

MEASURABLE STEINHAUS SETS DO NOT EXIST FOR FINITE SETS OR THE INTEGERS IN THE PLANE

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ABSTRACT. A Steinhaus set $S \subseteq \mathbb{R}^d$ for a set $A \subseteq \mathbb{R}^d$ is a set such that S has exactly one point in common with τA , for every rigid motion τ of \mathbb{R}^d . We show here that if A is a finite set of at least two points then there is no such set S which is Lebesgue measurable.

An old result of Komjáth says that there exists a Steinhaus set for $A = \mathbb{Z} \times \{0\}$ in \mathbb{R}^2 . We also show here that such a set cannot be Lebesgue measurable.

1. INTRODUCTION

A classical question of Steinhaus [20, 19] was if there exist two proper subsets $A, B \subseteq \mathbb{R}^2$ such that any rigid motion of A intersects B at exactly one point. This was answered in the affirmative by Sierpinski [20].

Let us rephrase the question. If $\rho \in O(2)$ denotes an arbitrary orthogonal transformation and $x \in \mathbb{R}^2$ then a general rigid motion of A is $x + \rho A$ and we are asking that

$$\sum_{b \in B} \mathbf{1}_{x + \rho A}(b) = 1, \quad \text{for all } \rho \in O(2) \text{ and all } x \in \mathbb{R}^2.$$

We can rewrite this (interpreting the infinite sum in the only possible way) as

$$(1) \quad \sum_{b \in B} \mathbf{1}_{\rho A}(x - b) = 1, \quad \text{for all } \rho \in O(2) \text{ and all } x \in \mathbb{R}^2.$$

In other words we are asking that for any $\rho \in O(2)$ the B -translates of ρA form a partition of \mathbb{R}^2 . This is, in other words, a *tiling* condition: all sets ρA tile space when translated by the same set B , a situation which we may denote by

$$\rho A \oplus B = \mathbb{R}^2.$$

Measurability in the Steinhaus question. Condition (1) may be altered by

- (a) demanding that the sets A and B are Lebesgue measurable (usually one of them is countable and hence automatically measurable and of measure 0) and
- (b) only asking that (1) holds for almost all $x \in \mathbb{R}^2$.

We call this a *measurable* Steinhaus problem.

A further question of Steinhaus was if there is a set $A \subseteq \mathbb{R}^2$ which intersects every rigid motion of \mathbb{Z}^2 at exactly one point. Let us call this a *lattice Steinhaus set*. Thus set B is given as \mathbb{Z}^2 and one asks for a set A which satisfies (1).

Komjáth [16] answered the Steinhaus question in the affirmative when $B = \mathbb{Z} \times \{0\}$ showing that there are Steinhaus sets for this B . Sierpinski [20] showed that a bounded set A which is either closed or open cannot have the lattice Steinhaus property (that is, intersect all rigid motions of \mathbb{Z}^2 at exactly one point). Croft [5] and Beck [2] showed that no bounded and measurable set A can have the lattice Steinhaus property. Kolountzakis [11, 10] and Kolountzakis and Wolff [15] proved that any measurable set in the plane that has the measurable Steinhaus property must necessarily have very slow decay at infinity (it is easy to see that any such set must have measure 1). In [15] it was also shown that there can be no measurable Steinhaus sets in dimension $d \geq 3$ (with the obvious definition) a fact that was also shown later by Kolountzakis and Papadimitrakis [14] by a very different method. See also [3, 17, 4, 21]. Kolountzakis [12] looks at the case where the linear transformation ρ is only required to take on finitely many values. In a major result Jackson and Mauldin [8, 7] proved the existence of lattice Steinhaus sets in the plane (not necessarily measurable). Their method does not extend to higher dimension $d \geq 3$. See also [18, 9].

In this paper we deal with the measurable Steinhaus problem for the case of finite sets B . So far there have been negative results without assuming measurability and only for small cardinalities of the set B . Gao, Miller and Weiss [6] prove that for a large class of 4-point sets B in the plane there is no corresponding Steinhaus set A . Xuan [22] proves this for all 4-point sets B .

New results. Our first result is that in the measurable case this is true for all finite sets.

Theorem 1. *If $B \subseteq \mathbb{R}^d$ is a finite set, $|B| > 1$, then there is no measurable Steinhaus set for B . That is, there is no Lebesgue measurable set $A \subseteq \mathbb{R}^d$*

such that for all $\rho \in O(d)$

$$\sum_{b \in B} \mathbf{1}_{\rho A}(x - b) = 1, \quad \text{for almost all } x \in \mathbb{R}^d.$$

Since [6, 22] do not deal with necessarily measurable sets A their results are not directly comparable to our result.

