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Notes on Harmonic Analysis

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Chapter 1

The Fourier transform on $L^1(\mathbb{R}^d)$.

We consider the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 1$. We denote

$$|x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$$

the Euclidean norm of $x = (x_1, \ldots, x_d)$ and we denote

$$x \cdot y = x_1 y_1 + \cdots + x_d y_d$$

the Euclidean inner product of $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. We denote

$$B(x; r) = \{x \,|\, |x| < r\}, \quad \overline{B}(x; r) = \{x \,|\, |x| \leq r\}$$

the open and the closed Euclidean balls in $\mathbb{R}^d$ of center x and radius $r > 0$.

In $\mathbb{R}^d$ we consider the Lebesgue measure m_d. We write

$$m_d(A)$$

for the Lebesgue measure of any Lebesgue measurable $A \subseteq \mathbb{R}^d$.

We also consider functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ or $f : \mathbb{R}^d \to \mathbb{C} \cup \{\infty\}$ which are integrable with respect to Lebesgue measure in $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} |f(x)| \, dx < +\infty.$$

The space of these functions is denoted

$$L^1(\mathbb{R}^d).$$

Observe that for simplicity we prefer to write the usual dx instead of $dm_d(x)$ in the integral with respect to Lebesgue measure.

Two functions $f, g \in L^1(\mathbb{R}^d)$ are considered equal if they differ only on a set of Lebesgue measure equal to 0, i.e. if they are equal almost everywhere (with respect to Lebesgue measure).

If $f \in L^1(\mathbb{R}^d)$ then f takes finite values almost everywhere in $\mathbb{R}^d$, i.e. in a set $A \subseteq \mathbb{R}^d$ with $m_d(A^c) = 0$, where $A^c = \mathbb{R}^d \setminus A$. Now we may consider a new function $\tilde{f} : \mathbb{R}^d \to \mathbb{C}$ which is equal to f in A and which has arbitrary finite values in A^c (for example, $\tilde{f} = 0$ in A^c). Then $\tilde{f} \in L^1(\mathbb{R}^d)$ and $f, \tilde{f}$ may differ only in A^c and so they are equal as elements of $L^1(\mathbb{R}^d)$. In other words, without loss of generality we may assume that all $f \in L^1(\mathbb{R}^d)$ are functions $f : \mathbb{R}^d \to \mathbb{C}$ (of course this includes the case $f : \mathbb{R}^d \to \mathbb{R}$).

We know that $L^1(\mathbb{R}^d)$ is a linear space: $kf + lg \in L^1(\mathbb{R}^d)$ if $f, g \in L^1(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$. We also know that if we define

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| \, dx, \qquad f \in L^1(\mathbb{R}^d),$$

then $\| \cdot \|_1$ if is a norm in $L^1(\mathbb{R}^d)$.

Finally, we know that $L^1(\mathbb{R}^d)$ with the norm $\| \cdot \|_1$ is complete, i.e. a Banach space. In other words, if (f_n) is a sequence in $L^1(\mathbb{R}^d)$ such that $\|f_n - f_m\|_1 \to 0$ when $n, m \to +\infty$, then there is some $f \in L^1(\mathbb{R}^d)$ such that $\|f_n - f\|_1 \to 0$ when $n \to +\infty$ and, moreover, there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere (i.e. $f_{n_k}(x) \to f(x)$ when $k \to +\infty$ for almost every x).

Definition 1.1. *Let $f \in L^1(\mathbb{R}^d)$. The **Fourier transform** of f is defined as the function*

$$\widehat{f} : \mathbb{R}^d \to \mathbb{C}$$

given by the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx, \qquad \xi \in \mathbb{R}^d.$$

We observe that $\widehat{f}(\xi)$ is a complex number for every $\xi \in \mathbb{R}^d$ since the function $e^{-2\pi i\,\xi\cdot x} f(x)$ of x is integrable: indeed,

$$\int_{\mathbb{R}^d} |e^{-2\pi i\,\xi\cdot x} f(x)|\,dx = \int_{\mathbb{R}^d} |e^{-2\pi i\,\xi\cdot x}|\,|f(x)|\,dx = \int_{\mathbb{R}^d} |f(x)|\,dx < +\infty$$

for every $\xi, x \in \mathbb{R}^d$.

Sometimes it is unavoidable to use the notation $\widehat{f(x)}(\xi)$, especially if the function f is very concrete and there is no special symbol for it.

