Computational Methods and Function Theory
Volume 00 (0000), No. 0, 1–?
XXYYYZZ

Condenser capacity under multivalent holomorphic functions

Michael Papadimitrakis and Stamatis Pouliasis

Abstract. We prove an inequality for the capacity of a condenser via a holomorphic function f, under a valency assumption on f, and we show that equality occurs if and only if f has finite constant valency. Also, we generalize a well known inequality for quasiregular mappings and we give a necessary condition for the case of equality.

Keywords. Condenser capacity, Green energy, valency, quasiregular mappings.

2000 MSC. Primary: 30C85. Secondary: 30C80, 31A15.

1. Introduction

A *condenser* in the complex plane $\mathbb{C}$ is a pair (D, K) where D is a proper subdomain of $\mathbb{C}$ and K is a compact subset of D. Let h be the solution of the generalized Dirichlet problem on $D \setminus K$ with boundary values 0 on ∂D and 1 on ∂K. The function h is the *equilibrium potential* of the condenser (D, K). The *capacity* of (D, K) is

$$\mathrm{Cap}(D, K) = \int_{D \setminus K} |\nabla h|^2.$$

When D is a Greenian domain, the *Green equilibrium energy* of (D, K) is defined by

$$I(D, K) = \inf_{\mu} \iint G_D(x, y) d\mu(x) d\mu(y),$$

where $G_D(x, y)$ is the Green function of D and the infimum is taken over all unit Borel measures μ supported on K. When $I(D, K) < +\infty$, the unique unit Borel measure for which the above infimum is attained is the *Green equilibrium measure*. See e.g. [5].

Let (D, K) be a condenser with positive capacity and f be a non-constant holomorphic function on the domain D. It is well known that

$$\mathrm{Cap}(f(D), f(K)) \leq \mathrm{Cap}(D, K)$$

Version 2012.

ISSN 1617-9447/\$ 2.50

and equality holds if and only if f is conformal (see [9]). In the present article, we prove an inequality for the capacity of a condenser via a holomorphic function f, which takes into account the valency of f on $f(K)$. Suppose that $I(f(D), f(K)) < +\infty$ and let ν be the equilibrium measure of $(f(D), f(K))$. We assume that each point on a particular subset of $f(K)$ has at least p distinct preimages on K and we prove that

$$\begin{aligned}\frac{1}{\mathrm{Cap}(D,K)} \leq & \frac{1}{p}\frac{1}{\mathrm{Cap}(f(D),f(K))} - \\ & -\frac{1}{2\pi p^2}\sum_{i=1}^{p}\iint \sum_{\substack{f(a)=u \\ a\in D\setminus L}} n(a) G_D(a, f_i^{-1}(v))d\nu(u)d\nu(v),\end{aligned}$$

where L is the union of p certain Borel subsets $L_i \subset K$ and the restriction f_i of f on L_i is injective, $i = 1, 2, ..., p$. Also, we show that equality

$$\mathrm{Cap}(f(D), f(K)) = \frac{1}{p}\mathrm{Cap}(D, K)$$

holds if and only if f is a p to 1 function except possibly on a subset of $f(D)$ with zero logarithmic capacity. See Theorem 1. A similar inequality with an equality statement has been proved by I. P. Mityuk [8] under different assumptions on the condenser and the holomorphic function (see also [1, p. 153]).

Finally, we generalize a well known inequality for quasiregular mappings and we give a necessary condition for the case of equality.

2. Background Material

We will say that a property holds *nearly everywhere*, if it holds everywhere except on a Borel set of zero logarithmic capacity.

2.1. Green energy. We will use the following formula which relates the Green energy with a Dirichlet integral:

$$\iint G_D(x,y)d\mu(x)d\mu(y) = \frac{1}{2\pi}\int_D |\nabla U_\mu^D(x)|^2 dx, \tag{2.1}$$

where

$$U_\mu^D(x) = \int G_D(x,y)d\mu(y)$$

is the Green potential of a measure μ with compact support on D; (see [5, p. 97]). From the boundary behavior of the Green potential of the Green equilibrium measure and (2.1) it follows that

$$\mathrm{Cap}(D,K) = \frac{2\pi}{I(D,K)}. \tag{2.2}$$

2.2. Lindelöf Principle. Let f be a non-constant holomorphic function on a Greenian domain D such that $f(D)$ is Greenian. We denote by $n(x)$ the multiplicity of f at $x \in D$ and by

