M. Papadimitrakis

Notes on Harmonic Analysis

Spring of 2024

Contents

Chapter 1

The Fourier transform on $L^1(\mathbb{R}^d)$.

We consider the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 1$. We denote

$$|x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}}$$

the Euclidean norm of $x = (x_1, \ldots, x_d)$ and we denote

$$x \cdot y = x_1 y_1 + \cdots + x_d y_d$$

the Euclidean inner product of $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. We denote

$$B(x; r) = \{x \,|\, |x| < r\}, \quad \overline{B}(x; r) = \{x \,|\, |x| \leq r\}$$

the open and the closed Euclidean balls in $\mathbb{R}^d$ of center x and radius $r > 0$.

In $\mathbb{R}^d$ we consider the Lebesgue measure m_d. We write

$$m_d(A)$$

for the Lebesgue measure of any Lebesgue measurable $A \subseteq \mathbb{R}^d$.

We also consider functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ or $f : \mathbb{R}^d \to \mathbb{C} \cup \{\infty\}$ which are integrable with respect to Lebesgue measure in $\mathbb{R}^d$, i.e.

$$\int_{\mathbb{R}^d} |f(x)| \, dx < +\infty.$$

The space of these functions is denoted

$$L^1(\mathbb{R}^d).$$

Observe that for simplicity we prefer to write the usual dx instead of $dm_d(x)$ in the integral with respect to Lebesgue measure.

Two functions $f, g \in L^1(\mathbb{R}^d)$ are considered equal if they differ only on a set of Lebesgue measure equal to 0, i.e. if they are equal almost everywhere (with respect to Lebesgue measure).

If $f \in L^1(\mathbb{R}^d)$ then f takes finite values almost everywhere in $\mathbb{R}^d$, i.e. in a set $A \subseteq \mathbb{R}^d$ with $m_d(A^c) = 0$, where $A^c = \mathbb{R}^d \setminus A$. Now we may consider a new function $\tilde{f} : \mathbb{R}^d \to \mathbb{C}$ which is equal to f in A and which has arbitrary finite values in A^c (for example, $\tilde{f} = 0$ in A^c). Then $\tilde{f} \in L^1(\mathbb{R}^d)$ and $f, \tilde{f}$ may differ only in A^c and so they are equal as elements of $L^1(\mathbb{R}^d)$. In other words, without loss of generality we may assume that all $f \in L^1(\mathbb{R}^d)$ are functions $f : \mathbb{R}^d \to \mathbb{C}$ (of course this includes the case $f : \mathbb{R}^d \to \mathbb{R}$).

We know that $L^1(\mathbb{R}^d)$ is a linear space: $kf + lg \in L^1(\mathbb{R}^d)$ if $f, g \in L^1(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$. We also know that if we define

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| \, dx, \qquad f \in L^1(\mathbb{R}^d),$$

then $\| \cdot \|_1$ if is a norm in $L^1(\mathbb{R}^d)$.

Finally, we know that $L^1(\mathbb{R}^d)$ with the norm $\| \cdot \|_1$ is complete, i.e. a Banach space. In other words, if (f_n) is a sequence in $L^1(\mathbb{R}^d)$ such that $\|f_n - f_m\|_1 \to 0$ when $n, m \to +\infty$, then there is some $f \in L^1(\mathbb{R}^d)$ such that $\|f_n - f\|_1 \to 0$ when $n \to +\infty$ and, moreover, there is a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere (i.e. $f_{n_k}(x) \to f(x)$ when $k \to +\infty$ for almost every x).

Definition 1.1. *Let $f \in L^1(\mathbb{R}^d)$. The* ***Fourier transform*** *of f is defined as the function*

$$\widehat{f} : \mathbb{R}^d \to \mathbb{C}$$

given by the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx, \qquad \xi \in \mathbb{R}^d.$$

We observe that $\widehat{f}(\xi)$ is a complex number for every $\xi \in \mathbb{R}^d$ since the function $e^{-2\pi i\,\xi\cdot x} f(x)$ of x is integrable: indeed,

$$\int_{\mathbb{R}^d} |e^{-2\pi i\,\xi\cdot x} f(x)|\,dx = \int_{\mathbb{R}^d} |e^{-2\pi i\,\xi\cdot x}|\,|f(x)|\,dx = \int_{\mathbb{R}^d} |f(x)|\,dx < +\infty$$

for every $\xi, x \in \mathbb{R}^d$.

Sometimes it is unavoidable to use the notation $\widehat{f(x)}(\xi)$, especially if the function f is very concrete and there is no special symbol for it.

Example. Let $d = 1$ and let $\chi_{[a,b]}$ be the characteristic function of the interval $[a,b]$. I.e. $\chi_{[a,b]}(x) = 1$ if $x \in [a,b]$ and $\chi_{[a,b]}(x) = 0$ if $x \notin [a,b]$. Then for every $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
\widehat{\chi_{[a,b]}}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i\,\xi x}\chi_{[a,b]}(x)\,dx = \int_a^b e^{-2\pi i\,\xi x}\,dx = \begin{cases} \frac{e^{-2\pi i\,\xi b} - e^{-2\pi i\,\xi a}}{-2\pi i\xi}, & \xi \neq 0, \\ b-a, & \xi = 0, \end{cases} \\
&= \begin{cases} e^{-\pi i\,\xi(a+b)}\,\frac{e^{-\pi i\,\xi(b-a)} - e^{\pi i\,\xi(b-a)}}{-2\pi i\xi}, & \xi \neq 0, \\ b-a, & \xi = 0, \end{cases} = \begin{cases} e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi}, & \xi \neq 0, \\ b-a, & \xi = 0. \end{cases}
\end{aligned}$$

We observe that the Fourier transform $\widehat{\chi_{[a,b]}}(\xi)$ is certainly continuous at every $\xi \neq 0$, and it is also continuous at $\xi = 0$, since

$$\lim_{\xi\to 0} e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi} = b-a.$$

Remark. From now on when we write $\frac{\sin k\xi}{\xi}$ we shall accept that this function is also defined at $\xi = 0$ with value $\lim_{\xi\to 0} \frac{\sin k\xi}{\xi} = k$ so that it is continuous everywhere. Hence we may write

$$\widehat{\chi_{[a,b]}}(\xi) = e^{-\pi i\,\xi(a+b)}\,\frac{\sin \pi\xi(b-a)}{\pi\xi}$$

for every $\xi \in \mathbb{R}$, interpreting $\frac{\sin \pi\xi(b-a)}{\pi\xi}$ as equal to $b-a$ when $\xi = 0$.

We observe that the calculation of the Fourier transform does not change if the interval $[a,b]$ becomes (a,b) or $[a,b)$ or $[a,b)$, since the sets $\{a\}$ and $\{b\}$ have Lebesgue measure equal to 0.
We continue with the same example but for the general dimension d. We consider an interval $R = [a_1,b_1] \times \cdots \times [a_d,b_d]$ in $\mathbb{R}^d$ and its characteristic function χ_R. Then using the identities

$$e^{-2\pi i\,\xi\cdot x} = \prod_{k=1}^{d} e^{-2\pi i\,\xi_k x_k}, \qquad \chi_R(x) = \prod_{k=1}^{d} \chi_{[a_k,b_k]}(x_k)$$

for $x = (x_1,\dots,x_d)$, $\xi = (\xi_1,\dots,\xi_d)$ and the theorem of Fubini, we get

$$\begin{aligned}
\widehat{\chi_R}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\chi_R(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{d} e^{-2\pi i\,\xi_k x_k}\chi_{[a_k,b_k]}(x_k)\,dx_1\cdots dx_d \\
&= \prod_{k=1}^{d} \int_{\mathbb{R}} e^{-2\pi i\,\xi_k x_k}\chi_{[a_k,b_k]}(x_k)\,dx_k = \prod_{k=1}^{d} \widehat{\chi_{[a_k,b_k]}}(\xi_k).
\end{aligned}$$

Hence

$$\widehat{\chi_R}(\xi) = \prod_{k=1}^{d} e^{-\pi i\,\xi_k(a_k+b_k)}\,\frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k} = e^{-\pi i\,\xi\cdot(a+b)} \prod_{k=1}^{d} \frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k},$$

where, according to our last remark, if $\xi_k = 0$ for some k, then we interpret the corresponding $\frac{\sin \pi\xi_k(b_k-a_k)}{\pi\xi_k}$ as equal to $b_k - a_k$.
Again we observe that $\widehat{\chi_R}(\xi)$ is continuous at every $\xi \in \mathbb{R}^d$.
We also observe that the Fourier transform of R remains the same even if some of the intervals $[a_k, b_k]$ change to open or closed-open or open-closed intervals with the same endpoints.

Now we consider functions $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ or $f : \mathbb{R}^d \to \mathbb{C} \cup \{\infty\}$ which are Lebesgue measurable and which are bounded almost everywhere in $\mathbb{R}^d$, i.e. for each such f there exists a number $M \geq 0$ (which depends on f) such that

$$|f(x)| \leq M$$

almost everywhere in $\mathbb{R}^d$. The space of these functions is denoted

$$L^\infty(\mathbb{R}^d).$$

Two functions $f, g \in L^\infty(\mathbb{R}^d)$ are considered equal if they are equal almost everywhere.