Example. Let $d = 1$ and let $\chi_{[a,b]}$ be the characteristic function of the interval $[a, b]$. I.e. $\chi_{[a,b]}(x) = 1$ if $x \in [a, b]$ and $\chi_{[a,b]}(x) = 0$ if $x \notin [a, b]$. Then for every $\xi \in \mathbb{R}$ we have

$$\widehat{\chi_{[a,b]}}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\,\xi x}\chi_{[a,b]}(x)\,dx = \int_a^b e^{-2\pi i\,\xi x}\,dx = \begin{cases} \frac{e^{-2\pi i\,\xi b} - e^{-2\pi i\,\xi a}}{-2\pi i\xi}, & \xi \neq 0, \\ b - a, & \xi = 0, \end{cases}$$
$$= \begin{cases} e^{-\pi i\,\xi(a+b)}\,\frac{e^{-\pi i\,\xi(b-a)} - e^{\pi i\,\xi(b-a)}}{-2\pi i\xi}, & \xi \neq 0, \\ b - a, & \xi = 0, \end{cases} = \begin{cases} e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi}, & \xi \neq 0, \\ b - a, & \xi = 0. \end{cases}$$

We observe that the Fourier transform $\widehat{\chi_{[a,b]}}(\xi)$ is certainly continuous at every $\xi \neq 0$, and it is also continuous at $\xi = 0$, since

$$\lim_{\xi\to 0} e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi} = b - a.$$

Remark. From now on when we write $\frac{\sin k\xi}{\xi}$ we shall accept that this function is also defined at $\xi = 0$ with value $\lim\limits_{\xi\to 0} \frac{\sin k\xi}{\xi} = k$ so that it is continuous everywhere. Hence we may write

$$\widehat{\chi_{[a,b]}}(\xi) = e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi}$$

for every $\xi \in \mathbb{R}$, interpreting $\frac{\sin \pi\xi(b-a)}{\pi\xi}$ as equal to $b - a$ when $\xi = 0$.

We observe that the calculation of the Fourier transform does not change if the interval $[a, b]$ becomes (a, b) or $[a, b)$ or $[a, b)$, since the sets $\{a\}$ and $\{b\}$ have Lebesgue measure equal to 0.
We continue with the same example but for the general dimension d. We consider an interval $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ in $\mathbb{R}^d$ and its characteristic function χ_R. Then using the identities

$$e^{-2\pi i\,\xi\cdot x} = \prod_{k=1}^{d} e^{-2\pi i\,\xi_k x_k}, \qquad \chi_R(x) = \prod_{k=1}^{d} \chi_{[a_k,b_k]}(x_k)$$

for $x = (x_1, \ldots, x_d)$, $\xi = (\xi_1, \ldots, \xi_d)$ and the theorem of Fubini, we get

$$\widehat{\chi_R}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\chi_R(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{d} e^{-2\pi i\,\xi_k x_k}\chi_{[a_k,b_k]}(x_k)\,dx_1 \cdots dx_d$$
$$= \prod_{k=1}^{d} \int_{\mathbb{R}} e^{-2\pi i\,\xi_k x_k}\chi_{[a_k,b_k]}(x_k)\,dx_k = \prod_{k=1}^{d} \widehat{\chi_{[a_k,b_k]}}(\xi_k).$$

Hence

$$\widehat{\chi_R}(\xi) = \prod_{k=1}^{d} e^{-\pi i\,\xi_k(a_k+b_k)}\,\frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k} = e^{-\pi i\,\xi\cdot(a+b)} \prod_{k=1}^{d} \frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k},$$

where, according to our last remark, if $\xi_k = 0$ for some k, then we interpret the corresponding $\frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k}$ as equal to $b_k - a_k$.
Again we observe that $\widehat{\chi_R}(\xi)$ is continuous at every $\xi \in \mathbb{R}^d$.
We also observe that the Fourier transform of R remains the same even if some of the intervals $[a_k, b_k]$ change to open or closed-open or open-closed intervals with the same endpoints.

Now we consider functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ or $f : \mathbb{R}^d \to \mathbb{C} \cup \{\infty\}$ which are Lebesgue measurable and which are bounded almost everywhere in $\mathbb{R}^d$, i.e. for each such f there exists a number $M \geq 0$ (which depends on f) such that

$$|f(x)| \leq M$$

almost everywhere in $\mathbb{R}^d$. The space of these functions is denoted

$$L^\infty(\mathbb{R}^d).$$

Two functions $f, g \in L^\infty(\mathbb{R}^d)$ are considered equal if they are equal almost everywhere.

If a function belongs to $L^\infty(\mathbb{R}^d)$ then obviously it takes finite values almost everywhere in $\mathbb{R}^d$. Therefore, exactly as we did for functions in $L^1(\mathbb{R}^d)$, we may assume without loss of generality that all $f \in L^\infty(\mathbb{R}^d)$ are functions $f : \mathbb{R}^d \to \mathbb{C}$.