$$\mathrm{v}(u) = \sum_{f(a)=u} n(a)$$

the valency of f at $u \in f(D)$. If p is a positive integer, we will say that f is *nearly* p *to* 1 if $\mathrm{v}(u) = p$ for nearly every $u \in f(D)$. The following inequality is known as the Lindelöf Principle (see e.g. [3])

$$G_{f(D)}(u_0, f(x)) \geq \sum_{f(a)=u_0} n(a) G_D(a, x), \tag{2.3}$$

where $x \in D$ and $u_0 \in f(D)$. If equality holds in (2.3) for a point $x \in D$, then it holds for every point in D. Following [3, p. 447], we will denote by BL_1 the class of holomorphic functions for which equality holds in (2.3) for a point $u_0 \in f(D)$ and for every $x \in D$. We will need the following result for the valency of a BL_1 function.

Theorem 2.1. [3, p. 469] *Let f be a non-constant holomorphic function on a Greenian domain D such that $f(D)$ is Greenian. If there exists $u_0 \in f(D)$ such that equality holds in* (2.3) *for every $x \in D$ and* $\mathrm{v}(u_0) = p < +\infty$ *then f is nearly p to* 1 *and* $\mathrm{v}(u) \leq p$ *for every $u \in f(D)$.*

3. Condenser capacity inequality for multivalent holomorphic functions

Let (D, K) be a condenser with positive capacity. Let f be a non-constant holomorphic function on the domain D such that the condenser $(f(D), f(K))$ has positive capacity, let ν be the Green equilibrium measure of $(f(D), f(K))$ and let

$$E := \mathrm{supp}(\nu) \setminus f(\{x \in K : n(x) \geq 2\}).$$

For every $u \in E$ let

$$R_f(u, K) := \{x \in K : f(x) = u\}.$$

For every $u \in E$, we denote by $N_f(u, K)$ the cardinality of the set $R_f(u, K)$. Since f is holomorphic on D and K is a compact subset of D, the set $\{x \in K : n(x) \geq 2\}$ has finite cardinality. We note that, since ν has finite Green energy, it does not have point charges and therefore ν is concentrated on E.

We introduce the number

$$V_f(K) := \min_{u \in E} N_f(u, K).$$

From the measurable selection theorem for compact-valued multifunctions (see [13, Corollary 5.2.5, p. 191]), there exist Borel sets $L_i \subset K$, $i = 1, 2, ..., V_f(K)$, such that

- $L_i \cap L_j = \varnothing$, for every $i \neq j$,
- the restriction f_i of f to L_i is an injective Borel measurable function, for every $i = 1, 2, ..., V_f(K)$,
- $f(L_i) = E$, for every $i = 1, 2, ..., V_f(K)$.

The system of pairs $(f_i, L_i)_{i=1}^{V_f(K)}$ will be called a $V_f(K)-$*selection of* f *on*

$$L := \bigcup_{i=1}^{V_f(K)} L_i.$$

Theorem 1. *Let* (D, K) *be a condenser and let* f *be a non-constant holomorphic function on the domain* D. *Suppose that* $(f(D), f(K))$ *has positive capacity and let* ν *be its Green equilibrium measure. Let* $p = V_f(K)$ *and let* $(f_i, L_i)_{i=1}^{p}$ *be a* $V_f(K)-$*selection of* f *on* $L = \cup_{i=1}^{p} L_i$. *Then*

$$\begin{aligned} \frac{1}{\operatorname{Cap}(D,K)} \quad &\leq \quad \frac{1}{p}\frac{1}{\operatorname{Cap}(f(D),f(K))} - \\ &-\frac{1}{2\pi p^2}\sum_{i=1}^{p}\iint \sum_{\substack{f(a)=u \\ a\in D\setminus L}} n(a)G_D(a, f_i^{-1}(v))d\nu(u)d\nu(v). \end{aligned} \tag{3.1}$$

Equality

$$\operatorname{Cap}(f(D), f(K)) = \frac{1}{p}\operatorname{Cap}(D, K) \tag{3.2}$$

holds if and only if f *is nearly* p *to* 1 *on* D.

