \sum_{n=0}^{+\infty} \frac{1}{(n+1)!} z^{3n} = z^3 g(z)$ for every z, where g is the function defined by the power series $\sum_{n=0}^{+\infty} \frac{1}{(n+1)!} z^{3n}$. Now g is holomorphic in $\mathbb{C}$ with $g(0) = 1 \neq 0$, hence 0 is a root of $e^{z^3} - 1$ of multiplicity 3.

Lemma 6.1. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and let $z_0 \in \Omega$ with $f(z_0) = 0$. If z_0 is a root of f of infinite multiplicity, then f is identically 0 in Ω.*

Proof. f is identically 0 in some disc with center z_0. We define

$$B = \{z \in \Omega \,|\, f \text{ is identically } 0 \text{ in some disc with center } z\}$$

and the complementary set $C = \Omega \setminus B$. Obviously, $B \cup C = \Omega$ and $B \cap C = \emptyset$. Also, $B \neq \emptyset$, since $z_0 \in B$.
If $z \in B$, then f is identically 0 in some disc $D_z(r)$, and if we take any $w \in D_z(r)$, then f is identically 0 in some small disc $D_w(r') \subseteq D_z(r)$. Thus every $w \in D_z(r)$ belongs to B, i.e. $D_z(r) \subseteq B$ and z is not a limit point of C.
Now, let $z \in C$. Then f is identically 0 in no disc with center z, and hence z is not a root of infinite multiplicity of f. Therefore, there is a disc $D_z(r)$ in which the only possible root of f is its center z. Then this disc contains no $w \in B$ and z is not a limit point of B.
Thus, none of B, C contains a limit point of the other. Since $B \neq \emptyset$, we must have $C = \emptyset$, otherwise B, C would form a decomposition of Ω. Hence $\Omega = B$ and f is identically 0 in Ω. $\square$

Principle of identity. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω. If the roots of f have an accumulation point in Ω, i.e. if there is a sequence (z_n) of roots of f so that $z_n \to z$ with $z \in \Omega$ and $z_n \neq z$ for every n, then f is identically 0 in Ω.*

Proof. Since f is continuous at z and $z_n \to z$, we have $0 = f(z_n) \to f(z)$ and hence $f(z) = 0$. If z is a root of finite multiplicity of f, then there would be some disc $D_z(r)$ in which the only root of f is its center z. This is wrong, since $D_z(r)$ contains, after some index, all roots z_n and these are different from z. Therefore, z is a root of infinite multiplicity of f, and lemma 6.1 implies that f is identically 0 in Ω. □

Lemma 6.1 and the principle of identity can be stated for a non-connected open set Ω. Then the result of lemma 6.1 holds in the connected component of Ω which contains the root of infinite multiplicity z_0 and the result of the principle of identity holds in the connected component of Ω which contains the accumulation point of the roots of f.

Instead of speaking only about the roots of f, i.e. the solutions of the equation $f(z) = 0$, we may state our results for the solutions of the equation $f(z) = w$ for any fixed w. The results are the same as before. We just consider the function $g(z) = f(z) - w$, and then the solutions of $f(z) = w$ are the same as the roots of g. For instance, if z_0 is a solution of $f(z) = w$ of infinite multiplicity, then f is constant w in some disc $D_{z_0}(R)$ and, if z_0 is a solution of $f(z) = w$ of finite multiplicity N, then in some disc $D_{z_0}(r)$ the function f takes the value w only at the center z_0. Then lemma 6.1 says that, if f is holomorphic in the region Ω and z_0 is a solution of $f(z) = w$ of infinite multiplicity, then f is constant w in Ω. And the principle of identity says that, if f is holomorphic in the region Ω and the solutions of $f(z) = w$ have an accumulation point in Ω, then f is constant w in Ω.

The principle of identity has another equivalent form.

Principle of identity. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω. If some compact $K \subseteq \Omega$ contains infinitely many roots of f, then f is identically 0 in Ω.*

Proof. Let us assume the previous principle of identity and let us suppose that some compact $K \subseteq \Omega$ contains infinitely many roots of f. Then there is a sequence (z_n) of roots of f in K with distinct terms. Since K is compact, there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some $z \in K$. But then $z \in \Omega$ is an accumulation point of roots of f and hence f is identically 0 in Ω.
Conversely, let us assume the present form of the principle of identity and let us suppose that the roots of f have an accumulation point in Ω. Then there is a sequence (z_n) of roots of f so that $z_n \to z$ with $z \in \Omega$ and $z_n \neq z$ for every n. We take a compact disc $\overline{D}_z(r) \subseteq \Omega$ and then this disc contains infinitely many of the roots z_n. Hence f is identically 0 in Ω. □

Example 6.8.2. Assume that there is f holomorphic in $\mathbb{C}$ so that $f(\frac{1}{n}) = \frac{n}{n+1}$ for every $n \in \mathbb{N}$. We write $f(\frac{1}{n}) = \frac{1}{1+\frac{1}{n}}$ and compare the functions $f(z)$ and $\frac{1}{1+z}$. Both are holomorphic in $\mathbb{C} \setminus \{-1\}$ and their difference $f(z) - \frac{1}{1+z}$ has roots at the points $\frac{1}{n}$ which have 0 as their accumulation point. Since $0 \in \mathbb{C} \setminus \{-1\}$ and $\mathbb{C} \setminus \{-1\}$ is connected, we have that $f(z) - \frac{1}{1+z}$ is identically 0 in this set, i.e. $f(z) = \frac{1}{1+z}$ for every $z \neq -1$. Since we assume that f is holomorphic at -1, we get $\lim_{z\to-1} \frac{1}{1+z} = \lim_{z\to-1} f(z) = f(-1)$ and we arrive at a contradiction.

Example 6.8.3. Assume that there is some f holomorphic in $\mathbb{C} \setminus \{0\}$ so that $f(x) = \sqrt{x}$ for every $x \in (0, +\infty)$ or even for every x in some subinterval (a, b) of $(0, +\infty)$.
We consider the continuous branch g of $z^{\frac{1}{2}}$ in the open set $A = \mathbb{C} \setminus (-\infty, 0]$ which has value 1 at $z = 1$. The function g is given by

$$g(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad \text{for } z = re^{i\theta} \text{ with } r > 0 \text{ and } -\pi < \theta < \pi.$$

We see that $f(x) = \sqrt{x} = g(x)$ for every $x \in (a,b)$. Hence $f-g$ is holomorphic in the region A and has roots at all points of (a,b). We conclude that $f-g$ is identically 0 in A. I.e.

$$f(z) = \sqrt{r}\, e^{i\frac{\theta}{2}} \qquad \text{for } z = re^{i\theta} \text{ with } r > 0 \text{ and } -\pi < \theta < \pi.$$

Since f is holomorphic in $\mathbb{C} \setminus \{0\}$, it is continuous at every point of $(-\infty, 0)$, e.g. at -1.
We take points $z = re^{i\theta}$ converging to -1 from the upper halfplane. This means that $r \to 1$ and $\theta \to \pi-$. Then we have $f(-1) = \lim_{r\to 1, \theta\to\pi-} \sqrt{r}\, e^{i\frac{\theta}{2}} = e^{i\frac{\pi}{2}} = i$.
Now we take points $z = re^{i\theta}$ converging to -1 from the lower halfplane. This means that $r \to 1$ and $\theta \to -\pi+$. Then we have $f(-1) = \lim_{r\to 1, \theta\to -\pi+} \sqrt{r}\, e^{i\frac{\theta}{2}} = e^{-i\frac{\pi}{2}} = -i$.
We arrive at a contradiction.

Exercises.

6.8.1. Let f be holomorphic in the disc $D_{z_0}(R)$ and let z_0 be a root of multiplicity $N \geq 1$ of f. Discuss the behavior of any primitive F of f at z_0.

6.8.2. Is there any f holomorphic in $\mathbb{C}$ which satisfies one of the following?
(i) $f(\frac{1}{n}) = (-1)^n$ for every $n \in \mathbb{N}$.
(ii) $f(\frac{1}{n}) = \frac{1+(-1)^n}{n}$ for every $n \in \mathbb{N}$.
(iii) $f(\frac{1}{2k}) = f(\frac{1}{2k+1}) = \frac{1}{k}$ for every $k \in \mathbb{N}$.

6.8.3. Is there any f holomorphic in $\mathbb{C} \setminus \{0\}$ so that $f(x) = |x|$ for every $x \in \mathbb{R} \setminus \{0\}$?

6.8.4. Let f, g be holomorphic in the region Ω and $0 \in \Omega$. If f, g have no root in Ω and $f'(\frac{1}{n})/f(\frac{1}{n}) = g'(\frac{1}{n})/g(\frac{1}{n})$ for every $n \in \mathbb{N}$, what do you conclude about f, g?

6.8.5. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic in the region Ω. If $fg = 0$ in Ω, prove that either $f = 0$ in Ω or $g = 0$ in Ω.

6.8.6. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic in the region Ω. If $\overline{f}\, g$ is holomorphic in Ω, prove that either $g = 0$ in Ω or f is constant in Ω.

6.8.7. (i) Let the region Ω be symmetric with respect to $\mathbb{R}$, i.e. $\overline{z} \in \Omega$ for every $z \in \Omega$. If $\Omega \neq \emptyset$, prove that $\Omega \cap \mathbb{R} \neq \emptyset$. Let also $f : \Omega \to \mathbb{C}$ be holomorphic in Ω and assume that $f(z) \in \mathbb{R}$ for every $z \in \Omega \cap \mathbb{R}$. Prove that $f(\overline{z}) = \overline{f(z)}$ for every $z \in \Omega$.
(ii) Let the region $\Omega \subseteq \mathbb{C} \setminus \{0\}$ be symmetric with respect to $\mathbb{T}$, i.e. $\frac{1}{\overline{z}} \in \Omega$ for every $z \in \Omega$. If $\Omega \neq \emptyset$, prove that $\Omega \cap \mathbb{T} \neq \emptyset$. Let also $f : \Omega \to \mathbb{C}$ be holomorphic in Ω and assume that $f(z) \in \mathbb{T}$ for every $z \in \Omega \cap \mathbb{T}$. Prove that $f(\frac{1}{\overline{z}}) = \frac{1}{\overline{f(z)}}$ for every $z \in \Omega$.
(iii) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic in $\mathbb{C}$ and let $f(z) \in \mathbb{T}$ for every $z \in \mathbb{T}$. Prove that there is c with $|c| = 1$ and $n \in \mathbb{N}_0$ so that $f(z) = cz^n$ for every z.

6.8.8. (i) Let $z_0 \in \mathbb{D}$ and $T : \overline{\mathbb{D}} \to \mathbb{C}$ be defined by $T(z) = \frac{z-z_0}{1-\overline{z_0}z}$ for $z \in \overline{\mathbb{D}}$. Prove that T is holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Also prove that $T(z) \in \mathbb{D}$ for every $z \in \mathbb{D}$ and that $T(z) \in \mathbb{T}$ for every $z \in \mathbb{T}$.
(ii) Let $z_1, \ldots, z_n \in \mathbb{D}$ and $|c| = 1$ and $B : \overline{\mathbb{D}} \to \mathbb{C}$ be defined by $B(z) = c\prod_{k=1}^n \frac{z-z_k}{1-\overline{z_k}z}$ for $z \in \overline{\mathbb{D}}$. Prove that B is holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$. Also prove that $B(z) \in \mathbb{D}$ for every $z \in \mathbb{D}$ and that $B(z) \in \mathbb{T}$ for every $z \in \mathbb{T}$.
(iii) Prove the converse of (ii). I.e. let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ and let $f(z) \in \mathbb{D}$ for every $z \in \mathbb{D}$ and $f(z) \in \mathbb{T}$ for every $z \in \mathbb{T}$. If f is non-constant, prove that there is $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{D}$ and c with $|c| = 1$ so that $f(z) = c\prod_{k=1}^n \frac{z-z_k}{1-\overline{z_k}z}$ for every $z \in \overline{\mathbb{D}}$.

6.8.9. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic in $\mathbb{C}$ and $|f(z)| \leq |g(z)|$ for every z. Prove that there is μ so that $f(z) = \mu g(z)$ for every z.

6.8.10. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic in $\mathbb{D}$. Prove that there is a sequence (z_n) in $\mathbb{D}$ so that $|z_n| \to 1$ and $(f(z_n))$ is bounded.

6.8.11. Many of the results of this section hold also for the point ∞.
(i) Let $\Omega \subseteq \mathbb{C}$ be an open set containing some ring $D_0(R, +\infty) = \{z \,|\, |z| > R\}$ and let $f : \Omega \cup \{\infty\} \to \mathbb{C}$ be holomorphic in Ω as well as at ∞. Then, according to exercice 6.7.7, the Laurent series of f in $D_0(R, +\infty)$ is of the form $\sum_{-\infty}^{n=-1} a_n z^n + a_0$ and also $f(\infty) = a_0$.
If $a_n = 0$ for every $n \leq 0$, we say that ∞ is a root of f of multiplicity $+\infty$, and in this case prove that f is identically 0 in the connected component of Ω which contains $D_0(R, +\infty)$.
If $a_0 = a_1 = \ldots = a_{-N+1} = 0$ and $a_{-N} \neq 0$, we say that ∞ is a root of f of multiplicity N, and in this case prove that ∞ is an isolated root of f, i.e. there is some $r \geq R$ so that f has no root in $D_0(r, +\infty)$.
Of course, if $a_0 \neq 0$, we say that ∞ is a root of f of multiplicity 0.
If ∞ is an accumulation point of roots of f, prove that f is identically 0 in the connected component of Ω which contains $D_0(R, +\infty)$.
Prove that ∞ is a root of f of multiplicity N if and only if 0 is a root of g of multiplicity N, where g is defined by $g(w) = f(\frac{1}{w})$.
(ii) Let $r = \frac{p}{q}$ be a rational function and let n be the degree of the polynomial p and m be the degree of the polynomial q. If $n \leq m$, prove that ∞ is a root of r of multiplicity $m - n$.

6.9 Isolated singularities.

Definition. *We say that z_0 is an* **isolated singularity** *of f if there is some disc $D_{z_0}(R)$ so that f is holomorphic in $D_{z_0}(R) \setminus \{z_0\}$.*

If z_0 is an isolated singularity of f, then f has a Laurent series in $D_{z_0}(0, R) = D_{z_0}(R) \setminus \{z_0\}$. I.e.

$$f(z) = \sum_{-\infty}^{+\infty} a_n (z - z_0)^n \qquad \text{for every } z \in D_{z_0}(R) \setminus \{z_0\}.$$

Definition. *Let z_0 be an isolated singularity of f and let $\sum_{-\infty}^{+\infty} a_n (z - z_0)^n$ be the Laurent series of f in $D_{z_0}(R) \setminus \{z_0\}$.*
If $a_n = 0$ for every $n < 0$, then we say that z_0 is a **removable singularity** *of f.*
If $a_n \neq 0$ for at least one $n < 0$ and there are only finitely many $n < 0$ such that $a_n \neq 0$, then we say that z_0 is a **pole** *of f.*
If $a_n \neq 0$ for infinitely many $n < 0$, then we say that z_0 is an **essential singularity** *of f.*

Let us start with the case of a removable singularity z_0. Then

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \text{for every } z \in D_{z_0}(R) \setminus \{z_0\}.$$

The power series $\sum_{n=0}^{+\infty} a_n (z - z_0)^n$ converges at every $z \in D_{z_0}(R)$ and defines a holomorphic function in $D_{z_0}(R)$ with value a_0 at z_0. The function f may not be defined at z_0 or it may be defined at z_0 with a value $f(z_0)$ either equal to a_0 or not equal to a_0. Now, in any case, we define (or redefine) f at z_0 to be $f(z_0) = a_0$. Then we have

$$f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \qquad \text{for every } z \in D_{z_0}(R).$$

and f becomes holomorphic in $D_{z_0}(R)$.

We summarize. *If $z_0 \in \Omega$ is a removable singularity of f, then f can be defined (or redefined) appropriately at z_0 so that it becomes holomorphic in a disc with center z_0. The Laurent series* *of f at z_0 reduces to a power series of first type and this power series is the Taylor series of the (extended) f in a disc with center z_0.*

Here is a useful test to decide if an isolated singularity is removable without calculating the Laurent series of the function.

Riemann's criterion. *Let z_0 be an isolated singularity of f. If $\lim_{z\to z_0}(z-z_0)f(z)=0$, then z_0 is a removable singularity of f.*

Proof. Let $f(z)=\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ for every $z\in D_{z_0}(R)\setminus\{z_0\}$. We take any $\epsilon>0$ and then there is $\delta>0$ so that $|z-z_0||f(z)|\leq\epsilon$ for every $z\in D_{z_0}(R)$ with $0<|z-z_0|<\delta$. Now, we consider any r with $0<r<\min\{\delta,R,1\}$ and any $n<0$. Then we have

$$|a_n|=\Big|\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta\Big|\leq\frac{1}{2\pi}\frac{\epsilon}{r^{n+2}}\,2\pi r=\epsilon r^{-n-1}=\epsilon r^{|n|-1}\leq\epsilon.$$

Since $\epsilon>0$ is arbitrary, we get $a_n=0$ for every $n<0$ and z_0 is a removable singularity of f. □

In the case of an isolated singularity z_0 for f, sometimes we know that the $\lim_{z\to z_0}f(z)$ exists and it is finite or that f is bounded close to z_0. In both cases we have that $\lim_{z\to z_0}(z-z_0)f(z)=0$ is satisfied and we conclude that z_0 is a removable singularity of f.

Example 6.9.1. The function $f(z)=\frac{z^2-3z+2}{z-2}$ is holomorphic in $\mathbb{C}\setminus\{2\}$. Since $\lim_{z\to 2}f(z)=1$, the point 2 is a removable singularity of f. If we define $f(2)=1$, then f, now defined in $\mathbb{C}$, is holomorphic in $\mathbb{C}$. The formula of the extended f is

$$f(z)=\begin{cases}\frac{z^2-3z+2}{z-2}, & \text{if } z\neq 2\\ 1, & \text{if } z=2\end{cases}=\begin{cases}z-1, & \text{if } z\neq 2\\ 1, & \text{if } z=2\end{cases}$$

I.e. the extended f is the simple function $z-1$ in $\mathbb{C}$.
To find the Laurent series of the original f with center 2 we write $f(z)=\frac{z^2-3z+2}{z-2}=z-1=1+(z-2)$ for every $z\in D_2(0,+\infty)$. The Laurent series has no negative powers of $z-2$ and we see again that 2 is a removable singularity of f.

Now we consider the case of a pole z_0 of f. Let $\sum_{-\infty}^{+\infty}a_n(z-z_0)^n$ be the Laurent series of f in the ring $D_{z_0}(R)\setminus\{z_0\}$ and then there is a largest $m\geq 1$ so that $a_{-m}\neq 0$. Let N be this largest m. Then we have

$$f(z)=\frac{a_{-N}}{(z-z_0)^N}+\cdots+\frac{a_{-1}}{z-z_0}+\sum_{n=0}^{+\infty}a_n(z-z_0)^n\qquad\text{for every } z\in D_{z_0}(R)\setminus\{z_0\}$$

with $a_{-N}\neq 0$. We may write this as

$$f(z)=\frac{1}{(z-z_0)^N}\sum_{n=0}^{+\infty}a_{n-N}(z-z_0)^n\qquad\text{for every } z\in D_{z_0}(R)\setminus\{z_0\}.$$

Since the power series $\sum_{n=0}^{+\infty}a_{n-N}(z-z_0)^n$ converges in the disc $D_{z_0}(R)$, it defines a holomorphic function $g:D_{z_0}(R)\to\mathbb{C}$ and we have

$$f(z)=\frac{g(z)}{(z-z_0)^N}\qquad\text{for every } z\in D_{z_0}(R)\setminus\{z_0\}.$$

Observe that

$$g(z_0)=a_{-N}\neq 0.$$

Definition. *Let $z_0\in\Omega$ be a pole of f and let N be the largest $m\geq 1$ such that $a_{-m}\neq 0$. Then we say that z_0 is a pole of f of order N or of multiplicity N.*

We saw that, if z_0 is a pole of f order N, then there is a function g holomorphic in some disc $D_{z_0}(R)$ so that

$$g(z_0) \neq 0 \qquad \text{and} \qquad f(z) = \frac{g(z)}{(z-z_0)^N} \quad \text{for every } z \in D_{z_0}(R) \setminus \{z_0\}. \tag{6.15}$$

It is easy to see the converse. Indeed, let g be holomorphic in $D_{z_0}(R)$ and let (6.15) hold. We consider the Taylor series $\sum_{n=0}^{+\infty} b_n(z-z_0)^n$ of g and then for $z \in D_{z_0}(R) \setminus \{z_0\}$ we have

$$f(z) = \frac{1}{(z-z_0)^N} \sum_{n=0}^{+\infty} b_n(z-z_0)^n = \frac{b_0}{(z-z_0)^N} + \cdots + \frac{b_{N-1}}{z-z_0} + \sum_{n=0}^{+\infty} b_{n+N}(z-z_0)^n.$$

The last power series is the Laurent series of f in $D_{z_0}(R) \setminus \{z_0\}$ and since $b_0 = g(z_0) \neq 0$, we have that z_0 is a pole of f of order N.

Since $g(z_0) \neq 0$ and g is continuous at z_0, we have that g does not vanish at any point of some disc $D_{z_0}(r)$ with $0 < r \leq R$. Then $h(z) = \frac{1}{g(z)}$ is holomorphic in $D_{z_0}(r)$ and (6.15) implies that $\frac{1}{f(z)} = (z-z_0)^N h(z)$ for every $z \in D_{z_0}(r) \setminus \{z_0\}$. Therefore, z_0 is a removable singularity of $\frac{1}{f}$. Moreover, if we define $\frac{1}{f}$ to take the value 0 at z_0, then we have $\frac{1}{f}(z) = (z-z_0)^N h(z)$ for every $z \in D_{z_0}(r)$ and, since $h(z_0) \neq 0$, then z_0 is a root of the extended $\frac{1}{f}$ of multiplicity N. It is easy to prove in a similar way the converse, and we conclude that z_0 *is a pole of f of order N if and only if it is a root of $\frac{1}{f}$ of mutiplicity N.*

Example 6.9.2. Many times we meet functions of the form $f = \frac{p}{q}$, where p and q are holomorphic in a neighborhood of z_0. If p and q are polynomials, then f is a rational function.
Let z_0 be a root of p and q of multiplicity $M \geq 0$ and $N \geq 0$, respectively. In this case we saw that there are holomorphic functions p_1 and q_1 in a neighborhood $D_{z_0}(R)$ of z_0 so that $p(z) = (z-z_0)^M p_1(z)$ and $q(z) = (z-z_0)^N q_1(z)$ for every $z \in D_{z_0}(R)$ and also $p_1(z_0) \neq 0$ and $q_1(z_0) \neq 0$. (Of course we consider the case that none of p, q is identically 0.) Then there is r with $0 < r \leq R$ so that $p_1(z) \neq 0$ and $q_1(z) \neq 0$ for every $z \in D_{z_0}(r)$, and then we have

$$f(z) = \frac{p(z)}{q(z)} = (z-z_0)^{M-N} \frac{p_1(z)}{q_1(z)} = (z-z_0)^{M-N} g(z) \qquad \text{for every } z \in D_{z_0}(r) \setminus \{z_0\},$$

where the function $g(z) = \frac{p_1(z)}{q_1(z)}$ is holomorphic in $D_{z_0}(r)$ and $g(z_0) = \frac{p_1(z_0)}{q_1(z_0)} \neq 0$. Now we have two cases. If $M \geq N$, then z_0 is a removable singularity of f, and f (after we extend it appropriately at z_0) is holomorphic at z_0 and z_0 is a root of f of multiplicity $M - N$. If $M < N$, then z_0 is a pole of order $N - M$ of f.

Here are some concrete instances of this example.

Example 6.9.3. The function $f(z) = \frac{z^2-3z+2}{(z-2)^2}$ is holomorphic in $\mathbb{C} \setminus \{2\}$.
Since $z^2 - 3z + 2 = (z-2)(z-1)$, we have $f(z) = \frac{z-1}{z-2}$ for $z \neq 2$. The function $g(z) = z - 1$ is holomorphic in $\mathbb{C}$ and $g(2) = 1 \neq 0$. Therefore, 2 is a pole of f of order 1. To find the Laurent series of f in $D_2(0, +\infty)$ we write $f(z) = \frac{1+(z-2)}{z-2} = \frac{1}{z-2} + 1$ and the Laurent series is $\frac{1}{z-2} + 1$.

Example 6.9.4. The function $f(z) = \frac{e^z-1}{z^3}$ is holomorphic in $\mathbb{C} \setminus \{0\}$.
The Taylor series of $e^z - 1$ with center 0 is $z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots$. Hence $e^z - 1 = zg(z)$ with $g(z) = 1 + \frac{1}{2!} z + \frac{1}{3!} z^2 + \cdots$. The function g is holomorphic in $\mathbb{C}$ and $g(0) = 1 \neq 0$ and we have $f(z) = \frac{g(z)}{z^2}$ for $z \neq 0$. Therefore, 0 is a pole of f of order 2. The Laurent series of f in $D_0(0, +\infty)$ is $\frac{1}{z^2} + \frac{1/(2!)}{z} + \frac{1}{3!} + \frac{1}{4!} z + \cdots$.

Example 6.9.5. The function $\cot z = \frac{\cos z}{\sin z}$ is holomorphic in $\mathbb{C} \setminus \{k\pi \,|\, k \in \mathbb{Z}\}$.
The points $k\pi$, $k \in \mathbb{Z}$, are isolated singularities of $\cot z$ and we shall prove that they are all poles

of order 1. We fix $k \in \mathbb{Z}$. The Taylor series of $\sin z$ with center $k\pi$ results from the Taylor series of $\sin z$ with center 0, as follows

$$\begin{aligned}\sin z &= \sin((z-k\pi)+k\pi) = \cos k\pi \sin(z-k\pi) = (-1)^k \sin(z-k\pi)\\ &= (-1)^k\Big((z-k\pi) - \frac{1}{3!}(z-k\pi)^3 + \cdots\Big) = (-1)^k(z-k\pi) - \frac{(-1)^k}{3!}(z-k\pi)^3 + \cdots .\end{aligned}$$

Therefore, $\sin z = (z-k\pi)q_1(z)$ for every z, where the function q_1 is holomorphic in $\mathbb{C}$ with $q_1(k\pi) = (-1)^k$. Hence, $\cot z = \frac{\cos z}{(z-k\pi)q_1(z)} = \frac{g(z)}{z-k\pi}$ with $g(z) = \frac{\cos z}{q_1(z)}$ and g is holomorphic in the disc $D_{k\pi}(\pi)$ and $g(k\pi) = \frac{\cos k\pi}{q_1(k\pi)} = 1$. Therefore, $k\pi$ is a pole of $\cot z$ of order 1.
(Observe that $D_{k\pi}(\pi)$ is the largest open disc with center $k\pi$ which is contained in the domain of holomorphy of g because it is the largest open disc with center $k\pi$ which does not contain any root of q_1. This is true because $q_1(z) = \frac{\sin z}{z-k\pi}$ vanishes at every $l\pi$ with $l \in \mathbb{Z}$, $l \neq k$.)
The Laurent series of $\cot z$ in $D_{k\pi}(0,\pi)$ is $\cot z = \frac{1}{z-k\pi} + g'(k\pi) + \frac{1}{2}g''(k\pi)(z-k\pi) + \cdots$.

For the determination of poles there is a criterion similar to the criterion of Riemann for removable singularities.

Proposition 6.4. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of f if and only if* $\lim_{z\to z_0} f(z) = \infty$.

Proof. There is a disc $D_{z_0}(R)$ so that f is holomorphic in $D_{z_0}(R) \setminus \{z_0\}$.
If z_0 is a pole of order N of f, then we saw that there is a function g holomorphic in $D_{z_0}(R)$ so that $g(z_0) \neq 0$ and $f(z) = \frac{g(z)}{(z-z_0)^N}$ for every $z \in D_{z_0}(R) \setminus \{z_0\}$. This implies $\lim_{z\to z_0} f(z) = \infty$.
Conversely, let $\lim_{z\to z_0} f(z) = \infty$. Then there is r with $0 < r \leq R$ so that $f(z) \neq 0$ for every $z \in D_{z_0}(r) \setminus \{z_0\}$. Then the function

$$h(z) = \frac{1}{f(z)} \qquad \text{for every } z \in D_{z_0}(r) \setminus \{z_0\} \tag{6.16}$$

is holomorphic in $D_{z_0}(r)\setminus\{z_0\}$. Since $\lim_{z\to z_0} h(z) = \lim_{z\to z_0} \frac{1}{f(z)} = 0$, the criterion of Riemann implies that z_0 is a removable singularity of h. Therefore, we may define h appropriately at z_0 so that it becomes holomorphic in $D_{z_0}(r)$: we set $h(z_0) = \lim_{z\to z_0} h(z) = 0$. It is clear that z_0 is the only root of (the extended) h in $D_{z_0}(r)$ and, if N is the multiplicity of this root, then

$$h(z) = (z-z_0)^N h_1(z) \qquad \text{for every } z \in D_{z_0}(r), \tag{6.17}$$

where h_1 is holomorphic in $D_{z_0}(r)$ and has no root in $D_{z_0}(r)$. Thus, the function

$$g(z) = \frac{1}{h_1(z)} \qquad \text{for } z \in D_{z_0}(r) \tag{6.18}$$

is holomorphic in $D_{z_0}(r)$ and, clearly, has no root in $D_{z_0}(r)$. Now (6.16), (6.17) and (6.18) imply $f(z) = \frac{g(z)}{(z-z_0)^N}$ for every $z \in D_{z_0}(r) \setminus \{z_0\}$ with $g(z_0) \neq 0$ and z_0 is a pole of f of order N. $\square$

There is one more test for the case of a pole which also determines the exact order of the pole.

Proposition 6.5. *Let z_0 be an isolated singularity of f. Then z_0 is a pole of f of order $N \geq 1$ if and only if the* $\lim_{z\to z_0}(z-z_0)^N f(z)$ *exists and it is finite and* $\neq 0$.

Proof. If z_0 is a pole of f of order N, then we repeat the beginning of the proof of proposition 6.4 and we get that $\lim_{z\to z_0}(z-z_0)^N f(z) = \lim_{z\to z_0} g(z) = g(z_0) \neq 0$.
Conversely, let $\lim_{z\to z_0}(z-z_0)^N f(z)$ be finite and $\neq 0$. Riemann's criterion implies that the function $g(z) = (z-z_0)^N f(z)$, which is holomorphic in some ring $D_{z_0}(R)\setminus\{z_0\}$, can be extended at z_0 by setting $g(z_0) = \lim_{z\to z_0} g(z) = \lim_{z\to z_0}(z-z_0)^N f(z) \neq 0$, and the extended g is holomorphic in $D_{z_0}(R)$. Therefore, there is a g holomorphic in $D_{z_0}(R)$ with $g(z_0) \neq 0$ so that $f(z) = \frac{g(z)}{(z-z_0)^N}$ for every $z \in D_{z_0}(R) \setminus \{z_0\}$ and z_0 is a pole of f of order N. $\square$

Finally, for the case of an essential singularity we have the following result.

Proposition 6.6. *Let z_0 be an isolated singularity of f. Then z_0 is an essential singularity of f if and only if the* $\lim_{z\to z_0} f(z)$ *does not exist.*

Proof. By the criterion of Riemann, z_0 is a removable singularity if and only if the $\lim_{z\to z_0} f(z)$ exists and it is finite. Proposition 6.4 says that z_0 is a pole if and only if $\lim_{z\to z_0} f(z) = \infty$. $\square$

Example 6.9.6. In example 6.7.8 we saw that $\sum_{-\infty}^{n=-1} \frac{1}{(-n)!} z^n + 1$ is the Laurent series of $e^{\frac{1}{z}}$ in $D_0(0, +\infty)$. Hence 0 is an essential singularity of $e^{\frac{1}{z}}$.
Therefore, the $\lim_{z\to 0} e^{\frac{1}{z}}$ does not exist. We can see this without proving first that 0 is an essential singularity of $e^{\frac{1}{z}}$. In fact, proving that the $\lim_{z\to 0} e^{\frac{1}{z}}$ does not exist is another way to see that 0 is an essential singularity of $e^{\frac{1}{z}}$. Indeed, if $z = x$ tends to 0 on the positive x-semiaxis, then $|e^{\frac{1}{z}}| = e^{\frac{1}{x}} \to +\infty$, and hence $e^{\frac{1}{z}} \to \infty$. If $z = x$ tends to 0 on the negative x-semiaxis, then $|e^{\frac{1}{z}}| = e^{\frac{1}{x}} \to 0$, and hence $e^{\frac{1}{z}} \to 0$. Thus, the $\lim_{z\to 0} e^{\frac{1}{z}}$ does not exist.

Definition. *Let z_0 be an isolated singularity of f and let $\sum_{-\infty}^{+\infty} a_n(z - z_0)^n$ be the Laurent series of f in the ring $D_{z_0}(0, R) = D_{z_0}(R) \setminus \{z_0\}$. Then $\sum_{-\infty}^{n=-1} a_n(z - z_0)^n$ is called the* **singular part** *of the Laurent series of f or, simply, the singular part of f at z_0. Also, $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$ is called the* **regular part** *of the Laurent series of f or, simply, the regular part of f at z_0.*

We have seen that in the case of a removable singularity z_0 the singular part of f at z_0 is zero and the Laurent series of f at z_0 consists only of its regular part. In the case of a pole z_0 of f of order N the singular part at z_0 is a finite sum of the form $\sum_{n=1}^{N} \frac{a_{-n}}{(z-z_0)^n}$ with $a_{-N} \neq 0$. In this case the singular part is a *rational function* whose denominator is $(z - z_0)^N$. In the case of an essential singularity z_0 the singular part at z_0 has infinitely many terms.

We should note that the power series $\sum_{-\infty}^{n=-1} a_n(z-z_0)^n$, i.e. the singular part of f, converges in $D_{z_0}(0, +\infty) \cup \{\infty\} = \widehat{\mathbb{C}} \setminus \{z_0\}$. Its value at ∞ is 0.

If we subtract from f its singular part at its singularity z_0, then we get

$$f(z) - \sum_{-\infty}^{n=-1} a_n(z - z_0)^n = \sum_{n=0}^{+\infty} a_n(z - z_0)^n,$$

which is a power series of first type and hence converges in the disc $D_{z_0}(R)$, including the center z_0. Therefore, z_0 is a removable singularity of the function $F(z) = f(z) - \sum_{-\infty}^{n=-1} a_n(z - z_0)^n$ and if we define F to have value $F(z_0) = a_0$ at z_0, then this function is holomorphic in $D_{z_0}(R)$.

We shall now establish the well known *analysis of a rational function into a sum of simple fractions*.

Proposition 6.7. *Let $r = \frac{p}{q}$ be a rational function. We assume that the polynomials p, q have no common roots (and hence no common factors), that the degree of p is n, the degree of q is m and that $z_1, \ldots, z_k$ are the roots of q with corresponding multiplicities $m_1, \ldots, m_k$. Then*

$$r(z) = p_1\Big(\frac{1}{z - z_1}\Big) + \cdots + p_k\Big(\frac{1}{z - z_k}\Big) + p_0(z),$$

where $p_1, \ldots, p_k$ are polynomials without constant terms and of degrees $m_1, \ldots, m_k$, respectively, and p_0 is either the null polynomial, if $n < m$, or a polynomial of degree $n - m$, if $n \geq m$.

