THE DISCREPANCY FUNCTION IN TWO DIMENSIONS

IOANNIS PARISSIS

ABSTRACT. Let $\mathcal{A}_N$ be an N-point set in the unit square and consider the Discrepancy function

$$D_N(\vec{x}) := \sharp\Big(\mathcal{A}_N \cap [\vec{0}, \vec{x})\Big) - N|[\vec{0}, \vec{x})|,$$

where $\vec{x} = (x_1, x_2) \in [0, 1]^2$, $[\vec{0}, \vec{x}) = [0, x_1) \times [0, x_2)$, and $|[\vec{0}, \vec{x})|$ denotes the Lebesgue measure of the rectangle. This is the difference between the actual number of points of $\mathcal{A}_N$ in such a rectangle and the expected number of points - Nx_1x_2 - in the rectangle. A basic theme of discrepancy theory is to study the "size" of this function in terms of N. It turns out that no matter how the N points are selected, their distribution must be far from uniform, i.e. the discrepancy function must be "large". In this talk I will give an overview of some classical results in discrepancy theory that quantify the principle described above. I will also give an example of an *extremal* set for Discrepancy, in particular the van der Corput point set. Finally, if time permits, I will discuss some size estimates for the discrepancy function obtained in a *joint work* with *D. Bilyk*, *M. Lacey* and *A. Vagharshakyan* . For example we prove that

$$\|D_N\|_{\mathrm{BMO}} \gtrsim (\log N)^{1/2}.$$

This estimate is sharp. For the van der Corput set, we have $\|D_N\|_{\mathrm{BMO}} \lesssim (\log N)^{1/2}$, whenever $N = 2^n$ for some positive integer n.

CENTER FOR MATHEMATICAL ANALYSIS, GEOMETRY, AND DYNAMICAL SYSTEMS, DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, 1049-001, LISBOA, PORTUGAL.

E-mail address: `ioannis.parissis@gmail.com`

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