If a function belongs to $L^\infty(\mathbb{R}^d)$ then obviously it takes finite values almost everywhere in $\mathbb{R}^d$. Therefore, exactly as we did for functions in $L^1(\mathbb{R}^d)$, we may assume without loss of generality that all $f \in L^\infty(\mathbb{R}^d)$ are functions $f : \mathbb{R}^d \to \mathbb{C}$.

We know that $L^\infty(\mathbb{R}^d)$ is a linear space: $kf + lg \in L^\infty(\mathbb{R}^d)$ if $f, g \in L^\infty(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$. We also know that $L^\infty(\mathbb{R}^d)$ has a norm denoted by $\|\cdot\|_\infty$ and defined as follows: for $f \in L^\infty(\mathbb{R}^d)$,

$$\|f\|_\infty \text{ is the smallest } M \geq 0 \text{ such that } |f(x)| \leq M \text{ for almost every } x.$$

In other words, $|f(x)| \leq \|f\|_\infty$ is true for almost every x, and for every $M < \|f\|_\infty$ we have that $m_d(\{x \,|\, |f(x)| > M\}) > 0$.

Finally, we know that $L^\infty(\mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ is a Banach space. In other words, if (f_n) is a sequence in $L^\infty(\mathbb{R}^d)$ such that $\|f_n - f_m\|_\infty \to 0$ when $n, m \to +\infty$, then there is some $f \in L^\infty(\mathbb{R}^d)$ such that $\|f_n - f\|_\infty \to 0$ when $n \to +\infty$ and, hence, there is some $A \subseteq \mathbb{R}^d$ so that $m_d(A^c) = 0$ and $f_n \to f$ uniformly in A.

Now, $L^\infty(\mathbb{R}^d)$ has some notable linear subspaces:

$$C_0(\mathbb{R}^d) \subseteq BUC(\mathbb{R}^d) \subseteq BC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d).$$

The space $BC(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are bounded and continuous in $\mathbb{R}^d$, the space $BUC(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are bounded and uniformly continuous in $\mathbb{R}^d$, and the space $C_0(\mathbb{R}^d)$ contains all $f : \mathbb{R}^d \to \mathbb{C}$ which are continuous in $\mathbb{R}^d$ and tend to 0 at ∞, i.e. $f(x) \to 0$ when $|x| \to +\infty$. Perhaps the only inclusion that needs some explanation is $C_0(\mathbb{R}^d) \subseteq BUC(\mathbb{R}^d)$. So let $f \in C_0(\mathbb{R}^d)$. Then there is some R so that $|f(x)| \leq 1$ when $|x| > R$. Now f is continuous and hence bounded in the compact ball $\overline{B}(0; R)$. Therefore f is bounded in $\mathbb{R}^d$. On the other hand, take any $\epsilon > 0$. Then there is some R so that $|f(x)| < \frac{\epsilon}{2}$ when $|x| > R$. Since f continuous and hence uniformly continuous on the compact ball $\overline{B}(0; R+1)$, there is δ with $0 < \delta \leq 1$ such that $|f(x) - f(y)| < \epsilon$ when $x, y \in \overline{B}(0; R+1)$ and $|x - y| < \delta$. Now take and x, y with $|x - y| < \delta \,(\leq 1)$. If both x, y are not in $\overline{B}(0; R)$ then $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. If at least one of x, y is in $\overline{B}(0; R)$ then both x, y are in $\overline{B}(0; R+1)$ and hence $|f(x) - f(y)| < \epsilon$. Therefore f is uniformly continuous in $\mathbb{R}^d$.

If we consider all these subspaces of $L^\infty(\mathbb{R}^d)$ with the norm $\|\cdot\|_\infty$ of the larger space $L^\infty(\mathbb{R}^d)$, then they are all closed subspaces of $L^\infty(\mathbb{R}^d)$ and so each of them is a Banach space.

Proposition 1.1. *If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in BUC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ and $\|\widehat{f}\|_\infty \leq \|f\|_1$.*

Proof. Let $f \in L^1(\mathbb{R}^d)$. Then for every $\xi, h \in \mathbb{R}^d$ we have

$$\begin{aligned} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| &= \Big| \int_{\mathbb{R}^d} e^{-2\pi i\,(\xi+h)\cdot x} f(x)\,dx - \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx \Big| \\ &= \Big| \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}(e^{-2\pi i\,h\cdot x} - 1) f(x)\,dx \Big| \leq \int_{\mathbb{R}^d} |e^{-2\pi i\,h\cdot x} - 1|\,|f(x)|\,dx. \end{aligned}$$

Since $|e^{-2\pi i\, h\cdot x} - 1| \to 0$ when $h \to 0$ and $|e^{-2\pi i\, h\cdot x} - 1| \leq 2$, an application of the Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^d} |e^{-2\pi i\, h\cdot x} - 1|\, |f(x)|\, dx \to 0 \quad \text{when } h \to 0$$

and so $|\widehat{f}(\xi + h) - \widehat{f}(\xi)| \to 0$ when $h \to 0$. Therefore $\widehat{f}$ is continuous at every $\xi \in \mathbb{R}^d$.
We also observe that $\int_{\mathbb{R}^d} |e^{-2\pi i\, h\cdot x} - 1|\, |f(x)|\, dx$ is independent of ξ and hence

$$\sup_{\xi\in\mathbb{R}^d} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \leq \int_{\mathbb{R}^d} |e^{-2\pi i\, h\cdot x} - 1|\, |f(x)|\, dx.$$

Again this implies that $\sup_{\xi\in\mathbb{R}^d} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| \to 0$ when $h \to 0$ and we conclude that $\widehat{f}$ is uniformly continuous in $\mathbb{R}^d$.
Finally, for every $\xi \in \mathbb{R}^d$ we have

$$|\widehat{f}(\xi)| = \Big| \int_{\mathbb{R}^d} e^{-2\pi i\, \xi\cdot x} f(x)\, dx \Big| \leq \int_{\mathbb{R}^d} |f(x)|\, dx = \|f\|_1.$$

Therefore $\|\widehat{f}\|_\infty \leq \|f\|_1$. □

Definition 1.2. *We define the **Fourier transform** operator*

$$\mathcal{F} : L^1(\mathbb{R}^d) \to BUC(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$$

by the formula

$$\mathcal{F}(f) := \widehat{f}, \qquad f \in L^1(\mathbb{R}^d).$$

We recall that if $T : X \to Y$ is a linear operator between the normed spaces X and Y with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and if there is some constant $M \geq 0$ so that

$$\|T(x)\|_Y \leq M\|x\|_X$$

for every $x \in X$, then we say that T is bounded and then we define the norm $\|T\|$ of T by

$$\|T\| \text{ is the smallest } M \geq 0 \text{ such that } \|T(x)\|_Y \leq M\|x\|_X \text{ for every } x \in X.$$

Proposition 1.2. $\mathcal{F} : L^1(\mathbb{R}^d) \to BUC(\mathbb{R}^d)$ *is a bounded linear operator with norm* $\|\mathcal{F}\| = 1$.

Proof. For every $f, g \in L^1(\mathbb{R}^d)$ and $k, l \in \mathbb{C}$ we have

$$\begin{aligned}\widehat{(kf + lg)}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\, \xi\cdot x}(kf(x) + lg(x))\, dx \\ &= k \int_{\mathbb{R}^d} e^{-2\pi i\, \xi\cdot x} f(x)\, dx + l \int_{\mathbb{R}^d} e^{-2\pi i\, \xi\cdot x} g(x)\, dx \\ &= k\widehat{f}(\xi) + l\widehat{g}(\xi)\end{aligned}$$

for every $\xi \in \mathbb{R}^d$. Hence

$$\mathcal{F}(kf + lg) = \widehat{kf + lg} = k\widehat{f} + l\widehat{g} = k\mathcal{F}(f) + l\mathcal{F}(g)$$

and so $\mathcal{F}$ is linear. Moreover, by Proposition 1.1.,

$$\|\mathcal{F}(f)\|_\infty = \|\widehat{f}\|_\infty \leq \|f\|_1$$

for all $f \in L^1(\mathbb{R}^d)$ which implies that $\|\mathcal{F}\| \leq 1$.
On the other hand, if we take any $f \in L^1(\mathbb{R}^d)$ such that $f(x) \geq 0$ for every x and so that $\|f\|_1 > 0$ (i.e. f is not equal to 0 almost everywhere), then

$$\widehat{f}(0) = \int_{\mathbb{R}^d} f(x)\, dx = \int_{\mathbb{R}^d} |f(x)|\, dx = \|f\|_1 > 0.$$