Proof. We saw in example 6.9.2 that every z_j is a pole of r of degree m_j. Therefore, the singular part of r at z_j has the form $\sum_{l=1}^{m_j} \frac{a_{-l}}{(z-z_0)^l}$ with $a_{-m_j} \neq 0$. This has the form $p_j(\frac{1}{z-z_j})$, where p_j is the polynomial $p_j(z) = \sum_{l=1}^{m_j} a_{-l} z^l$ without constant term and of degree m_j. Now we subtract from r all its singular parts at the points $z_1, \ldots, z_k$ and consider the function

$$p_0(z) = r(z) - \Big(p_1\Big(\frac{1}{z - z_1}\Big) + \cdots + p_k\Big(\frac{1}{z - z_k}\Big)\Big).$$

This function is a rational function defined in the set $\mathbb{C} \setminus \{z_1, \ldots, z_k\}$ and its only possible poles are the points $z_1, \ldots, z_k$. We observe, though, that every z_j is a removable singularity of $r(z) - p_j(\frac{1}{z-z_j})$ and that all terms $p_1(\frac{1}{z-z_1}), \ldots, p_k(\frac{1}{z-z_k})$, besides $p_j(\frac{1}{z-z_j})$, are holomorphic at z_j. Therefore, every z_j is a removable singularity of the function p_0. In other words, the rational function p_0 has *no poles* and hence it is a polynomial. Now, we have the identity

$$r(z) = p_1\Big(\frac{1}{z - z_1}\Big) + \cdots + p_k\Big(\frac{1}{z - z_k}\Big) + p_0(z)$$

and we consider two cases. If $n < m$, then $\lim_{z\to\infty} r(z) = 0$ and, since $\lim_{z\to\infty} p_j(\frac{1}{z-z_j}) = 0$ for every j, we have that $\lim_{z\to\infty} p_0(z) = 0$. Thus, p_0 is the null polynomial. If $n \geq m$, then $c = \lim_{z\to\infty} \frac{r(z)}{z^{n-m}}$ is a complex number $\neq 0$. Since $\lim_{z\to\infty} p_j(\frac{1}{z-z_j})/z^{n-m} = 0$ for every j, we have that $\lim_{z\to\infty} \frac{p_0(z)}{z^{n-m}} = c \neq 0$. Thus the polynomial p_0 has degree $n - m$. $\square$

Exercises.

6.9.1. Is 0 an isolated singularity of $\frac{1}{\sin(1/z)}$?

6.9.2. Find the isolated (non-removable) singularities of: $\frac{1}{z^2+5z+6}$, $\frac{1}{(z^2-1)^2}$, $\frac{e^z-1}{z}$, $\frac{e^z-1}{z^3}$, $\frac{z^2}{\sin z}$, $\frac{1}{\sin z}$, $\tan z$, $\frac{1}{\sin^2 z}$, $e^z + e^{1/z}$, $\frac{1}{e^z-1}$. Which of the singularities are poles and what is their order?

6.9.3. Find the initial four terms of the Laurent series at 0 of the functions: $\cot z$, $\frac{1}{\sin z}$, $\frac{z}{\sin^2 z}$, $\frac{1}{e^z-1}$.

6.9.4. Prove that an isolated singularity of $f(z)$ cannot be a pole of $e^{f(z)}$.

6.9.5. Let z_0 be an isolated singularity of f, which is not constant in any neighborhood of z_0. If there is $s \in \mathbb{R}$ so that $\lim_{z\to z_0} |z - z_0|^s |f(z)| \in [0, +\infty]$, prove that z_0 is either a removable singularity or a pole of f and that there is $m \in \mathbb{Z}$ so that

$$\lim_{z\to z_0} |z - z_0|^s |f(z)| \begin{cases} = 0, & \text{if } s > m \\ = +\infty, & \text{if } s < m \\ \in (0, +\infty), & \text{if } s = m \end{cases}$$

6.9.6. Let f be holomorphic in $D_{z_0}(R) \setminus \{z_0\}$ and let either $\mathrm{Re}\, f$ or $\mathrm{Im}\, f$ be bounded either from above or from below in $D_{z_0}(R) \setminus \{z_0\}$. Prove that z_0 is a removable singularity of f.

6.9.7. Let f be holomorphic in $D_0(R) \setminus \{z_0\}$, where $R > 1$ and $|z_0| = 1$, and let z_0 be a pole of f. If $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is the Taylor series of f in $D_0(1)$, prove that $\frac{a_n}{a_{n+1}} \to z_0$.

6.9.8. Let Ω be a region so that every point of Ω is either a point of holomorphy or an isolated singularity of f. If the roots of f have an accumulation point in Ω, which is not an essential singularity of f, prove that f is identically 0 in Ω.

6.9.9. (i) Let z_0 be an essential singularity of f and let $w \in \mathbb{C}$. Prove that for every $r > 0$ the function $\frac{1}{f-w}$ *is not* bounded in $D_{z_0}(r) \setminus \{z_0\}$.
(ii) Prove the **Casorati-Weierstrass theorem**. If z_0 is an essential singularity of f, then for every w there is a sequence (z_n) with $z_n \to z_0$ and $z_n \neq z_0$ for every n so that $f(z_n) \to w$.

6.9.10. (i) Prove that every $2k\pi i$, $k \in \mathbb{Z}$, is a pole of $\frac{1}{e^z-1}$ of order 1.
(ii) Prove that $\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$ for $|z| < 2\pi$, where the numbers B_k satisfy the recursive relations $\frac{1}{(2k+1)!} - \frac{1}{2(2k)!} + \sum_{\nu=1}^{k} \frac{(-1)^{\nu-1} B_\nu}{(2\nu)!(2k-2\nu+1)!} = 0$, $k \geq 1$. Evaluate B_1, B_2, B_3. The numbers B_k are called Bernoulli constants.

6.9.11. Look at exercises 6.7.7 and 6.8.11. We shall extend what we said in this section to the case of the point ∞.
(i) We say that ∞ is an isolated singularity of f if f is holomorphic in some ring $D_0(R, +\infty)$. Let $\sum_{-\infty}^{+\infty} a_n z^n$ be the Laurent series of f in this ring. If $a_n = 0$ for every $n \geq 1$, then we say that ∞ is a removable singularity of f. If $a_n \neq 0$ for at least one $n \geq 1$ and for only finitely many $n \geq 1$, then we say that ∞ is a pole of f. Finally, if $a_n \neq 0$ for infinitely many $n \geq 1$, then we say that ∞ is an essential singularity of f.
Prove that ∞ is a removable singularity of f if and only if $\lim_{z\to\infty} \frac{f(z)}{z} = 0$.
Prove that ∞ is a pole of f if and only if $\lim_{z\to\infty} f(z) = \infty$.
Let ∞ be a pole of f and let N be the largest $n \geq 1$ with $a_n \neq 0$. Then we say that ∞ is a pole of f of order N. Prove that ∞ is a pole of f of order N if and only if there is a g holomorphic in $D_0(R, +\infty) \cup \{\infty\}$ so that $g(\infty) \neq 0$ and $f(z) = z^N g(z)$ for every $z \in D_0(R, +\infty)$. Moreover, prove that ∞ is a pole of f of order N if and only if the $\lim_{z\to\infty} \frac{f(z)}{z^N}$ exists and it is finite and $\neq 0$.
Prove that ∞ is an essential singularity of f if and only if the $\lim_{z\to\infty} f(z)$ does not exist.
(ii) Let $r = \frac{p}{q}$ be a rational function and let n be the degree of the polynomial p and m be the degree of the polynomial q. Prove that ∞ is a removable singularity of r if $m \geq n$ and that it is a pole of r of order $n - m$ if $n > m$. In particular, a polynomial p of degree $n \geq 1$ has a pole of order n at ∞.
(iii) What kind of an isolated singularity is ∞ for the functions e^z, $e^{\frac{1}{z}}$, $z^2 e^{\frac{1}{z}}$, $\sin z$, $\sin \frac{1}{z}$, $z^5 \sin \frac{1}{z}$?
(iv) What kind of an isolated singularity is ∞ for any holomorphic branch of $((z-1)(z-2))^{\frac{1}{2}}$ in the region $\mathbb{C} \setminus [1, 2]$.
(v) Is ∞ an isolated singularity of $\frac{1}{\sin z}$ or of $\tan z$?

6.10 The open mapping theorem.

Open mapping theorem. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω. If f is not constant in Ω, then $f(U)$ is open for every open $U \subseteq \Omega$.*

Proof. Let $U \subseteq \Omega$ be open. We shall prove that $f(U)$ is also open, i.e. that every $w_0 \in f(U)$ is an interior point of $f(U)$.
Since $w_0 \in f(U)$ there is some $z_0 \in U$ so that $f(z_0) = w_0$. Since U is open, there is $r > 0$ so that $\overline{D}_{z_0}(r) \subseteq U$. Since f is non-constant in Ω, the solution z_0 of the equation $f(z) = w_0$ is isolated. Therefore, we may take r small enough so that $f(z) = w_0$ has no solution in $\overline{D}_{z_0}(r)$ except z_0. This means that $f(z) \neq w_0$ for every $z \in \overline{D}_{z_0}(r) \setminus \{z_0\}$. In particular, $f(z) \neq w_0$ for every $z \in C_{z_0}(r)$. Now, the function $|f(z) - w_0|$ is continuous and real-valued and the circle $C_{z_0}(r)$ is compact. Therefore, the restriction of $|f(z) - w_0|$ in $C_{z_0}(r)$ attains a minimum value at some point of $C_{z_0}(r)$ which is a *positive number*. We denote ϵ this minimum and we have:

$$\epsilon = \min_{z \in C_{z_0}(r)} |f(z) - w_0| > 0. \tag{6.19}$$

Now, we consider any $w \notin f(\overline{D}_{z_0}(r))$. Again, the function $|f(z) - w|$ is continuous and real-valued and the disc $\overline{D}_{z_0}(r)$ is compact. Therefore the restriction of $|f(z) - w|$ in $\overline{D}_{z_0}(r)$ attains a minimum value at some point of $\overline{D}_{z_0}(r)$ which is positive. But now we can say more: the function $\frac{1}{f(z)-w}$ is *holomorphic* in $D_{z_0}(r)$ and continuous in $\overline{D}_{z_0}(r)$. The second version of the maximum principle implies that the function $\frac{1}{|f(z)-w|}$, restricted in $\overline{D}_{z_0}(r)$, attains its maximum value at the boundary $C_{z_0}(r)$. Equivalently, the function $|f(z) - w|$, restricted in $\overline{D}_{z_0}(r)$, attains its minimum value at the boundary $C_{z_0}(r)$. Since one of the values of $|f(z) - w|$ in $\overline{D}_{z_0}(r)$ is $|f(z_0) - w| = |w_0 - w|$, we get

$$|w_0 - w| \geq \min_{z \in C_{z_0}(r)} |f(z) - w|. \tag{6.20}$$

Now, we have

$$|f(z) - w| \geq |f(z) - w_0| - |w_0 - w|. \tag{6.21}$$

We combine (6.19), (6.20) and (6.21) and get

$$|w_0 - w| \geq \min_{z \in C_{z_0}(r)} |f(z) - w| \geq \min_{z \in C_{z_0}(r)} |f(z) - w_0| - |w_0 - w| = \epsilon - |w_0 - w|.$$

Thus

$$|w_0 - w| \geq \frac{\epsilon}{2}.$$

We have proved that any $w \notin f(\overline{D}_{z_0}(r))$ satisfies $|w_0 - w| \geq \frac{\epsilon}{2}$. This implies that every $w \in D_{w_0}(\frac{\epsilon}{2})$ belongs to $f(\overline{D}_{z_0}(r))$. I.e.

$$D_{w_0}\Big(\frac{\epsilon}{2}\Big) \subseteq f(\overline{D}_{z_0}(r)) \subseteq f(U)$$

and w_0 is an interior point of $f(U)$. □

Exercises.

6.10.1. Prove the first maximum principle using the open mapping theorem.

6.11 Local mapping properties.

Proposition 6.8. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and let $z_0 \in \Omega$ with $f'(z_0) \neq 0$. Then there is an open set $U \subseteq \Omega$ containing z_0 so that $W = f(U)$ is an open set containing $w_0 = f(z_0)$ and the function $f : U \to W$ is one-to-one. Moreover, the function $f^{-1} : W \to U$ is holomorphic in W.*

Proof. We consider the Taylor series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ of f in its disc of convergence $D_{z_0}(R) \subseteq \Omega$. We know that the differentiated series $\sum_{n=1}^{+\infty} na_n(z-z_0)^{n-1}$ converges absolutely in the same disc, i.e. $\sum_{n=1}^{+\infty} n|a_n||z-z_0|^{n-1} < +\infty$ for every $z \in D_{z_0}(R)$. Therefore, the power series $\sum_{n=1}^{+\infty} n|a_n|(z-z_0)^{n-1}$ converges in the disc $D_{z_0}(R)$ and defines a continuous function in this disc. In particular, we have that $\lim_{z\to z_0} \sum_{n=1}^{+\infty} n|a_n|(z-z_0)^{n-1} = |a_1|$ or, equivalently,

$$\lim_{z\to z_0} \sum_{n=2}^{+\infty} n|a_n|(z-z_0)^{n-1} = 0.$$

Since $a_1 = f'(z_0) \neq 0$, there is a small enough $r > 0$ so that

$$\sum_{n=2}^{+\infty} n|a_n|r^{n-1} < |a_1|.$$

We shall see now that $f : D_{z_0}(r) \to \mathbb{C}$ is one-to-one. Assume that this is not the case and that there are $z_1, z_2 \in D_{z_0}(r)$ so that $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Then $\sum_{n=0}^{+\infty} a_n(z_1-z_0)^n = \sum_{n=0}^{+\infty} a_n(z_2-z_0)^n$ and hence

$$a_1 = \sum_{n=2}^{+\infty} a_n((z_1-z_0)^{n-1} + (z_1-z_0)^{n-2}(z_2-z_0) + \cdots + (z_1-z_0)(z_2-z_0)^{n-2} + (z_2-z_0)^{n-1}).$$

This implies $|a_1| \leq \sum_{n=2}^{+\infty} n|a_n|r^{n-1}$ and we arrive at a contradiction. Since f' is continuous at z_0 and $f'(z_0) \neq 0$, by taking a smaller $r > 0$ if necessary, we may suppose that $f'(z) \neq 0$ for every $z \in D_{z_0}(r)$. Now we take $U = D_{z_0}(r)$. From the open mapping theorem we have that the set $W = f(U) = f(D_{z_0}(r))$ is open and we have proved that $f : U \to W$ is one-to-one and hence the inverse mapping $f^{-1}: W \to U$ is defined. Now it is easy to see that this inverse mapping is continuous in W. Indeed let $w \in W$. Then there is (a unique) $z \in U$ so that $f(z) = w$. We take any $\epsilon > 0$ small enough so that $D_z(\epsilon) \subseteq U$. Then the set $f(D_z(\epsilon))$ is open and contains w. Hence there is $\delta > 0$ so that $D_w(\delta) \subseteq f(D_z(\epsilon))$. Then for every $w' \in D_w(\delta)$ the (unique) $z' \in U$ which satisfies $f(z') = w'$ is contained in $D_z(\epsilon)$. This says that for every $w' \in W$ with $|w' - w| < \delta$ we have $|f^{-1}(w') - f^{-1}(w)| = |z' - z| < \epsilon$ and the function $f^{-1}: W \to U$ is continuous at every $w \in W$. Now, proposition 4.4 implies that $f^{-1}: W \to U$ is holomorphic in W. □

Theorem 6.2. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the region Ω and let $z_0 \in \Omega$ and $w_0 = f(z_0)$. Let z_0 be a solution of $f(z) = w_0$ of multiplicity N. Then there is an open set $U \subseteq \Omega$ containing z_0 so that $W = f(U)$ is an open set containing $w_0 = f(z_0)$ and the function $f : U \to W$ is N-to-one.*

Proof. We know that there is a disc $D_{z_0}(R)$ and a function g holomorphic in $D_{z_0}(R)$ so that $f(z) - w_0 = (z - z_0)^N g(z)$ for every $z \in D_{z_0}(R)$ and $g(z_0) \neq 0$. By the continuity of g we have that there is $r \leq R$ so that $g(z) \neq 0$ for every $z \in D_{z_0}(r)$. Hence the function $\frac{g'}{g}$ is holomorphic in $D_{z_0}(r)$ and the theorem of Cauchy in convex regions implies that $\oint_\gamma \frac{g'(z)}{g(z)}\, dz = 0$ for every closed curve γ in $D_{z_0}(r)$. Now, proposition 5.24 implies that there is a holomorphic branch of $\log g$ in $D_{z_0}(r)$, i.e. that there is a holomorphic $k : D_{z_0}(r) \to \mathbb{C}$ so that $e^{k(z)} = g(z)$ for every $z \in D_{z_0}(r)$. Now the function $\phi : D_{z_0}(r) \to \mathbb{C}$ defined by $\phi(z) = e^{k(z)/N}$ for every $z \in D_{z_0}(r)$ is a holomorphic branch of $g^{1/N}$ in $D_{z_0}(r)$. Indeed, we have $\phi(z)^N = e^{k(z)} = g(z)$ for every $z \in D_{z_0}(r)$.
We have proved that there is a function $\phi : D_{z_0}(r) \to \mathbb{C}$ holomorphic in $D_{z_0}(r)$ so that $\phi(z)^N = g(z)$ for every $z \in D_{z_0}(r)$.
Now we consider the function $h : D_{z_0}(r) \to \mathbb{C}$ defined by $h(z) = (z - z_0)\phi(z)$. This is holomorphic in $D_{z_0}(r)$ and we have that $f(z) - w_0 = h(z)^N$ for every $z \in D_{z_0}(r)$. Moreover, $h'(z) = \phi(z) + (z - z_0)\phi'(z)$ and hence $h'(z_0) = \phi(z_0) \neq 0$. Therefore, we may apply proposition 6.8 and get that there is an open set $U_0 \subseteq D_{z_0}(r)$ containing z_0 so that $W_0 = h(U_0)$ is an open set containing $h(z_0) = 0$ and the function $h : U_0 \to W_0$ is one-to-one. Now, we consider a disc $D_0(r_0) \subseteq W_0$ and the open set $U = h^{-1}(D_0(r_0)) \subseteq U$. Then $h : U \to D_0(r_0)$ is holomorphic in U, onto $D_0(r_0)$ and one-to-one in U. Moreover, we have that $f(z) - w_0 = h(z)^N$ for every $z \in U$. Since the N-th power $w = \zeta^N$ maps the disc $D_0(r_0)$ onto the disc $D_0(r_0^N)$ and in an N-to-one manner, we conclude that $f : U \to W$ is N-to-one, where W is the disc $D_{w_0}(r_0^N)$. □

In the proof of theorem 6.2 if we take any linear segment $[w_0, w]$ in the disc $D_{w_0}(r_0^N)$, where w is a point of the circle $C_{w_0}(r_0^N)$, then, through the mapping $w = w_0 + \zeta^N$, this linear segment corresponds to N linear segments $[0, z_1], \dots [0, z_N]$ in the disc $D_0(r_0)$, where $z_1, \dots, z_N$ are N points on the circle $C_0(r_0)$. These N linear segments form N successive angles at 0 all equal to $\frac{2\pi}{N}$. Now the one-to-one function $h^{-1} : D_0(r_0) \to U$ maps these linear segments onto N curves $\gamma_1, \dots, \gamma_N$ with common initial endpoint z_0 and N corresponding final endpoints on ∂U. Since $h'(z_0) \neq 0$, the conformality of h at z_0 implies that $\gamma_1, \dots, \gamma_N$ form N successive angles at z_0 all equal to $\frac{2\pi}{N}$. The N successive "angular" regions $U_1, \dots, U_N$ in U between the curves $\gamma_1, \dots, \gamma_N$ are mapped by h onto the corresponding succesive angular regions $A_1, \dots, A_N$ in $D_0(r_0)$ between the linear segments $[0, z_1], \dots [0, z_N]$ and these are then mapped by the mapping $w = w_0 + \zeta^N$ onto the same region $B = D_{w_0}(r_0^N) \setminus [w_0, w]$. We conclude that f, which is the composition of the two mappings, maps each of $U_1, \dots, U_N$ in U onto B and in an one-to-one manner.

Exercises.

6.11.1. Let f be holomorphic in $D_0(R)$, $f'(0) \neq 0$ and $n \in \mathbb{N}$. Prove that there is $r > 0$ and there is g holomorphic in $D_0(r)$ so that $f(z^n) = f(0) + (g(z))^n$ for every $z \in D_0(r)$.

6.11.2. Let Ω_1, Ω_2 be two regions, let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \mathbb{C}$ be non-constant functions and let $h = g \circ f$.

(i) If f, h are holomorphic in Ω_1, is g holomorphic in Ω_2?
(ii) If g, h are holomorphic in Ω_2, Ω_1, respectively, is f holomorphic in Ω_1?

6.11.3. If f is holomorphic and one-to-one in $\mathbb{C}$, prove that there are $a \neq 0$ and b so that $f(z) = az + b$ for every z.

Chapter 7

Global behaviour of holomorphic functions.

7.1 Rotation number or index of a closed curve with respect to a point.

Definition. *Let $A \subseteq \mathbb{C}$, $h : A \to \mathbb{R}$ and $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A. We say that h is a* **continuous branch of arg g(w)** *in A if*
(i) h is continuous in A and
(ii) for every $w \in A$ we have $h(w) \in \arg g(w)$ or, equivalently, $g(w) = |g(w)|e^{ih(w)}$.

We recall the notion of a continuous branch of $\log g(w)$. We say that $f : A \to \mathbb{C}$ is a continuous branch of $\log g(w)$ if f is continuous in A and $f(w) \in \log g(w)$ or, equivalently, $e^{f(w)} = g(w)$ for every $w \in A$.

Proposition 7.1. *Let $A \subseteq \mathbb{C}$ and $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A. Then there is a one-to-one correspondence between continuous branches of $\log g(w)$ in A and continuous branches of $\arg g(w)$ in A.*

Proof. If $h : A \to \mathbb{R}$ is a continuous branch of $\arg g(w)$ in A, then the function $f : A \to \mathbb{C}$, defined by

$$f(w) = \ln|g(w)| + ih(w) \qquad \text{for every } w \in A, \tag{7.1}$$

is a continuous branch of $\log g(w)$ in A. Indeed, it is clear that f is continuous in A and also

$$e^{f(w)} = e^{\ln|g(w)|}e^{ih(w)} = |g(w)|e^{ih(w)} = g(w)$$

for every $w \in A$.
Conversely, if $f : A \to \mathbb{R}$ is a continuous branch of $\log g(w)$ in A, then the function $h : A \to \mathbb{C}$, defined through (7.1), is a continuous branch of $\arg g(w)$ in A. Indeed, it is clear that h is continuous in A and also

$$e^{ih(w)} = e^{f(w)}e^{-\ln|g(w)|} = \frac{g(w)}{|g(w)|}$$

and hence $g(w) = |g(w)|e^{ih(w)}$ for every $w \in A$. ☐

In other words, relation (7.1) says that, if we have a continuous branch f of $\log g(w)$ in A, then the imaginary part h of f is a continuous branch of $\arg g(w)$ in A. Conversely, if we have a continuous branch h of $\arg g(w)$ in A, then the function f with imaginary part h and real part $\ln|g(w)|$ is a continuous branch of $\log g(w)$ in A.
The next result is analogous to proposition 5.15 and their proofs are almost identical.

Proposition 7.2. *Let $A \subseteq \mathbb{C}$, $g : A \to \mathbb{C} \setminus \{0\}$ be continuous in A and $h_1, h_2 : A \to \mathbb{C}$.*
(i) If h_1 is a continuous branch of $\arg g(w)$ *in A and $h_2(w) - h_1(w) = k2\pi$ for every $w \in A$, where k is a fixed integer, then h_2 is also a continuous branch of* $\arg g(w)$ *in A.*
(ii) If, morever, A is connected and h_1, h_2 are continuous branches of $\arg g(w)$ *in A, then $h_2(w) - h_1(w) = k2\pi$ for every $w \in A$, where k is a fixed integer. In particular, if $h_1(w_0) = h_2(w_0)$ for some $w_0 \in A$, then $h_1(w) = h_2(w)$ for every $w \in A$.*

Proof. (i) The continuity of h_1 in A implies the continuity of h_2 in A. We also have $g(w) = |g(w)|e^{ih_1(w)}$ for every $w \in A$ and hence $|g(w)|e^{h_2(w)} = |g(w)|e^{ih_1(w)+ik2\pi} = g(w)$ for every $w \in A$. Therefore, h_2 is a continuous branch of $\arg g(w)$ in A.
(ii) We consider the function $k : A \to \mathbb{C}$ defined by

$$k(w) = \frac{1}{2\pi}(h_2(w) - h_1(w)) \qquad \text{for every } w \in A.$$

Since for every $w \in A$ both $h_2(w)$ and $h_1(w)$ are values of $\arg g(w)$, we have that $k(w)$ is an integer. I.e. $k : A \to \mathbb{Z}$. Also, since both h_1, h_2 are continuous in A, k is continuous in A. Now, k is a continuous real function in the connected set A, and hence it has the intermediate value property. But since its only values are integers, it is constant in A. Therefore, there is a fixed integer k so that $\frac{1}{2\pi}(h_2(w) - h_1(w)) = k$ or, equivalently, $h_2(w) - h_1(w) = k2\pi$ for every $w \in A$.
If $h_2(w_0) = h_1(w_0)$ for some $w_0 \in A$, then the integer k is 0 and we get that $h_2(w) = h_1(w)$ for every $w \in A$. □

We consider any curve

$$\gamma : [a, b] \to \mathbb{C}.$$

We recall that γ is continuous and piecewise continuously differentiable in $[a, b]$. I.e. there is a succession of points $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ so that γ is continuously differentiable in every $[t_{k-1}, t_k]$.

We consider an arbitrary fixed

$$z_0 \notin \gamma^* = \{\gamma(t) \,|\, t \in [a, b]\},$$

i.e. such that $z_0 \neq \gamma(t)$ for every $t \in [a, b]$ and we define the function $f : [a, b] \to \mathbb{C}$ by

$$f(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds \qquad \text{for every } t \in [a, b].$$

The function f is continuous in $[a, b]$ and differentiable at every point of continuity of $\frac{\gamma'(t)}{\gamma(t)-z_0}$, i.e. in every (t_{k-1}, t_k). In fact we have

$$f'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0} \qquad \text{for } t \text{ in any } (t_{k-1}, t_k).$$

Therefore, in each (t_{k-1}, t_k) we have

$$\frac{d}{dt}\big((\gamma(t) - z_0)e^{-f(t)}\big) = \gamma'(t)e^{-f(t)} - (\gamma(t) - z_0)f'(t)e^{-f(t)} = 0$$

and hence $(\gamma(t) - z_0)e^{-f(t)}$ is constant in each (t_{k-1}, t_k) with a constant value which *a priori* depends on k. Since this function is continuous on $[a, b]$, it is constant in $[a, b]$. Hence there is $c \in \mathbb{C}$ so that

$$(\gamma(t) - z_0)e^{-f(t)} = c \qquad \text{for every } t \in [a, b].$$

Since $c \neq 0$, there is $d \in \mathbb{C}$ so that $e^d = c$, and thus

$$e^{f(t)+d} = \gamma(t) - z_0 \qquad \text{for every } t \in [a, b].$$

Now we redefine the function $f : [a,b] \to \mathbb{C}$ by adding the constant d to the original f, i.e.

$$f(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds + d \qquad \text{for every } t \in [a,b] \tag{7.2}$$

and we have

$$e^{f(t)} = \gamma(t) - z_0 \qquad \text{for every } t \in [a,b].$$

In other words, the function $f(t)$ is a *continuous branch* of $\log(\gamma(t) - z_0)$ in $[a,b]$.

The real part of $f(t)$ is $\ln|\gamma(t) - z_0|$ and, if we denote $h(t)$ the imaginary part of $f(t)$, we have

$$f(t) = \ln|\gamma(t) - z_0| + ih(t) \qquad \text{for every } t \in [a,b] \tag{7.3}$$

and $h : [a,b] \to \mathbb{R}$ is a continuous branch of $\arg(\gamma(t) - z_0)$ in $[a,b]$.

We have proven the existence of a continuous branch h of $\arg(\gamma(t) - z_0)$ in $[a,b]$. Then the function $h + k2\pi$, where k is an arbitrary, but constant, integer, is also a continuous branch of $\arg(\gamma(t) - z_0)$ in $[a,b]$. Moreover, since $[a,b]$ is connected, these are all the continuous branches of $\arg(\gamma(t) - z_0)$ in $[a,b]$.

Now, let h be any continuous branch of $\arg(\gamma(t) - z_0)$ in $[a,b]$. We consider the expression

$$h(b) - h(a)$$

and we observe that this expression is independent of the particular choice of h. Indeed, if h_1 is another continuous branch of $\arg(\gamma(t) - z_0)$ in $[a,b]$, then there is a constant integer k so that $h_1 = h + k2\pi$ in $[a,b]$ and hence

$$h_1(b) - h_1(a) = (h(b) + k2\pi) - (h(a) + k2\pi) = h(b) - h(a).$$

Definition. *Let* $h : [a,b] \to \mathbb{R}$ *be any continuous branch of* $\arg(\gamma(t)-z_0)$ *in* $[a,b]$. *One such choice is given through (7.2) and (7.3). Then we call* **total increment of argument** *or* **total increment of angle** *over the curve* γ *with respect to* z_0 *the expression*

$$\Delta \arg(\gamma - z_0) = h(b) - h(a).$$

Now let us consider the important special case when the curve $\gamma : [a,b] \to \mathbb{C}$ is closed, i.e. when $\gamma(b) = \gamma(a)$. This implies that $\gamma(b)-z_0 = \gamma(a)-z_0$ and hence $\ln|\gamma(b)-z_0| = \ln|\gamma(a)-z_0|$ and that $h(b)$ and $h(a)$ differ by some integer multiple of 2π: indeed, both $h(b), h(a)$ are values of $\arg(\gamma(b) - z_0) = \arg(\gamma(a) - z_0)$. Therefore the expresion $\Delta \arg(\gamma - z_0) = h(b) - h(a)$ is an integer multiple of 2π.

Definition. *Let the curve* $\gamma : [a,b] \to \mathbb{C}$ *be closed and let* $z_0 \notin \gamma^*$. *Then the integer*

$$n(\gamma; z_0) = \frac{\Delta \arg(\gamma - z_0)}{2\pi}$$

is called **rotation number** *or* **index** *of* γ *with respect to* z_0.

It is easy to see the geometric content of the notion of rotation number or index of a closed curve γ with respect to z_0. For every $t \in [a,b]$ the number $h(t)$ is one of the values of the angle of $\gamma(t) - z_0$. When t increases in the parametric interval $[a,b]$, the angle $h(t)$ varies continuously from $h(a)$ to $h(b)$. Since $\Delta \arg(\gamma - z_0) = h(b) - h(a) = k2\pi$ for a certain integer k, the number $n(\gamma; z_0) = \frac{\Delta \arg(\gamma - z_0)}{2\pi} = \frac{h(b)-h(a)}{2\pi} = k$ shows the number of complete rotations of the continuously moving point $\gamma(t)$ around z_0.

Proposition 7.3. *Let the curve* $\gamma : [a,b] \to \mathbb{C}$ *be closed and let* $z_0 \notin \gamma^*$. *Then*

$$n(\gamma; z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z - z_0}\, dz.$$

Proof. We use the notation of the previous discussion and we get

$$n(\gamma; z_0) = \frac{\Delta \arg(\gamma - z_0)}{2\pi} = \frac{h(b) - h(a)}{2\pi}.$$

Since γ is closed, we have that $\ln|\gamma(b) - z_0| = \ln|\gamma(a) - z_0|$ and (7.2) and (7.3) imply

$$n(\gamma; z_0) = \frac{f(b) - f(a)}{2\pi i} = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_0}\, dz$$

and the proof is complete. □

Example 7.1.1. We take $n \in \mathbb{Z}$ and consider the closed curve $\gamma : [0, 2\pi] \to \mathbb{C}$ with parametric equation $\gamma(t) = z_0 + re^{int}$. It is clear that, if $n \neq 0$ and t increases in the interval $[0, 2\pi]$, then $\gamma(t)$ describes $|n|$ times the circle $C_{z_0}(r)$ in the positive direction, if $n > 0$, and in the negative direction, if $n < 0$. In the case $n = 0$, then $\gamma(t)$ describes $|n| = 0$ times the circle $C_{z_0}(r)$ since it stays still at the point $z_0 + r$. All these agree with the result of the calculation:

$$n(\gamma; z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z - z_0}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{re^{int}}\, rine^{int}\, dt = n.$$

The next result is just a special case, with $g(z) = z - z_0$, of proposition 5.24.

Proposition 7.4. *Let* $\Omega \subseteq \mathbb{C} \setminus \{z_0\}$ *be a region. Then a holomorphic branch of* $\log(z - z_0)$ *exists in* Ω *if and only if* $n(\gamma; z_0) = 0$ *for every closed curve* γ *in* Ω.

Proof. Obvious from proposition 5.24. □

Example 7.1.2. We consider the region $\Omega = \mathbb{C} \setminus l$, where l is any halfline with vertex z_0. We know that a holomorphic branch of $\log(z - z_0)$ exists in Ω and hence $\oint_\gamma \frac{1}{z - z_0}\, dz = 0$ for every closed curve γ in Ω. Of course, now this can be restated as $n(\gamma; z_0) = 0$ for every closed curve γ in Ω. Recalling the geometric content of the index, we conclude that every closed curve γ in Ω performs *no complete rotations* around z_0. This is geometrically clear: *since* γ *is in* Ω*, it does not intersect the halfline* l *with vertex* z_0*, and hence it cannot make any complete rotation around* z_0.

Proposition 7.5. *Let* γ *be any closed curve. Then the integer valued function*

$$n(\gamma; \cdot\,) : \mathbb{C} \setminus \gamma^* \to \mathbb{Z}$$

is constant in every connected component of the open set $\mathbb{C} \setminus \gamma^*$. *We also have that* $n(\gamma; z) = 0$ *for every* z *in the unbounded connected component of* $\mathbb{C} \setminus \gamma^*$.

Proof. Proposition 5.25 implies that $n(\gamma; z) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{\zeta - z}\, d\zeta$ is a holomorphic function of z in $\mathbb{C} \setminus \gamma^*$. Now, let Ω be any connected component of $\mathbb{C} \setminus \gamma^*$. Then Ω is a region and the function $n(\gamma; z)$ is continuous and integer valued in Ω. Since $n(\gamma; z)$ has the intermediate value property in Ω it has to be constant in Ω.
Since γ^* is bounded, it is contained in some disc $\overline{D}_0(R)$. Then the connected ring $D_0(R, +\infty)$ is contained in $\mathbb{C} \setminus \gamma^*$ and hence it is contained in (exactly) one of the connected components, call it Ω, of $\mathbb{C} \setminus \gamma^*$. Therefore, Ω is the unbounded connected component of $\mathbb{C} \setminus \gamma^*$ and we shall prove that $n(\gamma; z) = 0$ for every $z \in \Omega$. If $z \in D_0(R, +\infty)$, then the function $\frac{1}{\zeta - z}$ of ζ is holomorphic in the disc $D_0(R)$ which contains γ. Therefore, $c = n(\gamma; z) = \oint_\gamma \frac{1}{\zeta - z}\, d\zeta = 0$. □

Proposition 7.5 says that *if* z_1, z_2 *are in the same connected component of the complement of the trajectory of the closed curve* γ*, then the number of complete rotations of* γ *around* z_1 *is equal to the number of complete rotations of* γ *around* z_2.

Example 7.1.3. We take any $n \in \mathbb{Z}$, $n \neq 0$, and the closed curve $\gamma : [0, 2\pi] \to \mathbb{C}$ with $\gamma(t) = z_0 + re^{int}$. This is the curve in example 7.1.1. Then $\gamma^* = C_{z_0}(r)$ and its complement has two connected components: the disc $D_{z_0}(r)$ and the ring $D_{z_0}(r, +\infty)$. Observe that, if $z \neq z_0$, then it is *not trivial* to evaluate $n(\gamma; z)$ using the formula $n(\gamma; z) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - z} \, d\zeta$ and the parametric equation of γ. On the contrary, the calculation for $z = z_0$ is trivial and we did it in example 7.1.1: $n(\gamma, z_0) = 0$. Therefore,

$$n(\gamma; z) = n(\gamma; z_0) = n, \qquad \text{if } z \in D_{z_0}(r).$$

Also, $D_{z_0}(r, +\infty)$ is the unbounded connected component of the complement of γ^* and hence

$$n(\gamma; z) = 0, \qquad \text{if } z \in D_{z_0}(r, +\infty).$$

No calculation!

The proofs of the next three propositions are immediate applications of the integral representation of the index given in proposition 7.3. We omit them.