The method of proof follows a Fourier Analytic approach to tiling [13] accompanied by an interesting result about the zero set of multivariate trigonometric polynomials (Theorem 5). The proof is given in §2.

We also prove, without Fourier Analysis this time, that a set such as Komjath [16] constructed cannot be Lebesgue measurable.

Theorem 2. *There is no Lebesgue measurable set $S \subseteq \mathbb{R}^2$ such that for all $\rho \in O(2)$*

$$\sum_{n \in \mathbb{Z} \times \{0\}} \mathbf{1}_S(x - \rho n) = 1, \quad \text{for almost all } x \in \mathbb{R}^2.$$

A Fourier Analytic approach to Theorem 2, along the lines of the proof of Theorem 1, is made difficult by the fact that such a set S , if it exists, must necessarily have infinite measure.

Theorem 2 turns out to be a consequence of the following fact: there is no Lebesgue measurable set in the plane which intersects almost every line at the same measure (Theorem 7 and Theorem 8). The proof appears in §3.

2. NO MEASURABLE STEINHAUS SETS FOR FINITE SETS

Our first result is very similar (see e.g. [13]) to the corresponding result which covers tiling Euclidean space by a set of finite measure. The proof is not exactly the same, owing to the fact that our set S must necessarily have infinite measure.

Notation: For any discrete set of points B we denote by δ_B the measure which assigns a unit point mass to each point of B .

Theorem 3. *Suppose $S \subseteq \mathbb{R}^d$ is a measurable set and $B \subseteq \mathbb{R}^d$ is a finite set such that $\mathbf{1}_S * \delta_B = 1$ almost everywhere. Then for the tempered distribution $\widehat{\mathbf{1}}_S$ and the trigonometric polynomial*

$$\widehat{\delta}_B(x) = \sum_{b \in B} e^{-2\pi i b \cdot x}$$

we have

$$(2) \quad \text{supp } \widehat{\mathbf{1}}_S \subseteq \{0\} \cup \{\widehat{\delta}_B = 0\}.$$

Proof. Let $K = \{0\} \cup \{\widehat{\delta}_B = 0\}$, which is a closed set. Inclusion (2) means (by the definition of the support of a tempered distribution) that $\widehat{\mathbf{1}}_S(\psi) = 0$ for all smooth ψ supported in K^c . For such a ψ we have

$$(3) \quad \left(\widetilde{\delta}_B \psi\right)^\wedge(\lambda) = (\widetilde{\delta}_B * \widehat{\psi})(\lambda), \quad (\lambda \in \mathbb{R}^d)$$

where we use the notation $\widetilde{f}(x) = \overline{f(-x)}$, extended by duality to tempered distributions.

We must show $\widehat{\mathbf{1}}_S(\psi) = 0$. We have

$$\widehat{\mathbf{1}}_S(\psi) = \widehat{\mathbf{1}}_S \left(\widetilde{\delta}_B \cdot \frac{\psi}{\widetilde{\delta}_B} \right).$$

Notice that $\widetilde{\delta}_B = \widehat{\delta}_B$ (since δ_B is real), so the quotient $\phi = \psi/\widetilde{\delta}_B$ is a $C_0^\infty(K^c)$ function. We have

$$\begin{aligned} \widehat{\mathbf{1}}_S(\psi) &= \widehat{\mathbf{1}}_S \left(\widetilde{\delta}_B \phi \right) \\ &= \mathbf{1}_S \left(\left(\widetilde{\delta}_B \phi \right)^\wedge \right) \quad (\text{by the definition of the Fourier Transform for distributions}) \\ &= \int \mathbf{1}_S(\lambda) \left(\widetilde{\delta}_B \phi \right)^\wedge(\lambda) d\lambda \\ &= \int \mathbf{1}_S(\lambda) (\widetilde{\delta}_B * \widehat{\phi})(\lambda) d\lambda \quad (\text{by (3)}) \\ &= \int \mathbf{1}_S(\lambda) \int \widehat{\phi}(x) d\widetilde{\delta}_B(\lambda - x) d\lambda \\ &= \int \int \mathbf{1}_S(\lambda) d\delta_B(x - \lambda) \widehat{\phi}(x) dx \\ &= \int \widehat{\phi}(x) dx \quad (\text{since } \mathbf{1}_S * \delta_B = 1) \\ &= \phi(0) \\ &= 0 \quad (\text{as } 0 \notin \text{supp } \phi). \end{aligned}$$