Proof. We will pull back the measure ν on K p times via f_i, $i = 1, 2, ..., p$. For every $i = 1, 2, ..., p$, consider the measure

$$\mu_i(E) = \nu(f_i(E)), \qquad \text{Borel measurable } E \subset L_i.$$

Then μ_i is a unit Borel measure on K, $i = 1, 2, ..., p$. Let

$$\mu = \frac{1}{p}\sum_{i=1}^{p}\mu_i.$$

Then μ is a unit Borel measure on K. From the minimizing property of the Green equilibrium measure and Lindelöf's Principle, we obtain

$$\begin{aligned} & I(D,K) \\ \leq \quad & \iint_{K\times K} G_D(x,y)d\mu(x)d\mu(y) \\ = \quad & \frac{1}{p^2}\sum_{i,j=1}^{p}\iint_{L_i\times L_j} G_D(x,y)d\mu_i(x)d\mu_j(y) \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&= \frac{1}{p^2}\sum_{i,j=1}^{p}\iint_{E\times E} G_D(f_i^{-1}(u), f_j^{-1}(v))d\nu(u)d\nu(v)\\
&= \frac{1}{p^2}\sum_{j=1}^{p}\iint_{E\times E}\sum_{i=1}^{p} G_D(f_i^{-1}(u), f_j^{-1}(v))d\nu(u)d\nu(v)\\
&\le \frac{1}{p^2}\sum_{j=1}^{p}\iint_{E\times E}\Big[G_{f(D)}(u,v)\\
&\quad - \sum_{\substack{f(a)=u\\ a\in D\setminus L}} n(a)G_D(a, f_j^{-1}(v))\Big]d\nu(u)d\nu(v) \qquad (3.4)\\
&= \frac{1}{p}I(f(D), f(K))\\
&\quad -\frac{1}{p^2}\sum_{j=1}^{p}\iint_{E\times E}\sum_{\substack{f(a)=u\\ a\in D\setminus L}} n(a)G_D(a, f_j^{-1}(v))d\nu(u)d\nu(v).
\end{aligned}
$$

Equality in (3.1) occurs if and only if we have equality in (3.3) and (3.4), that is, if and only if μ is the equilibrium measure of (D, K) and $f \in BL_1$.

Suppose that equality (3.2) holds. Then we obtain that the inequality (3.1) must be equality, so $f \in BL_1$, and

$$\sum_{j=1}^{p}\iint_{E\times E}\sum_{\substack{f(a)=u\\ a\in D\setminus L}} n(a)G_D(a, f_j^{-1}(v))d\nu(u)d\nu(v) = 0. \qquad (3.5)$$

From equality (3.5) we obtain that $\{a \in D \setminus L : f(a) = u\} = \varnothing$ for $\nu-$almost every point $u \in E$. So $\mathrm{v}(u) = p$ for $\nu-$almost every point $u \in E$. Moreover, from Theorem 2.1, f is nearly p to 1 on D and $\mathrm{v}(u) \le p$ for every $u \in f(D)$.

Suppose that f is nearly p to 1 on D. Since f is p to 1 on L, $f^{-1}(E) = L \subset K$. Therefore, $U_\nu^{f(D)} \circ f$ is a bounded harmonic function on $D \setminus K$. Since f is nearly p to 1 on D, $f \in BL_1$ (see [3, Theorem 23.1, p. 472]). Let $u_0 \in f(D)$ be such that

$$G_{f(D)}(u_0, f(x)) = \sum_{i=1}^{p} G_D(a_i, x), \qquad (3.6)$$

for every $x \in D$, where $a_1, ..., a_p$ are the preimages of u_0. From equality (3.6) and the characterization of the regular boundary points via the Green function (see [10, Theorem 4.4.9, p. 111]), we conclude that for every regular boundary point ζ of D and every sequence $D \ni x_n \to \zeta$, every accumulation point of the sequence $f(x_n)$ must be a regular boundary point of $f(D)$. Also, if Z is the set of irregular boundary points of the open set $\mathbb{C} \setminus f(K)$, then $f^{-1}(Z)$ has zero logarithmic capacity (see [10, Corollary 3.6.6, p. 69]). Therefore, from the boundary behavior of the Green potential of ν, we obtain that $\lim_{x\to\zeta} U_\nu^{f(D)}(f(x)) = 0$ for nearly every point $\zeta \in \partial D$ and $\lim_{x\to\zeta} U_\nu^{f(D)}(f(x)) = I(f(D), f(K))$ for nearly every point $\zeta \in K$. If μ_K is the Green equilibrium measure of (D, K), then from the extended maximum principle,

$$U^D_{\mu_K}(x) = \frac{I(D,K)}{I(f(D), f(K))} U_\nu^{f(D)}(f(x)), \qquad x \in D \setminus K.$$