We take any $\epsilon > 0$. By Proposition 1.1. $\widehat{f}$ is continuous at 0, and so there is $\delta > 0$ such that $|\widehat{f}(\xi) - \widehat{f}(0)| < \epsilon$ for every $\xi \in B(0;\delta)$. This implies

$$|\widehat{f}(\xi)| > |\widehat{f}(0)| - \epsilon = \|f\|_1 - \epsilon$$

for every $\xi \in B(0;\delta)$. Thus $B(0;\delta) \subseteq \{\xi \,|\, |\widehat{f}(\xi)| > \|f\|_1 - \epsilon\}$ and so

$$m_d(\{\xi \,|\, |\widehat{f}(\xi)| > \|f\|_1 - \epsilon\}) \geq m_d(B(0;\delta)) > 0.$$

Therefore $\|\widehat{f}\|_\infty > \|f\|_1 - \epsilon$ and since $\epsilon > 0$ is arbitrary, we get $\|\widehat{f}\|_\infty \geq \|f\|_1$. So we get

$$\|f\|_1 \leq \|\widehat{f}\|_\infty = \|\mathcal{F}(f)\|_\infty \leq \|\mathcal{F}\| \, \|f\|_1.$$

Since $\|f\|_1 > 0$ we get $\|\mathcal{F}\| \geq 1$. □

For any $y \in \mathbb{R}^d$ we have the corresponding **translation operator** τ_y acting on functions $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\tau_y(f)(x) := f(x - y), \qquad x \in \mathbb{R}^d.$$

The function $\tau_y(f) : \mathbb{R}^d \to \mathbb{C}$, which is called translation of f by y, shares many properties of f. For example, if $f \in L^1(\mathbb{R}^d)$ then $\tau_y(f) \in L^1(\mathbb{R}^d)$ and the two functions have the same integrals:

$$\int_{\mathbb{R}^d} \tau_y(f)(x)\,dx = \int_{\mathbb{R}^d} f(x - y)\,dx = \int_{\mathbb{R}^d} f(x)\,dx.$$

If we work with the absolute values of the two functions we see that they also have the same norms:

$$\|\tau_y(f)\|_1 = \|f\|_1.$$

A property of functions in $L^1(\mathbb{R}^d)$ is the following. If $f \in L^1(\mathbb{R}^d)$ and $\epsilon > 0$ then there is some simple function $\phi = \sum_{k=1}^n l_k \chi_{R_k}$, where the l_k are complex numbers and the R_k are bounded intervals in $\mathbb{R}^d$, so that

$$\|f - \phi\|_1 < \epsilon.$$

Based on this we can prove the following.

Proposition 1.3. *If $f \in L^1(\mathbb{R}^d)$, then $\|\tau_h(f) - f\|_1 \to 0$ when $h \to 0$.*

Proof. We take any $\epsilon > 0$ and we consider a $\phi = \sum_{k=1}^n l_k \chi_{R_k}$, where the l_k are complex numbers and the R_k are bounded intervals in $\mathbb{R}^d$, so that

$$\|f - \phi\|_1 < \epsilon.$$

Then

$$\tau_h(\phi)(x) = \phi(x - h) = \sum_{k=1}^n l_k \chi_{R_k}(x - h) = \sum_{k=1}^n l_k \chi_{R_k + h}(x),$$

where $R_k + h$ is R_k translated by h. Now

$$\|\tau_h(\phi) - \phi\|_1 \leq \sum_{k=1}^n |l_k| \, \|\chi_{R_k+h} - \chi_{R_k}\|_1,$$

and for each k we have

$$\|\chi_{R_k+h} - \chi_{R_k}\|_1 = \int_{\mathbb{R}^d} |\chi_{R_k+h}(x) - \chi_{R_k}(x)|\,dx = m_d((R_k + h) \triangle R_k)$$

and we can make $m_d((R_k + h) \triangle R_k)$ as small as we like by taking $|h|$ small enough. Therefore, we can make

$$\|\tau_h(\phi) - \phi\|_1 < \epsilon$$

by taking $|h|$ small enough. Finally we get

$$\begin{aligned}
\|\tau_h(f) - f\|_1 &\leq \|\tau_h(f) - \tau_h(\phi)\|_1 + \|\tau_h(\phi) - \phi\|_1 + \|\phi - f\|_1 \\
&= \|\tau_h(f - \phi)\|_1 + \|\tau_h(\phi) - \phi\|_1 + \|\phi - f\|_1 \\
&= \|f - \phi\|_1 + \|\tau_h(\phi) - \phi\|_1 + \|\phi - f\|_1 < 3\epsilon
\end{aligned}$$

by taking $|h|$ small enough. □

Riemann-Lebesgue lemma. *If $f \in L^1(\mathbb{R}^d)$ then $\widehat{f} \in C_0(\mathbb{R}^d)$, i.e.*

$$\lim_{|\xi|\to+\infty} \widehat{f}(\xi) = 0.$$

First proof. Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be any bounded interval in $\mathbb{R}^d$. Then

$$|\widehat{\chi_R}(\xi)| = \prod_{k=1}^{d} \left| \tfrac{\sin \pi\xi_k(b_k - a_k)}{\pi\xi_k} \right|.$$

For each k we have

$$\left| \tfrac{\sin \pi\xi_k(b_k - a_k)}{\pi\xi_k} \right| \le \min\left\{ b_k - a_k, \tfrac{1}{\pi|\xi_k|} \right\} \le \min\left\{ M, \tfrac{1}{\pi|\xi_k|} \right\},$$

where $M = \max\{b_1 - a_1, \dots, b_d - a_d\}$. Also, since $|\xi|^2 = \xi_1^2 + \cdots + \xi_d^2$, for at least one k we have $|\xi_k| \ge \frac{|\xi|}{\sqrt{d}}$. Therefore,

$$|\widehat{\chi_R}(\xi)| \le \tfrac{\sqrt{d}M^{d-1}}{\pi|\xi|} \to 0$$

when $|\xi| \to +\infty$.
Take any $\epsilon > 0$. Then there is some $\phi = \sum_{k=1}^{n} l_k \chi_{R_k}$, where the l_k are complex numbers and the R_k are bounded intervals in $\mathbb{R}^d$, such that

$$\|f - \phi\|_1 < \tfrac{\epsilon}{2}.$$

Then

$$\widehat{\phi}(\xi) = \textstyle\sum_{k=1}^{n} l_k \widehat{\chi_{R_k}}(\xi) \to 0$$

when $|\xi| \to +\infty$, and so

$$|\widehat{\phi}(\xi)| < \tfrac{\epsilon}{2}$$

if $|\xi|$ is large enough. Therefore

$$|\widehat{f}(\xi)| \le |\widehat{(f - \phi)}(\xi)| + |\widehat{\phi}(\xi)| \le \|f - \phi\|_1 + |\widehat{\phi}(\xi)| < \epsilon$$

if $|\xi|$ is large enough.
Second proof. We set $h = \frac{\xi}{2|\xi|^2} \in \mathbb{R}^d$ so that $|h| = \frac{1}{2|\xi|} \to 0$ when $|\xi| \to +\infty$.
We also have $\xi \cdot h = \frac{1}{2}$ and

$$\begin{aligned}\widehat{\tau_h(f)}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} \tau_h(f)(x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x - h)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot(x+h)} f(x)\,dx \\ &= e^{-\pi i} \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx = -\widehat{f}(\xi).\end{aligned}$$

Thus,

$$2|\widehat{f}(\xi)| = |\widehat{\tau_h(f)}(\xi) - \widehat{f}(\xi)| \le \|\tau_h(f) - f\|_1 \to 0$$

when $|\xi| \to +\infty$. □

The Riemann-Lebesgue lemma says that

$$\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d).$$

For any $t > 0$ we have the corresponding **dilation operator** δ_t acting on functions $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\delta_t(f)(x) := \tfrac{1}{t^d} f\left(\tfrac{x}{t}\right), \qquad x \in \mathbb{R}^d.$$

The function $\delta_t(f) : \mathbb{R}^d \to \mathbb{C}$ is called dilation of f by t. If $f \in L^1(\mathbb{R}^d)$ then $\delta_t(f) \in L^1(\mathbb{R}^d)$ and the two functions have the same integrals and the same norms:

$$\int_{\mathbb{R}^d} \delta_t(f)(x)\,dx = \int_{\mathbb{R}^d} \tfrac{1}{t^d} f\left(\tfrac{x}{t}\right)\,dx = \int_{\mathbb{R}^d} f(x)\,dx.$$

In the same manner we get

$$\|\delta_t(f)\|_1 = \|f\|_1.$$

The next few propositions show the interaction of the Fourier transform with the translation and the dilation operators.