□

Theorem 4. *If $S \subseteq \mathbb{R}^d$ is a measurable Steinhaus set for the finite set $B \subseteq \mathbb{R}^d$, with cardinality $|B| > 1$, then the trigonometric polynomial $\widehat{\delta}_B$ vanishes on a sphere in \mathbb{R}^d centered at the origin.*

Proof. Every isometry of \mathbb{R}^d can be written as $\tau = x + \rho \cdot$, for some $x \in \mathbb{R}^d$ and some $\rho \in O(d)$, so for all $x \in \mathbb{R}^d, \rho \in O(d)$ we have

$$\sum_{b \in B} \mathbf{1}_S(x + \rho b) = 1.$$

This is in turn equivalent to

$$\sum_{b \in B} \mathbf{1}_{\rho S}(x - b) = 1, \quad \text{for all } x \in \mathbb{R}^d, \rho \in O(d).$$

This can be written as

$$(4) \quad \mathbf{1}_{\rho S} * \delta_B(x) = 1, \quad \text{for all } x \in \mathbb{R}^d.$$

We only need to assume that (4) holds for almost all $x \in \mathbb{R}^d$.

Since $|B| > 1$ the set ρS cannot be almost all of \mathbb{R}^d so the tempered distribution $\widehat{\mathbf{1}_{\rho S}}$ cannot have $\text{supp } \widehat{\mathbf{1}_{\rho S}} = \{0\}$. Suppose then that $0 \neq v \in \text{supp } \widehat{\mathbf{1}_{\rho S}}$. By Theorem 3 it follows that $\widehat{\delta_B}(v) = 0$. Varying ρ we conclude that $\widehat{\delta_B}$ vanishes on the entire sphere with radius $|v|$ centered at the origin. \square

Theorem 5. *If $\psi(x)$ is a trigonometric polynomial on \mathbb{R}^2 which vanishes on a circle then it is identically zero.*

Proof. Suppose, after appropriately modulating the coefficients and scaling the frequencies $b \in B$ (a finite set in the plane), that

$$\psi(x, y) = \sum_{b \in B} c_b e^{2\pi i b \cdot (x, y)}$$

vanishes on the unit circle centered at the origin. View the frequencies $b \in B$ as complex numbers $b = b_1 + ib_2$ and write also $z = x - iy$, so that for $|z| = 1$ the function

$$f(z) = \sum_{b \in B} c_b e^{2\pi i \Re(bz)} = \sum_{b \in B} c_b e^{\pi i (bz + \bar{b}/z)}$$

vanishes identically. The function $f(z)$ is defined and analytic on $\mathbb{C} \setminus \{0\}$, hence, by analytic continuation, vanishes identically for all $z \neq 0$.

Let $b_0 \in B$ have maximal modulus. We may suppose that b_0 is unique by appropriately translating the set B , an operation which does not change the zeros of $\psi(x, y)$. Write $z = \theta t$, where $|\theta| = 1$ and $t \rightarrow +\infty$ and select θ of modulus 1 to be such that

$$\pi i b_0 \theta = \pi |b_0|.$$

For all $b \in B \setminus \{b_0\}$ we have $\Re(\pi ib\theta) < \pi|b_0|$. We have, as $t \rightarrow +\infty$,

$$0 = f(\theta t) = c_{b_0} e^{\pi|b_0|t + O(1/t)} + \sum_{b \in B \setminus \{b_0\}} c_b e^{\pi ib\theta t + O(1/t)}$$

and the modulus of every term in the sum above is $|c_b| e^{\Re(\pi ib\theta)t + O(1/t)}$. Since the first term in the sum above is dominant we have reached a contradiction. \square

Corollary 6. *If $\psi(x)$ is a trigonometric polynomial on \mathbb{R}^d , $d > 1$, which vanishes on a sphere in \mathbb{R}^d then it is identically zero.*

Proof. Restricting ψ on any 2-plane in \mathbb{R}^d leaves us with a two-variable trigonometric polynomial that vanishes on a circle. It follows from Theorem 5 that the restriction of ψ on any 2-plane is identically 0, hence ψ is identically zero on \mathbb{R}^d . \square

From Theorem 4 and Corollary 6 we would have a contradiction if S were a Steinhaus set for B . The proof of Theorem 1 is complete.

