A CLASS OF NON-CONVEX POLYTOPES THAT ADMIT NO ORTHONORMAL BASIS OF EXPONENTIALS

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Abstract. A conjecture of Fuglede states that a bounded measurable set Ω ⊂ R^d, of measure 1, can tile R^d by translations if and only if the Hilbert space L^2(Ω) has an orthonormal basis consisting of exponentials e_λ(x) = exp{2πi⟨λ, x⟩}. If Ω has the latter property it is called spectral. Let Ω be a polytope in R^d with the following property: there is a direction ξ ∈ S^{d−1} such that, of all the polytope faces perpendicular to ξ, the total area of the faces pointing in the positive ξ direction is more than the total area of the faces pointing in the negative ξ direction. It is almost obvious that such a polytope Ω cannot tile space by translation. We prove in this paper that such a domain is also not spectral, which agrees with Fuglede’s conjecture. As a corollary, we obtain a new proof of the fact that a convex body that is spectral is necessarily symmetric, in the case where the body is a polytope.

Let Ω be a measurable subset of R^d, which we take for convenience to be of measure 1. Let also Λ be a discrete subset of R^d. We write

\[ e_λ(x) = \exp\{2\pi i⟨λ, x⟩\}, \quad (λ, x ∈ R^d), \]

\[ E_Λ = \{e_λ : λ ∈ Λ\} ⊂ L^2(Ω). \]

The inner product and norm on L^2(Ω) are

\[ ⟨f, g⟩_Ω = \int_Ω fg, \quad \|f\|_Ω^2 = \int_Ω |f|^2. \]

Definition 1. The pair (Ω, Λ) is called a spectral pair if E_Λ is an orthonormal basis for L^2(Ω). A set Ω will be called spectral if there is Λ ⊂ R^d such that (Ω, Λ) is a spectral pair. The set Λ is then called a spectrum of Ω.

Example. If Q_d = (−1/2, 1/2)^d is the cube of unit volume in R^d, then (Q_d, Z^d) is a spectral pair (d-dimensional Fourier series).

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We write $B_R(x) = \{ y \in \mathbb{R}^d : |x - y| < R \}$.

**Definition 2 (Density).** The discrete set $\Lambda \subset \mathbb{R}^d$ has density $\rho$, and we write $\rho = \text{dens} \Lambda$, if we have

$$\rho = \lim_{R \to \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|},$$

uniformly for all $x \in \mathbb{R}^d$.

We define translational tiling for complex-valued functions below.

**Definition 3.** Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable and $\Lambda \subset \mathbb{R}^d$ be a discrete set. We say that $f$ tiles with $\Lambda$ at level $w \in \mathbb{C}$, and sometimes write “$f + \Lambda = w \mathbb{R}^d$”, if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w,$$

for almost every (Lebesgue) $x \in \mathbb{R}^d$,

with the sum above converging absolutely a.e. If $\Omega \subset \mathbb{R}^d$ is measurable, we say that $\Omega + \Lambda$ is a tiling when $1_{\Omega} + \Lambda = w \mathbb{R}^d$ for some $w$. If $w$ is not mentioned it is understood to be equal to 1.

**Remark 1.** If $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, and $f + \Lambda = w \mathbb{R}^d$, then the set $\Lambda$ has density

$$\text{dens} \Lambda = \frac{w}{\int f}.$$ 

The following conjecture is still unresolved in all dimensions and in both directions.

**Conjecture (Fuglede [F74]).** If $\Omega \subset \mathbb{R}^d$ is bounded and has Lebesgue measure 1 then $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if there exists $\Lambda \subset \mathbb{R}^d$ such that $\Omega + \Lambda = \mathbb{R}^d$ is a tiling.

Fuglede’s conjecture has been confirmed in several cases.

1. Fuglede [F74] shows that if $\Omega$ tiles with $\Lambda$ being a lattice then it is spectral with the dual lattice $\Lambda^*$ being a spectrum. Conversely, if $\Omega$ has a lattice $\Lambda$ as a spectrum then it tiles by the dual lattice $\Lambda^*$.

2. If $\Omega$ is a convex non-symmetric domain (bounded, open set) then, as the first author of the present paper has proved [K00], it cannot be spectral. It has long been known that convex domains which tile by translation must be symmetric.

3. When $\Omega$ is a smooth convex domain it is clear that it admits no translational tilings. Iosevich, Katz and Tao [IKT] have shown that it is also not spectral.
(4) There has also been significant progress in dimension 1 (the conjecture is still open there as well) by Laba [La], [Lb]. For example, the conjecture has been proved in dimension 1 if the domain $\Omega$ is the union of two intervals.