Proposition 1.4. *Let* $f \in L^1(\mathbb{R}^d)$ *and* $y \in \mathbb{R}^d$. *Then*

$$\widehat{f(x-y)}(\xi) = e^{-2\pi i\,\xi\cdot y}\widehat{f(x)}(\xi), \qquad \widehat{e^{-2\pi i\,y\cdot x}f(x)}(\xi) = \widehat{f(x)}(\xi+y).$$

Proof. We have

$$\begin{aligned}\widehat{f(x-y)}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x-y)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot(x+y)} f(x)\,dx \\ &= e^{-2\pi i\,\xi\cdot y}\int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx = e^{-2\pi i\,\xi\cdot y}\widehat{f(x)}(\xi)\end{aligned}$$

for the first equality, and

$$\widehat{e^{-2\pi i\,y\cdot x}f(x)}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}e^{-2\pi i\,y\cdot x} f(x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,(\xi+y)\cdot x} f(x)\,dx = \widehat{f(x)}(\xi+y)$$

for the second equality. □

Proposition 1.5. *Let* $f \in L^1(\mathbb{R}^d)$ *and* $t > 0$. *Then*

$$\widehat{\tfrac{1}{t^d} f\big(\tfrac{x}{t}\big)}(\xi) = \widehat{f(x)}(t\xi), \qquad \widehat{f(tx)}(\xi) = \tfrac{1}{t^d}\widehat{f(x)}\big(\tfrac{\xi}{t}\big).$$

Proof. We have

$$\widehat{\tfrac{1}{t^d} f\big(\tfrac{x}{t}\big)}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\tfrac{1}{t^d} f\big(\tfrac{x}{t}\big)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot(tx)} f(x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,(t\xi)\cdot x} f(x)\,dx = \widehat{f(x)}(t\xi)$$

for the first equality, and we get the second using $\frac{1}{t}$ instead of t in the first equality. □

We may generalize the last proposition in the following way. We consider any linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ which is non-singular, i.e. $\det(T) \neq 0$. We know of course that then T is 1-1 and onto and hence invertible. Now we consider the operator δ_T acting on functions $f : \mathbb{R}^d \to \mathbb{C}$ by

$$\delta_T(f)(x) := |\det(T)|\, f(T(x)), \qquad x \in \mathbb{R}^d.$$

The function $\delta_T(f) : \mathbb{R}^d \to \mathbb{C}$ shares some properties with f. If $f \in L^1(\mathbb{R}^d)$ then $\delta_T(f) \in L^1(\mathbb{R}^d)$ and the two functions have the same integrals and the same norms:

$$\int_{\mathbb{R}^d} \delta_T(f)(x)\,dx = \int_{\mathbb{R}^d} |\det(T)|\, f(T(x))\,dx = \int_{\mathbb{R}^d} f(x)\,dx$$

and similarly

$$\|\delta_T(f)\|_1 = \|f\|_1.$$

In the particular case $T(x) = \frac{x}{t}$ for some $t > 0$, then T is a linear transformation with $\det(T) = \frac{1}{t^d}$ and hence the operator δ_T coincides with the dilation operator δ_t. For the more general T we have the following.

Proposition 1.6. *Let* $f \in L^1(\mathbb{R}^d)$ *and* $T : \mathbb{R}^d \to \mathbb{R}^d$ *be a non-singular linear transformation. Then*

$$\widehat{|\det(T)|\, f(T(x))}(\xi) = \widehat{f(x)}((T^{-1})^*(\xi)),$$

where $(T^{-1})^* : \mathbb{R}^d \to \mathbb{R}^d$ *is the adjoint of* T^{-1}.

Proof. We have

$$\widehat{|\det(T)|\,f(T(x))}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}|\det(T)|\,f(T(x))\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot T^{-1}(x)}f(x)\,dx$$
$$= \int_{\mathbb{R}^d} e^{-2\pi i\,(T^{-1})^*\xi\cdot x}f(x)\,dx = \widehat{f(x)}((T^{-1})^*(\xi)).$$

□

A linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ is called orthogonal if $T(x)\cdot T(y) = x\cdot y$ for every $x, y \in \mathbb{R}^d$. We know that an orthogonal linear transformation is non-singular, and that $T^{-1} = T^*$, $(T^{-1})^* = T$ and $|\det(T)| = 1$.

Corollary 1.1. *Let $f \in L^1(\mathbb{R}^d)$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ be an orthogonal linear transformation. Then*

$$\widehat{f(T(x))}(\xi) = \widehat{f(x)}(T(\xi)).$$

Proof. Immediate from Proposition 1.6. □

A function $f : \mathbb{R}^d \to \mathbb{C}$ is called **radial** if it depends only on the norm of the variable, i.e. if $f(x) = f(y)$ whenever $|x| = |y|$. This is equivalent to: $f(T(x)) = f(x)$ for every x and every orthogonal linear transformation T. Indeed, let f be radial. Since an orthogonal linear transformation satisfies $|T(x)| = |x|$ for every x, we get $f(T(x)) = f(x)$ for every x. Conversely, assume that $f(T(x)) = f(x)$ for every x and every orthogonal linear transformation T. If $|x| = |y|$ then there exists an orthogonal linear transformation T such that $T(x) = y$ and hence $f(x) = f(T(x)) = f(y)$. Therefore we get the following.

Corollary 1.2. *If $f \in L^1(\mathbb{R}^d)$ is radial then $\widehat{f}$ is also radial.*

Proof. If f is radial and T is any orthogonal linear transformation then from Corollary 1.1 we get

$$\widehat{f(x)}(\xi) = \widehat{f(T(x))}(\xi) = \widehat{f(x)}(T(\xi)).$$

This implies that $\widehat{f}$ is radial. □

Another very simple result is the following.

Proposition 1.7. *Let $f \in L^1(\mathbb{R}^d)$. Then*

$$\widehat{f(-x)}(\xi) = \widehat{f(x)}(-\xi), \qquad \widehat{\overline{f(x)}}(\xi) = \overline{\widehat{f(x)}(-\xi)}, \qquad \widehat{\overline{f(-x)}}(\xi) = \overline{\widehat{f(x)}(\xi)}.$$

Proof. For the first equality:

$$\widehat{f(-x)}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}f(-x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot(-x)}f(x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i\,(-\xi)\cdot x}f(x)\,dx = \widehat{f(x)}(-\xi).$$

For the second equality:

$$\widehat{\overline{f(x)}}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\overline{f(x)}\,dx = \overline{\int_{\mathbb{R}^d} e^{2\pi i\,\xi\cdot x}f(x)\,dx} = \overline{\int_{\mathbb{R}^d} e^{-2\pi i\,(-\xi)\cdot x}f(x)\,dx} = \overline{\widehat{f(x)}(-\xi)}.$$

The third equality can be proved either in the same manner or as a combination of the first two. □

We know that a function $f : \mathbb{R}^d \to \mathbb{C}$ is even or odd if, respectively, $f(-x) = f(x)$ for every x or $f(-x) = -f(x)$ for every x.

Corollary 1.3. *Let $f \in L^1(\mathbb{R}^d)$. If f is even or odd, then $\widehat{f}$ is even or odd respectively.*

Proof. Immediate from the first equality of Proposition 1.7. □

The following two lemmas are very useful when we want to prove that an integral depending on a parameter is continuous or differentiable with respect to the parameter. For both lemmas we have a function

$$f : X \times I \to \mathbb{C},$$

where I is an interval in $\mathbb{R}$ and X is a measurable space with a measure μ. We also assume that for every t the $f(x,t)$, as a function of x, is integrable with respect to μ so that the function $F : I \to \mathbb{C}$ given by the formula

$$F(t) = \int_X f(x,t)\, d\mu(x), \qquad t \in I,$$

is well defined.

The first lemma has to do with the continuity of F.

Lemma A. *Assume that for almost every $x \in X$ we have:*
(i) $f(x,t)$, as a function of t, is continuous in I,
(ii) $|f(x,t)| \le g(x)$ for every $t \in I$, where g is integrable in X.
Then F is continuous in I.

Proof. Take any $t \in I$ and any sequence (t_n) in I so that $t_n \to t$.
By assumption (i) we have $f(x,t_n) \to f(x,t)$ for almost every $x \in X$. Also, by assumption (ii), for every t_n we have $|f(x,t_n)| \le g(x)$ for almost every $x \in X$.
Then the Dominated Convergence Theorem implies

$$F(t_n) = \int_X f(x,t_n)\, d\mu(x) \to \int_X f(x,t)\, d\mu(x) = F(t).$$

Therefore F is continuous at t. □

The second lemma has to do with the differentiability of F.

Lemma B. *Assume that for almost every $x \in X$ we have:*
(i) $f(x,t)$, as a function of t, is differentiable in I,
(ii) $\left|\frac{df}{dt}(x,t)\right| \le g(x)$ for every $t \in I$, where g is integrable in X.
Then F is differentiable in I and

$$\frac{dF}{dt}(t) = \int_X \frac{df}{dt}(x,t)\, d\mu(x), \qquad t \in I.$$

Proof. Take any $t \in I$ and any sequence (t_n) in I so that $t_n \to t$ (and $t_n \neq t$ for every n).
By assumption (i) we have

$$\frac{f(x,t_n)-f(x,t)}{t_n-t} \to \frac{df}{dt}(x,t)$$

for almost every $x \in X$. Again by assumption (i) and the mean value theorem we have that for almost every x there is some t' between t_n and t so that

$$\frac{f(x,t_n)-f(x,t)}{t_n-t} = \frac{df}{dt}(x,t').$$

And then assumption (ii) implies that

$$\left|\frac{f(x,t_n)-f(x,t)}{t_n-t}\right| \le g(x),$$

for almost every x. Finally the Dominated Convergence Theorem implies

$$\frac{F(t_n)-F(t)}{t_n-t} = \int_X \frac{f(x,t_n)-f(x,t)}{t_n-t}\, d\mu(x) \to \int_X \frac{df}{dt}(x,t)\, d\mu(x).$$

Therefore F is differentiable at t and $\frac{dF}{dt}(t) = \int_X \frac{df}{dt}(x,t)\, d\mu(x)$. □

The folollowing two propositions are very usefull. They tell us (i) what the derivative of the Fourier transform is, and (ii) what the Fourier transform of the derivative is.