Proposition 7.6. *Let γ_1 and γ_2 be closed curves with the same endpoints. Then $\gamma_1 \dot{+} \gamma_2$ is defined and it is also a closed curve. Let z not be on the trajectory $(\gamma_1 \dot{+} \gamma_2)^* = \gamma_1^* \cup \gamma_2^*$. Then*

$$n(\gamma_1 \dot{+} \gamma_2; z) = n(\gamma_1; z) + n(\gamma_2; z).$$

Proposition 7.7. *If the closed curve γ_2 is a reparametrization of the closed curve γ_1 and z is not on the common tracectory $\gamma_1^* = \gamma_2^*$, then*

$$n(\gamma_2; z) = n(\gamma_1; z).$$

Proposition 7.8. *Let γ be a closed curve. Then $\neg\,\gamma$ is also a closed curve. Let z not be on the trajectory $(\neg\,\gamma)^* = \gamma^*$. Then*

$$n(\neg\,\gamma; z) = -n(\gamma; z).$$

Definition. *Let γ be a closed curve and $z \notin \gamma^*$. We say that γ* **surrounds** *z if $n(\gamma; z) \neq 0$.*

Cauchy's formula for derivatives and closed curves in convex regions. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the convex region Ω and γ be a closed curve in Ω. Then for all $n \in \mathbb{N}_0$ we have*

$$n(\gamma; z) f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \qquad \textit{for every } z \in \Omega \setminus \gamma^*.$$

Proof. The function $F : \Omega \setminus \{z\} \to \mathbb{C}$ defined by $F(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$ is holomorphic in $\Omega \setminus \{z\}$. Since z is a root of $f(\zeta) - f(z)$, the singularity z is removable. Therefore, we may define F at z with $F(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} = f'(z)$ and then F becomes holomorphic in Ω. Now we apply the theorem of Cauchy in convex regions and get

$$\oint_\gamma \frac{f(\zeta) - f(z)}{\zeta - z} \, d\zeta = \oint_\gamma F(\zeta) \, d\zeta = 0$$

for every $z \in \Omega \setminus \gamma^*$. This implies

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta = f(z) \oint_\gamma \frac{1}{\zeta - z} \, d\zeta = f(z) 2\pi i n(\gamma; z) \tag{7.4}$$

for every $z \in \Omega \setminus \gamma^*$. This is the result of the statement in the case $n = 0$.
Now, we consider a small disc $D_z(r) \subseteq \mathbb{C} \setminus \gamma^*$. This is possible, since z belongs to the open set $\mathbb{C} \setminus \gamma^*$. The disc $D_z(r)$ is connected and hence it is contained in one connected component of

$\mathbb{C} \setminus \gamma^*$. Therefore, the index $n(\gamma; w)$ is a constant function of w in $D_z(R)$, i.e. $n(\gamma; w) = n(\gamma; z)$ for every $w \in D_z(r)$. Now we write (7.4) with $w \in D_z(r)$ in the place of z:

$$\oint_\gamma \frac{f(\zeta)}{\zeta - w}\, d\zeta = f(w) 2\pi i n(\gamma; z).$$

We differentiate n times both sides with respect to w and get

$$n! \oint_\gamma \frac{f(\zeta)}{(\zeta - w)^{n+1}}\, d\zeta = f^{(n)}(w) 2\pi i n(\gamma; z).$$

Finally, we put $w = z$ and we have the result of the statement for general n. $\square$

A particular instance of the last result is Cauchy's formula for derivatives and circles. Indeed, in the case when the curve γ is the circle $C_{z_0}(R)$ described once in the positive direction we have $n(\gamma; z) = 1$ for all $z \in D_{z_0}(R)$. We originally proved the result in the case of a circle, using corollary 6.1. We now have a "new" proof using that z is a removable singularity of $\frac{f(\zeta)-f(z)}{\zeta - z}$. We have also introduced the notion of the index of a closed curve. This new proof together with the introduction of the notion of index allows us to generalize the case of a circle to the case of a closed curve. There is still a restriction in the sense that the curve has to be contained in a *convex* region in which the function is holomorphic. This is because our proof is based on Cauchy's theorem in *convex* regions. In this chapter we shall try to drop the restriction of convexity.

Exercises.

7.1.1. (i) Consider closed curves γ_1, γ_2 and z not on their trajectories. Assume that there are successive points $w_1^{(1)}, \ldots, w_n^{(1)}, w_{n+1}^{(1)} = w_1^{(1)}$ of γ_1^* and successive points $w_1^{(2)}, \ldots, w_n^{(2)}, w_{n+1}^{(2)} = w_1^{(2)}$ of γ_2^* and curves $\sigma_1, \ldots, \sigma_n, \sigma_{n+1} = \sigma_1$ so that every σ_j goes from $w_j^{(1)}$ to $w_j^{(2)}$ and so that, for each $j = 1, \ldots, n$, the part of γ_1 between $w_j^{(1)}, w_{j+1}^{(1)}$, the part of γ_2 between $w_j^{(2)}, w_{j+1}^{(2)}$, σ_j and σ_{j+1} are all in a convex subregion D_j of $\mathbb{C} \setminus \{z\}$. Prove that $n(\gamma_1; z) = n(\gamma_2; z)$.
(ii) Take a point z and two halflines l, m with vertex z. Let $A \in l$, $A \neq z$ and $B \in m$, $B \neq z$. Consider any curve γ_1 from A to B in one of the two angular regions defined by l, m and any curve γ_2 from B to A in the second angular region defined by l, m. Consider the closed curve $\gamma = \gamma_1 \dot{+} \gamma_2$. Using appropriately a small circle with center z, prove that $n(\gamma; z) = \pm 1$.

7.1.2. (i) Let $g_1, g_2 : A \to \mathbb{C} \setminus \{0\}$ be continuous in A and $h_1, h_2 : A \to \mathbb{C}$ be continuous branches of $\arg g_1, \arg g_2$ in A. Prove that $h_1 + h_2$ is a continuous branch of $\arg(g_1 g_2)$ in A.
(ii) If γ_1, γ_2 are closed curves in $\mathbb{C} \setminus \{0\}$ then $\gamma_1\gamma_2$ is a closed curve in $\mathbb{C} \setminus \{0\}$. Prove that $\Delta \arg(\gamma_1\gamma_2) = \Delta \arg \gamma_1 + \Delta \arg \gamma_2$.
(iii) Consider closed curves $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C} \setminus \{0\}$ such that $|\gamma_1(t) - \gamma_2(t)| < |\gamma_2(t)|$ for every $t \in [a, b]$. Prove that $n(\gamma_1; 0) = n(\gamma_2; 0)$.

7.2 Homotopy.

Definition. *Let $\gamma_0, \gamma_1 : [a, b] \to \mathbb{C}$ be two curves. We say that γ_1 is* **homotopic** *to γ_0 if there is a continuous function*

$$F : [a, b] \times [0, 1] \to \mathbb{C}$$

so that

$$F(t, 0) = \gamma_0(t), \quad F(t, 1) = \gamma_1(t) \qquad \text{for every } t \in [a, b].$$

The function F is called a **homotopy** *from γ_0 to γ_1.*

For each $s \in [0,1]$ the function $\gamma_s : [a,b] \to \mathbb{C}$, given by $\gamma_s(t) = F(t,s)$ for $t \in [a,b]$, is a curve. We shall call it *intermediate curve* between γ_0 and γ_1.

Since $[a,b] \times [0,1]$ is compact, the homotopy F is uniformly continuous. Thus for every $\epsilon > 0$ there is $\delta > 0$ so that $|F(t',s') - F(t'',s'')| < \epsilon$ when $\sqrt{(t'-t'')^2 + (s'-s'')^2} < \delta$. Therefore, is $|s'-s''| < \delta$ then we have $|\gamma_{s'}(t) - \gamma_{s''}(t)| < \epsilon$ for every $t \in [a,b]$, i.e. the curves $\gamma_{s'}$ and $\gamma_{s''}$ are uniformly close. We see that *when s increases in $[0,1]$ the curves γ_s form a continuously varying family of curves, starting with γ_0 and ending with γ_1.*

We have to observe that although we have agreed to assume that all our curves, and hence γ_0 and γ_1, are piecewise continuously differentiable, the intermediate curves γ_s of a homotopy need *not* be piecewise continuously differentiable: the homotopy F is only assumed to be continuous.

If all curves γ_s are closed, i.e. if $F(a,s) = F(b,s)$ for every $s \in [0,1]$, then we say that F is a **homotopy with closed intermediate curves**. If all curves γ_s have the same initial endpoint and the same final endpoint, i.e. if $F(a,s)$ is constant and $F(b,s)$ is constant for $s \in [0,1]$, then we say that F is a **homotopy with fixed endpoints**.

If all curves γ_s are in the same set A, then we say that F is a homotopy in A.

We may defined a relation between curves in a set A: we write $\gamma_0 \equiv \gamma_1$ if there is a homotopy from γ_0 to γ_1. It is easy to see that this is an equivalence relation. (i) Every curve $\gamma : [a,b] \to A$ is homotopic to itself through the homotopy $F : [a,b] \times [0,1] \to A$ given by $F(t,s) = \gamma(t)$. (ii) If $F : [a,b] \times [0,1] \to A$ is a homotopy from γ_0 to γ_1, i.e. if $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ for $t \in [a,b]$, then the function $G : [a,b] \times [0,1] \to A$ given by $G(t,s) = F(t,1-s)$ is a homotopy from γ_1 to γ_0. In fact G is continuous and $G(t,0) = \gamma_1(t)$ and $G(t,1) = \gamma_0(t)$ for $t \in [a,b]$. (iii) If $F : [a,b] \times [0,1] \to A$ is a homotopy from γ_0 to γ_1, i.e. if $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ for $t \in [a,b]$ and, if $G : [a,b] \times [0,1] \to A$ is a homotopy from γ_1 to γ_2, i.e. if $G(t,0) = \gamma_1(t)$ and $G(t,1) = \gamma_2(t)$ for $t \in [a,b]$, then $H : [a,b] \times [0,1] \to A$, given by

$$H(t,s) = \begin{cases} F(t,2s), & t \in [a,b],\, s \in [0,\frac{1}{2}] \\ G(t,2s-1), & t \in [a,b],\, s \in [\frac{1}{2},1] \end{cases}$$

is a homotopy from γ_0 to γ_2. Indeed, H is continuous and $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_2(t)$ for $t \in [a,b]$.

Furthermore, the previous argument shows that the relation of homotopy with closed intermediate curves and the relation of homotopy with fixed endpoints are both equivalence relations.

Example 7.2.1. If the set A is convex, every two curves in A are homotopic in A. Indeed, let $\gamma_0, \gamma_1 : [a,b] \to A$ be two curves in A. Since $\gamma_0(t), \gamma_1(t) \in A$ and A is convex, the linear segment $[\gamma_0(t), \gamma_1(t)]$ is contained in A. Now, if we define $F : [a,b] \times [0,1] \to \mathbb{C}$ by

$$F(t,s) = (1-s)\gamma_0(t) + s\gamma_1(t),$$

then F is continuous and all its values are in A. Moreover, $F(t,0) = \gamma_0(t)$ and $F(t,1) = \gamma_1(t)$ for $t \in [a,b]$. Therefore, F is a homotopy from γ_0 to γ_1 in A. It is easy to see that, if γ_0 and γ_1 are closed, then all intermediate curves are closed. Also, if γ_0 and γ_1 have the same initial endpoint and the same final endpoint, then all intermediate curves have the same initial endpoint and the same final endpoint.

Theorem 7.1. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω.*
(i) Let γ_0, γ_1 be two curves in Ω with the same initial endpoint and the same final endpoint. If there is a homotopy in Ω, with fixed endpoints, between γ_0 and γ_1, then

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

(ii) Let γ_0, γ_1 be two closed curves in Ω. If there is a homotopy in Ω, with closed intermediate curves, between γ_0 and γ_1, then

$$\oint_{\gamma_0} f(z)\,dz = \oint_{\gamma_1} f(z)\,dz.$$

Proof. (i) Let $F : [a, b] \times [0, 1] \to \Omega$ be the homotopy in Ω from γ_0 to γ_1. Then the subset $F([a, b] \times [0, 1])$ of Ω is compact and there is $\epsilon > 0$ so that

$$|z - w| \geq \epsilon \qquad \text{for every } z \in F([a, b] \times [0, 1]) \text{ and every } w \in \Omega^c. \tag{7.5}$$

Since F is uniformly continuous, there is $\delta > 0$ so that

$$|F(t', s') - F(t'', s'')| < \epsilon \qquad \text{if } |t' - t''| < \delta \text{ and } |s' - s''| < \delta. \tag{7.6}$$

We take points $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ and $0 = s_0 < s_1 < \ldots < s_{m-1} < s_m = 1$ so that $t_k - t_{k-1} < \delta$ and $s_l - s_{l-1} < \delta$ for all k and l. Then (7.5) and (7.6) imply that every rectangle $[t_{k-1}, t_k] \times [s_{l-1}, s_l]$ is mapped by F in the disc $D_{F(t_{k-1}, s_{l-1})}(\epsilon)$ which is contained in Ω. Thus, f is holomorphic in this disc and its curvilinear integral over any closed curve in this disc is equal to 0. If we denote $\gamma_{0,k}$ and $\gamma_{1,k}$ the restrictions of γ_0 and γ_1 in $[t_{k-1}, t_k]$, then for every $k = 1, \ldots, n$ we have

$$\int_{\gamma_{0,k}} f(z)\, dz + \int_{[F(t_k,s_1), F(t_{k-1},s_1)]} f(z)\, dz = -\int_{[F(t_{k-1},s_1), F(t_{k-1},0)]} f(z)\, dz - \int_{[F(t_k,0), F(t_k,s_1)]} f(z)\, dz$$

and

$$\int_{[F(t_{k-1},s_{l-1}), F(t_k,s_{1-1})]} f(z)\, dz + \int_{[F(t_k,s_l), F(t_{k-1},s_l)]} f(z)\, dz = -\int_{[F(t_{k-1},s_l), F(t_{k-1},s_{l-1})]} f(z)\, dz - \int_{[F(t_k,s_{l-1}), F(t_k,s_l)]} f(z)\, dz \qquad \text{for } l = 2, \ldots, m-1$$

and

$$\int_{[F(t_{k-1},s_{m-1}), F(t_k,s_{m-1})]} f(z)\, dz - \int_{\gamma_{1,k}} f(z)\, dz = -\int_{[F(t_{k-1},1), F(t_{k-1},s_{m-1})]} f(z)\, dz - \int_{[F(t_k,s_{m-1}), F(t_k,1)]} f(z)\, dz.$$

Adding these m equalities and then adding for $k = 1, \ldots, n$ and considering cancellations, we find

$$\int_{\gamma_0} f(z)\, dz - \int_{\gamma_1} f(z)\, dz = \sum_{l=1}^{m} \int_{[F(b,s_{l-1}), F(b,s_l)]} f(z)\, dz + \sum_{l=1}^{m} \int_{[F(a,s_l), F(a,s_{l-1})]} f(z)\, dz. \tag{7.7}$$

Since all intermediate curves have the same initial endpoint and the same final endpoint, we see that all linear segments $[F(b, s_{l-1}), F(b, s_l)]$ and $[F(a, s_l), F(a, s_{l-1})]$ are single point sets and hence all integrals in the right side of (7.7) are equal to 0. Hence $\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz$.
(ii) Since all intermediate curves are closed, we have $F(a, s) = F(b, s)$ for every $s \in [0, 1]$. Therefore, the linear segments $[F(b, s_{l-1}), F(b, s_l)]$ and $[F(a, s_l), F(a, s_{l-1})]$ are opposite and, again, the right side of (7.7) are equal to 0. Hence $\oint_{\gamma_0} f(z)\, dz = \oint_{\gamma_1} f(z)\, dz$. $\square$

Proposition 7.9. *Let γ_0, γ_1 be two closed curves in $\mathbb{C} \setminus \{z\}$. If there is a homotopy in $\mathbb{C} \setminus \{z\}$, with closed intermediate curves, between γ_0 and γ_1, then*

$$n(\gamma_0; z) = n(\gamma_1; z).$$

Proof. We apply theorem 7.1 to the function $f(\zeta) = \frac{1}{\zeta - z}$ which is holomorphic in $\mathbb{C} \setminus \{z\}$. $\square$

Exercises.

7.2.1. A set $A \subseteq \mathbb{C}$ is called **arcwise connected** if every two points in A can be joined by some curve in A.
Let A be arcwise connected and $\gamma_1(t) = z_1$ and $\gamma_2(t) = z_2$ be two constant curves in A. If a curve γ is homotopic in A to γ_1, prove that γ is homotopic in A to γ_2.

7.2.2. If γ is a closed curve in $\mathbb{C} \setminus \{0\}$, prove that γ is homotopic in $\mathbb{C} \setminus \{0\}$ to a closed curve whose trajectory is contained in the circle $C_0(1)$.

7.2.3. (i) Let f be continuous in $\overline{D}_0(R)$. We define $\gamma(t) = f(Re^{it})$ for every $t \in [0, 2\pi]$. Prove that, if $n(\gamma; w) \neq 0$, then $w \in f(D_0(R))$. I.e. $\{w \,|\, w \text{ is surrounded by } \gamma\} \subseteq f(D_0(R))$.
(ii) Using the result of (i), prove the fundamental theorem of algebra.

7.2.4. Let $p \in A$ and let $\mathcal{M}_p(A)$ be the set of all closed curves with both of their endpoints at p. If $\gamma_1, \gamma_2 \in \mathcal{M}_p(A)$, then clearly $\gamma_1 \dot{+} \gamma_2 \in \mathcal{M}_p(A)$. Also, if $\gamma \in \mathcal{M}_p(A)$, then $\neg\gamma \in \mathcal{M}_p(A)$. Hence $\mathcal{M}_p(A)$ is a group whose neutral element is the constant curve $\gamma_p(t) = p$.
(i) Prove that the relation of homotopy in A with closed intermediate curves and fixed endpoints ($= p$) is an equivalence relation in $\mathcal{M}_p(A)$. The set of all equivalence classes is denoted $\mathcal{H}_p(A) = \{[\gamma] \,|\, \gamma \in \mathcal{M}_p(A)\}$.
(ii) If $\gamma, \gamma_1, \gamma_2 \in \mathcal{M}_p(A)$, we define $[\gamma_1] + [\gamma_2] = [\gamma_1 \dot{+} \gamma_2]$ and $-[\gamma] = [\neg\gamma]$. Prove that these are well-defined and that $\mathcal{H}_p(A)$ with these operations is a group, whose neutral element is $[\gamma_p]$.
(iii) If A is arcwise connected (see exercise 7.2.1), prove that for every $p, q \in A$ the groups $\mathcal{H}_p(A)$ and $\mathcal{H}_q(A)$ are isomorphic. In this case we write $\mathcal{H}(A)$.
(iv) Prove that $\mathcal{H}(\mathbb{C}) \cong \{0\}$, $\mathcal{H}(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z}$, $\mathcal{H}(C_0(1)) \cong \mathbb{Z}$.

7.2.5. Let z_1, z_2, z_3, w_1, w_2, w_3 be distinct points. Is it possible to join every z_k with every w_j with simple curves γ_{kj} which are mutually disjoint?

7.3 Combinatorial results for curves and square nets.

Lemma 7.1. *Let $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ be a set of curves (not necessarily closed) and let $A = \{a_1, \dots, a_m\}$ be the set of their endpoints ($m \leq 2n$). We assume that for every point of A the number of the curves in Σ that arrive at this point is the same as the number of the curves in Σ that leave from this point. Then we can partition Σ into subsets $\Sigma_1, \dots, \Sigma_k$ so that each Σ_j consists of successive curves and the sum γ_j of the curves in Σ_j is a closed curve.*

Proof. We describe the following algorithm for the partitioning of Σ.
We start with σ_1. The final endpoint of σ_1 is the initial endpoint of at least one curve in Σ. If the final endpoint of σ_1 coincides with its initial endpoint, then σ_1 is closed and we stop the process. If this is not the case, then, renumbering if necessary the curves $\sigma_2, \dots, \sigma_n$, we may assume that the final endpoint of σ_1 coincides with the initial endpoint of σ_2. If the final endpoint of σ_2 coincides with the initial endpoint of σ_1, then the sum of σ_1, σ_2 is a closed curve and we stop the process. If the final endpoint of σ_2 coincides with its initial endpoint, then σ_2 is a closed curve and we stop the process. If the final endpoint of σ_2 is not the initial point of either σ_1 or σ_2, then renumbering if necessary the curves $\sigma_3, \dots, \sigma_n$, we may assume that the final endpoint of σ_2 coincides with the initial endpoint of σ_3. Then, exactly as before, we examine whether the final endpoint of σ_3 coincides with the initial endpoint of σ_1 or of σ_2 or of σ_3. Then, respectively, the sum of $\sigma_1, \sigma_2, \sigma_3$ or the sum of σ_2, σ_3 or σ_3 by itself is a closed curve and we stop the process. If the final endpoint of σ_3 is not the initial endpoint of either σ_1 or σ_2 or σ_3, then renumbering if necessary the curves $\sigma_4, \dots, \sigma_n$, we may assume that the final endpoint of σ_3 coincides with the initial endpoint of

σ_4. Now, it is clear that this process will eventually stop, because we have only finitely many curves. Therefore, we shall eventually find successive curves $\sigma_1, \sigma_2, \dots, \sigma_{k-1}, \sigma_k$ $(1 \leq k \leq n)$ so that the final endpoint of σ_k coincides with the initial endpoint of one of the same curves $\sigma_1, \sigma_2, \dots, \sigma_{k-1}, \sigma_k$. Let the final endpoint of σ_k coincide with the initial endpoint of σ_l for some l with $1 \leq l \leq k$. Then the sum of $\sigma_l, \sigma_{l+1}, \dots, \sigma_{k-1}, \sigma_k$ is a closed curve and we stop the process. Now we set $\Sigma_1 = \{\sigma_l, \sigma_{l+1}, \dots, \sigma_{k-1}, \sigma_k\}$ and call γ_1 the closed curve which is the sum of $\sigma_l, \sigma_{l+1}, \dots, \sigma_{k-1}, \sigma_k$. Then we drop the curves of Σ_1 from Σ, i.e. we consider the set $\Sigma' = \Sigma \setminus \Sigma_1 = \{\sigma_1, \dots, \sigma_{l-1}, \sigma_{k+1}, \dots, \sigma_n\}$.
Each endpoint of the curves in Σ' is one of the points of $A = \{a_1, \dots, a_m\}$, say it is a_j. Then the number of the curves in Σ that arrive at a_j is the same as the number of the curves in Σ that leave from a_j. But the curves $\sigma_l, \sigma_{l+1}, \dots, \sigma_{k-1}, \sigma_k$ are successive and hence if one of them arrives at a_j then the next one leaves from a_j. Therefore, the remaining curves of Σ' have the same property: the number of the curves in Σ' that arrive at a_j is the same as the number of the curves in Σ' that leave from a_j. Thus Σ' has the same property as the original Σ.
Now we continue our algorithm with Σ'. We find a subset Σ_2 of Σ' which consists of successive curves and we call γ_2 the closed curve which is the sum of the curves in Σ_2. Then we drop the curves of Σ_2 from Σ', i.e. we consider the set $\Sigma'' = \Sigma' \setminus \Sigma_2 = \Sigma \setminus (\Sigma_1 \cup \Sigma_2)$. We go on until we exhaust the original Σ. ☐

Lemma 7.2. *We take any $\delta > 0$ and two perpendicular lines. For each of them we consider all its parallel lines at distances equal to integer multiples of δ. The result is a net of closed square regions of sidelength δ which cover the plane and have disjoint interiors. We choose any of those closed square regions, say $Q_1, \dots, Q_l$. We consider the closed boundary curves $\partial Q_1, \dots, \partial Q_l$ with their positive direction. Each of them is the sum of four corresponding linear segments, considered as curves with the same direction. We drop the linear segments (with necessarily opposite directions) which are common to any two neighboring square regions from among the $Q_1, \dots, Q_l$ and we consider the set $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of all the remaining linear segments, i.e. those which belong to only one of $Q_1, \dots, Q_l$. Then we can partition Σ into subsets $\Sigma_1, \dots, \Sigma_k$ so that each Σ_j consists of successive linear segments and the sum γ_j of the linear segments in Σ_j is a closed curve.*

Proof. It is enough to prove that Σ has the property described in lemma 7.1, i.e. that for every point of intersection a of our lines the number of the curves in Σ that arrive at a is the same as the number of the curves in Σ that leave from a. This can be done easily, considering cases for the number, 0 or 1 or 2 or 3 or 4, of the squares among $Q_1, \dots, Q_l$ which have a as one of their corners. ☐

Lemma 7.3. *Let Ω be an open set and $K \subseteq \Omega$ be compact. Then there is $\delta > 0$ so that $|z - w| \geq 2\delta$ for every $z \in K$ and every $w \in \Omega^c$. For this $\delta > 0$ we consider the net of closed square regions of lemma 7.2 and we take all closed square regions $Q_1, \dots, Q_l$ of the net which intersect K. Then $Q_1, \dots, Q_l$ are contained in Ω. As in lemma 7.2, we consider the set $\Sigma = \{\sigma_1, \dots, \sigma_n\}$ of all linear segments which belong to only one of $Q_1, \dots, Q_l$ and we partition Σ into subsets $\Sigma_1, \dots, \Sigma_k$ so that each Σ_j consists of successive linear segments and the sum γ_j of the linear segments in Σ_j is a closed curve. Then $\gamma_1^* \cup \dots \cup \gamma_k^* \subseteq \Omega \setminus K$ (i.e. every γ_j is in $\Omega \setminus K$) and the distance of every $z \in \gamma_1^* \cup \dots \cup \gamma_k^*$ from K is $\leq \delta\sqrt{2}$. Moreover, for every $f : \Omega \to \mathbb{C}$ holomorphic in Ω and for every $z \in K$, we have*

$$f(z) = \sum_{j=1}^{k} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta. \tag{7.8}$$

Proof. Each of the closed square regions $Q_1, \dots, Q_l$, say Q_m, intersects K and its diameter is equal to $\sqrt{2}\delta$. Therefore, the distance of every point of Q_m from K is $\leq \sqrt{2}\delta$. Since $\sqrt{2}\delta < 2\delta$, we have that Q_m is contained in Ω.
Consider any of the linear segments $\sigma_1, \dots, \sigma_n$, say σ_j. Now, σ_j belongs to one of $Q_1, \dots, Q_l$,

say Q_m. Since Q_m is contained in Ω, we have that σ_j is also contained in Ω. Moreover, the distance of every point of Q_m from K is $\leq \sqrt{2}\delta$ and hence the distance of every point of σ_j from K is $\leq \sqrt{2}\delta$. If σ_j intersects K, then both closed square regions of our net which lie on the two sides of σ_j intersect K and hence both are among $Q_1, \ldots, Q_l$. This is impossible because σ_j belongs to only one of $Q_1, \ldots, Q_l$. Therefore, σ_j does not intersect K and hence it is contained in $\Omega \setminus K$. Finally, since each of $\gamma_1, \ldots, \gamma_k$ is the sum of certain of the $\sigma_1, \ldots, \sigma_n$, we have proved that $\gamma_1^* \cup \cdots \cup \gamma_k^* \subseteq \Omega \setminus K$ and that the distance of every $z \in \gamma_1^* \cup \cdots \cup \gamma_k^*$ from K is $\leq \delta\sqrt{2}$.
Now we take any $z \in K$. Then z belongs to one $Q_1, \ldots, Q_l$, say Q_m. Let us assume that z is an interior point of Q_m. Since the closed square region Q_m is contained in Ω, there is a slightly larger open square region Q' which is also contained in Ω. Now f is holomorphic in the convex region Q' and Cauchy's formula in section 7.1 says that

$$f(z) = \frac{1}{2\pi i} \oint_{\partial Q_m} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{7.9}$$

because the index of ∂Q_m with respect to z is equal to 1. Now we take any closed square region Q_p with $p \neq m$. Then z is not contained in Q_p and again we may find an open square region Q' slightly larger than Q_p which is contained in Ω and which does not contain z. Then $\frac{f(\zeta)}{\zeta - z}$ is a holomorphic function of ζ in the convex region Q' and hence

$$0 = \frac{1}{2\pi i} \oint_{\partial Q_p} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for } p \neq m. \tag{7.10}$$

We add (7.9) and (7.10) and we get

$$f(z) = \sum_{p=1}^{l} \frac{1}{2\pi i} \oint_{\partial Q_p} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{7.11}$$

Now we split the integral over each ∂Q_p in four integrals over the boundary linear segments of ∂Q_p and we get $4l$ integrals. We recall that if a linear segment belongs to two neighboring closed square regions, then it appears twice among the integrals with opposite directions and hence the two integrals cancel. Therefore, the remaining integrals will be only over the boundary linear segments which belong to exactly one of $Q_1, \ldots, Q_l$, i.e. the linear segments of the set $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$. Thus (7.11) becomes

$$f(z) = \sum_{\sigma \in \Sigma} \frac{1}{2\pi i} \int_{\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The subsets $\Sigma_1, \ldots, \Sigma_k$ form a partition of Σ and hence

$$f(z) = \sum_{j=1}^{k} \sum_{\sigma \in \Sigma_j} \frac{1}{2\pi i} \int_{\sigma} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Finally, since γ_j is the sum of the successive linear segments $\sigma \in \Sigma_j$, we end up with (7.8).
Now we consider the case when z is a boundary point of Q_m. Then we may consider a variable point z' in the interior of Q_m so that $z' \to z$. We have proved (7.8) for z', i.e.

$$f(z') = \sum_{j=1}^{k} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(\zeta)}{\zeta - z'}\, d\zeta.$$

Proposition 5.25 implies the continuity of the right side as a function of z'. Therefore, taking the limit as $z' \to z$, we end up again with (7.8). $\square$

7.4 The theorem of Cauchy in general open sets.

Definition. *Let $\sigma_1, \dots, \sigma_n$ be any curves (not necessarily closed) and $k_1, \dots, k_n$ be any integers. Then we say that the curves $\sigma_1, \dots, \sigma_n$ considered $k_1, \dots, k_n$ times, respectively, form a* **chain** Σ. *The integer k_j is called* **multiplicity** *of the corresponding σ_j in the chain Σ. If every σ_j is closed, then Σ is called* **closed chain** *or* **cycle**. *If every σ_j is in a set A, then we say that Σ is in A.*

If a curve σ is not among the curves which constitute a chain Σ, we may include it among those curves by assigning multiplicity 0 to σ. And now we may introduce the algebraic structure of a *module* in the set of all chains in the following manner. If Σ' and Σ'' are two chains, we may assume that they are formed by the same collection $\sigma_1, \dots, \sigma_n$ of curves (since some of the curves may have multiplicity 0 in one of the two chains). If $k_1', \dots, k_n'$ and $k_1'', \dots, k_n''$ are the corresponding multiplicities in the chains Σ' and Σ'', then we define $\Sigma' + \Sigma''$ to be the chain which consists of $\sigma_1, \dots, \sigma_n$ with multiplicities $k_1' + k_1'', \dots, k_n' + k_n''$. Moreover, if k is an integer and Σ is a chain formed by the collection $\sigma_1, \dots, \sigma_n$ of curves with multiplicities $k_1, \dots, k_n$, then we define $k\Sigma$ to be the chain formed by $\sigma_1, \dots, \sigma_n$ with multiplicities $kk_1, \dots, kk_n$. It is very easy to show that, under this addition of chains and this multiplication of chains and integers, the set of chains is a $\mathbb{Z}$-module. The opposite $-\Sigma$ of a chain Σ is $(-1)\Sigma$ and the neutral element of addition is the chain which contains no curve (or any curves with multiplicities 0).

If Σ is a chain formed by the curves $\sigma_1, \dots, \sigma_n$ with multiplicities $k_1, \dots, k_n$, we immediately see that, under the above definitions of addition and multiplication, we have $\Sigma = k_1\sigma_1 + \cdots + k_n\sigma_n$. Here we consider each σ_j as a chain consisting of only one curve with multiplicity 1.

We shall not go into this algebraic point of view, since it does not have much to offer in our study of complex analysis. We shall keep in mind, though, the definition and notation of $\Sigma' + \Sigma''$ and $k\Sigma$ and from time to time we shall feel free to make certain mild algebraic comments.

Definition. *Let Σ be a chain formed by the curves $\sigma_1, \dots, \sigma_n$ with multiplicities $k_1, \dots, k_n$ and let $\phi : \sigma_1^* \cup \cdots \cup \sigma_n^* \to \mathbb{C}$ be continuous on $\sigma_1^* \cup \cdots \cup \sigma_n^*$. We define the* **curvilinear integral** *of ϕ over Σ by*

$$\int_\Sigma \phi(z)\,dz = \sum_{j=1}^n k_j \int_{\sigma_j} \phi(z)\,dz.$$

If Σ is a cycle, we usually write

$$\oint_\Sigma \phi(z)\,dz.$$

It is very easy to show that

$$\int_{\Sigma'+\Sigma''} \phi(z)\,dz = \int_{\Sigma'} \phi(z)\,dz + \int_{\Sigma''} \phi(z)\,dz, \qquad \int_{k\Sigma} \phi(z)\,dz = k\int_\Sigma \phi(z)\,dz.$$

These two relations say that integration “respects” the “linearity” of the module structure of the set of chains.

Definition. *Let Σ be a cycle formed by the closed curves $\sigma_1, \dots, \sigma_n$ with multiplicities $k_1, \dots, k_n$. For every z which does not belong to $\sigma_1^* \cup \cdots \cup \sigma_n^*$ we define the* **rotation number** *or* **index** *of the chain Σ with respect to z by*

$$n(\Sigma; z) = \sum_{j=1}^n k_j n(\sigma_j; z).$$

We may say that $n(\Sigma; z)$ is the total number of rotations around z of the curves forming Σ, taking into account their multiplicities.

Combining the last two definitions, we easily see that the index of a cycle is given by the same integral form which gives the index of a closed curve:

$$n(\Sigma; z) = \frac{1}{2\pi i} \oint_{\Sigma} \frac{1}{\zeta - z}\, d\zeta.$$

Indeed,

$$n(\Sigma; z) = \sum_{j=1}^{n} k_j n(\sigma_j; z) = \sum_{j=1}^{n} k_j \frac{1}{2\pi i} \oint_{\sigma_j} \frac{1}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \oint_{\Sigma} \frac{1}{\zeta - z}\, d\zeta.$$

The following definition is basic.

Definition. *Let Σ be a cycle in the open set Ω. We say that Σ is* **null-homologous** *in Ω if $n(\Sigma; z) = 0$ for every $z \in \Omega^c$.*

In other words, a cycle Σ in Ω is null-homologous in Ω if Σ is in Ω and the total number of rotations of the curves forming Σ, taking into account their multiplicities, around every point of the complement of Ω is zero.

Lemma 7.4. *Let Ω be open, $\delta, R > 0$ and $K = \{z \in \Omega \,|\, |z| \leq R, |z - w| \geq \delta$ for every $w \in \Omega^c\}$. Then K is a compact subset of Ω.*

Proof. Since $K \subseteq \overline{D}_0(R)$, the set K is bounded.
Now, let $z_n \in K$ for every n and $z_n \to z$. If we prove that $z \in K$, then K is closed and hence compact. We have $|z_n| \leq R$ for every n and thus $|z| \leq R$. For every $w \in \Omega^c$ we have $|z_n - w| \geq \delta$ for every n and hence $|z - w| \geq \delta$. Therefore $z \in K$. $\square$

The set K of lemma 7.4 is the intersection of the closed disc $\overline{D}_0(R)$ and of the set of all points of Ω whose distance from Ω^c is $\geq \delta$.