3. NO MEASURABLE STEINHAUS SETS FOR THE INTEGERS IN THE PLANE

In this section we prove Theorem 2. We assume that S is a measurable Steinhaus set for $\mathbb{Z} \times \{0\}$ in \mathbb{R}^2 . First we show that such a set S must have a very strong property.

Lemma 1. *Suppose $S \subseteq \mathbb{R}^2$ is Lebesgue measurable and for all $\rho \in O(2)$ we have*

$$(5) \quad \sum_{n \in \mathbb{Z} \times \{0\}} \mathbf{1}_S(x - \rho n) = 1, \quad \text{for almost all } x \in \mathbb{R}^2.$$

Then for every $\rho \in O(2)$ and for almost all lines L in the plane parallel to $\rho(\mathbb{R} \times \{0\})$ we have

$$(6) \quad |S \cap L| = 1,$$

that is S has measure 1 when restricted on almost any straight line L of any slope.

We also have that for almost all points $(x_0, y_0) \in \mathbb{R}^2$ almost all lines L through (x_0, y_0) have property (6).

Proof. The function on the left of (5) is equal to 1 almost everywhere so for almost all lines L parallel to $\rho(\mathbb{R} \times \{0\})$ the function is measurable and equal to 1 almost everywhere as a function of one variable. For any such line L integrate (5) for x in a unit line segment on L to get that $|S \cap L| = 1$.

Ignoring lines parallel to the x -axis we can uniquely parametrize all lines by the pair (x, θ) , with $\theta \in (0, \pi)$ being the angle the line forms with the x -axis and $x \in \mathbb{R}$ being the point of intersection of the line with the x -axis. So far we have shown that for all θ and almost all x the corresponding line L satisfies (6).

We can also redundantly parametrize all lines not parallel to the x -axis by the triple (x_0, y_0, θ) , this line being the one through point (x_0, y_0) forming angle θ with the x -axis and let

$$\phi(x_0, y_0, \theta) = (x, \theta)$$

be the continuous function that translates from one parametrization to the other.

Let $E \subseteq \mathbb{R} \times (0, \pi)$ be the exceptional set for property (6) in the first parametrization

$$E = \{(x, \theta) : \text{the line } (x, \theta) \text{ fails (6)}\}.$$

Then E has measure 0 and therefore for any $\epsilon > 0$ we can find an open cover $E \subseteq \bigcup_n I_n$ of intervals (rectangles) with $\sum_n |I_n| < \epsilon$. Fix any $R > 0$ and let $Q_R = [-R, R] \times [-R, R] \times (0, \pi)$. By the simple geometry we deduce that for any interval I we have

$$|\phi^{-1}(I) \cap Q_R| = O(R)|I|.$$

In fact, if $I = J \times K$, then, fixing any $\theta \in K$, the set of (x_0, y_0) such that $\phi(x_0, y_0, \theta) \in J \times K$ has measure $O(R)|J|$. Therefore $|\phi^{-1}(I) \cap Q_R| = O(R)|J||K| = O(R)|I|$. Now, since $\epsilon > 0$ is arbitrary, we get that $|\phi^{-1}(E) \cap Q_R| = 0$, and, since $R > 0$ is arbitrary, $|\phi^{-1}(E)| = 0$. We have proved that for almost all triples $(x_0, y_0, \theta) \in \mathbb{R}^2 \times (0, \pi)$ the line indexed by (x_0, y_0, θ) has property (6). Applying Fubini's theorem we conclude that for almost all $(x_0, y_0) \in \mathbb{R}^2$ for almost all $\theta \in (0, \pi)$ the line L indexed by (x_0, y_0, θ) has property (6). \square

We now show that no Lebesgue measurable set S can have the property described in Lemma 1. Theorem 7 completes the proof of Theorem 2.

Theorem 7. *There is no Lebesgue measurable $S \subseteq \mathbb{R}^2$ which for almost all $(x_0, y_0) \in \mathbb{R}^2$ intersects almost all lines through (x_0, y_0) in measure 1.*

Proof. Define the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^+$ by

$$f(z) = \int_{\mathbb{R}^2} \mathbf{1}_S(w) |z - w|^{-1} dw.$$

Claim: $f(z) = \pi$ for almost all $z \in \mathbb{R}^2$.