Proposition 1.8. *Let $f \in L^1(\mathbb{R}^d)$ and $x_k f(x) \in L^1(\mathbb{R}^d)$. Then $\widehat{f}(\xi)$ has partial derivative with respect to ξ_k which is given by the formula*

$$\frac{\partial \widehat{f(x)}}{\partial \xi_k}(\xi) = \widehat{-2\pi i x_k f(x)}(\xi).$$

Proof. We have

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x)\, dx.$$

The function $e^{-2\pi i \xi \cdot x} f(x)$ is, as a function of ξ_k, differentiable and

$$\left|\frac{d(e^{-2\pi i \xi \cdot x} f(x))}{d\xi_k}\right| = \left|e^{-2\pi i \xi \cdot x}(-2\pi i x_k) f(x)\right| \le 2\pi |x_k f(x)|$$

for all x, ξ. Now Lemma B implies that $\widehat{f}(\xi)$ is differentiable with respect to ξ_k and its derivative is equal to

$$\int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x}(-2\pi i x_k) f(x)\, dx = \widehat{-2\pi i x_k f(x)}(\xi).$$

□

Proposition 1.9. *Let $f \in L^1(\mathbb{R}^d)$ and let $\frac{\partial f(x)}{\partial x_k}$ exist at every $x \in \mathbb{R}^d$ and $\frac{\partial f(x)}{\partial x_k} \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Then*

$$\widehat{\frac{\partial f(x)}{\partial x_k}}(\xi) = 2\pi i \xi_k \widehat{f(x)}(\xi).$$

Proof. Let e_k be the unit vector in the direction of the positive x_k-axis in $\mathbb{R}^d$. Then by the continuity of $\frac{\partial f(x)}{\partial x_k}$ we have

$$f(x + he_k) - f(x) = \int_0^h \frac{d}{dt} f(x + te_k)\, dt = \int_0^h \frac{\partial f}{\partial x_k}(x + te_k)\, dt$$

and hence

$$\left|\frac{f(x+he_k) - f(x)}{h} - \frac{\partial f}{\partial x_k}(x)\right| = \left|\frac{1}{h}\int_0^h \left(\frac{\partial f}{\partial x_k}(x + te_k) - \frac{\partial f}{\partial x_k}(x)\right) dt\right| \le \frac{1}{h}\int_0^h \left|\frac{\partial f}{\partial x_k}(x + te_k) - \frac{\partial f}{\partial x_k}(x)\right| dt$$

for every x and for $h > 0$. This implies

$$\int_{\mathbb{R}^d} \left|\frac{f(x+he_k) - f(x)}{h} - \frac{\partial f}{\partial x_k}(x)\right| dx \le \frac{1}{h}\int_0^h \left(\int_{\mathbb{R}^d} \left|\frac{\partial f}{\partial x_k}(x + te_k) - \frac{\partial f}{\partial x_k}(x)\right| dx\right) dt$$

i.e.

$$\left\|\frac{f(x+he_k) - f(x)}{h} - \frac{\partial f}{\partial x_k}(x)\right\|_1 \le \frac{1}{h}\int_0^h \left\|\frac{\partial f}{\partial x_k}(x + te_k) - \frac{\partial f}{\partial x_k}(x)\right\|_1 dt$$

for $h > 0$.

Now Proposition 1.3 implies that $\left\|\frac{\partial f}{\partial x_k}(x + te_k) - \frac{\partial f}{\partial x_k}(x)\right\|_1 \to 0$ when $t \to 0$ and this implies that the right side of the last inequality tends to 0 when $h \to 0+$. Therefore

$$\left\|\frac{f(x+he_k) - f(x)}{h} - \frac{\partial f}{\partial x_k}(x)\right\|_1 \to 0$$

when $h \to 0+$.

Now we take any $\xi \in \mathbb{R}^d$. Using the inequality $|\widehat{g}(\xi)| \le \|g\|_1$, we get that

$$\widehat{\frac{f(x+he_k) - f(x)}{h}}(\xi) \to \widehat{\frac{\partial f(x)}{\partial x_k}}(\xi)$$

when $h \to 0+$. By Proposition 1.4 we have that

$$\widehat{\frac{f(x+he_k) - f(x)}{h}}(\xi) = \frac{e^{2\pi i \xi_k h} - 1}{h}\, \widehat{f(x)}(\xi),$$

and so we get

$$\frac{e^{2\pi i \xi_k h} - 1}{h}\, \widehat{f(x)}(\xi) \to \widehat{\frac{\partial f(x)}{\partial x_k}}(\xi)$$

when $h \to 0+$. Of course this implies that $2\pi i \xi_k \widehat{f(x)}(\xi) = \widehat{\frac{\partial f(x)}{\partial x_k}}(\xi)$. □

Now let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be any d-tuple of non-negative integers. We define the *order* and the *factorial* of α to be

$$|\alpha| := \alpha_1 + \cdots + \alpha_d, \qquad \alpha! := \alpha_1! \cdots \alpha_d!$$

respectively. Also if $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ then we use the symbol x^α for the corresponding *monomial* of x:

$$x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}.$$

Finally, we use the symbol D^α for the corresponding *mixed derivative* of order $|\alpha|$:

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Now, taking linear combinations with complex coefficients we form polynomials

$$P(x) = \sum_{\alpha,\, |\alpha| \le k} c_\alpha x^\alpha$$

of degree $\le k$, and corresponding polynomial differential operators

$$P(D) = \sum_{\alpha,\, |\alpha| \le k} c_\alpha D^\alpha$$

of order $\le k$.

Example. If

$$P(x) = x_1^2 + \cdots + x_d^2 = |x|^2,$$

then

$$P(D) = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2} = \Delta$$

is the Laplacian.

We now have the following two corollaries of Propositions 1.8 and 1.9 for any polynomial $P(x)$ of degree k and for the corresponding differential operator $P(D)$ of order k.

Corollary 1.4. *If $x^\alpha f(x) \in L^1(\mathbb{R}^d)$ for $|\alpha| \le k$, then*

$$P(D)\widehat{f(x)}(\xi) = \widehat{P(-2\pi i\, x) f(x)}(\xi).$$

Corollary 1.5. *If $D^\alpha f(x) \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for $|\alpha| \le k$, then*

$$\widehat{P(D) f(x)}(\xi) = P(2\pi i \xi) \widehat{f(x)}(\xi).$$

Example. (i) If $x^\alpha f(x) \in L^1(\mathbb{R}^d)$ for $|\alpha| \le 2$, then

$$\Delta(\widehat{f(x)})(\xi) = \widehat{-4\pi^2 |x|^2 f(x)}(\xi).$$

(ii) If $D^\alpha f(x) \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ for $|\alpha| \le 2$, then

$$\widehat{\Delta f(x)}(\xi) = -4\pi^2 |\xi|^2 \widehat{f(x)}(\xi).$$

Proposition 1.10. *Let $f(x) = \prod_{k=1}^{d} f_k(x_k)$, where each f_k belongs to $L^1(\mathbb{R})$. Then f belongs to $L^1(\mathbb{R}^d)$ and*

$$\widehat{f}(\xi) = \prod_{k=1}^{d} \widehat{f_k}(\xi_k)$$

for every $\xi \in \mathbb{R}^d$.

Proof. Tonelli's theorem implies

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{d} |f_k(x_k)|\,dx_1 \cdots dx_d = \prod_{k=1}^{d} \int_{\mathbb{R}} |f_k(x_k)|\,dx_k < +\infty.$$

And then Fubini's theorem gives

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} f(x)\,dx = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{d} e^{-2\pi i\,\xi_k x_k} f_k(x_k)\,dx_1 \cdots dx_d \\ &= \prod_{k=1}^{d} \int_{\mathbb{R}} e^{-2\pi i\,\xi_k x_k} f_k(x_k)\,dx_k = \prod_{k=1}^{d} \widehat{f_k}(\xi_k).\end{aligned}$$

□

Recall that we have already proved Proposition 1.10 in a very particular setting when we calculated the Fourier transform of χ_R, the characteristic function of an interval R in $\mathbb{R}^d$. We shall see another instance of Proposition 1.10 in the following calculation of the Fourier transform of the **function of Gauss** $G : \mathbb{R}^d \to \mathbb{C}$ given by

$$G(x) := e^{-\pi|x|^2} = e^{-\pi(x_1^2+\cdots+x_d^2)} = \prod_{k=1}^{d} e^{-\pi x_k^2}, \qquad x \in \mathbb{R}^d.$$

Proposition 1.11. *We have $\widehat{G} = G$, i.e.*

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2}$$

for every $\xi \in \mathbb{R}^d$.