In this paper we describe a wide class of, generally non-convex, polytopes for which Fuglede's conjecture holds.

**Theorem 1.** Suppose $\Omega$ is a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that

$$\sum_i \sigma^*(\Omega_i) \neq 0.$$ 

Here the finite sum is extended over all faces $\Omega_i$ of $\Omega$ which are orthogonal to $\xi$ and $\sigma^*(\Omega_i) = \pm \sigma(\Omega_i)$, where $\sigma(\Omega_i)$ is the surface measure of $\Omega_i$ and the $\pm$ sign depends upon whether the outward unit normal vector to $\Omega_i$ is in the same or opposite direction with $\xi$.

Then $\Omega$ is not spectral.

Such polytopes cannot tile space by translation for the following, intuitively clear, reason. In any conceivable such tiling the set of positive-looking faces perpendicular to $\xi$ must be countered by an equal area of negatively-looking $\xi$-faces, which is impossible because there is more (say) area of the former than the latter.

The following corollary is a special case of the result in [K00], which says that all spectral convex domains are symmetric.

**Corollary 1.** If $\Omega$ is a spectral convex polytope then it is necessarily symmetric.

**Proof.** If $\Omega$ is spectral, then by Theorem 1 the area measure of $\Omega$ is symmetric. (See [S] for the definition of the area measure.) This implies that $\Omega$ is itself symmetric, as the area measure determines a convex body up to translation [S, Th. 4.3.1]. Therefore $\Omega$ and $-\Omega$, which have the same surface measure, are translates of each other. \qed

It has been observed in recent work on this problem (see, e.g., [K00]) that a domain (of volume 1) is spectral with spectrum $\Lambda$ if and only if $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling of Euclidean space at level 1. By Remark 1 this implies that $\Lambda$ has density 1.

By the orthogonality of $e_\lambda$ and $e_\mu$ for any two different $\lambda$ and $\mu$ in $\Lambda$, it follows that

$$\widehat{\chi_\Omega}(\lambda - \mu) = 0.$$ 

(2)

It is only this property, and the fact that any spectrum of $\Omega$ must have density 1, that are used in the proof.
Proof of Theorem 1. The quantities $P, Q, N, \ell$ and $K$, which are introduced in the proof below, will depend only on the domain $\Omega$. (The letter $K$ will denote several different constants.)

Suppose that $\Lambda$ is a spectrum of $\Omega$. Define the Fourier transform of $\chi_\Omega$ as

$$\hat{\chi}_\Omega(\eta) = \int_\Omega e^{-2\pi i (x, \eta)} \, dx.$$ 

By an easy application of the divergence theorem we get

$$\hat{\chi}_\Omega(\eta) = -\frac{1}{i|\eta|} \int_{\partial\Omega} e^{-2\pi i (x, \eta)} \left( \frac{\eta}{|\eta|} , \nu(x) \right) d\sigma(x), \quad \eta \neq 0,$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

From the last formula we easily see that for some $K \geq 1$

$$|\nabla \hat{\chi}_\Omega(\eta)| \leq \frac{K}{|\eta|}, \quad |\eta| \geq 1.$$

Without loss of generality we assume that $\xi = (0, \ldots, 0, 1)$. Hence

$$\hat{\chi}_\Omega(t\xi) = -\frac{1}{i\ell} \int_{\partial\Omega} e^{-2\pi i tx_d} \nu_d(x) d\sigma(x).$$

Now it is easy to see that each face of the polytope other than the faces $\Omega_i$ contributes $O(t^{-2})$ to $\hat{\chi}_\Omega(t\xi)$ as $t \to \infty$. Therefore

$$\left| \hat{\chi}_\Omega(t\xi) + \frac{1}{i\ell} \sum_i e^{-2\pi i \lambda_i t} \sigma^*(\Omega_i) \right| \leq \frac{K}{t^2}, \quad t \geq 1,$$

where $\lambda_i$ is the value of $x_d$ for $x = (x_1, \ldots, x_d) \in \Omega_i$.

Now define

$$f(t) = \sum_i \sigma^*(\Omega_i) e^{-2\pi i \lambda_i t}, \quad t \in \mathbb{R}.$$

$f$ is a finite trigonometric sum and has the following properties:

(i) $f$ is an almost-periodic function.

(ii) $f(0) \neq 0$ by assumption. Without loss of generality assume $f(0) = 1$.

(iii) $|f'(t)| \leq K$, for every $t \in \mathbb{R}$.