The theorem of Cauchy in general open sets. *Let $f : \Omega \to \mathbb{C}$ be holomorphic in the open set Ω. If the cycle Σ is null-homologous in Ω, then*

$$\oint_{\Sigma} f(z)\, dz = 0.$$

Proof. Let Σ consist of the closed curves $\sigma_1, \ldots, \sigma_n$ with multiplicities $k_1, \ldots, k_n$.
Since $\sigma_1^* \cup \cdots \cup \sigma_n^*$ is a compact subset of Ω, there is $\delta > 0$ so that every point of $\sigma_1^* \cup \cdots \cup \sigma_n^*$ has a distance $\geq 2\delta$ from Ω^c and there is $R > 0$ so that $\sigma_1^* \cup \cdots \cup \sigma_n^*$ is contained in the closed disc $\overline{D}_0(R)$. Now, we consider the set

$$K = \{z \in \Omega \,|\, |z| \leq R, |z - w| \geq 2\delta \text{ for every } w \in \Omega^c\}$$

of lemma 7.4 (with 2δ instead of δ). Then K is a compact subset of Ω which contains $\sigma_1^* \cup \cdots \cup \sigma_n^*$. With the same δ and with this set K we form the closed curves $\gamma_1, \ldots, \gamma_k$ in $\Omega \setminus K$, which are described in lemma 7.3. According to lemma 7.3 we have

$$f(z) = \sum_{l=1}^{k} \frac{1}{2\pi i} \oint_{\gamma_l} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for every } z \in \sigma_1^* \cup \cdots \cup \sigma_n^*.$$

Hence

$$\oint_{\Sigma} f(z)\, dz = \sum_{j=1}^{n} k_j \oint_{\sigma_j} f(z)\, dz = \sum_{j=1}^{n} k_j \oint_{\sigma_j} \Big(\sum_{l=1}^{k} \frac{1}{2\pi i} \oint_{\gamma_l} \frac{f(\zeta)}{\zeta - z}\, d\zeta \Big)\, dz.$$

Changing appropriately the order of sums and integrals, we end up with

$$\oint_\Sigma f(z)\,dz = -\sum_{l=1}^k \oint_{\gamma_l} \Big(\sum_{j=1}^n k_j\,\frac{1}{2\pi i}\oint_{\sigma_j}\frac{1}{z-\zeta}\,dz\Big)\,f(\zeta)\,d\zeta$$

and hence

$$\oint_\Sigma f(z)\,dz = -\sum_{l=1}^k \oint_{\gamma_l} \Big(\sum_{j=1}^n k_j\,n(\sigma_j;\zeta)\Big)\,f(\zeta)\,d\zeta = -\sum_{l=1}^k \oint_{\gamma_l} n(\Sigma;\zeta)f(\zeta)\,d\zeta. \qquad (7.12)$$

Now we consider the index $n(\Sigma;\zeta) = \sum_{j=1}^n k_j\,n(\sigma_j;\zeta)$ when, as in the integrals in (7.12), ζ belongs to any of $\gamma_1^*,\dots,\gamma_k^*$. Since every such ζ is in $\Omega\setminus K$, either $\zeta\notin\overline{D}_0(R)$ or the distance of ζ from Ω^c is $< 2\delta$. If $\zeta\notin\overline{D}_0(R)$, then, since Σ is in $\overline{D}_0(R)$, we have that $n(\Sigma;\zeta)=0$. If the distance of ζ from Ω^c is $<2\delta$, then there is $w\in\Omega^c$ so that $|\zeta-w|<2\delta$. Then every point of the linear segment $[\zeta,w]$ has distance $<2\delta$ from w and hence from Ω^c. Thus $[\zeta,w]$ is not contained in K which implies that $[\zeta,w]$ is in the complement of $\sigma_1^*,\dots,\sigma_n^*$. Since $[\zeta,w]$ is connected and it is contained in the complement of every σ_j^* we have that

$$n(\sigma_j;\zeta) = n(\sigma_j;w) \qquad \text{for every } j=1,\dots,n.$$

Therefore

$$n(\Sigma;\zeta) = \sum_{j=1}^n k_j\,n(\sigma_j;\zeta) = \sum_{j=1}^n k_j\,n(\sigma_j;w) = n(\Sigma;w) = 0$$

because $w\in\Omega^c$ and Σ is null-homologous in Ω. Now (7.12) implies

$$\oint_\Sigma f(z)\,dz = -\sum_{l=1}^k \oint_{\gamma_l} n(\Sigma;\zeta)f(\zeta)\,d\zeta = -\sum_{l=1}^k \oint_{\gamma_l} 0\,f(\zeta)\,d\zeta = 0,$$

because we proved that $n(\Sigma;\zeta)=0$ for every ζ in $\gamma_1^*,\dots,\gamma_k^*$. □

It is interesting to see that the assumption of our last result is at the same time a special case of it. Indeed, if we take any $w\in\Omega^c$, then the function $f(z)=\frac{1}{z-w}$ is holomorphic in Ω and the theorem of Cauchy implies that $\oint_\Sigma \frac{1}{z-w}\,dz = 0$. But this says that $n(\Sigma;w)=0$. In other words, we have the following situation. The assumption that Σ is null-homologous in Ω is equivalent to the validity of the theorem of Cauchy for the very particular holomorphic functions of the form $f(z)=\frac{1}{z-w}$ for every $w\in\Omega^c$. Therefore the real content of the theorem of Cauchy is that *the validity of $\oint_\Sigma f(z)\,dz=0$ for the special holomorphic functions in Ω of the form $f(z)=\frac{1}{z-w}$ for every $w\in\Omega^c$ implies its validity for every function f which is holomorphic in Ω.*

Example 7.4.1. Let γ be any closed curve in the *convex* region Ω and let $w\in\Omega^c$. Then w is contained in the unbounded connected component of $\mathbb{C}\setminus\gamma^*$ and proposition 7.5 implies that $n(\gamma;w)=0$. Hence γ is null-homologous in Ω. Now the theorem of Cauchy for general open sets says that $\oint_\gamma f(z)\,dz=0$ for every $f:\Omega\to\mathbb{C}$ which is holomorphic in Ω. We conclude that *the theorem of Cauchy for convex regions is a corrolary of the theorem of Cauchy for general open sets.*

Example 7.4.2. We consider the open set $D_{z_0}(R_1,R_2)$ with $0\le R_1<R_2\le+\infty$ and the closed curve γ which describes the circle $C_{z_0}(r)$, with $R_1<r<R_2$, once and in the positive direction. This curve is *not* null-homologous in $D_{z_0}(R_1,R_2)$. Indeed, z_0 is in the complement of $D_{z_0}(R_1,R_2)$ and $n(\gamma;z_0)=\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{1}{z-z_0}\,dz=1$. Therefore, we do not expect that $\oint_\gamma f(z)\,dz=0$ is true for *every* f which is holomorphic in $D_{z_0}(R_1,R_2)$. In fact, this is certainly not true for $f(z)=\frac{1}{z-z_0}$ which *is* holomorphic in $D_{z_0}(R_1,R_2)$.

Example 7.4.3. We consider the same open set $D_{z_0}(R_1, R_2)$ as in the previous example and an arbitrary closed curve γ in $D_{z_0}(R_1, R_2)$. We shall see how we can evaluate $\oint_\gamma f(z)\,dz$ with a minimum of effort for any f holomorphic in $D_{z_0}(R_1, R_2)$. It is clear that, depending on the specific curve γ, it may be difficult to evaluate the integral using a parametric equation of γ.
Let us assume that the shape of the trajectory and the direction of γ allow us to count the number of rotations of γ around z_0, i.e. we assume that we know the integer $k = n(\gamma; z_0)$. Since the disc $\overline{D}_{z_0}(R_1)$ is one of the two connected components of the complement of $D_{z_0}(R_1, R_2)$, we have that $n(\gamma; z) = k$ for every $z \in \overline{D}_{z_0}(R_1)$. On the other hand, we have that $n(\gamma; z) = 0$ for every z in the unbounded connected component of the complement of $D_{z_0}(R_1, R_2)$, which is $\overline{D}_{z_0}(R_2, +\infty)$. Now we take a closed curve γ_1 in $D_{z_0}(R_1, R_2)$ such that the $\oint_{\gamma_1} f(z)\,dz$ may be much easier to evaluate than the original $\oint_\gamma f(z)\,dz$. For instance, we may consider γ_1 to describe the circle $C_{z_0}(r)$ with $R_1 < r < R_2$ once and in the positive direction. In this case we have that $n(\gamma_1; z) = 1$ for every $z \in \overline{D}_{z_0}(R_1)$ and $n(\gamma_1; z) = 0$ for every $z \in \overline{D}_{z_0}(R_2, +\infty)$.
Now we form the cycle Σ from γ with mutiplicity 1 and from γ_1 with multiplicity $-k$, i.e.

$$\Sigma = 1\,\gamma + (-k)\,\gamma_1.$$

and we have:

$$n(\Sigma; z) = 1\,n(\gamma; z) + (-k)\,n(\gamma_1; z) = k + (-k) = 0 \quad \text{for every } z \in \overline{D}_{z_0}(R_1),$$
$$n(\Sigma; z) = 1\,n(\gamma; z) + (-k)\,n(\gamma_1; z) = 0 + 0 = 0 \quad \text{for every } z \in \overline{D}_{z_0}(R_2, +\infty).$$

Therefore, Σ is null-homologous in $D_{z_0}(R_1, R_2)$ and the theorem of Cauchy implies

$$0 = \oint_\Sigma f(z)\,dz = 1 \oint_\gamma f(z)\,dz + (-k) \oint_{\gamma_1} f(z)\,dz$$

and hence

$$\oint_\gamma f(z)\,dz = k \oint_{\gamma_1} f(z)\,dz = k \oint_{C_{z_0}(r)} f(z)\,dz.$$

We see that the evaluation of $\oint_\gamma f(z)\,dz$ has been reduced to the evaluation of the possibly much simpler integral $\oint_{C_{z_0}(r)} f(z)\,dz$ and the evaluation of the index $n(\gamma; z_0)$.
We shall generalize this technique in the following sections and chapters.

Exercises.

7.4.1. Let f be holomorphic in $D_0(1) \setminus \{0\}$. If the closed curve γ is in $D_0(1) \setminus \{0\}$ and $n(\gamma; 0) = 0$, evaluate $\oint_\gamma f(z)\,dz$.

7.4.2. Let f be holomorphic in $\mathbb{C}$ and let $f(1) = 6$, $f(-1) = 10$. Prove that, if γ is any closed curve in $\mathbb{C} \setminus \{-1, 1\}$, then $\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^2-1}\,dz$ can take every integral value.

7.4.3. Let $f(z) = (\frac{1}{z} + \frac{a}{z^3})e^z$ for $z \neq 0$. Find all the values of a so that $\oint_\gamma f(z)\,dz = 0$ for every closed curve γ in $\mathbb{C} \setminus \{0\}$.

7.4.4. (i) Find all possible values of $\oint_\gamma \frac{2z-1}{z^2-z}\,dz$, where γ is an arbitrary closed curve in $\mathbb{C} \setminus \{0, 1\}$.
(ii) Find all possible values of $\int_\gamma \frac{2z-1}{z^2-z}\,dz$, where γ is an arbitrary curve in $\mathbb{C} \setminus \{0, 1\}$ with initial endpoint $-i$ and final endpoint i.

7.4.5. Find all possible values of $\oint_\gamma \frac{\cos z}{z^2-\pi z}\,dz$, where γ is an arbitrary closed curve in $\mathbb{C} \setminus \{0, \pi\}$.

7.4.6. Let f be holomorphic in the open set Ω and γ be a closed curve null-homologous in Ω. If $|f(\zeta)| \leq 1$ for every $\zeta \in \gamma^*$, $z_0 \in \Omega$ and $n(\gamma; z_0) \neq 0$, prove that $|f(z_0)| \leq 1$.

7.5 The residue theorem.

Definition. *Let z_0 be an isolated singularity of f and let $\sum_{-\infty}^{+\infty} a_n(z-z_0)^n$ be the Laurent series of f in the ring $D_{z_0}(R) \setminus \{z_0\}$. Then the coefficient a_{-1} is called* **residue** *of f at z_0 and we denote*

$$\operatorname{Res}(f; z_0) = a_{-1}.$$

We know that

$$\operatorname{Res}(f; z_0) = a_{-1} = \frac{1}{2\pi i} \oint_{C_{z_0}(r)} f(\zeta)\, d\zeta \qquad \text{for } 0 < r < R.$$

Example 7.5.1. If z_0 is a removable singularity of f, then $a_n = 0$ for every $n < 0$ and in particular $\operatorname{Res}(f; z_0) = 0$.

Example 7.5.2. Every function of the form $f(z) = \frac{1}{(z-z_0)^N}$ with $N \geq 2$ has residue 0 at z_0.

Example 7.5.3. If z_0 is a pole of f of order $N \geq 1$, then we can find "easily" the residue of f at z_0. In this case there is a g holomorphic in a disc $D_{z_0}(R)$ so that $g(z_0) \neq 0$ and $f(z) = \frac{g(z)}{(z-z_0)^N}$ for every $z \in D_{z_0}(R) \setminus \{z_0\}$. From the Taylor series $\sum_{n=0}^{+\infty} b_n(z-z_0)^n$ of g we see that

$$\operatorname{Res}(f; z_0) = b_{N-1} = \frac{g^{(N-1)}(z_0)}{(N-1)!}.$$

For instance, if $N = 1$, then $\operatorname{Res}(f; z_0) = g(z_0)$ and, if $N = 2$, then $\operatorname{Res}(f; z_0) = g'(z_0)$.

Example 7.5.4. A particular case of the previous example is a rational function $r = \frac{p}{q}$ when z_0 is a root of the polynomial q of multiplicity N and not a root of the polynomial p. Then $q(z) = (z-z_0)^N q_1(z)$, where q_1 is a polynomial with $q_1(z_0) \neq 0$. The rational function $r_1 = \frac{p}{q_1}$ is holomorphic at z_0 and $r_1(z_0) \neq 0$. For some $R > 0$ we have

$$r(z) = \frac{r_1(z)}{(z-z_0)^N} \qquad \text{for every } z \in D_{z_0}(R) \setminus \{z_0\}.$$

According to the previous example,

$$\operatorname{Res}(r; z_0) = \frac{{r_1}^{(N-1)}(z_0)}{(N-1)!}.$$

If $N = 1$, then $\operatorname{Res}(r; z_0) = r_1(z_0) = \frac{p(z_0)}{q_1(z_0)}$. Differentiating $q(z) = (z-z_0)q_1(z)$, we get $q'(z) = q_1(z) + (z-z_0)q_1'(z)$, and hence $q_1(z_0) = q'(z_0)$. Thus

$$\operatorname{Res}(r; z_0) = \frac{p(z_0)}{q'(z_0)}.$$

If $N = 2$, then $\operatorname{Res}(r; z_0) = {r_1}'(z_0) = \frac{p'(z_0)q_1(z_0) - p(z_0){q_1}'(z_0)}{q_1(z_0)^2}$. We take $q(z) = (z-z_0)^2 q_1(z)$ and differentiate twice. We get $q''(z) = 2q_1(z) + 4(z-z_0)q_1'(z) + (z-z_0)^2 q_1''(z)$, and hence $q_1(z_0) = (1/2)q''(z_0)$. We differentiate a third time: $q'''(z) = 6q_1'(z) + 6(z-z_0)q_1''(z) + (z-z_0)^2 q_1'''(z)$. Hence $q_1'(z_0) = (1/6)q'''(z_0)$ and

$$\operatorname{Res}(r; z_0) = \frac{(1/2)p'(z_0)q''(z_0) - (1/6)p(z_0)q'''(z_0)}{(1/4)q''(z_0)^2}.$$

The residue theorem. *Let f be holomorphic, except for isolated singularities, in the open set Ω and Σ be a cycle null-homologous in Ω so that no isolated singularity of f is in the trajectory of any of the closed curves forming Σ. Then $n(\Sigma; z) \neq 0$ for at most finitely many isolated singularities z of f and hence the sum $\sum_{z \text{ sing. of } f} n(\Sigma; z) \operatorname{Res}(f; z)$, extended over all isolated singularities of f in Ω, is finite. Moreover,*

$$\frac{1}{2\pi i} \oint_{\Sigma} f(\zeta)\, d\zeta = \sum_{z \text{ sing. of } f} n(\Sigma; z) \operatorname{Res}(f; z).$$

Proof. Let us assume that $n(\Sigma; z) \neq 0$ for infinitely many isolated singularities z of f in Ω. Then there is a sequence (z_n) of distinct isolated singularities of f in Ω so that $n(\Sigma; z_n) \neq 0$ for every n. Since the trajectories of the curves which form Σ are bounded sets, Σ is in some disc $D_0(R)$. Hence $n(\Sigma; z) = 0$ for every z outside $D_0(R)$. Thus, the sequence (z_n) is in $D_0(R)$. The Bolzano-Weierstrass theorem implies that there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some z. Then z is a limit point of Ω.
If $z \in \Omega$, then either f is holomorphic at z or z is an isolated singularity of f. In any case, there are no isolated singularities of f in a neighborhood of z, except perhaps z itself. This contradicts z_{n_k} being distinct and $z_{n_k} \to z$.
If $z \in \partial\Omega$ and hence $z \in \Omega^c$, then $n(\Sigma; z) = 0$. Now there is a disc $D_z(r)$ which does not intersect any of the trajectories of the curves which form Σ. Since $D_z(r)$ is connected, we have that $n(\Sigma; w) = 0$ for every $w \in D_z(r)$. But this contradicts $z_{n_k} \to z$ and $n(\Sigma; z_{n_k}) \neq 0$ for all k.
In any case we arrive at a contradiction and thus $n(\Sigma; z) \neq 0$ for at most finitely many isolated singularities z of f in Ω. Therefore, the sum $\sum_{z \text{ sing. of } f} n(\Sigma; z)\ \mathrm{Res}(f; z)$ is finite.
Let $z_1, \dots, z_n$ be the isolated singularities of f in Ω with $n(\Sigma; z_k) \neq 0$ for $k = 1, \dots, n$. I.e. $n(\Sigma; z) = 0$ for every other isolated singularity z of f in Ω.
We define the integers

$$p_1 = n(\Sigma; z_1), \dots, p_n = n(\Sigma; z_n)$$

and then

$$\sum_{z \text{ sing. of } f} n(\Sigma; z)\ \mathrm{Res}(f; z) = \sum_{z \in \{z_1, \dots, z_n\}} n(\Sigma; z)\ \mathrm{Res}(f; z) = \sum_{k=1}^{n} p_k\ \mathrm{Res}(f; z_k).$$

Therefore, it is enough to prove

$$\frac{1}{2\pi i} \oint_{\Sigma} f(\zeta)\, d\zeta = \sum_{k=1}^{n} p_k\ \mathrm{Res}(f; z_k). \tag{7.13}$$

Since every $z_1, \dots, z_n$ is an isolated singularity, there are disjoint closed discs $\overline{D}_{z_k}(r_k)$ for $k = 1, \dots, n$ so that each of them contains no singularity of f except its center. We denote γ_k the closed curve which describes the circle $C_{z_k}(r_k)$ once and in the positive direction. We consider the cycle Σ' which is formed from Σ (i.e. from the closed curves of Σ with the same multiplicities) and from $\gamma_1, \dots, \gamma_n$ with multiplicities $-p_1, \dots, -p_n$. I.e.

$$\Sigma' = \Sigma + (-p_1)\, \gamma_1 + \cdots + (-p_n)\, \gamma_n.$$

Finally, we consider the open set

$$\Omega' = \Omega \setminus \{z \in \Omega \mid z \text{ singularity of } f\}.$$

Clearly, f is holomorphic in Ω' and we shall prove that the cycle Σ' is null-homologous in Ω', i.e. $n(\Sigma'; z) = 0$ for every $z \notin \Omega'$. If $z \notin \Omega'$, then either $z \notin \Omega$ or $z = z_1, \dots, z_n$ or z is any other isolated singularity of f in Ω.
If $z \notin \Omega$, then $n(\Sigma; z) = 0$ (because Σ is null-homologous in Ω) and $n(\gamma_k; z) = 0$ for every $k = 1, \dots, n$ (because $\overline{D}_{z_k}(r_k) \subseteq \Omega$ and $z \notin \Omega$). Therefore

$$n(\Sigma'; z) = n(\Sigma; z) - p_1 n(\gamma_1; z) - \cdots - p_n n(\gamma_n; z) = 0 - p_1 0 - \cdots - p_n 0 = 0.$$

If $z = z_{k_0}$ for some $k_0 = 1, \dots, n$, then $n(\Sigma; z) = n(\Sigma; z_{k_0}) = p_{k_0}$ and $n(\gamma_{k_0}; z) = n(\gamma_{k_0}; z_{k_0}) = 1$ and $n(\gamma_k; z) = n(\gamma_k; z_{k_0}) = 0$ for every $k = 1, \dots, k_0 - 1, k_0 + 1, \dots, n$. Therefore

$$\begin{aligned} n(\Sigma'; z) &= n(\Sigma; z) - p_1 n(\gamma_1; z) - \cdots - p_{k_0-1} n(\gamma_{k_0-1}; z) \\ &\qquad - p_{k_0} n(\gamma_{k_0}; z) - p_{k_0+1} n(\gamma_{k_0+1}; z) - \cdots - p_n n(\gamma_n; z) \\ &= p_{k_0} - p_1 0 - \cdots - p_{k_0-1} 0 - p_{k_0} 1 - p_{k_0+1} 0 - \cdots - p_n 0 = 0. \end{aligned}$$

If z is any isolated singularity of f in Ω different from $z_1, \dots, z_n$, then $n(\Sigma; z) = 0$ and $n(\gamma_k; z) = 0$ for every $k = 1, \dots, n$. Therefore

$$n(\Sigma'; z) = n(\Sigma; z) - p_1 n(\gamma_1; z) - \cdots - p_n n(\gamma_n; z) = 0 - p_1 0 - \cdots - p_n 0 = 0.$$

We proved that Σ' is null-homologous in Ω'.
Now, f is holomorphic in Ω' and the theorem of Cauchy implies $\oint_{\Sigma'} f(\zeta)\, d\zeta = 0$ and hence

$$\oint_{\Sigma} f(\zeta)\, d\zeta = \sum_{k=1}^{n} p_k \oint_{\gamma_k} f(\zeta)\, d\zeta = \sum_{k=1}^{n} p_k \oint_{C_{z_k}(r_k)} f(\zeta)\, d\zeta = 2\pi i \sum_{k=1}^{n} p_k \operatorname{Res}(f; z_k)$$

and we proved (7.13). □

The residue theorem is a powerful tool for the evaluation of integrals, because it reduces this evaluation to the location of the isolated sinularities of the function to be integrated, to the evaluation of the corresponding residues and to the calculation of the number of rotations of the cycle around each isolated singularity. In many cases all these are quite easy. Let us see some characteristic examples.

Example 7.5.5. *Evaluation of* $\int_{-\infty}^{+\infty} r(x)\, dx$*, where* $r = \frac{p}{q}$ *is a rational function,* $\deg q \geq \deg p + 2$ *and q has no real roots.*
Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$, with $a_n \neq 0$, and $q(x) = b_m x^m + \cdots + b_1 x + b_0$, with $b_m \neq 0$, and $m \geq n+2$. Then r is continuous in $\mathbb{R}$ and the generalized integral $\int_{-\infty}^{+\infty} r(x)\, dx$ converges. To see this, we observe that $\lim_{z\to\infty} \frac{p(z)}{a_n z^n} = 1$ and $\lim_{z\to\infty} \frac{q(z)}{b_m z^m} = 1$. Hence there is $R_0 > 0$ so that $\frac{1}{2} \leq |\frac{p(z)}{a_n z^n}| \leq 2$ and $\frac{1}{2} \leq |\frac{q(z)}{b_m z^m}| \leq 2$ when $|z| \geq R_0$. Therefore,

$$|r(z)| \leq 4\frac{|a_n|}{|b_m|}\frac{1}{|z|^{m-n}} = \frac{C}{|z|^{m-n}} \qquad \text{when } |z| \geq R_0, \tag{7.14}$$

where $C = 4\frac{|a_n|}{|b_m|}$. Now, since $m - n \geq 2$, we have that

$$\int_{-\infty}^{-R_0} |r(x)|\, dx \leq C \int_{-\infty}^{-R_0} \frac{1}{|x|^{m-n}}\, dx < +\infty, \qquad \int_{R_0}^{+\infty} |r(x)|\, dx \leq C \int_{R_0}^{+\infty} \frac{1}{x^{m-n}}\, dx < +\infty.$$

Thus, the integrals $\int_{-\infty}^{-R_0} r(x)\, dx$, $\int_{R_0}^{+\infty} r(x)\, dx$ converge absolutely and so they converge. Moreover, r is continuous in $[-R_0, R_0]$ and the integral $\int_{-\infty}^{+\infty} r(x)\, dx$ converges.
The roots of q are contained either in the upper halfplane or in the lower halfplane defined by the x-axis. We shall consider only the roots in the upper halfplane and let them be $z_1, \dots, z_M$, where $M \leq m$. We take any $R > R_0$ so that $z_1, \dots, z_M$ are contained in the disc $D_0(R)$, i.e.

$$R > \max\{R_0, |z_1|, \dots, |z_M|\}.$$

We apply the residue theorem with $r = \frac{p}{q}$ which is holomorphic in $\mathbb{C}$ except for the roots of q and with the closed curve γ_R which is the sum of the linear segment $[-R, R]$, with parametric equation $z = x$, $x \in [-R, R]$, and of the curve σ_R, with parametric equation $z = Re^{it}$, $t \in [0, \pi]$, which describes the upper semicircle of $C_0(R)$ from R to $-R$.
The trajectory of γ_R contains no isolated singularity of r. When we evaluate $n(\gamma_R; z) \operatorname{Res}(r; z)$ we consider only the isolated singularities z of r with $n(\gamma_R; z) \neq 0$: these are $z_1, \dots, z_M$. In fact, γ_R rotates around each of $z_1, \dots, z_M$ once and in the positive direction:

$$n(\gamma_R; z_1) = \ldots = n(\gamma_R; z_M) = 1.$$

Then the residue theorem implies

$$\frac{1}{2\pi i} \oint_{\gamma_R} r(z)\, dz = \operatorname{Res}(r; z_1) + \cdots + \operatorname{Res}(r; z_M).$$

We have $\oint_{\gamma_R} r(z)\,dz = \int_{[-R,R]} r(z)\,dz + \int_{\sigma_R} r(z)\,dz$ and hence

$$\int_{-R}^{R} r(x)\,dx = \int_{[-R,R]} r(z)\,dz = 2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M)) - \int_{\sigma_R} r(z)\,dz. \quad (7.15)$$

Since $R > R_0$, (7.14) together with $m-n-1 \geq 1$ imply

$$\Big|\int_{\sigma_R} r(z)\,dz\Big| \leq \frac{C}{R^{m-n}}\,\pi R = \frac{C\pi}{R^{m-n-1}} \to 0 \qquad \text{when } R\to+\infty$$

and from (7.15) we get

$$\int_{-\infty}^{+\infty} r(x)\,dx = 2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M)).$$

Thus, to evaluate $\int_{-\infty}^{+\infty} r(x)\,dx$ we need only to find the residues of r at the poles $z_1,\dots,z_M$ of r in the upper halfplane.

Example 7.5.6. *Evaluation of* pv $\int_{-\infty}^{+\infty} r(x)\,dx$, *where* $r = \frac{p}{q}$ *is a rational function,* $\deg q = \deg p + 1$ *and* q *has no real root.*
Let $p(x) = a_nx^n + \cdots + a_1x + a_0$, with $a_n \neq 0$, and $q(x) = b_{n+1}x^{n+1} + \cdots + b_1x + b_0$, with $b_{n+1} \neq 0$. It easy to see that the generalized integral $\int_{-\infty}^{+\infty} r(x)\,dx$ does not converge. Indeed, we recall the estimates in the previous example: $\frac{1}{2}|a_n||z|^n \leq |p(z)| \leq 2|a_n||z|^n$ and $\frac{1}{2}|b_{n+1}||z|^{n+1} \leq |q(z)| \leq 2|b_{n+1}||z|^{n+1}$ when $|z| \geq R_0$. Thus, $|r(z)| \geq \frac{1}{4}\frac{|a_n|}{|b_{n+1}|}\frac{1}{|z|} = \frac{c}{|z|}$ when $|z| \geq R_0$, where $c = \frac{1}{4}\frac{|a_n|}{|b_{n+1}|} > 0$. Therefore, for real $z = x$ we have that $|r(x)| \geq \frac{c}{x}$ when $x \geq R_0$. Now, r has constant sign in $[R_0,+\infty)$. If r is positive in $[R_0,+\infty)$, then $\int_{R_0}^{+\infty} r(x)\,dx \geq c\int_{R_0}^{+\infty}\frac{1}{x}\,dx = +\infty$ and, if r is negative in $[R_0,+\infty)$, then $\int_{R_0}^{+\infty} r(x)\,dx \leq -c\int_{R_0}^{+\infty}\frac{1}{x}\,dx = -\infty$. Similarly, $\int_{-\infty}^{-R_0} r(x)\,dx$ is either $+\infty$ or $-\infty$.
Since the generalized integral diverges, we examine its **principal value**, i.e.

$$\mathrm{pv}\int_{-\infty}^{+\infty} r(x)\,dx = \lim_{R\to+\infty}\int_{-R}^{R} r(x)\,dx.$$

We observe that

$$r(z) - \frac{a_n}{b_{n+1}}\,\frac{1}{z} = \frac{(a_{n-1}b_{n+1}-a_nb_n)z^n + \cdots + (a_0b_{n+1}-a_nb_1)z - a_nb_0}{b_{n+1}z^{n+2}+\cdots+b_1z^2+b_0z}.$$

This is a rational function whose denominator has degree two units larger than the degree of its numerator. According to the previous example, there is $R_0 > 0$ so that

$$\Big|r(z) - \frac{a_n}{b_{n+1}}\,\frac{1}{z}\Big| \leq 4\frac{|a_{n-1}b_{n+1}-a_nb_n|}{|b_{n+1}|}\,\frac{1}{|z|^2} = \frac{C}{|z|^2} \qquad \text{when } |z| \geq R_0, \quad (7.16)$$

with $C = 4\frac{|a_{n-1}b_{n+1}-a_nb_n|}{|b_{n+1}|}$. As in the previous example, we consider the roots $z_1,\dots,z_M$ of q in the upper halfplane and we take $R > \max\{R_0,|z_1|,\dots,|z_M|\}$. We apply the residue theorem with $r = \frac{p}{q}$ and with the same closed curve γ_R and we get

$$\frac{1}{2\pi i}\oint_{\gamma_R} r(z)\,dz = \mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M).$$

Now, $\oint_{\gamma_R} r(z)\,dz = \int_{[-R,R]} r(z)\,dz + \int_{\sigma_R} r(z)\,dz$ and hence

$$\begin{aligned}\int_{-R}^{R} r(x)\,dx &= 2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M)) - \int_{\sigma_R} r(z)\,dz\\ &= 2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M)) - \int_{\sigma_R}\Big(r(z)-\frac{a_n}{b_{n+1}}\,\frac{1}{z}\Big)\,dz \qquad (7.17)\\ &\quad - \frac{a_n}{b_{n+1}}\int_{\sigma_R}\frac{1}{z}\,dz.\end{aligned}$$

The last term is $\frac{a_n}{b_{n+1}}\int_{\sigma_R}\frac{1}{z}\,dz=\frac{a_n}{b_{n+1}}\int_0^\pi\frac{1}{Re^{it}}\,iRe^{it}\,dt=i\pi\,\frac{a_n}{b_{n+1}}$. Since $R>R_0$, we have from (7.16) that

$$\Big|\int_{\sigma_R}\Big(r(z)-\frac{a_n}{b_{n+1}}\,\frac{1}{z}\Big)\,dz\Big|\leq\frac{C}{R^2}\,\pi R=\frac{C\pi}{R}\to 0\qquad\text{when } R\to+\infty$$

and (7.17) implies

$$\mathrm{pv}\int_{-\infty}^{+\infty}r(x)\,dx=2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M))-i\pi\,\frac{a_n}{b_{n+1}}.$$

Example 7.5.7. *Evaluation of* pv $\int_{-\infty}^{+\infty}r(x)\,dx$, *where* $r=\frac{p}{q}$ *is a rational function,* $\deg q\geq\deg p+1$ *and* q *has real roots, all with multiplicity* 1.

Let $p(x)=a_nx^n+\cdots+a_1x+a_0$, with $a_n\neq 0$, and $q(x)=b_mx^m+\cdots+b_1x+b_0$, with $b_m\neq 0$, and $m\geq n+1$. We assume that the real roots of q are $x_1,\ldots,x_n$ with $x_1<\ldots<x_n$ and that these are not roots of p. We take $\epsilon_0>0$ so that the intervals $[x_1-\epsilon_0,x_1+\epsilon_0],\ldots,[x_n-\epsilon_0,x_n+\epsilon_0]$ around the real roots of q are disjoint. In order for $\int_{-\infty}^{+\infty}r(x)\,dx$ to converge, the generalized integrals $\int_{x_k-\epsilon_0}^{x_k}r(x)\,dx$ and $\int_{x_k}^{x_k+\epsilon_0}r(x)\,dx$ must converge for every x_k. This is not correct. Indeed, we write $r(z)=\frac{p(z)}{(z-x_k)q_k(z)}=\frac{g_k(z)}{z-x_k}$, where q_k is a polynomial with $q_k(x_k)\neq 0$ and where $g_k=\frac{p}{q_k}$ is a rational function holomorphic at x_k. Since $\lim_{z\to x_k}g_k(z)=g_k(x_k)\neq 0$, there is ϵ_k with $0<\epsilon_k\leq\epsilon_0$ so that $|g_k(z)|\geq\frac{1}{2}\,|g_k(x_k)|$ for every z with $|z-x_k|\leq\epsilon_k$. Hence, $|r(z)|\geq\frac{1}{2}\,\frac{|g_k(x_k)|}{|z-x_k|}$ for every z with $0<|z-x_k|\leq\epsilon_k$. The function r has constant sign in $(x_k,x_k+\epsilon_k]$. If r is positive in $(x_k,x_k+\epsilon_k]$, then $\int_{x_k}^{x_k+\epsilon_k}r(x)\,dx\geq\frac{|g_k(x_k)|}{2}\int_{x_k}^{x_k+\epsilon_k}\frac{1}{x-x_k}\,dx=+\infty$ and, similarly, if r is negative in $(x_k,x_k+\epsilon_k]$, then $\int_{x_k}^{x_k+\epsilon_k}r(x)\,dx\leq-\frac{|g_k(x_k)|}{2}\int_{x_k}^{x_k+\epsilon_k}\frac{1}{x-x_k}\,dx=-\infty$. Thus, the generalized integral $\int_{x_k}^{x_k+\epsilon_k}r(x)\,dx$ does not converge. Similarly, $\int_{x_k-\epsilon_k}^{x_k}r(x)\,dx$ does not converge either. This is why we examine the **principal value** of $\int_{-\infty}^{+\infty}r(x)\,dx$, i.e.

$$\begin{aligned}\mathrm{pv}\int_{-\infty}^{+\infty}r(x)\,dx&=\lim_{R\to+\infty,\epsilon\to 0+}\Big(\int_{-R}^{x_1-\epsilon}r(x)\,dx+\int_{x_1+\epsilon}^{x_2-\epsilon}r(x)\,dx+\cdots\\&\cdots+\int_{x_{n-1}+\epsilon}^{x_n-\epsilon}r(x)\,dx+\int_{x_n+\epsilon}^{R}r(x)\,dx\Big)=\lim_{R\to+\infty,\epsilon\to 0+}I(R,\epsilon).\end{aligned}\tag{7.18}$$

To evaluate $I(R,\epsilon)$ we use a variant of the curve γ_R in the two previous examples. We consider

$$R>\max\{R_0,|z_1|,\ldots,|z_M|,x_n+1,-x_1+1\},\quad\epsilon<\min\{1,\epsilon_0,\mathrm{Im}\,z_1,\ldots,\mathrm{Im}\,z_M\},$$

where $z_1,\ldots,z_M$ are the roots of q in the upper halfplane. Now, we consider the closed curve $\gamma_{R,\epsilon}$, the sum of the linear segments $[-R,x_1-\epsilon],[x_1+\epsilon,x_2-\epsilon],\ldots,[x_{n-1}+\epsilon,x_n-\epsilon],[x_n+\epsilon,R]$, of the curve σ_R, which describes the upper semicircle of $C_0(R)$ from R to $-R$, and of the curves $\sigma_{1,\epsilon},\ldots,\sigma_{n,\epsilon}$, where each $\sigma_{k,\epsilon}$ describes the upper semicircle of the corresponding $C_{x_k}(\epsilon)$ from $x_k-\epsilon$ to $x_k+\epsilon$. The curve $\gamma_{R,\epsilon}$ rotates once and in the positive direction around each of the roots $z_1,\ldots,z_M$ of q and no times around each of the remaining roots of q. The residue theorem implies that

$$\oint_{\gamma_{R,\epsilon}}r(z)\,dz=2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M)),$$

and hence

$$\begin{aligned}I(R,\epsilon)=2\pi i(\mathrm{Res}(r;z_1)+\cdots+\mathrm{Res}(r;z_M))-\int_{\sigma_R}r(z)\,dz\\-\int_{\sigma_{1,\epsilon}}r(z)\,dz-\cdots-\int_{\sigma_{n,\epsilon}}r(z)\,dz.\end{aligned}\tag{7.19}$$

Now, x_k is a pole of r of order 1 and r can be written $r(z) = \frac{c_k}{z-x_k} + f_k(z)$ for $z \neq x_k$ in a disc with center x_k, where f_k is holomorphic at x_k and $c_k = \text{Res}(r; x_k)$. Since f_k is bounded in a disc with center x_k, there is $M_k \geq 0$ and $\epsilon'_k > 0$ so that $|f_k(z)| \leq M_k$ for $|z - x_k| \leq \epsilon'_k$. Thus, $0 < \epsilon \leq \epsilon'_k$ implies $|\int_{\sigma_{k,\epsilon}} f_k(z)\,dz| \leq M_k \pi \epsilon$ and hence $\lim_{\epsilon \to 0+} \int_{\sigma_{k,\epsilon}} f_k(z)\,dz = 0$. Therefore,

$$\begin{aligned}\int_{\sigma_{k,\epsilon}} r(z)\,dz &= c_k \int_{\sigma_{k,\epsilon}} \frac{1}{z-x_k}\,dz + \int_{\sigma_{k,\epsilon}} f_k(z)\,dz \\ &= -\pi i c_k + \int_{\sigma_{k,\epsilon}} f_k(z)\,dz \to -\pi i c_k \qquad \text{when } \epsilon \to 0+\,.\end{aligned} \tag{7.20}$$

The limit of $\int_{\sigma_R} r(z)\,dz$ when $R \to +\infty$ has been evaluated in the previous two examples:

$$\lim_{R\to+\infty} \int_{\sigma_R} r(z)\,dz = \begin{cases} 0, & \text{if } m \geq n+2 \\ i\pi \frac{a_n}{b_{n+1}}, & \text{if } m = n+1 \end{cases} \tag{7.21}$$

Now, (7.18), (7.19), (7.20) and (7.21) imply

$$\begin{aligned}\text{pv}\int_{-\infty}^{+\infty} r(x)\,dx &= 2\pi i(\text{Res}(r; z_1) + \cdots + \text{Res}(r; z_M)) \\ &\quad + \pi i(\text{Res}(r; x_1) + \cdots + \text{Res}(r; x_n)) - \begin{cases} 0, & \text{if } m \geq n+2 \\ i\pi \frac{a_n}{b_{n+1}}, & \text{if } m = n+1 \end{cases}\end{aligned}$$

Example 7.5.8. *Evaluation of* $\int_{-\infty}^{+\infty} r(x)\cos x\,dx$, $\int_{-\infty}^{+\infty} r(x)\sin x\,dx$ *(or of their principal values), where* $r = \frac{p}{q}$ *is a rational function,* $\deg q \geq \deg p + 1$, *the real roots of* q *(if they exist) have multiplicity* 1 *and, also, the coefficients of* p, q *are real numbers.*

Since the coefficients of p, q are real, we have that $r(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$ which is not a root of q. Hence,

$$\int_{-\infty}^{+\infty} r(x)\cos x\,dx = \text{Re}\int_{-\infty}^{+\infty} r(x)e^{ix}\,dx, \qquad \int_{-\infty}^{+\infty} r(x)\sin x\,dx = \text{Im}\int_{-\infty}^{+\infty} r(x)e^{ix}\,dx$$

and we evaluate $\int_{-\infty}^{+\infty} r(x)e^{ix}\,dx$ (or its principal value).