First proof. We first consider the case $d = 1$. Checking that the proper assumptions are satisfied, we apply Propositions 1.8 (for the first equality) and 1.9 (for the third equality) and we get

$$\frac{d}{d\xi}\big(\widehat{e^{-\pi x^2}}\big)(\xi) = \widehat{-2\pi i x e^{-\pi x^2}}(\xi) = i\,\widehat{\tfrac{d}{dx}(e^{-\pi x^2})}(\xi) = i\,2\pi i\xi\, \widehat{e^{-\pi x^2}}(\xi) = -2\pi\xi\, \widehat{e^{-\pi x^2}}(\xi)$$

for every $\xi \in \mathbb{R}$. So the function $f(\xi) = \widehat{e^{-\pi x^2}}(\xi)$ satisfies the ordinary differential equation

$$f'(\xi) = -2\pi\xi f(\xi)$$

in $\mathbb{R}$. This implies easily that $f(\xi) = ce^{-\pi\xi^2}$ for every $\xi \in \mathbb{R}$ for some constant c. In other words,

$$\widehat{e^{-\pi x^2}}(\xi) = ce^{-\pi\xi^2}$$

for every $\xi \in \mathbb{R}$. Now, the constant c is given by

$$c = \widehat{e^{-\pi x^2}}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\,dx.$$

Now we go to the general d. By Proposition 1.10 we have that

$$\widehat{e^{-\pi|x|^2}}(\xi) = \prod_{k=1}^{d} \widehat{e^{-\pi x_k^2}}(\xi_k) = c^d \prod_{k=1}^{d} e^{-\pi\xi_k^2} = c^d e^{-\pi|\xi|^2},$$

for every $\xi \in \mathbb{R}^d$, with the same c as before. To calculate c we specialize to $d = 2$ and $\xi = 0$, and we get

$$\begin{aligned}c^2 &= \widehat{e^{-\pi|x|^2}}(0) = \int_{\mathbb{R}^2} e^{-\pi|x|^2}\,dx = \int_0^{2\pi}\int_0^{+\infty} e^{-\pi r^2}\, r\,dr d\theta \\ &= 2\pi \int_0^{+\infty} e^{-\pi r^2}\, r\,dr = -\int_0^{+\infty} \frac{d}{dr} e^{-\pi r^2}\,dr = 1.\end{aligned}$$

Thus, $c = 1$, and so

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2}$$

for every $\xi \in \mathbb{R}^d$.

Second proof. We take the following dilation of the function of Gauss:

$$\frac{1}{(\sqrt{d})^d}\, G\Big(\frac{x}{\sqrt{t}}\Big) = \frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}.$$

We consider it as a function of both $x = (x_1, \dots, x_d)$ and $t > 0$ in the upper half-space $\mathbb{R}^{d+1}_+ = \{(x,t)\,|\, x \in \mathbb{R}^d, t > 0\}$ of $\mathbb{R}^{d+1}$. It is easy to check that

$$\Delta\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big) = -\frac{2\pi d}{t^{\frac{d}{2}+1}}\, e^{-\pi\frac{|x|^2}{t}} + \frac{4\pi^2|x|^2}{t^{\frac{d}{2}+2}}\, e^{-\pi\frac{|x|^2}{t}},$$

$$\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big) = -\frac{d}{2t^{\frac{d}{2}+1}}\, e^{-\pi\frac{|x|^2}{t}} + \frac{\pi|x|^2}{t^{\frac{d}{2}+2}}\, e^{-\pi\frac{|x|^2}{t}},$$

where the Laplacian Δ is with respect to the variables $x_1, \dots, x_d$, i.e. $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$. This implies that our function satisfies the so-called **heat equation**

$$\Big(4\pi\,\frac{\partial}{\partial t} - \Delta\Big)\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big) = 0$$

in $\mathbb{R}^{d+1}_+$. Now for every $t > 0$ we take the Fourier transform of both sides (as functions of x) and we get

$$\widehat{\Big(4\pi\,\frac{\partial}{\partial t} - \Delta\Big)\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) = 0$$

i.e.

$$4\pi\,\widehat{\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) - \widehat{\Delta\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) = 0$$

i.e.

$$4\pi\,\widehat{\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) + 4\pi^2|\xi|^2\,\widehat{\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}}(\xi) = 0$$

by Proposition 1.9 (after we check that its assumptions are satisfied), i.e.

$$\widehat{\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) + \pi|\xi|^2\,\widehat{e^{-\pi|x|^2}}(\sqrt{t}\,\xi) = 0$$

by Proposition 1.5. Now we apply Lemma B (for the second equality) to get

$$\widehat{\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\,\frac{\partial}{\partial t}\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)\,dx = \frac{d}{dt}\Big(\int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x}\,\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\,dx\Big)$$

$$= \frac{d}{dt}\widehat{\Big(\frac{1}{t^{\frac{d}{2}}}\, e^{-\pi\frac{|x|^2}{t}}\Big)}(\xi) = \frac{d}{dt}\Big(\widehat{e^{-\pi|x|^2}}\Big)(\sqrt{t}\,\xi).$$

So our last equation becomes

$$\frac{d}{dt}\Big(\widehat{e^{-\pi|x|^2}}\Big)(\sqrt{t}\,\xi) + \pi|\xi|^2\,\widehat{e^{-\pi|x|^2}}(\sqrt{t}\,\xi) = 0.$$

In other words, the function $f_\xi(t) = \widehat{e^{-\pi|x|^2}}(\sqrt{t}\,\xi)$, as a function of t, satisfies the ordinary differential equation

$$f'_\xi(t) + \pi|\xi|^2 f_\xi(t) = 0$$

for $t > 0$. This implies that $f_\xi(t) = c_\xi e^{-\pi|\xi|^2 t}$ for $t > 0$ and for some constant c_ξ which may depend on $\xi \in \mathbb{R}^d$. Thus,

$$\widehat{e^{-\pi|x|^2}}(\sqrt{t}\,\xi) = c_\xi e^{-\pi|\xi|^2 t}$$

for every $\xi \in \mathbb{R}^d$ and every $t > 0$. When $t \to 0+$, by the continuity of the Fourier transform $\widehat{e^{-\pi|x|^2}}$ we get

$$c_\xi = \widehat{e^{-\pi|x|^2}}(0) = \int_{\mathbb{R}^d} e^{-\pi|x|^2}\,dx.$$

In other words, the constant c_ξ does not depend on $\xi \in \mathbb{R}^d$, and so we have that

$$\widehat{e^{-\pi|x|^2}}(\sqrt{t}\,\xi) = ce^{-\pi|\xi|^2 t}$$

for every $\xi \in \mathbb{R}^d$ and every $t > 0$. With $t = 1$ we get

$$\widehat{e^{-\pi|x|^2}}(\xi) = ce^{-\pi|\xi|^2}$$

for every $\xi \in \mathbb{R}^d$, where $c = \int_{\mathbb{R}^d} e^{-\pi|x|^2}\,dx$. For the calculation of c we repeat the tricks of the first proof. Very quickly: First we observe that the constant c actually depends on the dimension d, i.e. $c = c_d$. Then using Tonelli's theorem together with $e^{-\pi|x|^2} = \prod_{k=1}^{d} e^{-\pi x_k^2}$ we see that $c_d = c_1^d$. Then using polar coordinates in $\mathbb{R}^2$ we find that $c_2 = 1$. From $c_2 = c_1^2$ we get $c_1 = 1$ and finally $c = c_d = 1^d = 1$. □

Note that we have proved that

$$\int_{\mathbb{R}^d} G(x)\,dx = \int_{\mathbb{R}^d} e^{-\pi|x|^2}\,dx = 1.$$

We recall the *change into polar coordinates* for Lebesgue integrals. For any $x \in \mathbb{R}^d$, $x \neq 0$, we write $r = |x|$ and $x' = \frac{x}{|x|}$. Then $r \in (0, +\infty)$ and $x' \in \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere of center 0 in $\mathbb{R}^d$. Then x defines uniquely the pair (r, x') and conversely the pair (r, x') defines uniquely x, since $x = rx'$. We denote σ_{d-1} the rotationally invariant surface measure of the sphere $\mathbb{S}^{d-1}$. Then we have

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^{+\infty} \Big(\int_{\mathbb{S}^{d-1}} f(rx')\,d\sigma_{d-1}(x') \Big) r^{d-1}\,dr = \int_{\mathbb{S}^{d-1}} \Big(\int_0^{+\infty} f(rx') r^{d-1}\,dr \Big)\,d\sigma_{d-1}(x')$$

for every integrable function f in $\mathbb{R}^d$: this is the so-called *formula for change into polar coordinates* for Lebesgue integrals. It is also true for Lebesgue measurable $f \geq 0$ in $\mathbb{R}^d$.