We know that $L^\infty(\mathbb{R}^d)$ is a linear space: $kf + lg \in L^\infty(\mathbb{R}^d)$ if $f, g \in L^\infty(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$. We also know that $L^\infty(\mathbb{R}^d)$ has a norm denoted by $\|\cdot\|_\infty$ and defined as follows: for $f \in L^\infty(\mathbb{R}^d)$,

$$\|f\|_\infty \text{ is the smallest } M \geq 0 \text{ such that } |f(x)| \leq M \text{ for almost every } x.$$

In other words, $|f(x)| \leq \|f\|_\infty$ is true for almost every x, and for every $M < \|f\|_\infty$ we have that $m_d(\{x \,|\, |f(x)| > M\}) > 0$.

Finally, we know that $L^\infty(\mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ is a Banach space. In other words, if (f_n) is a sequence in $L^\infty(\mathbb{R}^d)$ such that $\|f_n - f_m\|_\infty \to 0$ when $n, m \to +\infty$, then there is some $f \in L^\infty(\mathbb{R}^d)$ such that $\|f_n - f\|_\infty \to 0$ when $n \to +\infty$ and, hence, there is some $A \subseteq \mathbb{R}^d$ so that $m_d(A^c) = 0$ and $f_n \to f$ uniformly in A.

Now, $L^\infty(\mathbb{R}^d)$ has some notable linear subspaces:

$$C_0(\mathbb{R}^d) \subseteq BUC(\mathbb{R}^d) \subseteq BC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d).$$

The space $BC(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are bounded and continuous in $\mathbb{R}^d$, the space $BUC(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are bounded and uniformly continuous in $\mathbb{R}^d$, and the space $C_0(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are continuous in $\mathbb{R}^d$ and tend to 0 at ∞, i.e. $f(x) \to 0$ when $|x| \to +\infty$. Perhaps the only inclusion that needs some explanation is $C_0(\mathbb{R}^d) \subseteq BUC(\mathbb{R}^d)$. So let $f \in C_0(\mathbb{R}^d)$. Then there is some R so that $|f(x)| \leq 1$ when $|x| > R$. Now f is continuous and hence bounded in the compact ball $\overline{B}(0; R)$. Therefore f is bounded in $\mathbb{R}^d$. On the other hand, take any $\epsilon > 0$. Then there is some R so that $|f(x)| < \frac{\epsilon}{2}$ when $|x| > R$. Since f continuous and hence uniformly continuous on the compact ball $\overline{B}(0; R+1)$, there is δ with $0 < \delta \leq 1$ such that $|f(x) - f(y)| < \epsilon$ when $x, y \in \overline{B}(0; R+1)$ and $|x - y| < \delta$. Now take and x, y with $|x - y| < \delta \, (\leq 1)$. If both x, y are not in $\overline{B}(0; R)$ then $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. If at least one of x, y is in $\overline{B}(0; R)$ then both x, y are in $\overline{B}(0; R+1)$ and hence $|f(x) - f(y)| < \epsilon$. Therefore f is uniformly continuous in $\mathbb{R}^d$.

If we consider all these subspaces of $L^\infty(\mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ of the larger space $L^\infty(\mathbb{R}^d)$, then they are all closed subspaces of $L^\infty(\mathbb{R}^d)$ and so each of them is a Banach space.

Proposition 1.1. *If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in BUC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$.*

Proof. Let $f \in L^1(\mathbb{R}^d)$. Then for every $\xi, h \in \mathbb{R}^d$ we have

$$\begin{aligned}|\widehat{f}(\xi + h) - \widehat{f}(\xi)| &= \Big| \int_{\mathbb{R}^d} e^{-2\pi i (\xi+h)\cdot x} f(x)\, dx - \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x)\, dx \Big| \\ &= \Big| \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x}(e^{-2\pi i h\cdot x} - 1) f(x)\, dx \Big| \leq \int_{\mathbb{R}^d} |e^{-2\pi i h\cdot x} - 1|\, |f(x)|\, dx.\end{aligned}$$

Since $|e^{-2\pi i h\cdot x} - 1| \to 0$ when $h \to 0$ and $|e^{-2\pi i h\cdot x} - 1| \leq 2$, an application of the Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^d} |e^{-2\pi i h\cdot x} - 1| \, |f(x)| \, dx \to 0 \quad \text{when } h \to 0$$

and so $|\widehat{f}(\xi + h) - \widehat{f}(\xi)| \to 0$ when $h \to 0$. Therefore $\widehat{f}$ is continuous at every $\xi \in \mathbb{R}^d$.
We also observe that $\int_{\mathbb{R}^d} |e^{-2\pi i h\cdot x} - 1| \, |f(x)| \, dx$ is independent of ξ and hence

$$\sup_{\xi\in\mathbb{R}^d} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \leq \int_{\mathbb{R}^d} |e^{-2\pi i h\cdot x} - 1| \, |f(x)| \, dx.$$