The method of evaluation has been described already in the previous three examples. We use either the curve γ_R or the curve $\gamma_{R,\epsilon}$ and we evaluate the residues of $r(z)e^{iz}$ at the roots of q.

We shall concentrate on the important specific generalized integral $\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$. (Equality holds because $\frac{\sin x}{x}$ is even.) We shall evaluate $\text{pv}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx$ instead of $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$. Observe that $\frac{e^{ix}}{x} = \frac{\cos x}{x} + i\frac{\sin x}{x}$ diverges at 0 because its real part $\frac{\cos x}{x}$ diverges at 0. The imaginary part $\frac{\sin x}{x}$ converges at 0 and, in fact, if we define $\frac{\sin x}{x}$ at 0 to have value $\lim_{x\to 0} \frac{\sin x}{x} = 1$, then it becomes continuous at 0.

The function $\frac{e^{iz}}{z}$ is holomorphic in $\mathbb{C}$ except for a pole at 0 of order 1. We consider the closed curve $\gamma_{R,\epsilon}$ which is the sum of the linear segments $[-R, -\epsilon]$ and $[\epsilon, R]$, of the curve σ_R, which describes the upper semicircle of $C_0(R)$ from R to $-R$, and of the curve σ_ϵ, which describes the upper semicircle of $C_0(\epsilon)$ from $-\epsilon$ to ϵ. Then $\gamma_{R,\epsilon}$ does not rotate around the pole 0 of $\frac{e^{iz}}{z}$. The residue theorem implies $\oint_{\gamma_{R,\epsilon}} \frac{e^{iz}}{z}\,dz = 0$ and hence

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx = -\int_{\sigma_R} \frac{e^{iz}}{z}\,dz - \int_{\sigma_\epsilon} \frac{e^{iz}}{z}\,dz. \tag{7.22}$$

Now, $\int_{\sigma_R} \frac{e^{iz}}{z}\,dz = \int_0^\pi \frac{e^{iRe^{it}}}{Re^{it}}\,iRe^{it}\,dt = i\int_0^\pi e^{-R\sin t + iR\cos t}\,dt$ and

$$\begin{aligned}\Big|\int_{\sigma_R} \frac{e^{iz}}{z}\,dz\Big| &\leq \int_0^\pi e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt \leq 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt \\ &= \frac{\pi}{R}(1 - e^{-R}) \to 0 \qquad \text{when } R \to +\infty.\end{aligned} \tag{7.23}$$

For the second inequality we used the well known inequality $\sin t \geq \frac{2t}{\pi}$ for $0 \leq t \leq \frac{\pi}{2}$. From the Laurent series of $\frac{e^{iz}}{z}$ at 0 we see that $\frac{e^{iz}}{z} = \frac{1}{z} + h(z)$ for $z \neq 0$, where h is holomorphic in $\mathbb{C}$. Now, h is bounded in a neighborhood of 0, i.e. there is $M \geq 0$ so that $|h(z)| \leq 1$ when $|z| \leq 1$. Hence, for $\epsilon \leq 1$ we have $|\int_{\sigma_\epsilon} h(z)\,dz| \leq M\pi\epsilon \to 0$ when $\epsilon \to 0+$. Therefore

$$\begin{aligned}\int_{\sigma_\epsilon} \frac{e^{iz}}{z}\,dz &= \int_{\sigma_\epsilon} \frac{1}{z}\,dz + \int_{\sigma_\epsilon} h(z)\,dz \\ &= -\pi i + \int_{\sigma_\epsilon} h(z)\,dz \to -\pi i \qquad \text{when } \epsilon \to 0+\,.\end{aligned} \tag{7.24}$$

Finally, (7.22), (7.23) and (7.24) imply

$$\mathrm{pv}\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx = \lim_{\epsilon\to 0+, R\to+\infty} \Big(\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx\Big) = \pi i.$$

Since $\frac{\cos x}{x}$ is odd and $\frac{\sin x}{x}$ is even, we get $\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx = 2i\int_{\epsilon}^{R} \frac{\sin x}{x}\,dx$ and hence

$$\int_0^{+\infty} \frac{\sin x}{x}\,dx = \lim_{\epsilon\to 0+, R\to+\infty} \int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Example 7.5.9. We shall evaluate $\int_0^{+\infty} \frac{\ln x}{x^2+4}\,dx$.
We consider the holomorphic branch of the logarithm, which we shall denote $\log z$, in the open region A which is $\mathbb{C}$ without the negative y-semiaxis (with 0) and which takes the value 0 at 1. This branch is given by

$$\log z = \ln r + i\theta \qquad \text{for } z = re^{i\theta} \text{ with } r > 0 \text{ and } -\frac{\pi}{2} < \theta < \frac{3\pi}{2}.$$

The function $\frac{\log z}{z^2+4}$ is holomorphic in A except for the point $2i$ which is a pole of order 1. Indeed, we write $\frac{\log z}{z^2+4} = \frac{(\log z)/(z+2i)}{z-2i} = \frac{g(z)}{z-2i}$ and we have that $g(z) = \frac{\log z}{z+2i}$ is holomorphic in A with $g(2i) = \frac{\pi}{8} - \frac{\ln 2}{4}\,i$. Moreover, $\mathrm{Res}(\frac{\log z}{z^2+4}; 2i) = g(2i) = \frac{\pi}{8} - \frac{\ln 2}{4}\,i$.
Now we consider the closed curve $\gamma_{R,\epsilon}$ of the previous example. The residue theorem implies

$$\oint_{\gamma_{R,\epsilon}} \frac{\log z}{z^2+4}\,dz = 2\pi i\,\mathrm{Res}\Big(\frac{\log z}{z^2+4}; 2i\Big) = \frac{\pi\ln 2}{2} + \frac{\pi^2}{4}\,i,$$

and hence

$$\int_{-R}^{-\epsilon} \frac{\ln|x| + i\pi}{x^2+4}\,dx + \int_{\epsilon}^{R} \frac{\ln x}{x^2+4}\,dx = \frac{\pi\ln 2}{2} + \frac{\pi^2}{4}\,i - \int_{\sigma_R} \frac{\log z}{z^2+4}\,dz - \int_{\sigma_\epsilon} \frac{\log z}{z^2+4}\,dz.$$

Thus,

$$2\int_{\epsilon}^{R} \frac{\ln x}{x^2+4}\,dx + i\pi\int_{\epsilon}^{R} \frac{1}{x^2+4}\,dx = \frac{\pi\ln 2}{2} + \frac{\pi^2}{4}\,i - \int_{\sigma_R} \frac{\log z}{z^2+4}\,dz - \int_{\sigma_\epsilon} \frac{\log z}{z^2+4}\,dz.$$

Taking real parts, we get

$$2\int_{\epsilon}^{R} \frac{\ln x}{x^2+4}\,dx = \frac{\pi\ln 2}{2} - \mathrm{Re}\int_{\sigma_R} \frac{\log z}{z^2+4}\,dz - \mathrm{Re}\int_{\sigma_\epsilon} \frac{\log z}{z^2+4}\,dz. \tag{7.25}$$

We estimate:

$$\Big|\int_{\sigma_R} \frac{\log z}{z^2+4}\,dz\Big| \leq \frac{\ln R + \pi}{R^2-4}\,\pi R \to 0 \qquad \text{when } R \to +\infty \tag{7.26}$$

and

$$\Big|\int_{\sigma_\epsilon}\frac{\log z}{z^2+4}\,dz\Big|\leq\frac{\ln\epsilon+\pi}{4-\epsilon^2}\,\pi\epsilon\to 0\qquad\text{when }\epsilon\to 0+\,. \tag{7.27}$$

From (7.25), (7.26) and (7.27) we have

$$\int_0^{+\infty}\frac{\ln x}{x^2+4}\,dx=\lim_{\epsilon\to 0+,R\to+\infty}\int_\epsilon^R\frac{\ln x}{x^2+4}\,dx=\frac{\pi\ln 2}{4}.$$

Example 7.5.10. We shall evaluate $\int_0^{+\infty}\frac{x^{a-1}}{x+1}\,dx$ when $0<a<1$.
We write x^2 instead of x:

$$\int_0^{+\infty}\frac{x^{a-1}}{x+1}\,dx=2\int_0^{+\infty}\frac{x^{2a-1}}{x^2+1}\,dx=2\int_0^{+\infty}\frac{x^b}{x^2+1}\,dx \tag{7.28}$$

with $b=2a-1$ and $-1<b<1$.
We consider the holomorphic branch $\log z$ of the previous example in the same region A. The function $h(z)=e^{b\log z}$ is holomorphic in A and, if $z=x$ is real, we have $h(x)=e^{b\ln x}=x^b$. The function $\frac{h(z)}{z^2+1}$ is holomorphic in A except for a pole at i of order 1. Indeed, we write $\frac{h(z)}{z^2+1}=\frac{h(z)/(z+i)}{z-i}=\frac{g(z)}{z-i}$ and we have that $g(z)=\frac{h(z)}{z+i}$ is holomorphic in A with $g(i)=\frac{h(i)}{2i}=\frac{e^{\frac{b\pi}{2}i}}{2i}$. Moreover, $\mathrm{Res}(\frac{h(z)}{z^2+1};i)=g(i)=\frac{e^{\frac{b\pi}{2}i}}{2i}$. Now we consider the same closed curve $\gamma_{R,\epsilon}$ of the previous example. The residue theorem implies $\oint_{\gamma_{R,\epsilon}}\frac{h(z)}{z^2+1}\,dz=2\pi i\,\mathrm{Res}(\frac{h(z)}{z^2+1};i)=\pi e^{\frac{b\pi}{2}i}$, and hence

$$\int_{-R}^{-\epsilon}\frac{h(x)}{x^2+1}\,dx+\int_\epsilon^R\frac{h(x)}{x^2+1}\,dx=\pi e^{\frac{b\pi}{2}i}-\int_{\sigma_R}\frac{h(z)}{z^2+1}\,dz-\int_{\sigma_\epsilon}\frac{h(z)}{z^2+1}\,dz.$$

Thus,

$$\int_\epsilon^R\frac{h(-x)}{x^2+1}\,dx+\int_\epsilon^R\frac{h(x)}{x^2+1}\,dx=\pi e^{\frac{b\pi}{2}i}-\int_{\sigma_R}\frac{h(z)}{z^2+1}\,dz-\int_{\sigma_\epsilon}\frac{h(z)}{z^2+1}\,dz$$

and

$$(e^{b\pi i}+1)\int_\epsilon^R\frac{x^b}{x^2+1}\,dx=\pi e^{\frac{b\pi}{2}i}-\int_{\sigma_R}\frac{h(z)}{z^2+1}\,dz-\int_{\sigma_\epsilon}\frac{h(z)}{z^2+1}\,dz. \tag{7.29}$$

Now we estimate:

$$\Big|\int_{\sigma_R}\frac{h(z)}{z^2+1}\,dz\Big|\leq\frac{R^b}{R^2-1}\,\pi R\to 0\qquad\text{when }R\to+\infty \tag{7.30}$$

and

$$\Big|\int_{\sigma_\epsilon}\frac{h(z)}{z^2+1}\,dz\Big|\leq\frac{\epsilon^b}{1-\epsilon^2}\,\pi\epsilon\to 0\qquad\text{when }\epsilon\to 0+\,. \tag{7.31}$$

From (7.29), (7.30) and (7.31) we get

$$\int_0^{+\infty}\frac{x^b}{x^2+1}\,dx=\lim_{\epsilon\to 0+,R\to+\infty}\int_\epsilon^R\frac{x^b}{x^2+1}\,dx=\frac{\pi e^{\frac{b\pi}{2}i}}{e^{b\pi i}+1}=\frac{\pi}{2\cos(\frac{b\pi}{2})}.$$

Finaly, (7.28) implies

$$\int_0^{+\infty}\frac{x^{a-1}}{x+1}\,dx=\frac{\pi}{\sin a\pi}.$$

We shall evaluate $\int_0^{+\infty}\frac{x^{a-1}}{x+1}\,dx$ in a different way.
We consider the holomorphic branch of the logarithm, which we shall denote $\log z$ again, in the

region B which is $\mathbb{C}$ without the positive x-semiaxis (with 0) and which takes the value $i\pi$ at -1. This branch is given by

$$\log z = \ln r + i\theta \qquad \text{for } z = re^{i\theta} \text{ with } r > 0 \text{ and } 0 < \theta < 2\pi.$$

The function $h(z) = e^{(a-1)\log z}$ is holomorphic in B, and hence $\frac{h(z)}{z+1}$ is holomorphic in B except at the point -1 which is a pole of order 1. Indeed, we have $\mathrm{Res}(\frac{h(z)}{z+1}; -1) = h(-1) = e^{(a-1)\pi i}$. We also consider the closed curve $\gamma_{R,\epsilon,\delta}$ which is the sum of the curve $\sigma_{R,\delta}$, which describes the arc of $C_0(R)$ from $Re^{i\delta}$ to $Re^{i(2\pi-\delta)}$ in the positive direction, of the curve $\sigma_{\epsilon,\delta}$, which describes the arc of $C_0(\epsilon)$ from $\epsilon e^{i(2\pi-\delta)}$ to $\epsilon e^{i\delta}$ in the negative direction, of the linear segment $[\epsilon e^{i\delta}, Re^{i\delta}]$ and of the linear segment $[Re^{i(2\pi-\delta)}, \epsilon e^{i(2\pi-\delta)}]$. The residue theorem implies that $\oint_{\gamma_{R,\epsilon,\delta}} \frac{h(z)}{z+1}\, dz = 2\pi i \,\mathrm{Res}(\frac{h(z)}{z+1}; -1) = 2\pi i e^{(a-1)\pi i}$ and hence

$$\begin{aligned}\int_{[\epsilon e^{i\delta}, Re^{i\delta}]} \frac{h(z)}{z+1}\, dz &+ \int_{[Re^{i(2\pi-\delta)}, \epsilon e^{i(2\pi-\delta)}]} \frac{h(z)}{z+1}\, dz \\ &= 2\pi i e^{(a-1)\pi i} - \int_{\sigma_{R,\delta}} \frac{h(z)}{z+1}\, dz - \int_{\sigma_{\epsilon,\delta}} \frac{h(z)}{z+1}\, dz.\end{aligned} \tag{7.32}$$

We have $\int_{[\epsilon e^{i\delta}, Re^{i\delta}]} \frac{h(z)}{z+1}\, dz = e^{ia\delta} \int_\epsilon^R \frac{r^{a-1}}{re^{i\delta}+1}\, dr$. Keeping ϵ and R fixed, we take the limit when $\delta \to 0+$. Clearly, $e^{ia\delta} \to 1$. Also, $\int_\epsilon^R \frac{r^{a-1}}{re^{i\delta}+1}\, dr \to \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr$. We can see this as follows:

$$\begin{aligned}\Big| \int_\epsilon^R \frac{r^{a-1}}{re^{i\delta}+1}\, dr - \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr \Big| &\le \int_\epsilon^R \Big| \frac{r^{a-1}}{re^{i\delta}+1} - \frac{r^{a-1}}{r+1} \Big|\, dr \\ = |e^{i\delta} - 1| \int_\epsilon^R \frac{r^a}{|re^{i\delta}+1|(r+1)}\, dr &\le |e^{i\delta} - 1| \int_\epsilon^R r^a\, dr \to 0 \qquad \text{when } \delta \to 0+.\end{aligned}$$

Therefore,

$$\int_{[\epsilon e^{i\delta}, Re^{i\delta}]} \frac{h(z)}{z+1}\, dz \to \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr \qquad \text{when } \delta \to 0+. \tag{7.33}$$

We also have $\int_{[Re^{i(2\pi-\delta)}, \epsilon e^{i(2\pi-\delta)}]} \frac{h(z)}{z+1}\, dz = -e^{ia(2\pi-\delta)} \int_\epsilon^R \frac{r^{a-1}}{re^{-i\delta}+1}\, dr$. Keeping ϵ and R fixed, we take the limit when $\delta \to 0+$. Clearly, $e^{ia(2\pi-\delta)} \to e^{i2a\pi}$. Also, $\int_\epsilon^R \frac{r^{a-1}}{re^{-i\delta}+1}\, dr \to \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr$. This is proved in the same way as the previous analogous limit. Therefore,

$$\int_{[Re^{i(2\pi-\delta)}, \epsilon e^{i(2\pi-\delta)}]} \frac{h(z)}{z+1}\, dz \to -e^{i2a\pi} \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr \qquad \text{when } \delta \to 0+. \tag{7.34}$$

Now we estimate:

$$\Big| \int_{\sigma_{R,\delta}} \frac{h(z)}{z+1}\, dz \Big| \le \frac{R^{a-1}}{R-1} (2\pi - 2\delta) R \le \frac{2\pi R^a}{R-1}. \tag{7.35}$$

and

$$\Big| \int_{\sigma_{\epsilon,\delta}} \frac{h(z)}{z+1}\, dz \Big| \le \frac{\epsilon^{a-1}}{1-\epsilon} (2\pi - 2\delta) \epsilon \le \frac{2\pi \epsilon^a}{1-\epsilon}. \tag{7.36}$$

Using (7.35) and (7.36), from (7.32) we get

$$\Big| \int_{[\epsilon e^{i\delta}, Re^{i\delta}]} \frac{h(z)}{z+1}\, dz + \int_{[Re^{i(2\pi-\delta)}, \epsilon e^{i(2\pi-\delta)}]} \frac{h(z)}{z+1}\, dz - 2\pi i e^{(a-1)\pi i} \Big| \le \frac{2\pi R^a}{R-1} + \frac{2\pi \epsilon^a}{1-\epsilon}.$$

Now, letting $\delta \to 0+$, from (7.33) and (7.34) we get

$$\Big| (1 - e^{i2a\pi}) \int_\epsilon^R \frac{r^{a-1}}{r+1}\, dr - 2\pi i e^{(a-1)\pi i} \Big| \le \frac{2\pi R^a}{R-1} + \frac{2\pi \epsilon^a}{1-\epsilon}.$$

Finally, we let $\epsilon \to 0+$ and $R \to +\infty$ and we conclude that

$$\int_0^{+\infty} \frac{x^{a-1}}{x+1}\,dx = \lim_{\epsilon\to 0+, R\to+\infty} \int_\epsilon^R \frac{r^{a-1}}{r+1}\,dr = \frac{2\pi i e^{(a-1)\pi i}}{1-e^{i2a\pi}} = \frac{\pi}{\sin a\pi}.$$

Example 7.5.11. *Evaluation of $\int_0^{2\pi} r(\sin\theta,\cos\theta)\,d\theta$, where $r(s,t)$ is a rational function of two variables.*
We parametrize $C_0(1)$ with $z = e^{i\theta}$, $\theta \in [0,2\pi]$, and we have $\cos\theta = \frac{1}{2}(z+\frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z-\frac{1}{z})$ and $\frac{dz}{d\theta} = ie^{i\theta} = iz$. Hence

$$\int_0^{2\pi} r(\sin\theta,\cos\theta)\,d\theta = \frac{1}{i}\oint_{C_0(1)} r\Big(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\Big)\,\frac{1}{z}\,dz.$$

The function $s(z) = r(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz})\frac{1}{z}$ is a rational function of z. We apply the residue theorem after we evaluate the residues of s at its poles in the disc $D_0(1)$.

Exercises.

7.5.1. Find the singular parts as well as the residues of $\frac{1}{z^2+5z+6}$, $\frac{1}{(z^2-1)^2}$, $e^z + e^{1/z}$, $\frac{\cos z - 1}{z^4}$ at their isolated singularities.

7.5.2. Find the residues of $\frac{1}{\sin z}$, $\tan z$, $\frac{1}{\sin^2 z}$, $\frac{1}{e^z-1}$ at their isolated singularities.

7.5.3. If $f = gh$, where g is holomorphic at z_0 and h has a pole of order 1 at z_0, prove that $\mathrm{Res}(f;z_0) = g(z_0)\,\mathrm{Res}(h;z_0)$.

7.5.4. Evaluate $\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx$, $\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$, $\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)^2}\,dx$, $\int_{-\infty}^{+\infty} \frac{x^4}{1+x^8}\,dx$.

7.5.5. Evaluate pv $\int_{-\infty}^{+\infty} \frac{x+1}{x^2+1}\,dx$, pv $\int_{-\infty}^{+\infty} \frac{x^3}{x^4-4x^2+5}\,dx$, pv $\int_{-\infty}^{+\infty} \frac{x^2+3}{x(x^2+1)}\,dx$.

7.5.6. Evaluate $\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx$, $\int_{-\infty}^{+\infty} \frac{x^3\sin x}{x^4+1}\,dx$, pv $\int_{-\infty}^{+\infty} \frac{\sin x}{x(x^2+1)}\,dx$.

7.5.7. Evaluate $\int_0^{2\pi} \frac{1}{(1-a\cos\theta)^2}\,d\theta$, $\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta$ when $0 < a < 1$, $\int_0^{\pi/2} \frac{1}{a+\sin^2 x}\,dx$ when $|a| > 1$.

7.5.8. Evaluate $\int_0^{+\infty} \frac{x^a}{x^2+3x+2}\,dx$ when $-1 < a < 1$.

7.5.9. Evaluate $\int_0^{+\infty} \frac{\ln x}{(x^2+1)(x^2+4)}\,dx$, $\int_0^{+\infty} \frac{\ln^2 x}{x^2+1}\,dx$, $\int_0^{+\infty} \frac{\ln(1+x^2)}{x^{1+a}}\,dx$ when $0 < a < 2$.

7.5.10. Evaluate $\int_{-\infty}^{+\infty} \frac{\cos x}{e^x+e^{-x}}\,dx$.

7.5.11. Evaluate $\int_0^{+\infty} \frac{1}{x^3+8}\,dx$, $\int_0^{+\infty} \frac{x}{x^4+16}\,dx$, using $\int_0^{+\infty} \frac{x^{a-1}}{x+1}\,dx = \frac{\pi}{\sin a\pi}$.

7.5.12. Evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta$.

7.5.13. If $z_1,\dots,z_N \in D_0(R)$ are distinct and f is holomorphic in an open set containing $\overline{D}_0(R)$, prove that $\oint_{C_0(R)} \frac{f(z)}{(z-z_1)\cdots(z-z_n)}\,dz = 2\pi i(f(z_1)+\cdots+f(z_N))$.

7.5.14. If $n \in \mathbb{N}$, evaluate $\oint_{C_0(n)} \tan\pi z\,dz$.

7.5.15. Let $r = \frac{p}{q}$ be a rational function with $\deg q \geq \deg p + 2$. If $z_1,\dots,z_n$ are the distinct roots of q, prove that $\sum_{k=1}^n \mathrm{Res}(r;z_k) = 0$.
What is the value of $\sum_{k=1}^n \mathrm{Res}(r;z_k)$ if $\deg q = \deg p + 1$?

7.5.16. If $f(z) = e^{z+(1/z)}$, prove that $\mathrm{Res}(f;0) = \sum_{n=0}^{+\infty} \frac{1}{n!(n+1)!}$.

7.5.17. (i) If $n \in \mathbb{N}$, prove that there is $M \geq 0$ *independent of* n so that $|\cot z| \leq M$ for every $z \in \partial R_n$, where R_n is the square region with corners at the points $\pm(n+\frac{1}{2})\pi \pm i(n+\frac{1}{2})\pi$.
(ii) Prove that $\oint_{\partial R_n} \frac{\cot z}{z^2}\,dz \to 0$ when $n \to +\infty$.
(iii) Prove that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.
(iv) Let f be holomorphic in $\mathbb{C}$ except for poles $z_1, \ldots, z_N$ and let $\lim_{z\to\infty} zf(z) = 0$. Prove that $\lim_{n\to+\infty} \sum_{k=-n}^{n} f(k) = -\sum_{j=1}^{N} \mathrm{Res}(f(z)\cot z; z_j)$.
(v) Find the sum $\sum_{n=1}^{+\infty} \frac{1}{n^2+a^2}$, where $a > 0$, and the sum $\sum_{-\infty}^{+\infty} \frac{1}{(n+a)^2}$, where $a \notin \mathbb{Z}$.

7.5.18. Let p, q be polynomials with $\deg q \geq \deg p + 2$. For each $m \in \mathbb{N}$ let I_m be the square region with corners at $(m+\frac{1}{2})(\pm 1 \pm i)$.
(i) Prove that $\oint_{\partial I_m} \frac{1}{\sin \pi z} \frac{p(z)}{q(z)}\,dz \to 0$ when $m \to +\infty$.
(ii) Prove the same result if $\deg q = \deg p + 1$.

7.5.19. Let $-1 < \nu < 1$ and $n \in \mathbb{N}$. Prove that $\oint_{C_0(n+\frac{1}{2})} \frac{e^{i\nu\pi z}}{\sin \pi z} \frac{1}{z-a}\,dz \to 0$ when $n \to +\infty$.

7.5.20. (i) Let p, q be polynomials with $\deg q \geq \deg p + 1$ and $q(k) \neq 0$ for every $k \in \mathbb{Z}$. Prove that $\lim_{m\to+\infty} \sum_{k=-m}^{m} \frac{p(k)}{q(k)}$ is equal to the sum of the residues of $-\pi \cot \pi z \frac{p(z)}{q(z)}$ at the roots of q. Also, prove that $\lim_{m\to+\infty} \sum_{k=-m}^{m} (-1)^k \frac{p(k)}{q(k)}$ is equal to the sum of the residues of $-\pi \frac{1}{\sin \pi z} \frac{p(z)}{q(z)}$ at the roots of q.
(ii) Prove $\pi \cot \pi w = \lim_{m\to+\infty} \sum_{k=-m}^{m} \frac{1}{w-k} = \frac{1}{w} + \sum_{k=-\infty}^{+\infty} (\frac{1}{w-k} + \frac{1}{k})$ if $w \notin \mathbb{Z}$.
(iii) Prove $\frac{\pi^2}{\sin^2 \pi w} = \sum_{k=-\infty}^{+\infty} \frac{1}{(w-k)^2}$ if $w \notin \mathbb{Z}$.
(iv) Prove $\sum_{k=1}^{+\infty} \frac{1}{a+bk^2} = -\frac{1}{2a} + \frac{\pi}{2\sqrt{ab}} \frac{e^{\pi\sqrt{a/b}} + e^{-\pi\sqrt{a/b}}}{e^{\pi\sqrt{a/b}} - e^{-\pi\sqrt{a/b}}}$ if $ab > 0$.
(v) Prove $\sum_{k=1}^{+\infty} \frac{(-1)^k}{a+bk^2} = -\frac{1}{2a} - \frac{\pi}{\sqrt{ab}} \frac{1}{e^{\pi\sqrt{a/b}} - e^{-\pi\sqrt{a/b}}}$ if $ab > 0$.

7.6 The argument principle. The theorem of Rouché.

Definition. *A function f is called* **meromorphic** *in the open set Ω if it is holomorphic in Ω except at certain points in Ω which are poles of f.*

Let f be meromorphic in the open set Ω. If $w \in \mathbb{C}$, we shall denote A_w the set of solutions of $f(z) = w$, i.e.

$$A_w = \{z \in \Omega \,|\, f(z) = w\}.$$

If f is not constant in any connected component of Ω, then the solutions of $f(z) = w$ are isolated points.

Also, letting f have the value ∞ at each of its poles in Ω, so that f becomes continuous at its poles considered as a function from Ω to $\widehat{\mathbb{C}}$, we denote A_∞ the set of solutions of $f(z) = \infty$, i.e.

$$A_\infty = \{z \in \Omega \,|\, f(z) = \infty\} = \{z \in \Omega \,|\, z \text{ is a pole of } f\}.$$

The argument principle. *Let $w \in \mathbb{C}$. We assume that f is meromorphic in the open set Ω and that it is not constant in any connected component of Ω. We also consider Σ to be a cycle null-homologous in Ω so that no element of $A_w \cup A_\infty$ is in the trajectory of any of the closed curves forming Σ. Then $n(\Sigma; z) \neq 0$ for at most finitely many elements of $A_w \cup A_\infty$ and hence the sums $\sum_{z\in A_w} n(\Sigma; z)\, m(z)$ and $\sum_{z\in A_\infty} n(\Sigma; z)\, m(z)$, where $m(z)$ is the corresponding multiplicity of $z \in A_w \cup A_\infty$, are finite. Moreover,*

$$n(f(\Sigma); w) = \frac{1}{2\pi i} \oint_\Sigma \frac{f'(\zeta)}{f(\zeta) - w}\,d\zeta = \sum_{z\in A_w} n(\Sigma; z)\, m(z) - \sum_{z\in A_\infty} n(\Sigma; z)\, m(z). \tag{7.37}$$

Proof. We apply the residue theorem to the function $\frac{f'}{f-w}$. The isolated singularities of this function are the elements of $A_w \cup A_\infty$.
If $m(z)$ is the multiplicity of $z \in A_w$, then there is a g holomorphic in some neighborhood $D_z(r)$ of z so that $f(\zeta) - w = (\zeta - z)^{m(z)} g(\zeta)$ when $\zeta \in D_z(r)$ and also $g(z) \neq 0$. We have $f'(\zeta) = m(z)(\zeta - z)^{m(z)-1} g(\zeta) + (\zeta - z)^{m(z)} g'(\zeta)$ when $\zeta \in D_z(r)$. Also, since $g(z) \neq 0$, we may assume that r is small enough so that $g(\zeta) \neq 0$ when $\zeta \in D_z(r)$. Therefore

$$\frac{f'(\zeta)}{f(\zeta) - w} = \frac{m(z)}{\zeta - z} + \frac{g'(\zeta)}{g(\zeta)} \qquad \text{when } \zeta \in D_z(r) \setminus \{z\}.$$

Since $\frac{g'}{g}$ is holomorphic in $D_z(r)$, we have that z is a pole of $\frac{f'}{f-w}$ of order 1 with residue $m(z)$.
If $m(z)$ is the order of $z \in A_\infty$, there is a g holomorphic in some neighborhood $D_z(r)$ of z so that $f(\zeta) - w = \frac{g(\zeta)}{(\zeta - z)^{m(z)}}$ when $\zeta \in D_z(r)$ and also $g(z) \neq 0$. We have $f'(\zeta) = -m(z) \frac{g(\zeta)}{(\zeta - z)^{m(z)+1}} + \frac{g'(\zeta)}{(\zeta - z)^{m(z)}}$ when $\zeta \in D_z(r)$. Also, since $g(z) \neq 0$, we may assume that r is small enough so that $g(\zeta) \neq 0$ when $\zeta \in D_z(r)$. Hence

$$\frac{f'(\zeta)}{f(\zeta) - w} = \frac{-m(z)}{\zeta - z} + \frac{g'(\zeta)}{g(\zeta)} \qquad \text{when } \zeta \in D_z(r) \setminus \{z\}.$$

Since $\frac{g'}{g}$ is holomorphic in $D_z(r)$, we have that z is a pole of $\frac{f'}{f-w}$ of order 1 with residue $-m(z)$.
Now, the residue theorem implies the second equality in (7.37). The first equality is a matter of a simple change of variable. If $\zeta = \gamma(t)$, $t \in [a, b]$, is the parametric equation of any curve γ forming Σ, then the parametric equation of $f(\gamma)$ is $\eta = f(\gamma(t))$, $t \in [a, b]$, and hence:

$$n(f(\gamma); w) = \frac{1}{2\pi i} \oint_{f(\gamma)} \frac{1}{\eta - w}\, d\eta = \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)) - w}\, dt = \frac{1}{2\pi i} \oint_\gamma \frac{f'(\zeta)}{f(\zeta) - w}\, d\zeta.$$

The rest is simple if we recall that $\Sigma = n_1\gamma_1 + \cdots + n_k\gamma_k$ and $f(\Sigma) = n_1 f(\gamma_1) + \cdots + n_k f(\gamma_k)$. $\square$

The geometric content of the argument principle is described as follows. *The number of rotations of $f(\Sigma)$ around w is equal to the total number of rotations of Σ around the solutions of $f(z) = w$ minus the total number of rotations of Σ around the poles of f.* When we count the solutions of $f(z) = w$ and the poles of f we take into account their multiplicities. We count $m(z)$ points at every point $z \in A_w \cup A_\infty$ which has multiplicity $m(z)$.

If f has no pole in Ω, i.e. if f is holomorphic in Ω, then the argument principle says that *the number of rotations of $f(\Sigma)$ around w is equal to the total number of rotations of Σ around the solutions of $f(z) = w$.* In fact, if Σ is such that for every z not in the trajectories of the curves forming Σ we have either $n(\Sigma; z) = 1$ or $n(\Sigma; z) = 0$, then *the number of rotations of $f(\Sigma)$ around w is equal to the number of solutions of $f(z) = w$ which are surrounded by Σ.*

The theorem of Rouché. *Let $w \in \mathbb{C}$. We assume that $f, g : \Omega \to \mathbb{C}$ are holomorphic in the open set Ω and that they are not constant in any connected component of Ω. We also consider Σ to be a cycle null-homologous in Ω. If $|f(z) - g(z)| < |g(z) - w|$ for every z in the trajectories of the closed curves forming Σ, then*

$$\sum_{z \in A_{w,f}} n(\Sigma; z)\, m_f(z) = \sum_{z \in A_{w,g}} n(\Sigma; z)\, m_g(z),$$

where $m_f(z)$ and $m_g(z)$ are the corresponding multiplicities and $A_{w,f} = \{z \in \Omega \mid f(z) = w\}$, $A_{w,g} = \{z \in \Omega \mid g(z) = w\}$.