Since the set $\{u \in f(D) : \mathrm{v}(u) < p\}$ has zero logarithmic capacity, it has zero two dimensional Lebesgue measure. Also, the set $f(\{x \in D : n(x) \geq 2\})$ has zero two dimensional Lebesgue measure, since it is at most countable. We conclude that almost every $u \in f(D)$ has exactly p distinct preimages on D. Therefore, by (2.1) and the nonunivalent change of variables formula (see [2, p. 99]),

$$\begin{aligned}
I(D,K) &= \frac{1}{2\pi}\int_{D\setminus K} |\nabla U^D_{\mu_K}(x)|^2 dx \\
&= \frac{I(D,K)^2}{I(f(D), f(K))^2}\frac{1}{2\pi}\int_{D\setminus K} |\nabla (U_\nu^{f(D)} \circ f)(x)|^2 dx \\
&= \frac{I(D,K)^2}{I(f(D), f(K))^2}\frac{1}{2\pi}\int_{D\setminus K} |\nabla U_\nu^{f(D)}(f(x))|^2 |f'(x)|^2 dx \\
&= \frac{I(D,K)^2}{I(f(D), f(K))^2}\frac{1}{2\pi}\int_{f(D)\setminus f(K)} \sum_{x\in f^{-1}(u)} |\nabla U_\nu^{f(D)}(f(x))|^2 du \\
&= \frac{I(D,K)^2}{I(f(D), f(K))^2}\frac{1}{2\pi}\int_{f(D)\setminus f(K)} p|\nabla U_\nu^{f(D)}(u)|^2 du \\
&= p\,\frac{I(D,K)^2}{I(f(D), f(K))}.
\end{aligned}$$

Therefore, $I(f(D), f(K)) = p\, I(D, K)$ and equality (3.2) follows from (2.2). ■

4. Condenser capacity inequality for two dimensional quasiregular mappings

O. Martio [7] and J. Väisälä [14] proved similar inequalities for the conformal capacity of condensers in $\mathbb{R}^n$. If (Ω, C) is a condenser in $\mathbb{R}^n$ and $F : \Omega \mapsto \mathbb{R}^n$ is a nonconstant quasiregular mapping, then

$$\mathrm{Cap}(F(\Omega), F(C)) \leq \frac{K_I(F)}{M(F,C)} \mathrm{Cap}(\Omega, C), \tag{4.1}$$

where Cap is the conformal capacity in $\mathbb{R}^n$, $K_I(F)$ is the inner dilatation of F and $M(F, C)$ is the minimal multiplicity of F on C (see [11, Theorem 10.11, p. 57]). Also, see [4] and [12] for related results. For $n = 2$, we will prove a condenser capacity inequality for quasiregular mappings which is a consequence of Theorem 1 and generalizes (4.1).

Let (Ω, C) be a condenser in $\mathbb{C}$ and let $g : \Omega \mapsto \mathbb{C}$ be a $\mathcal{K}-$quasiregular mapping. It is well known that g can be expressed as a composition of a $\mathcal{K}-$quasiconformal mapping ϕ on Ω with a holomorphic function f on $\phi(\Omega)$ (see [6, Chapter VI]). From Theorem 1, we obtain the following capacity inequality for quasiregular mappings and a corresponding necessary condition for the case of equality.

Corollary 4.1. *Let (Ω, C) be a condenser and let $g = f \circ \phi : \Omega \mapsto \mathbb{C}$ be a non-constant $\mathcal{K}-$quasiregular mapping, where ϕ is a $\mathcal{K}-$quasiconformal mapping on Ω and f is a holomorphic function on $\phi(\Omega)$. Suppose that the condenser $(g(\Omega), g(C))$ has positive capacity and let ν be its Green equilibrium measure. Let $p = V_f(\phi(C))$ and let $(f_i, L_i)_{i=1}^p$ be a $V_f(\phi(C))-$selection of f on $L = \cup_{i=1}^p L_i$. Then*

$$\begin{aligned}\frac{1}{\mathrm{Cap}(\Omega, C)} \leq \; & \frac{\mathcal{K}}{p}\frac{1}{\mathrm{Cap}(g(\Omega), g(C))} - \\ & -\frac{\mathcal{K}}{2\pi p^2}\sum_{i=1}^{p}\iint \sum_{\substack{f(a)=u \\ a\in\phi(\Omega)\setminus L}} n(a) G_{\phi(\Omega)}(a, f_i^{-1}(v)) d\nu(u) d\nu(v).\end{aligned}$$