Again this implies that $\sup_{\xi\in\mathbb{R}^d} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \to 0$ when $h \to 0$ and we conclude that $\widehat{f}$ is uniformly continuous in $\mathbb{R}^d$.
Finally, for every $\xi \in \mathbb{R}^d$ we have

$$|\widehat{f}(\xi)| = \left| \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x) \, dx \right| \leq \int_{\mathbb{R}^d} |f(x)| \, dx = \|f\|_1.$$

Therefore $\|\widehat{f}\|_\infty \leq \|f\|_1$. □

Definition 1.2. *We define the* ***Fourier transform*** *operator*

$$\mathcal{F} : L^1(\mathbb{R}^d) \to BUC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$$

by the formula

$$\mathcal{F}(f) := \widehat{f}, \qquad f \in L^1(\mathbb{R}^d).$$

We recall that if $T : X \to Y$ is a linear operator between the normed spaces X and Y with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and if there is some constant $M \geq 0$ so that

$$\|T(x)\|_Y \leq M\|x\|_X$$

for every $x \in X$, then we say that T is bounded and then we define the norm $\|T\|$ of T by

$$\|T\| \text{ is the smallest } M \geq 0 \text{ such that } \|T(x)\|_Y \leq M\|x\|_X \text{ for every } x \in X.$$

Proposition 1.2. *$\mathcal{F} : L^1(\mathbb{R}^d) \to BUC(\mathbb{R}^d)$ is a bounded linear operator with norm $\|\mathcal{F}\| = 1$.*

Proof. For every $f, g \in L^1(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$ we have

$$\begin{aligned}
(\widehat{kf + lg})(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x}(kf(x) + lg(x)) \, dx \\
&= k \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x) \, dx + l \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} g(x) \, dx \\
&= k\widehat{f}(\xi) + l\widehat{g}(\xi)
\end{aligned}$$

for every $\xi \in \mathbb{R}^d$. Hence

$$\mathcal{F}(kf + lg) = (\widehat{kf + lg}) = k\widehat{f} + l\widehat{g} = k\mathcal{F}(f) + l\mathcal{F}(g)$$

and so $\mathcal{F}$ is linear. Moreover, by Proposition 1.1.,

$$\|\mathcal{F}(f)\|_\infty = \|\widehat{f}\|_\infty \leq \|f\|_1$$

for all $f \in L^1(\mathbb{R}^d)$ which implies that $\|\mathcal{F}\| \leq 1$.
On the other hand, if we take any $f \in L^1(\mathbb{R}^d)$ such that $f(x) \geq 0$ for every x and so that $\|f\|_1 > 0$ (i.e. f is not equal to 0 almost everywhere), then

$$\widehat{f}(0) = \int_{\mathbb{R}^d} f(x) \, dx = \int_{\mathbb{R}^d} |f(x)| \, dx = \|f\|_1 > 0.$$

We take any $\epsilon > 0$. By Proposition 1.1. $\widehat{f}$ is continuous at 0, and so there is $\delta > 0$ such that $|\widehat{f}(\xi) - \widehat{f}(0)| < \epsilon$ for every $\xi \in B(0;\delta)$. This implies

$$|\widehat{f}(\xi)| > |\widehat{f}(0)| - \epsilon = \|f\|_1 - \epsilon$$

for every $\xi \in B(0;\delta)$. Thus $B(0;\delta) \subseteq \{\xi \mid |\widehat{f}(\xi)| > \|f\|_1 - \epsilon\}$ and so

$$m_d(\{\xi \mid |\widehat{f}(\xi)| > \|f\|_1 - \epsilon\}) \geq m_d(B(0;\delta)) > 0.$$

Therefore $\|\widehat{f}\|_\infty > \|f\|_1 - \epsilon$ and since $\epsilon > 0$ is arbitrary, we get $\|\widehat{f}\|_\infty \geq \|f\|_1$. Therefore

$$\|f\|_1 \leq \|\widehat{f}\|_\infty = \|\mathcal{F}(f)\|_\infty \leq \|\mathcal{F}\| \, \|f\|_1.$$

Since $\|f\|_1 > 0$ we get $\|\mathcal{F}\| \geq 1$. □