Proof. We observe that the condition $|f(z) - g(z)| < |g(z) - w|$ for every z in the trajectories of the closed curves forming Σ implies that no element of $A_{w,f} \cup A_{w,g}$ is in these trajectories. Now (7.37) implies

$$\frac{1}{2\pi i} \oint_\Sigma \frac{f'(\zeta)}{f(\zeta) - w}\, d\zeta = \sum_{z \in A_{w,f}} n(\Sigma; z)\, m_f(z), \qquad \frac{1}{2\pi i} \oint_\Sigma \frac{g'(\zeta)}{g(\zeta) - w}\, d\zeta = \sum_{z \in A_{w,g}} n(\Sigma; z)\, m_g(z).$$

Hence

$$\sum_{z\in A_{w,f}} n(\Sigma;z)\,m_f(z)-\sum_{z\in A_{w,g}} n(\Sigma;z)\,m_g(z)=\frac{1}{2\pi i}\oint_\Sigma\Big(\frac{f'(\zeta)}{f(\zeta)-w}-\frac{g'(\zeta)}{g(\zeta)-w}\Big)\,d\zeta. \quad (7.38)$$

The function $h=\frac{f-w}{g-w}$ is holomorphic in Ω except for the elements of $A_{w,g}$, which are either poles or removable singularities of h. From (7.37) again we have

$$n(h(\Sigma);0)=\frac{1}{2\pi i}\oint_\Sigma \frac{h'(\zeta)}{h(\zeta)}\,d\zeta. \quad (7.39)$$

Since $\frac{h'}{h}=\frac{f'}{f-w}-\frac{g'}{g-w}$, from (7.38) and (7.39) we find

$$\sum_{z\in A_{w,f}} n(\Sigma;z)\,m_f(z)-\sum_{z\in A_{w,g}} n(\Sigma;z)\,m_g(z)=n(h(\Sigma);0).$$

Now, our hypothesis says that $|h(z)-1|<1$ for every z in the trajectories of the curves forming Σ. I.e. the curves forming $h(\Sigma)$ are in the disc $D_1(1)$ and hence $n(h(\Sigma);0)=0$. $\square$

Example 7.6.1. We shall find the number of roots of $f(z)=z^7-2z^5+6z^3-z+1$ in $\mathbb{D}$.
We consider $g(z)=6z^3$ and we have

$$|f(z)-g(z)|=|z^7-2z^5-z+1|\le|z|^7+2|z|^5+|z|+1=5<6|z|^3=|g(z)|$$

for every $z\in\mathbb{T}$. Now we apply the theorem of Rouché with $w=0$ and Σ consisting of only the curve γ which describes $\mathbb{T}$ once and in the positive direction. We have $n(\gamma;z)=1$ for every $z\in\mathbb{D}$ and $n(\gamma;z)=0$ for every $z\notin\overline{\mathbb{D}}$. The only solution of $g(z)=0$ in $\mathbb{D}$ is $z=0$ with multiplicity $m_g(0)=3$. Therefore

$$\sum_{z\in A_{0,g}} n(\gamma;z)\,m_g(z)=\sum_{z\in A_{0,g}\cap\mathbb{D}} m_g(z)=3.$$

Also

$$\sum_{z\in A_{0,f}} n(\gamma;z)\,m_f(z)=\sum_{z\in A_{0,f}\cap\mathbb{D}} m_f(z).$$

The theorem of Rouché implies $\sum_{z\in A_{0,f}\cap\mathbb{D}} m_f(z)=3$ and hence f has three roots in $\mathbb{D}$.

Exercises.

7.6.1. Let f be holomorphic in $D_{z_0}(R)$, let $0<r<R$ and assume that there is no solution of $f(z)=w$ in $C_{z_0}(r)$. If $k\in\mathbb{N}$, what is the content of $\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f'(z)}{f(z)-w}\,z^k\,dz$?

7.6.2. Let f be holomorphic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$ and let $|f(z)|<1$ for every $z\in\mathbb{T}$. Prove that the equation $f(z)=z^n$ has exactly n solutions in $\mathbb{D}$.

7.6.3. Find the number of roots of
(i) z^4-6z+3 in $D_0(1,2)$.
(ii) $z^4+8z^3+3z^2+8z+3$ in $\{z\,|\,\mathrm{Re}\,z>0\}$.

7.6.4. Let $z_1,\dots,z_n\in\mathbb{D}$. In $\mathbb{C}\setminus\big\{\frac{1}{\overline{z_1}},\dots,\frac{1}{\overline{z_k}}\big\}$ we consider the function $f(z)=\prod_{k=1}^n\frac{z-z_k}{1-\overline{z_k}\,z}$.
(i) Prove that $f(z)\in\mathbb{D}$ for every $z\in\mathbb{D}$ and that $f(z)\in\mathbb{T}$ for every $z\in\mathbb{T}$.
(ii) Find the index with respect to 0 of the curve with parametric equation $z=f(e^{it})$, $t\in[0,2\pi]$.
(iii) Prove that for every $w\in\mathbb{D}$ the equation $f(z)=w$ has exactly n solutions in $\mathbb{D}$.

7.6.5. Let $\lambda>1$. Prove that the equation $\lambda-z=e^{-z}$ has exactly one solution in $\{z\,|\,\mathrm{Re}\,z>0\}$ and that this solution is real. How does this solution behave when $\lambda\to1+$?

7.6.6. Prove that the set of all meromorphic functions in the region Ω is an algebraic field.

7.6.7. Let $f(z)=\sum_{n=0}^{+\infty}a_nz^n$ for $z\in\mathbb{D}$ and let $F\subseteq\mathbb{D}$ be compact with $0\in F$. If $\mu=\inf\{|f(z)|\,|\,z\in\partial F\}$ and m is the number of roots of f in F, prove that $\mu\le|a_0|+|a_1|+\dots+|a_m|$.

Chapter 8

Simply connected regions and the theorem of Riemann.

8.1 Uniform convergence in compact sets and holomorphy.

Lemma 8.1. *Let $\Omega \subseteq \mathbb{C}$ be open and $K \subseteq \Omega$ be compact. Then there is $\epsilon > 0$ so that $|z-w| \geq 2\epsilon$ for every $z \in K$ and every $w \in \Omega^c$. Then the set $K' = \{z' \,|\, |z'-z| \leq \epsilon$ for some $z \in K\}$ is a compact subset of Ω which contains K.*

Proof. It is clear that $K \subseteq K' \subseteq \Omega$.
Since K is bounded, there is M so that $|z| \leq M$ for every $z \in K$. If $z' \in K'$, then there is $z \in K$ so that $|z'-z| \leq \epsilon$ and then $|z'| \leq |z'-z| + |z| \leq \epsilon + M$. Hence K' is bounded.
Finally, let (z'_n) be a sequence in K' and $z'_n \to z'$. For each n there is $z_n \in K$ so that $|z'_n - z_n| \leq \epsilon$. Since K is compact, there is a subsequence (z_{n_k}) so that $z_{n_k} \to z$ for some $z \in K$. From $|z'_{n_k} - z_{n_k}| \leq \epsilon$ we get $|z'-z| \leq \epsilon$ and hence $z' \in K'$. Therefore K' is closed. $\square$

The theorem of Weierstrass. *Let every f_n be holomorphic in the open set $\Omega \subseteq \mathbb{C}$. If $f_n \to f$ uniformly in every compact subset of Ω, then f is also holomorphic in Ω and for every $k \in \mathbb{N}$ we have that $f_n^{(k)} \to f^{(k)}$ uniformly in every compact subset of Ω.*

Proof. We take any $z_0 \in \Omega$. Then there is a closed disc $\overline{D}_{z_0}(R)$ contained in Ω and for every n we have

$$f_n(z) = \frac{1}{2\pi i} \oint_{C_{z_0}(R)} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for every } z \in D_{z_0}(R).$$

Since $C_{z_0}(R)$ is a compact subset of Ω, we have that $f_n \to f$ uniformly in $C_{z_0}(R)$. We also have that $f_n(z) \to f(z)$ for every $z \in D_{z_0}(R)$. Hence

$$f(z) = \frac{1}{2\pi i} \oint_{C_{z_0}(R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for every } z \in D_{z_0}(R).$$

The right side of this equality is a holomorphic function of z in $D_{z_0}(R)$ and hence the left side, $f(z)$, is also holomorphic in $D_{z_0}(R)$. Therefore, f is holomorphic at every $z_0 \in \Omega$.
Now let $K \subseteq \Omega$ be compact. Then there is $\epsilon > 0$ so that $|z-w| \geq 2\epsilon$ for every $z \in K$ and every $w \in \Omega^c$. We consider the set $K' = \{z' \,|\, |z'-z| \leq \epsilon$ for some $z \in K\}$ of lemma 8.1. This K' is a compact subset of Ω and hence $f_n \to f$ uniformly in K'. We also observe that for every $z \in K$ we have $C_z(\epsilon) \subseteq K'$. Then we apply the formula of Cauchy for derivatives and we get for every $z \in K$ that

$$\begin{aligned} |f_n^{(k)}(z) - f^{(k)}(z)| &= \Big| \frac{k!}{2\pi i} \oint_{C_z(\epsilon)} \frac{f_n(\zeta) - f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta \Big| \leq \frac{k!}{2\pi} \frac{\sup_{\zeta \in C_z(\epsilon)} |f_n(\zeta) - f(\zeta)|}{\epsilon^{k+1}} 2\pi\epsilon \\ &\leq \frac{k!}{\epsilon^k} \sup_{\zeta \in K'} |f_n(\zeta) - f(\zeta)|. \end{aligned}$$

Hence,
$$\sup_{z\in K}|f_n^{(k)}(z)-f^{(k)}(z)|\leq\frac{k!}{\epsilon^k}\sup_{\zeta\in K'}|f_n(\zeta)-f(\zeta)|\to 0$$
and so $f_n^{(k)}\to f^{(k)}$ uniformly in K. $\square$

The theorem of Hurwitz. *Let every f_n be holomorphic in the region $\Omega\subseteq\mathbb{C}$ and $f_n\to f$ uniformly in every compact subset of Ω. If $f_n(z)\neq 0$ for every n and every $z\in\Omega$, then either $f(z)\neq 0$ for every $z\in\Omega$ or $f(z)=0$ for every $z\in\Omega$.*

Proof. The theorem of Weierstras implies that f is holomorphic in Ω. We assume that f is not identically 0 in Ω and we shall prove that $f(z)\neq 0$ for every $z\in\Omega$.
We take any $z_0\in\Omega$. Even if $f(z_0)=0$, we know that z_0 is an isolated root of f and hence there is $r>0$ so that $f(\zeta)\neq 0$ for every $\zeta\in C_{z_0}(r)$. By the continuity of f we get that there is some $\delta>0$ so that $|f(\zeta)|\geq\delta$ for every $\zeta\in C_{z_0}(r)$. Now, we have that $f_n\to f$ uniformly in $C_{z_0}(r)$ and the theorem of Weierstrass implies that also $f_n'\to f'$ uniformly in $C_{z_0}(r)$. Therefore, $\frac{f_n'}{f_n}\to\frac{f'}{f}$ uniformly in $C_{z_0}(r)$ and hence
$$\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f_n'(\zeta)}{f_n(\zeta)}\,d\zeta\to\frac{1}{2\pi i}\oint_{C_{z_0}(r)}\frac{f'(\zeta)}{f(\zeta)}\,d\zeta.$$
By the argument principle, the left side is equal to the number of roots of f_n in the disc $D_{z_0}(r)$ and hence it is equal to 0. Thus, the right side is also equal to 0 and, by the argument principle again, there is no root of f in the disc $D_{z_0}(r)$. In particular, $f(z_0)\neq 0$. $\square$

Definition. *Let $A\subseteq\mathbb{C}$ and $\mathcal{F}$ be a collection of functions defined in the set A.*
(i) We say that $\mathcal{F}$ is **bounded** *at some $z\in A$ if there is M so that $|f(z)|\leq M$ for every $f\in\mathcal{F}$.*
(ii) We say that $\mathcal{F}$ is **equicontinuous** *at some $z\in A$ if for every $\epsilon>0$ there is $\delta>0$ so that $|f(w)-f(z)|<\epsilon$ for every $w\in A$ with $|w-z|<\delta$ and for every $f\in\mathcal{F}$.*

We observe that if $\mathcal{F}$ is equicontinuous at some $z\in A$, then every $f\in\mathcal{F}$ is continuous at z and that δ depends on ϵ but not on f, i.e. δ is *uniform* over $f\in\mathcal{F}$.

Lemma 8.2. *Let $A\subseteq\mathbb{C}$ and (f_n) be a sequence of continuous functions in A. If $f_n\to f$ uniformly in every compact subset of A, then f is continuous in A.*

Proof. Take any $z\in A$ and a sequence (z_m) in A with $z_m\to z$. Then $K=\{z_m\,|\,m\in\mathbb{N}\}\cup\{z\}$ is a compact subset of A and hence $f_n\to f$ uniformly in K. Since every f_n is continuous in K, we have that f is also continuous in K. Thus, $f(z_m)\to f(z)$ and so f is continuous at z. $\square$

Lemma 8.3. *Let $A\subseteq\mathbb{C}$. Then there is a countable $L\subseteq A$ so that for every $z\in A$ and every $\delta>0$ there is $w\in L$ with $|w-z|<\delta$. In other words, there is a countable $L\subseteq A$ so that $A\subseteq\overline{L}$.*

Proof. We consider the countable set $M=\{x+iy\,|\,x,y\in\mathbb{Q}\}$ and let $M=\{z_n\,|\,n\in\mathbb{N}\}$ be any enumeration of M. For every $n,m\in\mathbb{N}$ such that $D_{z_n}(\frac{1}{m})\cap A\neq\emptyset$ we take a point $w_{n,m}\in D_{z_n}(\frac{1}{m})\cap A$. Then the set L of all such $w_{n,m}$ is a countable subset of A.
Now take any $z\in A$ and any δ so that $0<\delta\leq 3$. Then there is $z_n\in M$ so that $|z-z_n|<\frac{\delta}{3}$ and there is $m\in\mathbb{N}$ so that $\frac{1}{m+1}<\frac{\delta}{3}\leq\frac{1}{m}$. Then $z\in D_{z_n}(\frac{1}{m})$ and hence $D_{z_n}(\frac{1}{m})\cap A\neq\emptyset$. Now we have $|z-w_{n,m}|\leq|z-z_n|+|z_n-w_{n,m}|<\frac{\delta}{3}+\frac{1}{m}\leq\frac{\delta}{3}+\frac{2}{m+1}<\delta$. $\square$

The theorem of Arzela-Ascoli. *Let $A\subseteq\mathbb{C}$ and $\mathcal{F}$ be a collection of continuous functions in A. Then the following are equivalent:*
(i) For every sequence (f_n) in $\mathcal{F}$ there is a subsequence (f_{n_k}) and a function f continuous in A so that $f_{n_k}\to f$ uniformly in every compact subset of A.
(ii) $\mathcal{F}$ is equicontinuous and bounded at every $z\in A$.

Proof. (i) ⇒ (ii) Assume that $\mathcal{F}$ is not bounded at some $z \in A$. Then there is a sequence (f_n) in $\mathcal{F}$ so that $|f_n(z)| \to +\infty$. Then there is a subsequence (f_{n_k}) and a function f continuous in A so that $f_{n_k} \to f$ uniformly in every compact subset of A. One such compact set is $\{z\}$ and we get that $f_{n_k}(z) \to f(z)$, arriving at a contradiction.
Now assume that $\mathcal{F}$ is not equicontinuous at some $z \in A$. Then there is $\epsilon > 0$ so that for every $n \in \mathbb{N}$ there is $z_n \in A$ and $f_n \in \mathcal{F}$ so that $|z_n - z| < \frac{1}{n}$ and $|f_n(z_n) - f_n(z)| \geq \epsilon$. Then there is a subsequence (f_{n_k}) and a function f continuous in A so that $f_{n_k} \to f$ uniformly in every compact subset of A. Since $z_{n_k} \to z$, the set $K = \{z_{n_k} \,|\, k \in \mathbb{N}\} \cup \{z\}$ is a compact subset of A and hence $f_{n_k} \to f$ uniformly in K. Now

$$\begin{aligned}\epsilon &\leq |f_{n_k}(z_{n_k}) - f_{n_k}(z)| \leq |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(z)| + |f(z) - f_{n_k}(z)| \\ &\leq \sup_{\zeta \in K} |f_{n_k}(\zeta) - f(\zeta)| + |f(z_{n_k}) - f(z)| + \sup_{\zeta \in K} |f_{n_k}(\zeta) - f(\zeta)|.\end{aligned}$$

We arrive at a contradiction because $\sup_{\zeta \in K} |f_{n_k}(\zeta) - f(\zeta)| \to 0$ and $f(z_{n_k}) \to f(z)$.
(ii) ⇒ (i) Let (f_n) be a sequence in $\mathcal{F}$. We consider the countable set $L \subseteq A$ which is described in Lemma 8.3. Let

$$L = \{w_m \,|\, m \in \mathbb{N}\}.$$

The set $\{f_n(w_1) \,|\, n \in \mathbb{N}\}$ is bounded. Hence there is a subsequence $(f_{n,1})$ of (f_n) so that $f_{n,1}(w_1) \to \zeta_1$ for some $\zeta_1 \in \mathbb{C}$. Similarly, the set $\{f_{n,1}(w_2) \,|\, n \in \mathbb{N}\}$ is bounded. Hence there is a subsequence $(f_{n,2})$ of $(f_{n,1})$ so that $f_{n,2}(w_2) \to \zeta_2$ for some $\zeta_2 \in \mathbb{C}$. Similarly, the set $\{f_{n,2}(w_3) \,|\, n \in \mathbb{N}\}$ is bounded. Hence there is a subsequence $(f_{n,3})$ of $(f_{n,2})$ so that $f_{n,3}(w_3) \to \zeta_3$ for some $\zeta_3 \in \mathbb{C}$. We continue inductively and we find

$$\begin{array}{cccccc} f_1 & f_2 & f_3 & \cdots & f_n & \cdots \\ f_{1,1} & f_{2,1} & f_{3,1} & \cdots & f_{n,1} & \cdots \\ f_{1,2} & f_{2,2} & f_{3,2} & \cdots & f_{n,2} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ f_{1,m} & f_{2,m} & f_{3,m} & \cdots & f_{n,m} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{array}$$

so that the sequence in every row is a subsequence of the sequence in the previous row and so that $f_{n,m}(w_m) \to \zeta_m$ for some $\zeta_m \in \mathbb{C}$. Now we consider the *diagonal sequence* $(f_{n,n})$. For every m, $(f_{n,n})$ is, after the value m of the index n, a subsequence of $(f_{n,m})$ and hence $f_{n,n}(w_m) \to \zeta_m$. Therefore, $(f_{n,n})$ is a subsequence of (f_n) and $\lim_{n\to+\infty} f_{n,n}(w)$ exists in $\mathbb{C}$ for every $w \in L$. Now we take any $z \in A$. Since $\mathcal{F}$ is equicontinuous at z, for every $\epsilon > 0$ there is $\delta > 0$ so that $|f(w) - f(z)| < \frac{\epsilon}{3}$ for every $w \in A$ with $|w - z| < \delta$ and every $f \in \mathcal{F}$. From the basic property of L we have that there is $w \in L$ so that $|w - z| < \delta$. Since $(f_{n,n}(w))$ is a Cauchy sequence, there is n_0 so that $|f_{n',n'}(w) - f_{n'',n''}(w)| < \frac{\epsilon}{3}$ for every $n', n'' \geq n_0$. Therefore, for every $n', n'' \geq n_0$ we have

$$\begin{aligned}|f_{n',n'}(z) - f_{n'',n''}(z)| &\leq |f_{n',n'}(z) - f_{n',n'}(w)| + |f_{n',n'}(w) - f_{n'',n''}(w)| \\ &\quad + |f_{n'',n''}(w) - f_{n'',n''}(z)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This means that $(f_{n,n}(z))$ is a Cauchy sequence in $\mathbb{C}$ and hence $\lim_{n\to+\infty} f_{n,n}(z)$ exists in $\mathbb{C}$. Now we define the function $f : A \to \mathbb{C}$ by

$$f(z) = \lim_{n\to+\infty} f_{n,n}(z) \qquad \text{for every } z \in A.$$

We consider any compact set $K \subseteq A$ and we shall prove that $f_{n,n} \to f$ uniformly in K. Assume on the contrary that this is not true. Then there is $\epsilon > 0$ and a subsequence (f_{n_k,n_k}) of $(f_{n,n})$ and a sequence (z_k) in K so that

$$|f_{n_k,n_k}(z_k) - f(z_k)| \geq \epsilon \qquad \text{for every } k. \tag{8.1}$$

Since K is compact, there is a subsequence (z_{k_l}) of z_k which converges to some $z \in K$. For simplicity of notation we shall ignore all the terms of (z_k) which do not belong to (z_{k_l}) and we shall assume that $z_k \to z$ for some $z \in K$. Now, since $\mathcal{F}$ is equicontinuous at z, there is $\delta > 0$ so that $|f_{n_k,n_k}(w) - f_{n_k,n_k}(z)| < \frac{\epsilon}{3}$ for every $w \in A$ with $|w - z| < \delta$ and every k. Taking the limit as $k \to +\infty$, we get that $|f(w) - f(z)| \leq \frac{\epsilon}{3}$ for every $w \in A$ with $|w - z| < \delta$. Now, if k is large enough, we have that $|z_k - z| < \delta$ and $|f_{n_k,n_k}(z) - f(z)| < \frac{\epsilon}{3}$ and then we get

$$\begin{aligned}|f_{n_k,n_k}(z_k) - f(z_k)| &\leq |f_{n_k,n_k}(z_k) - f_{n_k,n_k}(z)| + |f_{n_k,n_k}(z) - f(z)| + |f(z_k) - f(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

This contradicts (8.1) and hence $f_{n,n} \to f$ uniformly in K. Finally, lemma 8.2 implies that f is continuous in A. $\square$

Definition. *Let $A \subseteq \mathbb{C}$ and $\mathcal{F}$ be a collection of functions defined in the set A. We say that $\mathcal{F}$ is* **locally bounded** *at some $z \in A$ if there are $\delta > 0$ and M so that $|f(w)| \leq M$ for every $w \in A$ with $|w - z| < \delta$ and for every $f \in \mathcal{F}$.*

The theorem of Montel. *Let $\Omega \subseteq \mathbb{C}$ be open and $\mathcal{F}$ be a collection of holomorphic functions in Ω. Then the following are equivalent:*
(i) For every sequence (f_n) in $\mathcal{F}$ there is a subsequence (f_{n_k}) and a function f holomorphic in Ω so that $f_{n_k} \to f$ uniformly in every compact subset of Ω.
(ii) $\mathcal{F}$ is locally bounded at every $z \in \Omega$.

Proof. (i) ⇒ (ii) Assume that $\mathcal{F}$ is not locally bounded at some $z \in \Omega$. Then for every $n \in \mathbb{N}$ there is $z_n \in \Omega$ and $f_n \in \mathcal{F}$ with $|z_n - z| < \frac{1}{n}$ and $|f_n(z_n)| > n$. Now, there is a subsequence (f_{n_k}) of (f_n) and a function f holomorphic in Ω so that $f_{n_k} \to f$ uniformly in every compact subset of Ω. Since $z_n \to z$, the set $K = \{z_n \,|\, n \in \mathbb{N}\} \cup \{z\}$ is a compact subset of Ω and hence $f_{n_k} \to f$ uniformly in K. But

$$\sup_{\zeta \in K} |f_{n_k}(\zeta) - f(\zeta)| \geq |f_{n_k}(z_{n_k}) - f(z_{n_k})| \geq |f_{n_k}(z_{n_k})| - |f(z_{n_k})| \to +\infty - |f(z)| = +\infty$$

and we arrive at a contradiction.
(ii) ⇒ (i) By the Arzela-Ascoli theorem and by the theorem of Weierstrass it is enough to prove that $\mathcal{F}$ is bounded and equicontinuous at every $z \in \Omega$. Obviously, local boundedness of $\mathcal{F}$ implies that $\mathcal{F}$ is bounded at every $z \in \Omega$. Now we take any $z_0 \in \Omega$ and then there is $r > 0$ and M so that $|f(z)| \leq M$ for every $z \in \overline{D}_{z_0}(r)$ and every $f \in \mathcal{F}$. Thus, for every $z \in D_{z_0}(\frac{r}{2})$ and every $f \in \mathcal{F}$ we have

$$|f'(z)| = \Big| \frac{1}{2\pi i} \oint_{C_{z_0}(r)} \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta \Big| \leq \frac{1}{2\pi} \frac{M}{(r/2)^2} \, 2\pi r = \frac{4M}{r}.$$

This implies that for every $z \in D_{z_0}(\frac{r}{2})$ and every $f \in \mathcal{F}$ we have

$$|f(z) - f(z_0)| = \Big| \int_{[z_0,z]} f'(\zeta) \, d\zeta \Big| \leq \frac{4M}{r} \, |z - z_0|.$$

Hence, for every $\epsilon > 0$ we may take $\delta = \min\{\frac{r\epsilon}{4M}, \frac{r}{2}\}$ and then for every $z \in D_{z_0}(\delta)$ and every $f \in \mathcal{F}$ we get $|f(z) - f(z_0)| \leq \frac{4M}{r} \, |z - z_0| < \frac{4M}{r} \, \delta \leq \epsilon$. Thus, $\mathcal{F}$ is equicontinuous at z_0. $\square$

Exercises.

8.1.1. Let (f_n) be a sequence of functions holomorphic in the region $\Omega \subseteq \mathbb{C}$ which is locally bounded at every $z \in \Omega$. If every f_n has no roots in Ω and $f_n(z_0) \to 0$ for some $z_0 \in \Omega$, prove that $f_n \to 0$ uniformly in every compact subset of Ω.

8.1.2. Let (f_n) be a sequence of functions holomorphic in the region $\Omega \subseteq \mathbb{C}$ which is locally bounded at every $z \in \Omega$ and let $E \subseteq \Omega$ have an accumulation point in Ω. If $\lim_{n\to+\infty} f_n(z)$ is a complex number for every $z \in E$, prove that (f_n) converges to some function uniformly in every compact subset of Ω.

8.1.3. Let (f_n) be a sequence of functions holomorphic in the open set $\Omega \subseteq \mathbb{C}$. If $\lim_{n\to+\infty} f_n(z)$ is a complex number for every $z \in \Omega$, prove that there is an open set $H \subseteq \Omega$ so that $\Omega \subseteq \overline{H}$ and so that (f_n) converges to some function uniformly in every compact subset of H. (*Note:* This needs the theorem of Baire.)

8.1.4. Let $\Omega \subseteq \mathbb{C}$ be a region and (f_n) be a sequence of functions holomorphic in Ω with $\mathrm{Re}\, f_n > 0$ in Ω for every n.
(i) If $(f_n(z_0))$ is bounded for some $z_0 \in \Omega$, prove that there is a subsequence (f_{n_k}) which converges to some function uniformly in every compact subset of Ω.
(ii) If $(f_n(z_0))$ is unbounded for some $z_0 \in \Omega$, prove that there is a subsequence (f_{n_k}) so that $f_{n_k} \to \infty$ uniformly in every compact subset of Ω.

8.1.5. Let f_n, f be holomorphic in $D_{z_0}(R)$ and $f_n \to f$ uniformly in every compact subset of $D_{z_0}(R)$. If $f_n(z) = \sum_{k=0}^{+\infty} a_{k,n}(z-z_0)^k$ and $f(z) = \sum_{k=0}^{+\infty} a_k(z-z_0)^k$ are the corresponding Taylor series, prove that $a_{k,n} \to a_k$ for every k.

8.1.6. Let $\mathcal{F}$ be a collection of functions holomorphic in $D_{z_0}(R)$. We denote $a_k(f) = \frac{f^{(k)}(z_0)}{k!}$ the k-th Taylor coefficient of each $f \in \mathcal{F}$. Prove that the following are equivalent:
(i) For every sequence (f_n) in $\mathcal{F}$ there is a subsequence (f_{n_j}) which converges to some function uniformly in every compact subset of $D_{z_0}(R)$.
(ii) There are $M_k \geq 0$ so that $\limsup_{k\to+\infty} \sqrt[k]{M_k} \leq \frac{1}{R}$ and $|a_k(f)| \leq M_k$ for every k and every $f \in \mathcal{F}$.

8.1.7. A theorem of Montel. Let $-\infty < a < x_0 < b < +\infty$ and f be bounded and holomorphic in the vertical hafzone $\Omega = \{z = x+iy \,|\, a < x < b, y > 0\}$. If $f(x_0+iy) \to A \in \mathbb{C}$ when $y \to +\infty$, prove that for every $\epsilon > 0$ we have $\sup_{x\in[a+\epsilon,b-\epsilon]} |f(x+iy) - A| \to 0$ when $y \to +\infty$.

8.1.8. Let $M \geq 0$ and $\Omega \subseteq \mathbb{C}$ be open and $\mathcal{F}$ be the collection of all functions f holomorphic in Ω with $\iint_\Omega |f(x,y)|^2\, dxdy \leq M$. Prove that $\mathcal{F}$ is locally bounded at every $z \in \Omega$.

8.1.9. Let $\mathcal{F}$ be a collection of holomorphic functions in the open set $\Omega \subseteq \mathbb{C}$ with the property: for every sequence (f_n) in $\mathcal{F}$ there is a subsequence (f_{n_k}) which converges to some function uniformly in every compact subset of Ω. Prove that the collection $\mathcal{F}' = \{f' \,|\, f \in \mathcal{F}\}$ has the same property. Is the converse true?

8.1.10. Let $\Omega \subseteq \mathbb{C}$ be open, $\overline{D}_{z_0}(r) \subseteq \Omega$, f_n, f be holomorphic in Ω and $f_n \to f$ uniformly in $C_{z_0}(r)$. If f has no root in $C_{z_0}(r)$ and has exactly k roots in $D_{z_0}(r)$, prove that every f_n, after some value of the index n, has exactly k roots in $D_{z_0}(r)$.

8.1.11. Let (f_n) be a sequence of holomorphic functions in the region $\Omega \subseteq \mathbb{C}$ so that $f_n \to f$ uniformly in every compact subset of Ω. If every f_n has at most k roots in Ω, prove that either f has also at most k roots in Ω or that f is identically 0 in Ω.

8.1.12. Prove that for every $R > 0$ there is N so that for every $n \geq N$ the polynomial $1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$ has no root in $D_0(R)$.

8.1.13. Let f_n, f be holomorphic in the open set $\Omega \subseteq \mathbb{C}$ and $f_n \to f$ uniformly in every compact subset of Ω. Prove that $\{z \in \Omega \,|\, f(z) = 0\} = \Omega \cap \bigcap_{n=1}^{+\infty} \big(\overline{\bigcup_{k=n}^{+\infty} \{z \in \Omega \,|\, f_k(z) = 0\}}\big)$.

8.2 Conformal equivalence.

If $\Omega \subseteq \mathbb{C}$ is a region and f is holomorphic and not constant in Ω, then, by the open mapping theorem, $f(\Omega)$ is also a region.

Proposition 8.1. *Let f be holomorphic and one-to-one in the region $\Omega \subseteq \mathbb{C}$. Then $f(\Omega)$ is also a region, $f'(z) \neq 0$ for every $z \in \Omega$ and f^{-1} is holomorphic in $f(\Omega)$.*

Proof. If $f'(z_0) = 0$ for some $z \in \Omega$, then theorem 6.2 implies that there is $N \geq 2$ so that f is N-to-one in some open set $U \subseteq \Omega$ containing z_0. Hence $f'(z) \neq 0$ for every $z \in \Omega$.
Now let $w_0 \in f(\Omega)$ and consider the unique $z_0 \in \Omega$ so that $f(z_0) = w_0$. Then proposition 6.8 implies that there are two open sets, $U \subseteq \Omega$ and $W \subseteq f(\Omega)$ with $z_0 \in U$ and $w_0 \in W$ so that $f^{-1} : W \to U$ is holomorphic. Thus f^{-1} is holomorphic at every $w_0 \in f(\Omega)$. $\square$

Definition. *Let f be holomorphic and one-to-one in the region $\Omega \subseteq \mathbb{C}$. Since $f'(z) \neq 0$ for every $z \in \Omega$, we say that f is a* **conformal mapping** *of Ω.*

Definition. *Two regions $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ are called* **conformally equivalent** *if there is $f : \Omega_1 \to \Omega_2$ holomorphic and one-to-one from Ω_1 onto Ω_2 or, equivalently, if there is a conformal mapping of Ω_1 onto Ω_2.*

If $f : \Omega_1 \to \Omega_2$ is holomorphic and one-to-one from Ω_1 onto Ω_2, then $f^{-1} : \Omega_2 \to \Omega_1$ is also holomorphic and one-to-one from Ω_2 onto Ω_1. It is easy to see that conformal equivalence between regions in $\mathbb{C}$ is an equivalence relation.

The Schwarz lemma. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic in $\mathbb{D}$ and $f(0) = 0$. Then*
(i) $|f(z)| \leq |z|$ for every $z \in \mathbb{D}$,
(ii) $|f'(0)| \leq 1$.
If equality holds in (i) for at least one $z \in \mathbb{D} \setminus \{0\}$ or in (ii), then there is a constant c with $|c| = 1$ so that $f(z) = cz$ for every $z \in \mathbb{D}$.

Proof. Since $f(0) = 0$, the function $\frac{f(z)}{z}$ has a removable singularity at 0 and we may define the function g by

$$g(z) = \begin{cases} \frac{f(z)}{z}, & \text{if } z \in \mathbb{D}, z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases}$$

Then g is holomorphic in $\mathbb{D}$.
We take any $z \in \mathbb{D}$ and we take any r so that $|z| < r < 1$. By the maximum principle we have $|g(z)| \leq \max\{|g(\zeta)| \,|\, |\zeta| = r\}$. But, when $|\zeta| = r$ we have $|g(\zeta)| = \frac{|f(\zeta)|}{|\zeta|} \leq \frac{1}{r}$. Hence, $|g(z)| \leq \frac{1}{r}$ and since this is true for every r with $|z| < r < 1$, we conclude that $|g(z)| \leq 1$. Of course this implies (i) and (ii).
Now, assume that equality holds in (i) for at least one $z \in \mathbb{D} \setminus \{0\}$ or in (ii). Then $|g(z)| = 1$ for at least one $z \in \mathbb{D}$ and the maximum principle implies that g is a constant c in $\mathbb{D}$ with $|c| = 1$. Hence $f(z) = cz$ for every $z \in \mathbb{D}$. $\square$

Example 8.2.1. Let $z_0 \in \mathbb{D}$ and $|\lambda| = 1$. We consider the function $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$T(z) = \begin{cases} \lambda \frac{z - z_0}{1 - \overline{z_0}\, z}, & \text{if } z \in \mathbb{C}, z \neq \frac{1}{\overline{z_0}} \\ \infty, & \text{if } z = \frac{1}{\overline{z_0}} \\ -\frac{\lambda}{\overline{z_0}}, & \text{if } z = \infty \end{cases}$$

Then T is a linear fractional transformation and hence it is one-to-one from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ and holomorphic in $\widehat{\mathbb{C}} \setminus \{\frac{1}{\overline{z_0}}\}$. The inverse function $T^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is given by

$$T^{-1}(w) = \begin{cases} \mu \frac{w - w_0}{1 - \overline{w_0}\, w}, & \text{if } w \in \mathbb{C}, w \neq \frac{1}{\overline{w_0}} \\ \infty, & \text{if } w = \frac{1}{\overline{w_0}} \\ -\frac{\mu}{\overline{w_0}}, & \text{if } w = \infty \end{cases}$$

where $\mu = \frac{1}{\lambda}$ and $w_0 = -\lambda z_0$. Since $|\mu| = 1$ and $w_0 \in \mathbb{D}$, the inverse function T^{-1} is of the same form as T.
For simplicity, we shall follow the same practice as with all l.f.t. and we shall write

$$T(z) = \lambda \, \frac{z - z_0}{1 - \overline{z_0}\, z}$$

the formula of T, understanding that $T(\frac{1}{\overline{z_0}}) = \infty$ and $T(\infty) = -\frac{\lambda}{\overline{z_0}}$ whenever this is needed.
We easily see that

$$T(\mathbb{D}) = \mathbb{D}, \qquad T(\mathbb{T}) = \mathbb{T}.$$

Indeed,

$$1 - |T(z)|^2 = 1 - \frac{|z - z_0|^2}{|1 - \overline{z_0}\, z|^2} = \frac{1 + |z|^2|z_0|^2 - |z|^2 - |z_0|^2}{|1 - \overline{z_0}\, z|^2} = \frac{(1 - |z|^2)(1 - |z_0|^2)}{|1 - \overline{z_0}\, z|^2},$$

from which we have that $|T(z)| < 1$ if $|z| < 1$, that $|T(z)| = 1$ if $|z| = 1$ and that $|T(z)| > 1$ if $|z| > 1$. These imply that $T(\mathbb{D}) \subseteq \mathbb{D}$, $T(\mathbb{T}) \subseteq \mathbb{T}$ and $T(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \subseteq \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. But, since T is onto $\widehat{\mathbb{C}}$, all these inclusions are equalities.
Another simple property of T is

$$T(z_0) = 0.$$

We also have $T'(z) = \lambda \frac{1 - |z_0|^2}{(1 - \overline{z_0}\, z)^2}$ for every $z \neq \frac{1}{\overline{z_0}}$. Thus, $T'(z_0) = \frac{\lambda}{1 - |z_0|^2}$ and hence

$$\mathrm{Arg}(T'(z_0)) = \mathrm{Arg}\, \lambda.$$

If we restrict T in $\mathbb{D}$ we see that T is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$.