By (i), for every $\epsilon > 0$ there exists an $\ell > 0$ such that every interval of $\mathbb{R}$ of length $\ell$ contains a translation number $\tau$ of $f$ belonging to $\epsilon$:

$$\sup_t |f(t + \tau) - f(t)| \leq \epsilon$$

(see [B32]).

Fix $\epsilon > 0$ to be determined later ($\epsilon = 1/6$ will do) and the corresponding $\ell$. Fix the partition of $\mathbb{R}$ in consecutive intervals of length $\ell$, one of them being
[0, ℓ]. Divide each of these ℓ-intervals into \( N \) consecutive equal intervals of length \( \ell/N \), where

\[
N > \frac{6K\ell\sqrt{d-1}}{\epsilon}.
\]

In each ℓ-interval there is at least one \((\ell/N)\)-interval containing a number \( \tau \) satisfying (5). For example, in \([0, \ell]\) we may take \( \tau = 0 \) and the corresponding \((\ell/N)\)-interval to be \([0, \ell/N]\).

Define the set \( L \) to be the union of all these \((\ell/N)\)-intervals in \( \mathbb{R} \). Then \( L \) is a copy of \( L \) on the \( x_d \)-axis. Construct \( M \) by translating copies of the cube \([0, \ell/N]^d\) along the \( x_d \)-axis so that they have their \( x_d \)-edges on the \((\ell/N)\)-intervals of \( L \).

The point now is that there can be no two elements \( \lambda \) of \( \Lambda \) in the same translate of \( M \), at distance \( D > 2K/\epsilon \) from each other. Suppose, on the contrary, that \( \lambda_1, \lambda_2 \in \Lambda \), \( |\lambda_1 - \lambda_2| \geq D \), \( \lambda_1, \lambda_2 \in M + \eta \).

Then \( \lambda_1 = t_1\xi + \eta_1 + \eta_2 \), \( \lambda_2 = t_2\xi + \eta_1 + \eta_2 \), for some \( t_1, t_2 \in L \), \( \eta_1, \eta_2 \in \mathbb{R}^d \) with

\[
|\eta_1|, |\eta_2| < \frac{\ell}{N}\sqrt{d-1} < \frac{\epsilon}{6K}.
\]

Hence, \( \lambda_1 - \lambda_2 = (t_1 - t_2)\xi + \eta_1 - \eta_2 \), and an application of the mean value theorem together with (2) and (3) gives

\[
|\hat{\chi}_\Omega((t_1 - t_2)\xi)| \leq \frac{3K}{|t_1 - t_2|}|\eta_1 - \eta_2|.
\]

From (4) we get

\[
|f(t_1 - t_2)| \leq 3K|\eta_1 - \eta_2| + \frac{K}{|t_1 - t_2|} < 2\epsilon.
\]

Now, since \( t_1, t_2 \in L \), there exist \( \tau_1, \tau_2 \) satisfying (5) so that

\[
|\tau_1 - t_1|, |\tau_2 - t_2| < \frac{\ell}{N}
\]

and hence (by (iii))

\[
|f(\tau_1 - \tau_2) - f(\tau_1 - t_2)|, |f(\tau_1 - t_2) - f(t_1 - t_2)| < K\frac{\ell}{N} < \epsilon.
\]

Therefore

\[
2\epsilon > |f(t_1 - t_2)|
\]

\[
\geq |f(0)| - |f(0) - f(-\tau_2)| - |f(-\tau_2) - f(\tau_1 - \tau_2)|
\]

\[
- |f(\tau_1 - \tau_2) - f(\tau_1 - t_2)| - |f(\tau_1 - t_2) - f(t_1 - t_2)|
\]

\[
\geq 1 - \epsilon - \epsilon - \epsilon - \epsilon.
\]

It suffices to take \( \epsilon = 1/6 \) for a contradiction.
Therefore, as the distance between any two \( \lambda \)'s is bounded below by the modulus of the zero of \( \widehat{\chi}_\Omega \) that is nearest to the origin, there exists a natural number \( P \) so that every translate of \( M \) contains at most \( P \) elements of \( \Lambda \). Hence there exists a natural number \( Q \) (we may take \( Q = 2NP \)) so that every translate of

\[
\mathbb{R} \xi + [0, \ell/N]^{d}
\]

contains at most \( Q \) elements of \( \Lambda \).

It follows that \( \Lambda \) cannot have positive density, a contradiction as any spectrum of \( \Omega \) (which has volume 1) must have density equal to 1. \( \square \)

**REFERENCES**


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