If equality

$$\mathrm{Cap}(g(\Omega), g(C)) = \frac{\mathcal{K}}{p}\mathrm{Cap}(\Omega, C) \tag{4.2}$$

holds then g is nearly p to 1 on Ω.

Proof. Since

$$\mathrm{Cap}(\phi(\Omega), \phi(C)) \geq \mathrm{Cap}(f(\phi(\Omega)), f(\phi(C))) = \mathrm{Cap}(g(\Omega), g(C)) > 0,$$

$\phi(\Omega)$ is a Greenian domain. From the quasi-invariance of condenser capacity under quasiconformal mappings and Theorem 1,

$$\begin{aligned}\frac{1}{\mathrm{Cap}(\Omega, C)} \leq \; & \frac{\mathcal{K}}{\mathrm{Cap}(\phi(\Omega), \phi(C))} \\ \leq \; & \frac{\mathcal{K}}{p}\frac{1}{\mathrm{Cap}(g(\Omega), g(C))} - \\ & -\frac{\mathcal{K}}{2\pi p^2}\sum_{i=1}^{p}\iint \sum_{\substack{f(a)=u \\ a\in\phi(\Omega)\setminus L}} n(a) G_{\phi(\Omega)}(a, f_i^{-1}(v)) d\nu(u) d\nu(v).\end{aligned}$$

If equality (4.2) holds then equality

$$\mathrm{Cap}(\phi(\Omega), \phi(C)) = \frac{1}{p}\mathrm{Cap}(f(\phi(\Omega)), f(\phi(C)))$$

must be true and from Theorem 1 we obtain that f is nearly p to 1 on $\phi(\Omega)$. Therefore, since ϕ is a homeomorphism, $g = f \circ \phi$ is nearly p to 1 on Ω. ■

References

[1] V. N. Dubinin, *On the results of I. P. Mityuk on the behavior of the inner radius of a domain and condenser capacity under regular mappings*, (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov, (POMI) 371 (2009); translation in J. Math. Sci. (N. Y.) 166 (2010), no. 2, 145-154.

[2] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[3] M. Heins, *On the Lindelöf principle*, Ann. Math. (2) 61, 440–473 (1955).

[4] P. Koskela and J. Onninen, *Mappings of finite distortion: capacity and modulus inequalities*, J. Reine Angew. Math. 599 (2006), 1-26.

[5] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, 1972.

[6] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, Second Edition, Die Grundlehren der mathematischen Wissenschaften, Band 126, Springer-Verlag, 1973.

[7] O. Martio, *A capacity inequality for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 474 (1970), 1–18.

[8] I. P. Mityuk, *Certain properties of functions regular in a multiply connected region*, (Russian), Dokl. Akad. Nauk SSSR 164 (1965), 495–498.

[9] S. Pouliasis, *Condenser capacity and meromorphic functions*, Comput. Methods Funct. Theory 11 (2011), No. 1, 237–245.

[10] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, 1995.

[11] S. Rickman, *Quasiregular Mappings*, Results in Mathematics and Related Areas (3), 26, Springer-Verlag, 1993.

[12] E. Sevost'yanov, *The Väisälä inequality for mappings with finite length distortion*, Complex Var. Elliptic Equ. 55 (2010), no. 1-3, 91-101.

[13] S. M. Srivastava, *A Course on Borel Sets*, Graduate Texts in Mathematics, 180, Springer-Verlag, 1998.

[14] J. Väisälä, *Modulus and capacity inequalities for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 509 (1972), 1–14.

Michael Papadimitrakis Address: *Department of Mathematics, University of Crete, Knossos Ave., 71409 Iraklio, Greece.*
E-mail: papadim@math.uoc.gr

Stamatis Pouliasis Address: *Department of Mathematics, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece.*
E-mail: spoulias@math.auth.gr