The next proposition describes all conformal mappings of $\mathbb{D}$ onto $\mathbb{D}$: they are just the functions T of example 8.2.1.

Proposition 8.2. *Let $z_0 \in \mathbb{D}$ and $\theta_0 \in (-\pi, \pi]$. Then the function $T : \mathbb{D} \to \mathbb{D}$ given by*

$$T(z) = e^{i\theta_0}\, \frac{z - z_0}{1 - \overline{z_0}\, z} \qquad \textit{for every } z \in \mathbb{D}$$

is a conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$. Moreover, T is the unique conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ satisfying $T(z_0) = 0$ and $\mathrm{Arg}(T'(z_0)) = \theta_0$.

Proof. From the discussion in example 8.2.1 we have all properties of the function T. Therefore, we only have to prove the uniqueness of T. So let S be another conformal mapping of $\mathbb{D}$ onto $\mathbb{D}$ satisfying $S(z_0) = 0$ and $\mathrm{Arg}(S'(z_0)) = \theta_0$.
Then the function $f = S \circ T^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic in $\mathbb{D}$ and satisfies $f(0) = 0$ and $f'(0) = \frac{S'(z_0)}{T'(z_0)} > 0$. By the Schwarz lemma we get $|f'(0)| \leq 1$.
But also the function $g = T \circ S^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic in $\mathbb{D}$ and satisfies $g(0) = 0$ and $g'(0) > 0$. Again, by the Schwarz lemma we get $|g'(0)| \leq 1$.
Now, the functions f and g are mutually inverse and hence $g'(0) = \frac{1}{f'(0)}$. Therefore, $|f'(0)| = |g'(0)| = 1$ and the Schwarz lemma implies that there is some c with $|c| = 1$ so that $f(w) = cw$ for every $w \in \mathbb{D}$. Now, $c = f'(0) > 0$ implies $c = 1$. Hence, $f(w) = w$ for every $w \in \mathbb{D}$ and finally $S(z) = T(z)$ for every $z \in \mathbb{D}$. $\square$

Exercises.

8.2.1. Let f be a conformal mapping of the region $\Omega \subseteq \mathbb{C}$ onto $\mathbb{D}$ with $f(z_0) = 0$ for some $z_0 \in \Omega$ and let $g : \Omega \to \mathbb{D}$ be holomorphic in Ω with $g(z_0) = 0$. Prove that $|g'(z_0)| \leq |f'(z_0)|$. What can you conclude if $|g'(z_0)| = |f'(z_0)|$?

8.2.2. Let $f:\mathbb{D}\to\mathbb{D}$ be holomorphic in $\mathbb{D}$. Prove that:
(i) $\left|\frac{f(z_1)-f(z_2)}{1-\overline{f(z_2)}f(z_1)}\right|\leq\left|\frac{z_1-z_2}{1-\overline{z_2}z_1}\right|$ for every $z_1,z_2\in\mathbb{D}$.
(ii) $\frac{|f'(z)|}{1-|f(z)|^2}\leq\frac{1}{1-|z|^2}$ for every $z\in\mathbb{D}$.
Prove that, if equality holds in (i) for at least one pair of $z_1,z_2\in\mathbb{D}$ with $z_1\neq z_2$ or in (ii) for at least one $z\in\mathbb{D}$, then there is $z_0\in\mathbb{D}$ and λ with $|\lambda|=1$ so that $f(z)=\lambda\,\frac{z-z_0}{1-\overline{z_0}\,z}$ for every $z\in\mathbb{D}$ and then equalities in (i) and (ii) hold identically.

8.2.3. (See exercise 8.2.2.) For every curve $\gamma:[a,b]\to\mathbb{D}$ we define the **hyperbolic length** of γ by $l_h(\gamma)=\int_a^b\frac{|\gamma'(t)|}{1-|\gamma(t)|^2}\,dt$.
(i) If $f:\mathbb{D}\to\mathbb{D}$ is holomorphic in $\mathbb{D}$, and γ is a curve in $\mathbb{D}$, prove that $l_h(f(\gamma))\leq l_h(\gamma)$. If, moreover, f is conformal onto $\mathbb{D}$, prove that $l_h(f(\gamma))=l_h(\gamma)$.
(ii) If $z_1,z_2\in\mathbb{D}$ and $z_1\neq z_2$, prove that among all curves in $\mathbb{D}$ with endpoints z_1 and z_2 the one with the smallest hyperbolic length is the arc of the circle which contains z_1,z_2 and which is orthogonal to $\mathbb{T}$. This smallest hyperbolic length is called **hyperbolic distance** of z_1,z_2 and it is equal to $d_h(z_1,z_2)=\frac{1}{2}\ln\frac{1+\left|\frac{z_1-z_2}{1-\overline{z_2}z_1}\right|}{1-\left|\frac{z_1-z_2}{1-\overline{z_2}z_1}\right|}$.

8.2.4. (See exercise 8.2.3.) Let $f:\mathbb{D}\to\mathbb{D}$ be holomorphic in $\mathbb{D}$. Consider sequences (z_n') and (z_n'') in $\mathbb{D}$ so that $z_n'\to1$ and so that $d_h(z_n',z_n'')\leq M$ for every n. Prove that $z_n''\to1$. Also, if $f(z_n')\to1$, prove that $f(z_n'')\to1$.

8.2.5. Find all $f:\mathbb{D}\to\mathbb{D}$ holomorphic in $\mathbb{D}$ with $f(0)=\frac{1}{2}$ and $f'(0)=\frac{3}{4}$.

8.2.6. Prove that for every M,N with $0<M<N$ there is $P=P(M,N)<N$ with this property: if f is holomorphic in $D_{z_0}(R)$ with $|f(z_0)|<M$ and $|f(z)|<N$ for every $z\in D_{z_0}(R)$, then $|f(z)|<P$ for every $z\in D_{z_0}(\frac{R}{2})$.

8.3 Simply connected regions and the theorem of Riemann.

Definition. *The region $\Omega\subseteq\mathbb{C}$ is called* **simply connected** *if $n(\gamma;z)=0$ for every closed curve γ in Ω and every $z\in\Omega^c$.*

If the region Ω is simply connected, then, of course, $n(\Sigma;z)=0$ for every cycle Σ in Ω and every $z\in\Omega^c$.

Example 8.3.1. Every convex region Ω is simply connected.

Example 8.3.2. If l is any halfline, then the region $\mathbb{C}\setminus l$ is simply connected.

Example 8.3.3. The region $\Omega=D_{z_0}(R_1,R_2)$ with $0\leq R_1<R_2\leq+\infty$ is not simply connected. Indeed, if $R_1<r<R_2$, the closed curve γ in Ω which describes the circle $C_{z_0}(r)$ once in the positive direction has $n(\gamma;z_0)=1$.

Example 8.3.4. A set $A\subseteq\mathbb{C}$ is called **star-shaped** if there is a specific $z_0\in A$ so that $[z_0,z]\subseteq A$ for every $z\in A$ (see also exercise 1.6.6). The point z_0 is called *center* of A. A star-shaped set A may have many centers, but this does not mean that all points of A are centers. For example, every convex set A is star-shaped and every point of A is a center of A. On the other hand, if l is a halfline, then the set $A=\mathbb{C}\setminus l$ is star-shaped and the centers of A are only the points of the halfline which is opposite to l and with the same vertex.
Now, let Ω be any open star-shaped set and let z_0 be a center of Ω. If γ is a closed curve in Ω and $z\in\Omega^c$, then we consider the halfline l_z with vertex z which is opposite to the halfine with vertex z going through z_0. The halfline l_z is contained in Ω^c and hence in the unbounded connected component of $\mathbb{C}\setminus\gamma^*$. So z is contained in the unbounded connected component of $\mathbb{C}\setminus\gamma^*$ and thus $n(\gamma;z)=0$. Therefore every open star-shaped set is simply connected.

Example 8.3.5. The region $\Omega = \mathbb{C} \setminus \big(\overline{D}_0(1) \cup (-\infty, -1]\big)$ is not star-shaped but it is simply connected. Indeed, if γ is a closed curve in Ω, then $\Omega^c = \overline{D}_0(1) \cup (-\infty, -1]$ is connected and it is contained in the unbounded connected component of $\mathbb{C} \setminus \gamma^*$. Hence $n(\gamma; z) = 0$ for every $z \in \Omega^c$.

The theorem of Cauchy in simply connected regions. *If f is holomorphic in the simply connected region $\Omega \subseteq \mathbb{C}$, then for every cycle Σ in Ω we have*

$$\int_\Sigma f(z)\, dz = 0.$$

Proof. Immediate from the theorem of Cauchy in general open sets. □

In the same manner we have versions of Cauchy's formulas for derivatives of any order, of the residue theorem, of the argument principle and of the theorem of Rouché for regions Ω which are simply connected. In all these cases we do not have to assume that the cycles Σ in Ω are null-homologous in Ω: every cycle in a simply connected region Ω is automatically null-homologous in Ω.

Proposition 8.3. *Let the region $\Omega \subseteq \mathbb{C}$ be simply connected. Then*
(i) every f holomorphic in Ω has a primitive in Ω.
(ii) for every holomorphic $g : \Omega \to \mathbb{C} \setminus \{0\}$ there is a holomorphic branch of $\log g$ *in Ω.*

Proof. (i) An application of proposition 5.22 and the theorem of Cauchy in simply connected regions.
(ii) An application of proposition 5.24 and the theorem of Cauchy in simply connected regions. □

Proposition 8.4. *Let the regions $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ be conformally equivalent. If Ω_1 is simply connected, then Ω_2 is also simply connected.*

Proof. Let $f : \Omega_1 \to \Omega_2$ be holomorphic and one-to-one from Ω_1 onto Ω_2.
Assume that Ω_2 is not simply connected. Then there is some closed curve γ in Ω_2 and some $w_0 \in \Omega_2^c$ so that $n(\gamma; w_0) \neq 0$. This implies that the holomorphic function h in Ω_2, given by $h(w) = \frac{1}{w - w_0}$, has no primitive in Ω_2. We consider the holomorphic function g in Ω_1, given by $g(z) = h(f(z))f'(z)$. Since Ω_1 is simply connected, proposition 8.3 implies that there is a primitive, say G, of g in Ω_1, i.e $G'(z) = g(z)$ for every $z \in \Omega_1$. Then we define the holomorphic function H in Ω_2 by $H(w) = G(f^{-1}(w))$ and we have

$$\begin{aligned} H'(w) &= G'(f^{-1}(w))(f^{-1})'(w) = g(f^{-1}(w))(f^{-1})'(w) \\ &= h(w)f'(f^{-1}(w))(f^{-1})'(w) = h(w) \end{aligned}$$

for every $w \in \Omega_2$. We have arrived at a contradiction. □

The theorem of Riemann. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected region, $z_0 \in \Omega$ and $\theta_0 \in (-\pi, \pi]$. Then there is a unique conformal mapping f of Ω onto $\mathbb{D}$ with $f(z_0) = 0$ and* $\mathrm{Arg}(f'(z_0)) = \theta_0$.

Proof. Step 1. We take any $a \in \Omega^c$. Since the function $z - a$ is holomorphic in Ω and has no root in Ω, proposition 8.3 implies that there is a holomorphic branch g of $\log(z - a)$ in Ω. I.e. $g : \Omega \to \mathbb{C}$ is holomorphic in Ω and $e^{g(z)} = z - a$ for every $z \in \Omega$.
Now, g is one-to-one in Ω. Indeed, if $g(z_1) = g(z_2)$, then $e^{g(z_1)} = e^{g(z_2)}$ and hence $z_1 = z_2$.
We consider $w_0' = g(z_0) + 2\pi i$ and then we have $w_0' \notin \overline{g(\Omega)}$. Indeed, if $w_0' \in \overline{g(\Omega)}$, then there are $z_n \in \Omega$ so that $g(z_n) \to w_0'$. Hence

$$z_n - a = e^{g(z_n)} \to e^{w_0'} = e^{g(z_0) + 2\pi i} = e^{g(z_0)} = z_0 - a$$

and thus $z_n \to z_0$. Then $g(z_n) \to g(z_0)$ which implies $w_0' = g(z_0)$ and we arrive at a contradiction. Since $w_0' \notin \overline{g(\Omega)}$, there is $r_0 > 0$ so that $|g(z) - w_0'| > r_0$ for every $z \in \Omega$.
We consider the function $\chi : \Omega \to \mathbb{D}$ given by

$$\chi(z) = \frac{r_0}{g(z) - w_0'} \qquad \text{for every } z \in \Omega.$$

Then χ is holomorphic and one-to-one in Ω. In particular, $\chi'(z_0) \neq 0$.
Now we consider the function $R : \mathbb{D} \to \mathbb{D}$ given by

$$R(w) = \frac{|\chi'(z_0)|}{\chi'(z_0)}\, e^{i\theta_0}\, \frac{w - \chi(z_0)}{1 - \overline{\chi(z_0)}\, w} \qquad \text{for every } w \in \mathbb{D}.$$

(Look again at example 8.2.1 and at proposition 8.2 for the properties of this kind of functions. They appear many times in this proof.) Then the function $h = R \circ \chi : \Omega \to \mathbb{D}$ is holomorphic and one-to-one in Ω and satisfies $h(z_0) = R(\chi(z_0)) = 0$ and $h'(z_0) = R'(\chi(z_0))\chi'(z_0) = \frac{|\chi'(z_0)|e^{i\theta_0}}{1-|\chi(z_0)|^2}$ and hence $\mathrm{Arg}(h'(z_0)) = \theta_0$.
Step 2. We consider the set

$$\mathcal{F} = \{h \,|\, h : \Omega \to \mathbb{D}, h \text{ is holomorphic and one-to-one in } \Omega, h(z_0) = 0, \mathrm{Arg}(h'(z_0)) = \theta_0\}.$$

The result of step 1 implies that $\mathcal{F}$ is non-empty. We also define

$$\alpha = \sup\{|h'(z_0)| \,|\, h \in \mathcal{F}\}.$$

Since, $h'(z_0) \neq 0$ for every $h \in \mathcal{F}$, we have that $\alpha > 0$ (but perhaps $\alpha = +\infty$).
There is a sequence (h_n) in $\mathcal{F}$ so that $|h_n'(z_0)| \to \alpha$. For every $h \in \mathcal{F}$ we have that $|h(z)| < 1$ for every $z \in \Omega$ and hence $\mathcal{F}$ is obviously locally bounded at every $z \in \Omega$. Montel's theorem implies that there is a subsequence (h_{n_k}) and a function f holomorphic in Ω so that $h_{n_k} \to f$ uniformly in every compact subset of Ω. Since $h_{n_k}(z_0) = 0$ for every n_k, we get $f(z_0) = 0$. The theorem of Weierstrass implies that $h_{n_k}' \to f'$ uniformly in every compact subset of Ω. Hence, $h_{n_k}'(z_0) \to f'(z_0)$ and thus $|f'(z_0)| = \alpha$ and $\mathrm{Arg}(f'(z_0)) = \theta_0$. Since $f'(z_0) \neq 0$, we have that f is not constant in Ω. Now, for every $z \in \Omega$ we have $|h_{n_k}(z)| < 1$ for every n_k and hence $|f(z)| \leq 1$. If $|f(z)| = 1$ for some $z \in \Omega$, the maximum principle implies that f is constant in Ω and we just saw that this is wrong. Therefore, $f : \Omega \to \mathbb{D}$. Next, we take any $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$. Since $h_{n_k}(z_2) \to f(z_2)$, we get that $h_{n_k} - h_{n_k}(z_2) \to f - f(z_2)$ uniformly in every compact subset of Ω and hence in every compact subset of $\Omega \setminus \{z_2\}$. Each h_{n_k} is one-to-one in Ω and so $h_{n_k} - h_{n_k}(z_2)$ has no root in $\Omega \setminus \{z_2\}$. Since $f - f(z_2)$ is not identically 0 in $\Omega \setminus \{z_2\}$, the theorem of Hurwitz implies that $f - f(z_0)$ has no root in $\Omega \setminus \{z_2\}$. Thus $f(z_1) - f(z_2) \neq 0$ and we conclude that f is one-to-one in Ω.
We proved that there is $f \in \mathcal{F}$ with $|f'(z_0)| = \alpha$.
Step 3. Assume that there is some $b \in \mathbb{D} \setminus f(\Omega)$.
We consider the function $T : \mathbb{D} \to \mathbb{D}$ given by

$$T(w) = \frac{w - b}{1 - \bar{b}\, w} \qquad \text{for every } w \in \mathbb{D}$$

and then the function $\phi = T \circ f : \Omega \to \mathbb{D}$. Then ϕ is holomorphic and one-to-one in Ω. Since $f(z) \neq b$ for every $z \in \Omega$, we have that $\phi(z) \neq 0$ for every $z \in \Omega$. But Ω is simply connected and so proposition 8.3 implies that there is a holomorphic branch of $\log \phi$ and hence a holomorphic branch ψ of $\phi^{1/2}$ in Ω. I.e. there is $\psi : \Omega \to \mathbb{D}$ which is holomorphic in Ω and satisfies

$$\psi(z)^2 = \phi(z) \qquad \text{for every } z \in \Omega.$$

It is easy to see that ψ is one-to-one in Ω, because ϕ is one-to-one in Ω.
Now we consider the function $S : \mathbb{D} \to \mathbb{D}$ given by

$$S(w) = \frac{|\psi'(z_0)|}{\psi'(z_0)}\, e^{i\theta_0}\, \frac{w - \psi(z_0)}{1 - \overline{\psi(z_0)}\, w} \qquad \text{for every } w \in \mathbb{D}$$

and then the function $h = S \circ \psi : \Omega \to \mathbb{D}$. Then h is holomorphic and one-to-one in Ω. We also see easily that $h(z_0) = S(\psi(z_0)) = 0$ and $h'(z_0) = S'(\psi(z_0))\psi'(z_0) = \frac{|\psi'(z_0)|e^{i\theta_0}}{1-|\psi(z_0)|^2}$ and hence $\mathrm{Arg}(h'(z_0)) = \theta_0$. Thus, $h \in \mathcal{F}$.
Now we have altogether that $f, \phi, \psi, h : \Omega \to \mathbb{D}$, that $T, S : \mathbb{D} \to \mathbb{D}$ and that

$$\phi = T \circ f, \qquad h = S \circ \psi, \qquad \phi = F \circ \psi,$$

where $F : \mathbb{D} \to \mathbb{D}$ is given by $F(w) = w^2$ for every w. All these functions, except F, are one-to-one. We consider now the holomorphic function $\Phi : \mathbb{D} \to \mathbb{D}$, given by

$$\Phi = T^{-1} \circ F \circ S^{-1},$$

and then we have

$$f = \Phi \circ h.$$

Now, $\Phi(0) = (T^{-1} \circ F \circ S^{-1})(0) = (T^{-1} \circ F)(\psi(z_0)) = T^{-1}(\phi(z_0)) = f(z_0) = 0$ and

$$|f'(z_0)| = |\Phi'(h(z_0))||h'(z_0)| = |\Phi'(0)||h'(z_0)|. \tag{8.2}$$

Then the Schwartz lemma implies that $|\Phi'(0)| \leq 1$.
If $|\Phi'(0)| = 1$, then there is c with $|c| = 1$ so that $\Phi(z) = cz$ for every $z \in \mathbb{D}$. This implies that $F(w) = T(cS(w))$ for every $w \in \mathbb{D}$. This is wrong because the right side is one-to-one in $\mathbb{D}$. We conclude that $|\Phi'(0)| < 1$ and (8.2) implies that

$$|h'(z_0)| > |f'(z_0)| = \alpha.$$

This contradicts the definition of α and the fact that $h \in \mathcal{F}$. Therefore, there is no $b \in \mathbb{D} \setminus f(\Omega)$ and hence f is onto $\mathbb{D}$.
We proved the existence of a function $f : \Omega \to \mathbb{D}$ which is conformal from Ω onto $\mathbb{D}$ and which satisfies $f(z_0) = 0$ and $\mathrm{Arg}(f'(z_0)) = \theta_0$.
Step 4. To prove the uniqueness of f, we repeat the argument in the proof of proposition 8.2. Let $f_1, f_2 : \Omega \to \mathbb{D}$ be conformal from Ω onto $\mathbb{D}$ with $f_1(z_0) = f_2(z_0) = 0$ and $\mathrm{Arg}(f_1'(z_0)) = \mathrm{Arg}(f_2'(z_0)) = \theta_0$.
Then the function $f = f_2 \circ f_1^{-1} : \mathbb{D} \to \mathbb{D}$ is holomorphic in $\mathbb{D}$ and satisfies $f(0) = 0$ and $f'(0) = \frac{f_2'(z_0)}{f_1'(z_0)} > 0$. By the Schwarz lemma we get $|f'(0)| \leq 1$.
The function $g = f_1 \circ f_2^{-1} : \mathbb{D} \to \mathbb{D}$ is also holomorphic in $\mathbb{D}$ and satisfies $g(0) = 0$ and $g'(0) = \frac{f_1'(z_0)}{f_2'(z_0)} > 0$. Again, by the Schwarz lemma we get $|g'(0)| \leq 1$.
But the functions f and g are mutually inverse and hence $g'(0) = \frac{1}{f'(0)}$. Therefore, $|f'(0)| = |g'(0)| = 1$ and the Schwarz lemma implies that there is some c with $|c| = 1$ so that $f(w) = cw$ for every $w \in \mathbb{D}$. Now, $c = f'(0) > 0$ implies $c = 1$. Hence, $f(w) = w$ for every $w \in \mathbb{D}$ and finally $f_2(z) = f_1(z)$ for every $z \in \mathbb{D}$. □

Proposition 8.5. *Every simply connected region $\Omega \subsetneqq \mathbb{C}$ is conformally equivalent with $\mathbb{D}$. The simply connected region $\mathbb{C}$ is conformally equivalent only with itself.*

Proof. The first part is a simple application of the theorem of Riemann.
If $\mathbb{C}$ is conformally equivalent with some simply connected region $\Omega \subsetneqq \mathbb{C}$, then, by the first part, $\mathbb{C}$ is conformally equivalent with $\mathbb{D}$. Thus, there is a holomorphic $f : \mathbb{C} \to \mathbb{D}$ which is one-to-one in $\mathbb{C}$. But Liouville's theorem implies that f is constant and we arrive at a contradiction. □

Exercises.

8.3.1. Are the regions $D_0(1,3)\setminus[1,3]$ and $\mathbb{C}\setminus\big((-\infty,-2]\cup[-\frac{1}{2},\frac{1}{2}]\cup[2,+\infty)\big)$ simply connected? Which are the possible values of $\int_\gamma(z+\frac{1}{z})\,dz$, where γ is a closed curve (i) in the first set? (ii) in the second set?

8.3.2. Let f be holomorphic in the simply connected region Ω except for isolated singularities in Ω. Prove that (i) and (ii) are equivalent:
(i) $e^{\int_\gamma f(z)\,dz}=1$ for every closed curve γ in Ω whose trajectory contains no isolated singularity of f.
(ii) $\mathrm{Res}(f;z)\in\mathbb{Z}$ for every isolated singularity z of f in Ω.
If f satisfies (i), (ii) and it is holomorphic at $z_0\in\Omega$, define $F(z)=e^{\int_\gamma f(\zeta)\,d\zeta}$ for every $z\in\Omega$, where γ is any curve in Ω from z_0 to z and whose trajectory contains no isolated singularity of f. Prove that F is well-defined and holomorphic in Ω except for the isolated singularities of f.
Prove that every point in Ω is either a point of holomorphy or a pole of F if and only if all isolated singularities of f in Ω are simple poles of f.

8.3.3. Let $\mathbb{H}_+=\{z\mid \mathrm{Im}\,z>0\}$, $z_0\in\mathbb{H}_+$, $\theta_0\in(-\pi,\pi]$. Find the unique conformal mapping f of $\mathbb{H}_+$ onto $\mathbb{D}$ with $f(z_0)=0$ and $\mathrm{Arg}(f'(z_0))=\theta_0$.

8.3.4. Find a conformal mapping of $\{z\mid \mathrm{Re}\,z>0,\mathrm{Im}\,z>0\}$ onto $\mathbb{D}$.

8.3.5. (i) Find a conformal mapping between two angular regions.
(ii) Find a conformal mapping between an angular region and an open zone.
(iii) Find a conformal mapping between an angular region and the intersection of two open discs or the intersection of an open disc and an open halfplane.

8.3.6. Find a conformal mapping f
(i) between $\widehat{\mathbb{C}}\setminus[-1,1]$ and $\mathbb{D}$, with $f(\infty)=0$.
(ii) between $\widehat{\mathbb{C}}\setminus\tau$ and $\mathbb{D}$, with $f(\infty)=0$, where τ is the arc of $\mathbb{T}$ with endpoints e^{-ia} and e^{ia} $(0<a<\pi)$.

8.3.7. Find a conformal mapping of $[-1,1]\times[-1,1]$ onto $D_0(1)$.

8.3.8. Prove that there is no conformal mapping of $\mathbb{D}$ onto $\mathbb{D}\setminus\{0\}$.

8.3.9. Let $\mathbb{H}_+=\{z\mid \mathrm{Im}\,z>0\}$ and let $f:\mathbb{H}_+\to\mathbb{H}_+$ be holomorphic in $\mathbb{H}_+$. Prove that:
(i) $\big|\frac{f(z_1)-f(z_2)}{f(z_1)-\overline{f(z_2)}}\big|\le\big|\frac{z_1-z_2}{z_1-\overline{z_2}}\big|$ for every $z_1,z_2\in\mathbb{H}_+$.
(ii) $\frac{|f'(z)|}{\mathrm{Im}\,f(z)}\le\frac{1}{\mathrm{Im}\,z}$ for every $z\in\mathbb{H}_+$.
Prove that, if equality holds in (i) for at least one pair of $z_1,z_2\in\mathbb{H}_+$ with $z_1\ne z_2$ or in (ii) for at least one $z\in\mathbb{H}_+$, then there is $z_0\in\mathbb{H}_+$ and λ with $|\lambda|=1$ so that $\frac{f(z)-i}{f(z)+i}=\lambda\,\frac{z-z_0}{z-\overline{z_0}}$ for every $z\in\mathbb{H}_+$ and then equalities in (i) and (ii) hold identically.

8.3.10. Let $\mathbb{H}_+=\{z\mid \mathrm{Im}\,z>0\}$ and let $f:\mathbb{H}_+\to\mathbb{D}$ be holomorphic in $\mathbb{H}_+$ with $f(i)=0$. Prove that $|f(z)|\le|\frac{i-z}{i+z}|$ for every $z\in\mathbb{H}_+$ and $|f'(i)|\le\frac{1}{2}$.

8.3.11. Let $\Omega\subseteq\mathbb{C}$ be a simply connected region, $z_0\in\Omega$ and f,g be conformal mappings of Ω onto $\mathbb{D}$ with $f(z_0)=g(z_0)$ for some $z_0\in\Omega$. Find a relation between f,g.

8.3.12. Let $\Omega_1,\Omega_2\subseteq\mathbb{C}$ be two regions and f be a conformal mapping of Ω_1 onto Ω_2. If (z_n) is in Ω_1 and $z_n\to z\in\partial\Omega_1$, prove that every limit point of $(f(z_n))$ belongs to $\partial\Omega_2$. Is it necessary for $(f(z_n))$ to converge?

8.3.13. (i) Let $f,g:\mathbb{D}\to\Omega$ be holomorphic in $\mathbb{D}$ so that f is one-to-one in $\mathbb{D}$ and onto Ω. If $f(0)=g(0)$, prove that $g(D_0(r))\subseteq f(D_0(r))$ for every r with $0<r<1$.
(ii) Let $\Omega=\{w=u+iv\mid -1<u<1\}$. Find the conformal mapping f of $\mathbb{D}$ onto Ω with $f(0)=0$ and $f'(0)>0$. If $g:\mathbb{D}\to\Omega$ is holomorphic in $\mathbb{D}$ with $g(0)=0$, prove that $|g(z)|\le\frac{2}{\pi}\ln\frac{1+|z|}{1-|z|}$ for every $z\in\mathbb{D}$.

8.3.14. Let $\Omega \subseteq \mathbb{D}$ be a simply connected region, $0 \in \Omega$ and let $\mathcal{F}$ be the collection of all holomorphic $f : \Omega \to \mathbb{D}$ with $f(0) = 0$ and which are one-to-one in Ω. We fix $a \in \mathbb{D} \setminus \{0\}$ and we define $m = \sup\{|f(a)| \mid f \in \mathcal{F}\}$. Prove that there is $f_0 \in \mathcal{F}$ so that $|f_0(a)| = m$ and that such a f_0 is a conformal mapping of Ω onto $\mathbb{D}$.

8.3.15. Let $\Omega \subsetneqq \mathbb{C}$ be a simply connected region so that $\overline{z} \in \Omega$ for every $z \in \Omega$. Let $z_0 \in \Omega \cap \mathbb{R}$ and let f be the conformal mapping of Ω onto $\mathbb{D}$ with $f(z_0) = 0$ and $f'(z_0) > 0$. Let $\Omega_+ = \{z \in \Omega \mid \operatorname{Im} z > 0\}$, $\Omega_- = \{z \in \Omega \mid \operatorname{Im} z < 0\}$, $\mathbb{D}_+ = \{z \in \mathbb{D} \mid \operatorname{Im} z > 0\}$ and $\mathbb{D}_- = \{z \in \mathbb{D} \mid \operatorname{Im} z < 0\}$. Prove that $f(\Omega_+) = \mathbb{D}_+$, $f(\Omega_-) = \mathbb{D}_-$ and $f(\Omega \cap \mathbb{R}) = (-1, 1)$.

Chapter 9

Isolated singularities and roots.

9.1 Isolated singularities in the complex plane.

Let us generalize slightly the argument at the end of section 6.9. We consider a function f in $\mathbb{C}$ with a finite number of isolated singularities $z_1, \dots, z_n$ and holomorphic in the rest of $\mathbb{C}$. The singular part of f at z_j has the form

$$s_j(z) = \sum_{-\infty}^{k=-1} a_{j,k}(z - z_j)^k = \sum_{k=1}^{+\infty} \frac{a_{j,-k}}{(z - z_j)^k}$$

and converges in $\widehat{\mathbb{C}} \setminus \{z_j\}$. We consider the function

$$h(z) = f(z) - (s_1(z) + \cdots + s_n(z)).$$

Then h is holomorphic in the set $\mathbb{C} \setminus \{z_1, \dots, z_n\}$ and its only possible singularities are the points $z_1, \dots, z_n$. We observe that every z_j is a removable singularity of $f(z) - s_j(z)$ and that all terms $s_1(z), \dots, s_n(z)$, besides $s_j(z)$, are holomorphic at z_j. Therefore, every z_j is a removable singularity of the function h. So the function h has *no isolated singularities* and hence it is holomorphic in $\mathbb{C}$. Now, we have the identity

$$f(z) = s_1(z) + \cdots + s_n(z) + h(z),$$

which gives *the general form of a holomorphic function in $\mathbb{C}$ with the exception of finitely many isolated singularities*.

We shall generalize this to the case of a holomorphic function f in $\mathbb{C}$ with the exception of infinitely many isolated singularities. In this case, i.e. if the terms of the sequence (z_n) are the distinct isolated singularities of f in $\mathbb{C}$, it is necessary that $z_n \to \infty$. In the opposite case there would be a subsequence of (z_n) converging to some $z \in \mathbb{C}$ and then this z would be a non-isolated singularity of f.

We may obviously try to form the infinite sum

$$\sum_{n=1}^{+\infty} s_n(z),$$

but this is doomed to failure in the general case since there is no guarantee that this series converges. The next theorem shows that we may subtract a suitable “correction term” from each $s_n(z)$ so as to make the series convergent.

The theorem of Mittag-Leffler. *Let the terms of the sequence (z_n) be distinct with $z_n \to \infty$. For each z_n we consider a power series of the form $s_n(z) = \sum_{-\infty}^{k=-1} a_{n,k}(z - z_n)^k$, which converges*

in $\widehat{\mathbb{C}} \setminus \{z_n\}$.
(i) Then there are polynomials q_n so that the series of functions

$$\sum_{n=1}^{+\infty}(s_n - q_n)$$

has the property: for every compact set K there is n_0 so that $\sum_{n=n_0+1}^{+\infty}(s_n - q_n)$ converges uniformly in K.
(ii) If the polynomials q_n satisfy (i) then the function $F = \sum_{n=1}^{+\infty}(s_n - q_n)$ is holomorphic in $\mathbb{C}$ with the exception of the terms of (z_n) and its singular part at each z_n is s_n. Moreover, the most general holomorphic function in $\mathbb{C}$ with the exception of the terms of (z_n) and whose singular part at each z_n is s_n is of the form

$$f = F + h = \sum_{n=1}^{+\infty}(s_n - q_n) + h,$$

where h is an arbitrary function holomorphic in $\mathbb{C}$. We also have that

$$f' = \sum_{n=1}^{+\infty}(s_n' - q_n') + h'.$$

Proof. (i) If $z_n = 0$, we just take $q_n = 0$. If $z_n \neq 0$, then the function s_n is holomorphic in the disc $D_0(|z_n|)$ and so its Taylor series at 0 converges to it uniformly in the smaller disc $D_0(|z_n|/2)$. Hence there is a partial sum q_n of this Taylor series so that

$$\sup_{z \in D_0(|z_n|/2)} |s_n(z) - q_n(z)| \leq \frac{1}{2^n}.$$

Of course q_n is a polynomial.
Now let K be any compact set. Since K is bounded, there is $R > 0$ so that $K \subseteq D_0(R)$. Since $z_n \to \infty$, there is n_0 so that $|z_n| \geq 2R$ and hence $K \subseteq D_0(|z_n|/2)$ for every $n \geq n_0 + 1$. Thus

$$\sup_{z \in K} |s_n(z) - q_n(z)| \leq \frac{1}{2^n} \qquad \text{for every } n \geq n_0 + 1.$$

The test of Weierstrass implies that $\sum_{n=n_0+1}^{+\infty}(s_n - q_n)$ converges uniformly in K.
(ii) We assume that the polynomials q_n satisfy (i) and we take any $z \in \mathbb{C}$. Since $\{z\}$ is compact, there is n_0 so that $\sum_{n=n_0+1}^{+\infty}(s_n(z) - q_n(z))$ converges. So if z is not equal to any of $z_1, \dots, z_{n_0-1}$, then the sum $\sum_{n=1}^{+\infty}(s_n(z) - q_n(z))$ is finite and we define the function $F : \mathbb{C} \setminus \{z_n \,|\, n \in \mathbb{N}\} \to \mathbb{C}$ by

$$F = \sum_{n=1}^{+\infty}(s_n - q_n).$$

If z is not equal to any of the terms of (z_n), then, because of $z_n \to \infty$, there is a closed disc $\overline{D}_z(r)$ which contains no term of (z_n). Then there is n_0 so that $\sum_{n=n_0+1}^{+\infty}(s_n - q_n)$ converges uniformly in $\overline{D}_z(r)$ and so it defines a function holomorphic in $D_z(r)$. But the finite sum $\sum_{n=1}^{n_0}(s_n - q_n)$ is also holomorphic in $D_z(r)$ and hence F is holomorphic in $D_z(r)$. Moreover, by the uniform convergence of $\sum_{n=n_0+1}^{+\infty}(s_n - q_n)$ in $D_z(r)$, we have that the series of the derivatives also converges uniformly in $D_z(r)$ and hence

$$F'(z) = \sum_{n=1}^{+\infty}(s_n'(z) - q_n'(z)).$$

This equality holds at every z which is not equal to any of the terms of $\{z_n\}$.
If $z = z_k$ for some k, then there is a closed disc $\overline{D}_{z_k}(r)$ which contains only the term z_k of (z_n). Then there is n_0 so that $\sum_{n=n_0+1}^{+\infty}(p_n - q_n)$ converges uniformly in $\overline{D}_{z_k}(r)$ and so it defines a function holomorphic in $D_{z_k}(r)$. But the finite sum $\sum_{n=1}^{n_0}(s_n - q_n)$ is holomorphic in $D_{z_k}(r)\setminus\{z_k\}$ with singular part s_k at z_k. So F has the singular part s_k at z_k.
We conclude that F is holomorphic in $\mathbb{C}$ with the exception of the terms of (z_n) and that its singular part at each z_n is s_n.
Now let us consider an arbitrary holomorphic function f in $\mathbb{C}$ with the exception of the terms of (z_n) and whose singular part at each z_n is s_n. Then the function $h = f - F$ is holomorphic in $\mathbb{C}$ and hence $f = F + h$. □

The theorem of Mittag-Leffler describes the most general holomorphic function in $\mathbb{C}$ with the exception of preassigned isolated singularities and corresponding preassigned singular parts. In fact, the actual theorem of Mittag-Leffler is restricted to the case of meromorphic functions, i.e. to the case that all isolated singularities are poles.