This is particularly useful when $f : \mathbb{R}^d \to \mathbb{C}$ is a radial function, i.e. if $f(x) = f(y)$ whenever $|x| = |y|$. Then we may consider the function $\tilde{f} : (0, +\infty) \to \mathbb{C}$ defined by $\tilde{f}(r) := f(x)$ for any x with $|x| = r$. Observe that this defines uniquely the value $\tilde{f}(r)$. Indeed, if we have two different x, y so that $|x| = r$ and $|y| = r$, then $|x| = |y|$ and so the radiality of f implies $f(x) = f(y)$. So the radial function $f : \mathbb{R}^d \to \mathbb{C}$ determines the function $\tilde{f} : (0, +\infty) \to \mathbb{C}$. Conversely, a function $\tilde{f} : (0, +\infty) \to \mathbb{C}$ determines the function $f : \mathbb{R}^d \to \mathbb{C}$ given by the formula $f(x) = \tilde{f}(|x|)$ for $x \neq 0$ (we assign any value for f at 0), and then f is obviously radial: if $|x| = |y|$ then $f(x) = \tilde{f}(|x|) = \tilde{f}(|y|) = f(y)$.
For example, the function of Gauss $G(x) = e^{-\pi|x|^2}$ is clearly radial in $\mathbb{R}^d$ and we get the corresponding $\tilde{G}(r) = e^{-\pi r^2}$ in $(0, +\infty)$.
Now, in the formula for change into polar coordinates with a radial function f we shall have $f(rx') = \tilde{f}(|rx'|) = \tilde{f}(r)$ and hence

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^{+\infty} \Big(\int_{\mathbb{S}^{d-1}} \tilde{f}(r)\,d\sigma_{d-1}(x') \Big) r^{d-1}\,dr = \int_{\mathbb{S}^{d-1}} \Big(\int_0^{+\infty} \tilde{f}(r) r^{d-1}\,dr \Big)\,d\sigma_{d-1}(x').$$

The two last integrals are both equal to $\sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \tilde{f}(r) r^{d-1}\,dr$ and so we get

$$\int_{\mathbb{R}^d} f(x)\,dx = \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \tilde{f}(r) r^{d-1}\,dr.$$

This is a very useful formula for the integral of a radial function.

Now we shall find the Fourier transform of another important function, the **function of Poisson** $P : \mathbb{R}^d \to \mathbb{C}$ given by

$$P(x) := c_d \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}}, \qquad x \in \mathbb{R}^d.$$

The c_d is a positive constant depending only on the dimension d such that

$$\int_{\mathbb{R}^d} P(x)\, dx = 1.$$

To begin with, we must see that P is integrable. Clearly, P is radial in $\mathbb{R}^d$ and we have the corresponding $\tilde{P}(r) = c_d \frac{1}{(1+r^2)^{\frac{d+1}{2}}}$ in $(0, +\infty)$. Since $P \geq 0$ in $\mathbb{R}^d$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} P(x)\, dx &= \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \tilde{P}(r) r^{d-1}\, dr = c_d \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \frac{r^{d-1}}{(1+r^2)^{\frac{d+1}{2}}}\, dr \\ &\leq c_d \sigma_{d-1}(\mathbb{S}^{d-1}) \Big(\int_0^1 r^{d-1}\, dr + \int_1^{+\infty} \frac{1}{r^2}\, dr \Big) < +\infty.\end{aligned}$$

To find the value of the constant c_d we introduce the **gamma function** $\Gamma : (0, +\infty) \to (0, +\infty)$ given by

$$\Gamma(s) := \int_0^{+\infty} e^{-t} t^{s-1}\, dt, \qquad s > 0.$$

One can easily see that $\Gamma(1) = \int_0^{+\infty} e^{-t}\, dt = 1$, and, using integration py parts, that

$$\Gamma(s+1) = s\Gamma(s)$$

for every $s > 0$. Then, by induction, we get that

$$\Gamma(n) = (n-1)!, \qquad n \in \mathbb{N}.$$

Another interesting value of the gamma function is

$$\Gamma\big(\tfrac{1}{2}\big) = \pi^{\frac{1}{2}}.$$

To see this, we make a change from t to πt^2 and get

$$\Gamma\big(\tfrac{1}{2}\big) = \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}}\, dt = 2\pi^{\frac{1}{2}} \int_0^{+\infty} e^{-\pi t^2}\, dt = \pi^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\pi t^2}\, dt = \pi^{\frac{1}{2}}.$$

Now we go for the calculation of c_d. Using the change of variables from t to $t(1+|x|^2)$, we get

$$\Gamma\big(\tfrac{d+1}{2}\big) = \int_0^{+\infty} e^{-t} t^{\frac{d-1}{2}}\, dt = (1+|x|^2)^{\frac{d+1}{2}} \int_0^{+\infty} e^{-t(1+|x|^2)} t^{\frac{d-1}{2}}\, dt.$$

Hence

$$\begin{aligned}\Gamma\big(\tfrac{d+1}{2}\big) \int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}}\, dx &= \int_0^{+\infty} e^{-t} t^{\frac{d-1}{2}} \Big(\int_{\mathbb{R}^d} e^{-t|x|^2}\, dx \Big)\, dt \\ &= \int_0^{+\infty} e^{-t} t^{\frac{d-1}{2}} \big(\tfrac{\pi}{t}\big)^{\frac{d}{2}} \Big(\int_{\mathbb{R}^d} e^{-\pi|x|^2}\, dx \Big)\, dt \\ &= \pi^{\frac{d}{2}} \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}}\, dt = \pi^{\frac{d}{2}} \Gamma\big(\tfrac{1}{2}\big) \\ &= \pi^{\frac{d+1}{2}}.\end{aligned}$$

From this we find

$$c_d = \frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{\frac{d+1}{2}}}.$$

In other words, the function of Poisson is given by

$$P(x) = \frac{\Gamma\big(\frac{d+1}{2}\big)}{\pi^{\frac{d+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}}, \qquad x \in \mathbb{R}^d.$$

Now we shall evaluate the surface of the unit sphere $\mathbb{S}^{d-1}$, i.e. the number $\sigma_{d-1}(\mathbb{S}^{d-1})$. Using the function of Gauss $G(x) = e^{-\pi|x|^2}$, which is radial, and the corresponding $\tilde{G}(r) = e^{-\pi r^2}$, we get

$$1 = \int_{\mathbb{R}^d} e^{-\pi|x|^2}\, dx = \sigma_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} e^{-\pi r^2} r^{d-1}\, dr = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{2\pi^{\frac{d}{2}}} \int_0^{+\infty} e^{-t} t^{\frac{d}{2}-1}\, dt = \frac{\sigma_{d-1}(\mathbb{S}^{d-1})}{2\pi^{\frac{d}{2}}} \Gamma\big(\tfrac{d}{2}\big).$$

Therefore

$$\sigma_{d-1}(\mathbb{S}^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\big(\frac{d}{2}\big)}.$$

Proposition 1.12. *We have $\widehat{P}(\xi) = e^{-2\pi|\xi|}$, i.e.*

$$c_d\, \widehat{\frac{1}{(1+|x|^2)^{\frac{d+1}{2}}}}(\xi) = e^{-2\pi|\xi|}$$

for every $\xi \in \mathbb{R}^d$.

Proof. We apply the method of the second proof for the Fourier transform of the function of Gauss. We consider the dilation

$$\frac{1}{t^d} P\big(\tfrac{x}{t}\big) = c_d \frac{t}{(t^2+|x|^2)^{\frac{d+1}{2}}}$$

of the function of Poisson as a function of both $x = (x_1, \dots, x_d)$ and $t > 0$ in the upper half-space $\mathbb{R}^{d+1}_+ = \{(x,t)\,|\, x \in \mathbb{R}^d, t > 0\}$ of $\mathbb{R}^{d+1}$. Easy calculations show that our function satisfies the so-called **wave equation**

$$\big(\tfrac{\partial^2}{\partial t^2} + \Delta\big)\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big) = 0$$

in $\mathbb{R}^{d+1}_+$. For every $t > 0$ we take the Fourier transform of both sides (as functions of x) and we get

$$\widehat{\big(\tfrac{\partial^2}{\partial t^2} + \Delta\big)\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) = 0$$

i.e.

$$\widehat{\tfrac{\partial^2}{\partial t^2}\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) + \widehat{\Delta\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) = 0$$

i.e.

$$\widehat{\tfrac{\partial^2}{\partial t^2}\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) - 4\pi^2|\xi|^2\, \widehat{\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)}(\xi) = 0$$

by Proposition 1.9, i.e.