Indeed with $z \in \mathbb{R}^2$ we have

$$\begin{aligned}
f(z) &= \int_{\mathbb{R}^2} \mathbf{1}_S(w) \frac{dw}{|z-w|} \\
&= \int_{\mathbb{R}^2} \mathbf{1}_S(z+w) \frac{dw}{|w|} \quad (\text{change of variable}) \\
&= \int_{[0,\pi]} \int_{\mathbb{R}} \mathbf{1}_S(z+r(\cos \theta, \sin \theta)) dr d\theta \quad (\text{polar coordinates}) \\
&= \int_{[0,\pi]} 1 d\theta \quad (\text{the integral is 1 for almost all } z \text{ and } \theta) \\
&= \pi.
\end{aligned}$$

Claim: The function $f(z)$ is continuous in \mathbb{R}^3 .

To prove the claim we consider the orthogonal projection z'_0 of $z_0 \in \mathbb{R}^3$ onto the plane \mathbb{R}^2 of the first two coordinates and the disk D in \mathbb{R}^2 centered at z'_0 with radius 1. Then

$$f(z) = \int_D \mathbf{1}_S(w) \frac{dw}{|z-w|} + \int_{D^c} \mathbf{1}_S(w) \frac{dw}{|z-w|} = g(z) + h(z).$$

Now, if $z \rightarrow z_0$, then $g(z) \rightarrow g(z_0)$.

To show this we let $\delta = |z - z_0|$ and we write

$$|g(z) - g(z_0)| \leq \int_D \left| \frac{1}{|z-w|} - \frac{1}{|z_0-w|} \right| dw = \int_{D(2\delta)} \dots + \int_{D \setminus D(2\delta)} \dots$$

where $D(2\delta)$ is the disk centered at z'_0 with radius 2δ .

Now

$$\int_{D(2\delta)} \dots = \int_{D(2\delta)} \frac{1}{|z-w|} dw + \int_{D(2\delta)} \frac{1}{|z_0-w|} dw \leq 2 \int_{|u| \leq 3\delta} \frac{1}{|u|} du = O(\delta).$$

Also

$$\int_{D \setminus D(2\delta)} \dots \leq \int_{D \setminus D(2\delta)} \frac{\delta}{|z-w||z_0-w|} dw \leq \int_{\delta \leq |u| \leq 1} \frac{\delta}{|u|^2} du = O(\delta \log \frac{1}{\delta}).$$

Hence $g(z) \rightarrow g(z_0)$.

To prove that $h(z) \rightarrow h(z_0)$ when $z \rightarrow z_0$ we may consider a good point $z''_0 \in \mathbb{R}^2$ such that $|z''_0 - z'_0| < 1/2$ and we may also assume that $|z - z_0| < 1/2$. (We call a point of \mathbb{R}^2 *good* if almost all lines through the point intersect S at measure 1. Almost every point of \mathbb{R}^2 is good

by Lemma 1.) This implies that

$$\frac{1}{|z - w|} \leq \frac{3}{|z_0'' - w|}$$

for all $w \in D^c$. Since

$$\int_{D^c} \mathbf{1}_S(w) \frac{dw}{|z_0'' - w|} \leq \int_{\mathbb{R}^2} \mathbf{1}_S(w) \frac{dw}{|z_0'' - w|} = \pi,$$

by dominated convergence we get $h(z) \rightarrow h(z_0)$ and the claim has been proved.

As a result we now have that $f(z) = \pi$ for all $z \in \mathbb{R}^2$.

Claim: The function $f(z)$ is harmonic in the upper half-space

$$H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}.$$

It is enough to show that $f(z)$ satisfies the mean value property in H .

Suppose $z_0 \in H$ and let Σ be the surface of a sphere centered at z_0 and contained in H . Let also σ denote the surface measure of Σ . We have

$$\begin{aligned} \frac{1}{\sigma(\Sigma)} \int_{\Sigma} f(z) d\sigma(z) &= \frac{1}{\sigma(\Sigma)} \int_{\Sigma} \int_{\mathbb{R}^2} \mathbf{1}_S(w) \frac{dw}{|z - w|} d\sigma(z) \\ &= \int_{\mathbb{R}^2} \mathbf{1}_S(w) \frac{1}{\sigma(\Sigma)} \int_{\Sigma} \frac{d\sigma(z)}{|z - w|} dw \\ &= \int_{\mathbb{R}^2} \mathbf{1}_S(w) \frac{dw}{|z_0 - w|} \quad (\text{as } 1/|\cdot - w| \text{ is harmonic in } \bar{\Sigma}) \\ &= f(z_0), \end{aligned}$$

so that $f(z)$ has the mean value property at z_0 .

Considering the orthogonal projection z' of $z \in \mathbb{R}^3$ onto the plane \mathbb{R}^2 and using $f(z') = \pi$ we get

$$0 \leq f(z) \leq \pi$$

for all $z \in \mathbb{R}^3$.