Example 9.1.1. We consider the function $\frac{1}{\sin z}$, which is meromorphic in $\mathbb{C}$. Its poles are the points $n\pi$, $n \in \mathbb{Z}$. Since $\sin z = z - \frac{z^3}{3!} + \cdots$ is the Taylor series of $\sin z$ at 0 we have, for each $n \in \mathbb{Z}$, that

$$\sin z = (-1)^n \sin(z - n\pi) = (-1)^n(z - n\pi) - \frac{(-1)^n(z - n\pi)^3}{3!} + \cdots = (z - n\pi)g(z)$$

where g is holomorphic in $\mathbb{C}$ with $g(n\pi) = (-1)^n$. So the function $h = \frac{1}{g}$ is holomorphic at $n\pi$ with $h(n\pi) = (-1)^n$ and

$$\frac{1}{\sin z} = \frac{h(z)}{z - n\pi} = \frac{(-1)^n}{z - n\pi} + h'(0) + \frac{h''(0)}{2!}(z - n\pi) + \cdots$$

in a neighborhood of $n\pi$. This says that $n\pi$ is a pole of order 1 of $\frac{1}{\sin z}$ and the singular part at $n\pi$ is $s_n(z) = \frac{(-1)^n}{z-n\pi}$.
Now we take $n \neq 0$ and we write the Taylor series of $\frac{(-1)^n}{z-n\pi}$ at 0:

$$\frac{(-1)^n}{z - n\pi} = \frac{(-1)^{n+1}}{n\pi}\,\frac{1}{1 - z/(n\pi)} = \frac{(-1)^{n+1}}{n\pi} + \frac{(-1)^{n+1}}{(n\pi)^2}\,z + \frac{(-1)^{n+1}}{(n\pi)^3}\,z^2 + \cdots .$$

We consider the polynomial q_n to be the constant term of this Taylor series, i.e. $q_n(z) = \frac{(-1)^{n+1}}{n\pi}$.
If $n = 0$, we just take $q_n = 0$.
Now we examine the uniform convergence of the series

$$\sum_{n\in\mathbb{Z}}^{+\infty}(s_n(z) - q_n(z)) = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\Big(\frac{1}{z - n\pi} + \frac{1}{n\pi}\Big).$$

If K is a compact set, then there is R so that $K \subseteq D_0(R)$. Now, if $n_0 + 1 \geq \frac{2R}{\pi}$ and $z \in K$, then for every n with $|n| \geq n_0 + 1$ we have that $|z - n\pi| \geq |n|\pi - |z| \geq |n|\pi - R \geq \frac{|n|\pi}{2}$ and hence

$$\Big|(-1)^n\Big(\frac{1}{z - n\pi} + \frac{1}{n\pi}\Big)\Big| = \frac{|z|}{|n|\pi|z - n\pi|} \leq \frac{2R}{n^2\pi^2} \qquad \text{when } z \in K \text{ and } |n| \geq n_0 + 1.$$

By the test of Weierstrass, $\sum_{n\in\mathbb{Z},|n|\geq n_0+1}^{+\infty}(-1)^n\big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\big)$ converges uniformly in K.
Now the theorem of Mittag-Leffler implies that

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}}(-1)^n\Big(\frac{1}{z - n\pi} + \frac{1}{n\pi}\Big) + h(z),$$

where h is holomorphic in $\mathbb{C}$. We shall determine the function h.
We differentiate the series to get

$$\frac{\cos z}{\sin^2 z} = \frac{1}{z^2} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{(z-n\pi)^2} + h'(z) = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n\pi)^2} + h'(z).$$

The function $\frac{\cos z}{\sin^2 z}$ is 2π-periodic and it is easy to prove that $\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n\pi)^2}$ is also 2π-periodic. Indeed,

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z+2\pi-n\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-(n-2)\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{(-1)^{n+2}}{(z-n\pi)^2} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n\pi)^2}.$$

Therefore, h' is 2π-periodic.
We restrict now our investigation in a *period-zone* $A = \{z + iy \mid -\pi \le x \le \pi\}$. Again, it is easy to prove that $\frac{\cos z}{\sin^2 z} \to 0$ when $z \to \infty$ in A. Indeed, if $z = x + iy$ and $|x| \le \pi$, then we have

$$\Big|\frac{\cos z}{\sin^2 z}\Big| = \frac{\sinh^2 y + \cos^2 x}{(\sinh^2 y + \sin^2 x)^2} \le \frac{\sinh^2 y + 1}{\sinh^4 y} \to 0 \qquad \text{when } y \to \pm\infty.$$

The same is true for $\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n\pi)^2}$. To see this we take $\epsilon > 0$ and then there is n_0 so that

$$\sum_{n\in\mathbb{Z},|n|\ge n_0+1} \frac{1}{(|n|-1)^2\pi^2} < \frac{\epsilon}{2}.$$

If $z \in A$, i.e. if $z = x + iy$ and $|x| \le \pi$, then $|z - n\pi| \ge |x - n\pi| \ge (|n|-1)\pi$ and hence

$$\Big|\sum_{n\in\mathbb{Z},|n|\ge n_0+1} \frac{(-1)^n}{(z-n\pi)^2}\Big| \le \sum_{n\in\mathbb{Z},|n|\ge n_0+1} \frac{1}{|z-n\pi|^2} \le \sum_{n\in\mathbb{Z},|n|\ge n_0+1} \frac{1}{(|n|-1)^2\pi^2} < \frac{\epsilon}{2}.$$

Since $\frac{1}{(z-n\pi)^2} \to 0$ when $z \to \infty$, we have that there is $y_0 > 0$ so that

$$\Big|\sum_{n\in\mathbb{Z},|n|\le n_0} \frac{(-1)^n}{(z-n\pi)^2}\Big| < \frac{\epsilon}{2}$$

when $z = x + iy$ and $|x| \le \pi$, $|y| > y_0$. Thus

$$\Big|\sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(z-n\pi)^2}\Big| \le \Big|\sum_{n\in\mathbb{Z},|n|\le n_0} \frac{(-1)^n}{(z-n\pi)^2}\Big| + \Big|\sum_{n\in\mathbb{Z},|n|\ge n_0+1} \frac{(-1)^n}{(z-n\pi)^2}\Big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

when $z = x + iy$ and $|x| \le \pi$, $|y| > y_0$.
We conclude that $h'(z) \to 0$ when $z \to \infty$ in A. This implies that h' is bounded in the period-zone A and since h' is 2π-periodic we have that h' is bounded in C. By the theorem of Liouville, h' is constant in $\mathbb{C}$. But since $h'(z) \to 0$ when $z \to \infty$ in A, we find that $h' = 0$ in $\mathbb{C}$. This implies that h is constant in $\mathbb{C}$.
Now we go back to $\frac{1}{\sin z} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n\big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\big) + h(z)$. We observe that the terms $\frac{1}{\sin z}$ and $\frac{1}{z}$ are odd functions. The same is true for $\sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n\big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\big)$. Indeed,

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n\Big(\frac{1}{-z-n\pi} + \frac{1}{n\pi}\Big) &= -\sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n\Big(\frac{1}{z+n\pi} - \frac{1}{n\pi}\Big) \\ &= -\sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^{-n}\Big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\Big) \\ &= -\sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n\Big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\Big).\end{aligned}$$

Hence h is an odd constant function and this implies that $h = 0$ in $\mathbb{C}$. So we end up with the identity

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n \Big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\Big).$$

In exactly the same manner we can prove the identity

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \Big(\frac{1}{z-n\pi} + \frac{1}{n\pi}\Big)$$

from which, by differentiation, we get

$$\frac{1}{\sin^2 z} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n\pi)^2}.$$

Exercises.

9.1.1. Express $\sum_{n\in\mathbb{Z}} \frac{1}{z^2-n^2\pi^2}$ in closed form.

9.1.2. Find the values of $\sum_{n=1}^{+\infty} \frac{1}{n^2}$, $\sum_{n=1}^{+\infty} \frac{1}{n^4}$, $\sum_{n=1}^{+\infty} \frac{1}{n^6}$ through the Laurent series of $\cot z$.

9.1.3. Express $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n\pi)^2+a^2}$ in closed form.

9.2 Infinite products.

Let (z_n) be a sequence in $\mathbb{C}$. The expression

$$\prod_{n=1}^{+\infty} z_n$$

is called **infinite product** of the $z_1, z_2, \ldots$. We consider three cases.
First case. $z_n \neq 0$ for every n.
We denote $p_n = z_1 \cdots z_n$ the n-th *partial product* of the $z_1, z_2, \ldots$. If $p_n \to p$ for some $p \in \widehat{\mathbb{C}}$, we write

$$\prod_{n=1}^{+\infty} z_n = p$$

and we say that p is the product of the $z_1, z_2, \ldots$. If $p \neq 0$ and $p \neq \infty$, we say that the infinite product **converges** to p. If $p = 0$ or $p = \infty$, we say that the infinite product **diverges** to 0 or to ∞, respectively. If the sequence (p_n) does not have a limit, we say that the infinite product **diverges**.

Example 9.2.1. Let $z_n = 1 + \frac{1}{n}$ for every n. Then

$$p_n = \Big(1+\frac{1}{1}\Big)\cdots\Big(1+\frac{1}{n}\Big) = \frac{2}{1}\frac{3}{2}\cdots\frac{n}{n-1}\frac{n+1}{n} = n+1.$$

Hence $p_n \to \infty$ and so $\prod_{n=1}^{+\infty}(1+\frac{1}{n}) = \infty$. In this case the infinite product diverges to ∞.
Since all p_n are real, we may also say that $p_n \to +\infty$ and that the infinite product diverges to $+\infty$.

Example 9.2.2. Let $z_n = 1 - \frac{1}{n+1}$ for every n. Then

$$p_n = \Big(1-\frac{1}{2}\Big)\cdots\Big(1-\frac{1}{n+1}\Big) = \frac{1}{2}\frac{2}{3}\cdots\frac{n-1}{n}\frac{n}{n+1} = \frac{1}{n+1}.$$

So $p_n \to 0$ and $\prod_{n=1}^{+\infty}(1-\frac{1}{n+1}) = 0$. In this case the infinite product diverges to 0.

Example 9.2.3. Let $z_n = 1 - \frac{1}{(n+1)^2}$ for every n. Then

$$p_n = \Big(1 - \frac{1}{2^2}\Big) \cdots \Big(1 - \frac{1}{(n+1)^2}\Big) = \frac{1 \cdot 3}{2^2} \frac{2 \cdot 4}{3^2} \cdots \frac{(n-1)(n+1)}{n^2} \frac{n(n+2)}{(n+1)^2} = \frac{1}{2} \frac{n+2}{n+1}.$$

Therefore $p_n \to \frac{1}{2}$ and so $\prod_{n=1}^{+\infty}(1 - \frac{1}{(n+1)^2}) = \frac{1}{2}$. In this case the infinite product converges to $\frac{1}{2}$.

Second case. There is m so that $z_n \neq 0$ for every $n \geq m+1$ and $z_n = 0$ for at least one $n \leq m$. If the infinite product $\prod_{n=m+1}^{+\infty} z_n$ does not have a limit we say that the infinite product $\prod_{n=1}^{+\infty} z_n$ **diverges**. Now let $\prod_{n=m+1}^{+\infty} z_n = p'$ for some $p' \in \widehat{\mathbb{C}}$. If $p' \neq 0$ and $p' \neq \infty$, then $\prod_{n=m+1}^{+\infty} z_n$ converges to p', and we say that $\prod_{n=1}^{+\infty} z_n$ **converges** to $p = \big(\prod_{n=1}^{m} z_n\big)p' = 0p' = 0$. If $p' = 0$, then $\prod_{n=m+1}^{+\infty} z_n$ diverges to 0, and we say that $\prod_{n=1}^{+\infty} z_n$ **diverges** to $p = \big(\prod_{n=1}^{m} z_n\big)0 = 00 = 0$. If $p' = \infty$, then $\prod_{n=m+1}^{+\infty} z_n$ diverges to ∞, and we say that $\prod_{n=1}^{+\infty} z_n$ **diverges**.
Third case. There are infinitely many n so that $z_n = 0$.
Then we say that $\prod_{n=1}^{+\infty} z_n$ **diverges**.

Therefore, the infinite product $\prod_{n=1}^{+\infty} z_n$ converges if and only if there is m so that $z_n \neq 0$ for every $n \geq m+1$ and the partial products $z_{m+1} \cdots z_n$ converge (as $n \to +\infty$) to some complex number $\neq 0$. Moreover, if $\prod_{n=1}^{+\infty} z_n$ converges, its value is equal to 0 if and only if $z_n = 0$ for at least one n.

Proposition 9.1. *If $\prod_{n=1}^{+\infty} z_n$ converges, then $z_n \to 1$.*

Proof. There is m so that $z_n \neq 0$ for every $n \geq m+1$ and $\prod_{n=m+1}^{+\infty} z_n = p'$ where $p' \neq 0, \infty$. If $n \geq m+1$, we set $p'_n = z_{m+1} \cdots z_n$ and we have that $p'_n \to p'$. Thus, $z_n = \frac{p'_n}{p'_{n-1}} \to \frac{p'}{p'} = 1$. □

From now on we shall use the symbol

$$\prod_{n=1}^{+\infty}(1 + a_n)$$

for the infinite product. According to the previous discussion, convergence of the infinite product implies that $a_n \to 0$.

There are two simple inequalities which play some role in the study of infinite products. The first is:

$$1 + a_1 + \cdots + a_n \leq (1 + a_1) \cdots (1 + a_n) \leq e^{a_1 + \cdots + a_n} \qquad \text{when } 0 \leq a_1, \ldots, a_n. \tag{9.1}$$

The left is easily proved by induction and the right is based on the well-known $1 + x \leq e^x$. The second inequality is:

$$1 - a_1 - \cdots - a_n \leq (1 - a_1) \cdots (1 - a_n) \qquad \text{when } 0 \leq a_1, \ldots, a_n \leq 1. \tag{9.2}$$

This is proved also by induction.

Lemma 9.1. *Let $a_n \geq 0$ for every n. Then $\prod_{n=1}^{+\infty}(1 + a_n)$ converges if and only if $\sum_{n=1}^{+\infty} a_n$ converges.*

Proof. We set $p_n = (1 + a_1) \cdots (1 + a_n)$ for every n. Then the sequence (p_n) is increasing and we have $p_n \geq 1$ for every n. Then $p = \lim_{n \to +\infty} p_n$ exists and $1 \leq p \leq +\infty$. We also denote $s = \sum_{n=1}^{+\infty} a_n$ and we have $0 \leq s \leq +\infty$. Taking the limit in (9.1) we find

$$1 + s \leq p \leq e^s.$$

Thus, $p < +\infty$ if and only if $s < +\infty$. □

Example 9.2.4. $\prod_{n=1}^{+\infty}(1+\frac{1}{n})=+\infty$, because $\sum_{n=1}^{+\infty}\frac{1}{n}=+\infty$.
$\prod_{n=1}^{+\infty}(1+\frac{1}{n^2})$ converges, because $\sum_{n=1}^{+\infty}\frac{1}{n^2}<+\infty$.

We say that the infinite product $\prod_{n=1}^{+\infty}(1+a_n)$ **converges absolutely** if the infinite product $\prod_{n=1}^{+\infty}(1+|a_n|)$ converges or, equivalently, if the series $\sum_{n=1}^{+\infty}|a_n|$ converges.

Criterion of absolute convergence. *If $\prod_{n=1}^{+\infty}(1+a_n)$ converges absolutely, then it converges.*

Proof. Since $\sum_{n=1}^{+\infty}|a_n|<+\infty$, we have that $a_n\to 0$ and so at most finitely many a_n are equal to -1.
At first we assume that $\sum_{n=1}^{+\infty}|a_n|<1$. Then $a_n\neq -1$ for every n. We denote

$$p_n=(1+a_1)\cdots(1+a_n),\qquad P_n=(1+|a_1|)\cdots(1+|a_n|).$$

Then, if $n<m$, we have

$$\begin{aligned}|p_m-p_n|&=\Big|\prod_{k=1}^{m}(1+a_k)-\prod_{k=1}^{n}(1+a_k)\Big|=\Big|\prod_{k=1}^{n}(1+a_k)\Big(\prod_{k=n+1}^{m}(1+a_k)-1\Big)\Big|\\&=\prod_{k=1}^{n}|1+a_k|\Big|\prod_{k=n+1}^{m}(1+a_k)-1\Big|\leq\prod_{k=1}^{n}(1+|a_k|)\Big(\prod_{k=n+1}^{m}(1+|a_k|)-1\Big)\\&=\prod_{k=1}^{m}(1+|a_k|)-\prod_{k=1}^{n}(1+|a_k|)=P_m-P_n.\end{aligned}\tag{9.3}$$

Since $\prod_{n=1}^{+\infty}(1+|a_n|)$ converges, we have that (P_n) is a Cauchy sequence. So the last inequality implies that (p_n) is also a Cauchy sequence and hence converges. We only have to show that $\lim_{n\to+\infty}p_n\neq 0$. We write

$$|p_n|=\prod_{k=1}^{n}|1+a_k|\geq\prod_{k=1}^{n}(1-|a_k|)\geq 1-\sum_{k=1}^{n}|a_k|\geq 1-\sum_{k=1}^{+\infty}|a_k|>0,$$

where for the second inequality we use (9.2). Thus, $\lim_{n\to+\infty}|p_n|\geq 1-\sum_{k=1}^{+\infty}|a_k|>0$.
Now we consider the case $\sum_{n=1}^{+\infty}|a_n|\geq 1$. Then there is m so that $\sum_{n=m+1}^{+\infty}|a_n|<1$ and from the first case we have that $\prod_{n=m+1}^{+\infty}(1+a_n)$ converges. Hence $\prod_{n=1}^{+\infty}(1+a_n)$ also converges. $\square$

Proposition 9.2. *Let $a_n:A\to\mathbb{C}$ be bounded functions in A and let $\sum_{n=1}^{+\infty}|a_n|$ converge uniformly in A. Then $\prod_{n=1}^{+\infty}(1+a_n)$ converges uniformly in A.*

Proof. Since $\sum_{n=1}^{+\infty}|a_n(z)|$ converges for every $z\in A$, we have that $\prod_{n=1}^{+\infty}(1+a_n(z))$ converges absolutely and so it converges for every $z\in A$. We define $p:A\to\mathbb{C}$ by

$$p(z)=\prod_{n=1}^{+\infty}(1+a_n(z))\qquad\text{for every } z\in A.$$

The uniform convergence of $\sum_{n=1}^{+\infty}|a_n|$ in A implies that there is M so that $\sum_{n=1}^{+\infty}|a_n(z)|\leq M$ for every $z\in A$.
We set $p_n(z)=\prod_{k=1}^{n}(1+a_k(z))$, $S_n(z)=\sum_{k=1}^{n}|a_k(z)|$ and $S(z)=\sum_{k=1}^{+\infty}|a_k(z)|$. We apply (9.3) and we get

$$|p_m(z)-p_n(z)|\leq\prod_{k=1}^{n}(1+|a_k(z)|)\Big(\prod_{k=n+1}^{m}(1+|a_k(z)|)-1\Big)$$

for $n<m$ and $z\in A$. We apply the right side of (9.1) and then we let $m\to+\infty$ to find

$$|p(z)-p_n(z)|\leq e^{S_n(z)}\big(e^{S(z)-S_n(z)}-1\big)\leq e^{M}\big(e^{S(z)-S_n(z)}-1\big)=e^{M}\big(e^{|S(z)-S_n(z)|}-1\big)$$

for every n and $z \in A$. Therefore,

$$\sup_{z\in A} |p(z) - p_n(z)| \leq e^M \left(e^{\sup_{z\in A} |S(z)-S_n(z)|} - 1\right)$$

for every n. Since $S_n \to S$ uniformly in A, we have that $p_n \to p$ uniformly in A. □

Now we state the analogue of the theorem of Weierstrass for the uniform convergence of series of holomorphic functions in compact sets.

Theorem 9.1. *Let $\Omega \subseteq \mathbb{C}$ be open and (a_n) be a sequence of holomorphic functions in Ω. If $\prod_{n=1}^{+\infty}(1+a_n)$ converges uniformly in every compact subset of Ω then it defines a function*

$$p = \prod_{n=1}^{+\infty}(1+a_n),$$

which is holomorphic in Ω. Moreover, $p(z) = 0$ if and only if $a_n(z) = -1$ for at least one n. Finally, if none of the a_n is identically -1 in any connected component of Ω, we have that

$$\frac{p'}{p} = \sum_{n=1}^{+\infty} \frac{a_n'}{1+a_n} \tag{9.4}$$

at every point in Ω which is not a root of p. The series in (9.4) has the property: for every compact $K \subseteq \Omega$ there is n_0 so that $\sum_{n=n_0+1}^{+\infty} \frac{a_n'}{1+a_n}$ converges uniformly in K.

Proof. Every $p_n = \prod_{k=1}^{n}(1+a_k)$ is holomorphic in Ω. Since $p_n \to p$ uniformly in every compact subset of Ω the theorem of Weierstrass implies that p is holomorphic in Ω. Moreover, for every $z \in \Omega$ we have $p(z) = \prod_{n=1}^{+\infty}(1+a_n(z))$ and, since the product converges, we have that $p(z) = 0$ if and only if $a_n(z) = -1$ for at least one n.
Now, let us assume that none of the a_n is identically -1 in any connected component of Ω. Then every root of the function $1 + a_n$ is isolated and hence the set of the roots of $1 + a_n$ is countable. From the first part of the theorem we have that the set of the roots of p is also countable and hence p is not identically 0 in any connected component of Ω. In particular, the roots of p are isolated and if we take any compact $K \subseteq \Omega$ then there are only finitely many roots, say $z_1, \ldots, z_m$, of p in K. Now, by the convergence of the infinite product, for each $j = 1, \ldots, m$, there is n_j so that $a_n(z_j) \neq -1$ for every $n \geq n_j + 1$. If we set $n_0 = \max\{n_1, \ldots, n_m\}$, then we have that $a_n(z_j) \neq -1$ for every $n \geq n_0 + 1$ and for every $j = 1, \ldots, m$. Moreover, since p has no root in K other than $z_1, \ldots, z_m$, we have that $a_n(z) \neq -1$ for every $n \geq n_0 + 1$ and for every $z \in K$.
Now we consider the infinite product $q = \prod_{n=n_0+1}^{+\infty}(1 + a_n)$ and the partial products $q_n = \prod_{k=n_0+1}^{n}(1 + a_k)$. Of course, we have that $q_n \to q$ uniformly in K and also $q_n' \to q'$ uniformly in K. We also have that q has no root in K and so there is $\delta > 0$ so that $|q(z)| \geq \delta$ for every $z \in K$. These imply that $\frac{q_n'}{q_n} \to \frac{q'}{q}$ uniformly in K. On the other hand, it is trivial to show that

$$\frac{q_n'}{q_n} = \sum_{k=n_0+1}^{n} \frac{a_k'}{1+a_k} \tag{9.5}$$

and hence we have that $\sum_{n=n_0+1}^{+\infty} \frac{a_n'}{1+a_n}$ converges uniformly in K.
At last, from $p_n = \prod_{k=1}^{n_0}(1+a_k)\, q_n$ and from $p = \prod_{k=1}^{n_0}(1+a_k)\, q$, we also get

$$\frac{p_n'}{p_n} = \sum_{k=1}^{n_0} \frac{a_k'}{1+a_k} + \frac{q_n'}{q_n}, \qquad \frac{p'}{p} = \sum_{k=1}^{n_0} \frac{a_k'}{1+a_k} + \frac{q'}{q} \tag{9.6}$$

at every point in K which is not a root of p. From (9.5) and (9.6) we get (9.4) at every $z \in K$ which is not a root of p. Since K is an arbitrary compact subset of Ω, we conclude that (9.4) holds at every point in Ω which is not a root of p. □

As we have shown in the proof of the argument principle, the roots of p are simple poles of $\frac{p'}{p}$ and at the same time they are simple poles of finitely many of the terms $\frac{a_n'}{1+a_n}$. Hence, we may say that (9.4) holds also at the roots of p.

9.3 Holomorphic functions in the complex plane.

We know that every non-zero polynomial of degree n can be written as

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k}$$

where $z_1, \ldots, z_k$ are the distinct roots of p and $m_1, \ldots, m_k$ are the corresponding multiplicities. In particular, $m_1 + \cdots + m_k = n$.

Let f be a non-zero function holomorphic in the region Ω and let $z_1, \ldots, z_k$ be *all* the roots of f in Ω with corresponding multiplicities $m_1, \ldots, m_k$. We know that we can factorize $(z - z_1)^{m_1}$ from f, i.e. that $f(z) = (z - z_1)^{m_1} g(z)$ for every $z \in \Omega$, where g is holomorphic in Ω with $g(z_1) \neq 0$. Now g has roots $z_2, \ldots, z_n$ with corresponding multiplicities $m_2, \ldots, m_n$. Similarly, $g(z) = (z - z_2)^{m_2} h(z)$ for every $z \in \Omega$, where h is holomorphic in Ω with $h(z_1), h(z_2) \neq 0$. Now h has roots $z_3, \ldots, z_n$ with corresponding multiplicities $m_3, \ldots, m_n$. Continuing inductively, we get that

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_k)^{m_k} F(z)$$

for every $z \in \Omega$, where F is holomorphic and has no roots in Ω. If we do not want to show the multiplicities of the roots except for the (possible) root at 0 we may simply write

$$f(z) = z^m (z - z_1) \cdots (z - z_n) F(z),$$

where $m \geq 0$ is the multiplicity of the root 0 and $z_1, \ldots, z_n$ are the remaining (not necessarily distinct) non-zero roots of f in Ω.

The question now is to generalize this situation in case f has infinitely many roots $0, z_1, z_2, \ldots$. In this case the corresponding infinite product $z^m (z - z_1)(z - z_2) \cdots$ may not converge.

To prepare for what will follow, we rewrite the last identity in the form

$$f(z) = z^m \Big(1 - \frac{z}{z_1}\Big) \cdots \Big(1 - \frac{z}{z_n}\Big) F(z),$$

where the new F is the previous F multiplied by the non-zero number $(-1)^n z_1 \cdots z_n$. We also note that if the region is *simply connected*, e.g. if $\Omega = \mathbb{D}$ or $\Omega = \mathbb{C}$, then, since F has no roots in Ω, there is a holomorphic branch g of F in Ω. So the last identity becomes

$$f(z) = z^m \Big(1 - \frac{z}{z_1}\Big) \cdots \Big(1 - \frac{z}{z_n}\Big) e^{g(z)}$$

for every $z \in \Omega$, where g is holomorphic in Ω. This is the most general form of a holomorphic function in the simply connected region Ω with finitely many preassigned roots (and no other roots).

In the following discussion we shall concentrate only in the case $\Omega = \mathbb{C}$.

Lemma 9.2. *We have* $|e^z - 1| \leq \frac{8}{7} |z|$ *for every* z *with* $|z| \leq \frac{1}{4}$.

Proof. Since $2^{k-1} \leq k!$ when $k \geq 1$, we get

$$|e^z - 1| = \Big| \sum_{k=1}^{+\infty} \frac{z^k}{k!} \Big| \leq \sum_{k=1}^{+\infty} \frac{|z|^k}{k!} = |z| \sum_{k=1}^{+\infty} \frac{|z|^{k-1}}{k!} \leq |z| \sum_{k=1}^{+\infty} \frac{|z|^{k-1}}{2^{k-1}} = \frac{|z|}{1 - \frac{|z|}{2}} \leq \frac{8}{7} |z|$$

when $|z| \leq 1$. □

We set

$$p_0(z) = 1 - z, \qquad p_m(z) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}} \quad \text{when } m \geq 1.$$

Lemma 9.3. *For every $m \geq 0$ we have*

$$|p_m(z) - 1| \leq \frac{3|z|^{m+1}}{m+1} \qquad \text{when } |z| \leq \frac{1}{2}.$$

Proof. For $m = 0$ we have $|p_0(z) - 1| = |z| \leq 3|z|$.
Now let $m \geq 1$. If $|z| \leq \frac{1}{2}$, then by the Taylor series of $-\operatorname{Log}(1-z)$ in the disc $D_0(1)$ we get

$$\Big|\operatorname{Log}(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^m}{m}\Big| = \Big|\sum_{k=m+1}^{+\infty} \frac{z^k}{k}\Big| \leq \sum_{k=m+1}^{+\infty} \frac{|z|^k}{k} \leq \frac{|z|^{m+1}}{m+1} \sum_{k=0}^{+\infty} |z|^k$$
$$= \frac{|z|^{m+1}}{(m+1)(1-|z|)} \leq \frac{2|z|^{m+1}}{m+1}.$$

Since $\frac{2|z|^{m+1}}{m+1} \leq \frac{1}{(m+1)2^m} \leq \frac{1}{4}$ when $|z| \leq \frac{1}{2}$, lemma 9.2 implies

$$|p_m(z) - 1| = |e^{\operatorname{Log}(1-z)+z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}} - 1| \leq \frac{8}{7}\,\frac{2|z|^{m+1}}{m+1} \leq \frac{3|z|^{m+1}}{m+1}.$$

□

Theorem 9.2. *Let (z_n) be a sequence of non-zero numbers so that $z_n \to \infty$.*
(i) Then there are integers $m_n \geq 0$ so that

$$\sum_{n=1}^{+\infty} \frac{1}{m_n+1}\Big(\frac{R}{|z_n|}\Big)^{m_n+1} < +\infty$$

for every $R > 0$.
(ii) If the integers m_n satisfy (i) then the function $F(z) = \prod_{n=1}^{+\infty} p_{m_n}(\frac{z}{z_n})$ is holomorphic in $\mathbb{C}$ and its only roots are the terms of (z_n). The multiplicity of each z_k as a root of F is the same as the number of its appearances as a term of (z_n). Moreover, the most general holomorphic function in $\mathbb{C}$, whose only roots, besides 0, are the terms of (z_n) and so that the multiplicity of each z_k as a root of f is the same as the number of its appearances as a term of (z_n), is of the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{+\infty} p_{m_n}\Big(\frac{z}{z_n}\Big) = z^m e^{g(z)} \prod_{n=1}^{+\infty} \Big(1 - \frac{z}{z_n}\Big) e^{\frac{z}{z_n}+\frac{1}{2}(\frac{z}{z_n})^2+\cdots+\frac{1}{m_n}(\frac{z}{z_n})^{m_n}},$$

where $m \geq 0$ and g is an arbitrary function holomorphic in $\mathbb{C}$. We also have that

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + g'(z) + \sum_{n=1}^{+\infty} \Big(\frac{1}{z-z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} + \cdots + \frac{z^{m_n-1}}{z_n^{m_n}}\Big)$$

at every z which is not a root of f.

Proof. (i) We may consider $m_n = n$ and then, since $z_n \to \infty$, for every $R > 0$ there is n_0 so that $|z_n| \geq 2R$ for every $n \geq n_0 + 1$. This implies that

$$\sum_{n=n_0+1}^{+\infty} \frac{1}{n+1}\Big(\frac{R}{|z_n|}\Big)^{n+1} \leq \sum_{n=n_0+1}^{+\infty} \frac{1}{(n+1)2^{n+1}} < +\infty.$$

(ii) Let the integers m_n satisfy (i). We consider any compact $K \subseteq \mathbb{C}$ and then there is $R > 0$ so that $K \subseteq \overline{D}_0(R)$. Now lemma 9.3 implies that for every $z \in K$ we have

$$\sum_{n=1}^{+\infty} \Big| p_{m_n}\Big(\frac{z}{z_n}\Big) - 1 \Big| \leq 3 \sum_{n=1}^{+\infty} \frac{1}{m_n + 1} \Big(\frac{R}{|z_n|}\Big)^{m_n+1} < +\infty.$$

From proposition 9.2 we get that the infinite product

$$\prod_{n=1}^{+\infty} p_{m_n}\Big(\frac{z}{z_n}\Big) = \prod_{n=1}^{+\infty} \Big(1 + \Big(p_{m_n}\Big(\frac{z}{z_n}\Big) - 1\Big)\Big)$$

converges uniformly in K. Since this is true for an arbitrary compact $K \subseteq \mathbb{C}$, theorem 9.1 implies that the infinite product defines a function

$$F(z) = \prod_{n=1}^{+\infty} p_{m_n}\Big(\frac{z}{z_n}\Big)$$

holomorphic in $\mathbb{C}$. It is clear that the roots of F are the roots of p_{m_n}, i.e. the terms of (z_n). Also, the multiplicity of each z_k as a root of F is the same as the number of its appearances as a term of (z_n). Theorem 9.1 also implies that

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{+\infty} \Big(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} + \cdots + \frac{z^{m_n-1}}{z_n^{m_n}}\Big)$$

at every z which is not a root of F.
Now let f be any holomorphic function in $\mathbb{C}$, whose only roots, besides 0, are the terms of (z_n) and so that the multiplicity of each z_k as a root of f is the same as the number of its appearances as a term of (z_n). Let $m \geq 0$ be the multiplicity of 0 as a root of f. Then the function $\frac{f(z)}{z^m F(z)}$ is holomorphic in $\mathbb{C}$ and has no roots. So there is some function g holomorphic in $\mathbb{C}$ so that $\frac{f(z)}{z^m F(z)} = e^{g(z)}$ for every z.
Finally, from $f(z) = z^m e^{g(z)} F(z)$ we easily get that

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + g'(z) + \frac{F'(z)}{F(z)}$$

and the proof is over. □

There is an important special case of theorem 9.2. It is the case when all integers $m_n \geq 0$ can be taken to be equal to the same integer $h \geq 0$. This means that

$$\frac{1}{|z_n|^{h+1}} < +\infty.$$

If this is true for some integer $h \geq 0$ and we consider the *smallest* such h then the most general holomorphic function in $\mathbb{C}$, whose only roots, besides 0, are the terms of (z_n) and so that the multiplicity of each z_k as a root of f is the same as the number of its appearances as a term of (z_n), is of the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{+\infty} p_h\Big(\frac{z}{z_n}\Big) = z^m e^{g(z)} \prod_{n=1}^{+\infty} \Big(1 - \frac{z}{z_n}\Big) e^{\frac{z}{z_n} + \frac{1}{2}(\frac{z}{z_n})^2 + \cdots + \frac{1}{h}(\frac{z}{z_n})^h},$$

where $m \geq 0$ and g is an arbitrary function holomorphic in $\mathbb{C}$.

Example 9.3.1. The function $\sin z$ is holomorphic in $\mathbb{C}$ and its roots are the numbers $n\pi$, $n \in \mathbb{Z}$. Each root is of multiplicity 1. For the non-zero roots we have that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|n\pi|} = +\infty, \qquad \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{|n\pi|^2} < +\infty.$$

Thus, we may use $h = 1$ in order to apply theorem 9.2 and we get that

$$\sin z = z e^{g(z)} \prod_{n\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{n\pi}\Big) e^{\frac{z}{n\pi}}$$

for some g holomorphic in $\mathbb{C}$. We also have that

$$\frac{\cos z}{\sin z} = \frac{1}{z} + g'(z) + \sum_{n\in\mathbb{Z}\setminus\{0\}} \Big(\frac{1}{z - n\pi} + \frac{1}{n\pi}\Big).$$

Now, one of the last formulas of section 9.1 implies that $g'(z) = 0$ for every z and so g is constant in $\mathbb{C}$. Then e^g is a constant, say c, and then we have that

$$\frac{\sin z}{z} = c \prod_{n\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{n\pi}\Big) e^{\frac{z}{n\pi}}$$

for every z. Both sides of this equality are holomorphic in $\mathbb{C}$ and setting $z = 0$ to it we get $c = 1$. Therefore,

$$\sin z = z \prod_{n\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{n\pi}\Big) e^{\frac{z}{n\pi}}$$