$$\widehat{\tfrac{\partial^2}{\partial t^2}\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) - 4\pi^2|\xi|^2\, \widehat{P(x)}(t\xi) = 0$$

by Proposition 1.5. We apply Lemma B to get

$$\begin{aligned}\widehat{\tfrac{\partial^2}{\partial t^2}\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} \tfrac{\partial^2}{\partial t^2}\big(\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\big)\, dx = \tfrac{d^2}{dt^2}\Big(\int_{\mathbb{R}^d} e^{-2\pi i\,\xi\cdot x} \tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)\, dx\Big)\\ &= \tfrac{d^2}{dt^2}\big(\widehat{\tfrac{1}{t^d} P\big(\tfrac{x}{t}\big)}\big)(\xi) = \tfrac{d^2}{dt^2}\big(\widehat{P(x)}\big)(t\xi).\end{aligned}$$

So our last equation becomes

$$\tfrac{d^2}{dt^2}\big(\widehat{P(x)}\big)(t\xi) - 4\pi^2|\xi|^2\, \widehat{P(x)}(t\xi) = 0.$$

In other words, the function $f_\xi(t) = \widehat{P(x)}(t\xi)$, as a function of t, satisfies the ordinary differential equation

$$f_\xi''(t) - 4\pi^2|\xi|^2 f_\xi(t) = 0$$

for $t > 0$. This implies that $f_\xi(t) = c_\xi e^{-2\pi|\xi|t} + c_\xi' e^{2\pi|\xi|t}$ for $t > 0$ and for some constants c_ξ, c_ξ' which may depend on $\xi \in \mathbb{R}^d$. Thus,

$$\widehat{P(x)}(t\xi) = c_\xi e^{-2\pi|\xi|t} + c_\xi' e^{2\pi|\xi|t}$$

for every $\xi \in \mathbb{R}^d$ and every $t > 0$. Since $\widehat{P(x)}(t\xi)$ is a bounded function, we see that $c'_\xi = 0$. Thus,

$$\widehat{P(x)}(t\xi) = c_\xi e^{-2\pi|\xi|t}$$

for every $\xi \in \mathbb{R}^d$ and every $t > 0$. When $t \to 0+$, by the continuity of $\widehat{P(x)}$ we get

$$c_\xi = \widehat{P(x)}(0) = \int_{\mathbb{R}^d} P(x)\, dx = 1.$$

Hence

$$\widehat{P(x)}(t\xi) = e^{-2\pi|\xi|t}$$

for every $\xi \in \mathbb{R}^d$ and every $t > 0$. With $t = 1$ we get

$$\widehat{P(x)}(\xi) = e^{-2\pi|\xi|}$$

for every $\xi \in \mathbb{R}^d$. □

Now we consider two functions $f, g \in L^1(\mathbb{R}^d)$. Then the function of the two variables x, y given by

$$f(x-y)g(y)$$

is measurable with respect to the Lebesgue measure of $\mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$ (see for instance "Real Analysis" by Folland or "Measure and Integral" by Wheeden and Zygmund). Moreover, by Tonelli's theorem, this function is integrable in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dxdy &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |f(x-y)g(y)|\, dx \Big)\, dy = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |f(x-y)|\, dx \Big)\, |g(y)|\, dy \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |f(x)|\, dx \Big)\, |g(y)|\, dy = \int_{\mathbb{R}^d} |f(x)|\, dx \int_{\mathbb{R}^d} |g(y)|\, dy < +\infty.\end{aligned}$$

And now Fubini's theorem says that, for almost every $x \in \mathbb{R}^d$, $f(x-y)g(y)$, as a function of $y \in \mathbb{R}^d$, is integrable in $\mathbb{R}^d$, and that the function $\int_{\mathbb{R}^d} f(x-y)g(y)\, dy$, which is thus defined for almost every $x \in \mathbb{R}^d$, is integrable in $\mathbb{R}^d$, and also that the integral in $\mathbb{R}^d$ of the last function is

$$\begin{aligned}\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x-y)g(y)\, dy \Big)\, dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(y)\, dxdy = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x-y)g(y)\, dx \Big)\, dy \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x-y)\, dx \Big)\, g(y)\, dy = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(x)\, dx \Big)\, g(y)\, dy \\ &= \int_{\mathbb{R}^d} f(x)\, dx \int_{\mathbb{R}^d} g(y)\, dy.\end{aligned}$$

Based on the last discussion we give the following definition.

Definition 1.3. *If $f, g \in L^1(\mathbb{R}^d)$, we define the function $f * g : \mathbb{R}^d \to \mathbb{C}$ by*

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\, dy$$

for the almost all $x \in \mathbb{R}^d$ for which $f(x-y)g(y)$ is integrable as a function of y. For all other $x \in \mathbb{R}^d$ (which form a set of measure equal to 0) we assign an arbitrary value to $(f * g)(x)$.
The function $f * g$ ia called **convolution** *of f, g.*

So we know that $f * g \in L^1(\mathbb{R}^d)$ and, as our last calculation shows, that

$$\int_{\mathbb{R}^d} (f * g)(x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \int_{\mathbb{R}^d} g(x)\, dx.$$

Moreover, we have that

$$|(f * g)(x)| = \Big| \int_{\mathbb{R}^d} f(x-y)g(y)\, dy \Big| \le \int_{\mathbb{R}^d} |f(x-y)|\, |g(y)|\, dy = (|f| * |g|)(x)$$

and hence

$$\|f * g\|_1 \le \|\, |f| * |g|\, \|_1 = \|f\|_1 \|g\|_1.$$

Now it is very easy to show the following properties of the convolution:

$$f * g = g * f,$$

$$f * (g * h) = (f * g) * h$$

$$f * (kg + lh) = k f * g + l f * h$$

for all $f, g, h \in L^1(\mathbb{R}^d)$ and all $k, l \in \mathbb{C}$. In other words, the operation of convolution in $L^1(\mathbb{R}^d)$ is commutative, associative and distributive, and so $L^1(\mathbb{R}^d)$ is a *commutative algebra*. Since we also have that $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ for all $f, g \in L^1(\mathbb{R}^d)$ we see that $L^1(\mathbb{R}^d)$ is a normed algebra, and, since $L^1(\mathbb{R}^d)$ is complete, it is a Banach algebra.

Proposition 1.13. *If $f, g \in L^1(\mathbb{R}^d)$, then*

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

for every $\xi \in \mathbb{R}^d$.

Proof. Again this is an application of Fubini's theorem:

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x}(f * g)(x)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x}\Big(\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\Big)\,dx \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x-y)\,dx\Big)g(y)\,dy = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot (x-y)} f(x-y)\,dx\Big)e^{-2\pi i \xi\cdot y}g(y)\,dy \\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x)\,dx\Big)e^{-2\pi i \xi\cdot y}g(y)\,dy = \int_{\mathbb{R}^d} \widehat{f}(\xi)e^{-2\pi i \xi\cdot y}g(y)\,dy \\ &= \widehat{f}(\xi)\int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot y}g(y)\,dy = \widehat{f}(\xi)\widehat{g}(\xi).\end{aligned}$$

□

The linear space $L^\infty(\mathbb{R}^d)$ is also a commutative algebra in which the "product" operation on two functions $f, g \in L^\infty(\mathbb{R}^d)$ is the usual product fg of the two functions, i.e.

$$(fg)(x) = f(x)g(x)$$

for all $x \in \mathbb{R}^d$. And again, since $\|f * g\|_\infty \leq \|f\|_\infty\|g\|_\infty$ for all $f, g \in L^\infty(\mathbb{R}^d)$ we have that $L^\infty(\mathbb{R}^d)$ is a Banach algebra. The same is true for all smaller spaces $C_0(\mathbb{R}^d)$, $BUC(\mathbb{R}^d)$, $BC(\mathbb{R}^d)$. All of them are Banach sub-algebras of $L^\infty(\mathbb{R}^d)$.

We already know that

$$\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$$

satisfies

$$\mathcal{F}(kf + lg) = k\mathcal{F}(f) + l\mathcal{F}(g)$$

for all $f, g \in L^1(\mathbb{R}^d)$ and all $k, l \in \mathbb{C}$, and Proposition 1.13 says that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

for all $f, g \in L^1(\mathbb{R}^d)$. This means that $\mathcal{F}$ is a *homomorphism* between the algebras $L^1(\mathbb{R}^d)$ and $C_0(\mathbb{R}^d)$.

Proposition 1.14. *The algebra $L^1(\mathbb{R}^d)$ does not have a unit.*

Proof. Let us assume that f is a unit element of $L^1(\mathbb{R}^d)$, i.e. that $f * g = g$ for every $g \in L^1(\mathbb{R}^d)$. Then

$$\widehat{f}(\xi)\widehat{g}(\xi) = \widehat{g}(\xi)$$

for every $g \in L^1(\mathbb{R}^d)$ and every $\xi \in \mathbb{R}^d$. In particular, taking $g = G$, the function of Gauss, we have that $\widehat{g}(\xi) \neq 0$ for every $\xi \in \mathbb{R}^d$, and we get

$$\widehat{f}(\xi) = 1$$

for every $\xi \in \mathbb{R}^d$. This is impossible since $\widehat{f}$ belongs to $C_0(\mathbb{R}^d)$. □