Any function which is harmonic in the open upper half-space, bounded and continuous in the closed upper half-space is the Poisson integral of its restriction on the plane (see e.g. [1, Theorem 7.5]). Since f is constant on the plane we deduce that $f \equiv \pi$ everywhere in the upper half-space. But this is a contradiction as clearly $\lim_{z \rightarrow +\infty} f(x, y, z) = 0$ for all $(x, y) \in \mathbb{R}^2$.

□

Remark. It is not hard to see that with the same proof as Theorem 7 we can prove the following stronger result.

Theorem 8. *There is no Lebesgue measurable set $S \subseteq \mathbb{R}^2$ which intersects almost all lines of the plane at a measure bounded below by $m > 0$ and above by $M < \infty$.*

REFERENCES

- [1] S. Axler, P. Bourdon, and R. Wade. *Harmonic function theory, 2nd ed.*, volume 137. Springer, 2001.
- [2] J. Beck. On a lattice point problem of H. Steinhaus. *Studia Sci. Math. Hung.*, 24:263–268, 1989.
- [3] W. K. Chan and R. Mauldin. Steinhaus tiling problem and integral quadratic forms. *Proceedings of the American Mathematical Society*, 135(2):337–342, 2007.
- [4] M. Ciucu. A remark on sets having the Steinhaus property. *Combinatorica*, 16(3):321–324, 1996.
- [5] H. T. Croft. Three lattice-point problems of Steinhaus. *The Quarterly Journal of Mathematics*, 33(1):71–83, 1982.
- [6] S. Gao, A. W. Miller, and W. A. Weiss. Steinhaus sets and Jackson sets. *Contemporary Mathematics*, 425:127–145, 2007.
- [7] S. Jackson and R. Mauldin. On a lattice problem of H. Steinhaus. *Journal of the American Mathematical Society*, 15(4):817–856, 2002.
- [8] S. Jackson and R. D. Mauldin. Sets meeting isometric copies of the lattice \mathbb{Z}^2 in exactly one point. *Proceedings of the National Academy of Sciences*, 99(25):15883–15887, 2002.
- [9] S. Jackson and R. D. Mauldin. Survey of the Steinhaus tiling problem. *Bulletin of Symbolic Logic*, 9(03):335–361, 2003.
- [10] M. N. Kolountzakis. A new estimate for a problem of Steinhaus. *International Mathematics Research Notices*, 1996(11):547–555, 1996.
- [11] M. N. Kolountzakis. A problem of Steinhaus: Can all placements of a planar set contain exactly one lattice point? *Progress in Mathematics*, 139:559–566, 1996.
- [12] M. N. Kolountzakis. Multi-lattice tiles. *International Mathematics Research Notices*, 1997(19):937–952, 1997.
- [13] M. N. Kolountzakis. The study of translational tiling with fourier analysis. In *Fourier analysis and convexity*, pages 131–187. Springer, 2004.
- [14] M. N. Kolountzakis and M. Papadimitrakis. The Steinhaus tiling problem and the range of certain quadratic forms. *Illinois Journal of Mathematics*, 46(3):947–951, 2002.
- [15] M. N. Kolountzakis and T. Wolff. On the Steinhaus tiling problem. *Mathematika*, 46(02):253–280, 1999.
- [16] P. Komjáth. A lattice-point problem of Steinhaus. *The Quarterly Journal of Mathematics*, 43(2):235–241, 1992.
- [17] R. Mauldin and A. Yingst. Comments about the Steinhaus tiling problem. *Proceedings of the American Mathematical Society*, 131(7):2071–2079, 2003.
- [18] R. D. Mauldin. Some problems in set theory, analysis and geometry. *Paul Erdos and his Mathematics I*, pages 493–505, 2001.

- [19] W. Moser. *Research problems in discrete geometry*. Department of Mathematics, McGill University, 1981.
- [20] W. Sierpiński. Sur un probleme de H. Steinhaus concernant les ensembles de points sur le plan. *Fundamenta Mathematicae*, 2(46):191–194, 1958.
- [21] S. Srivastava and R. Thangadurai. On Steinhaus sets. *Expositiones Mathematicae*, 23(2):171–177, 2005.
- [22] M. Xuan. *On Steinhaus Sets, Orbit Trees and Universal Properties of Various Subgroups in the Permutation Group of Natural Numbers*. PhD thesis, University of North Texas, 2012.

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