# NORMAL SUBGROUPS OF ODD ORDER MONOMIAL $P^A Q^B$-GROUPS

BY

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THESIS

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# Abstract

A finite group $G$ is called monomial if every irreducible character of $G$ is induced from a linear character of some subgroup of $G$. One of the main questions regarding monomial groups is whether or not a normal subgroup $N$ of a monomial group $G$ is itself monomial. In the case that $G$ is a group of even order, it has been proved (Dade, van der Waall) that $N$ need not be monomial. Here we show that, if $G$ is a monomial group of order $p^a q^b$, where $p$ and $q$ are distinct odd primes, then any normal subgroup $N$ of $G$ is also monomial.

to my parents Anna and Giannis,

and

to my teacher, Everett C. Dade,
without whom none of this would have been done,
and everything would have been written faster.

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# Chapter 1

# Introduction

## 1.1 The problem

Around 1930, Taketa (see [20]) introduced the notion of a monomial finite group, i.e., a group for which every irreducible character is induced from a linear character of some subgroup. He also proved that any monomial group is solvable. As any supersolvable group is monomial, Taketa's theorem places the class of monomial groups between the classes of solvable and of supersolvable groups.

In 1967, Dornhoff (in [6]) proved that every normal Hall subgroup of a monomial group is monomial. Furthermore, he asked whether or not every normal subgroup of a monomial group is monomial, a question that arises very naturally after his result on normal Hall subgroups. A negative answer to Dornhoff's question was found independently by both Dade [2] and van der Waall [21] in 1973. But their counterexamples have even order. Furthermore, the prime 2 plays an important role in their construction, so that the examples can't be modified to give an answer to Dornhoff's question in the case of an odd order group. Thus Dornhoff's question remains open in the case of odd order monomial groups. In the 1980's Dade and Isaacs, among others, tried to solve the remaining part of the problem. They produced many beautiful theorems suggesting that the following conjecture might be true:

**Conjecture 1.1.** *Let $G$ be a monomial group of odd order. Assume that $N$ is a normal subgroup of $G$. Then $N$ is also monomial.*

Among their results of that period, the following are the most useful for this thesis.

**Theorem 1.2 (Isaacs).** [Problem 6.11 in [12]] *Assume that $G$ is a monomial group and that $A$ is any normal subgroup of $G$. Let $\lambda$ be a linear character of $A$. We write $G(\lambda)$ for the stabilizer of $\lambda$ in $G$. Assume that $\chi$ is any irreducible character of $G$ that lies over $\lambda$. Let $\chi_\lambda$ be its Clifford correspondent, i.e., the unique irreducible character of $G(\lambda)$ that lies above $\lambda$ and induces $\chi$ in $G$. Then $\chi_\lambda$ is monomial.*

**Theorem 1.3 (Dade).** [Theorem 3.2 in [1]] *Let $G$ be a $p$-solvable group for some odd prime $p$. Assume that $U$ is a $\mathbb{Z}_p(G)$-module that affords a nondegenerate alternating $G$-invariant $\mathbb{Z}_p$-bilinear form. Let $H$ be a subgroup of $G$ that has $p$-power index in $G$. Then $U$ is $\mathbb{Z}_p(G)$-hyperbolic if and only if $U$ is $\mathbb{Z}_p(H)$-hyperbolic.*

(For the definition of hyperbolic modules, see Section 1.2.)

Based on this theorem Dade was able to prove

**Theorem 1.4 (Dade).** [Theorem (0) in [1]] *Let $G$ be a $p$-solvable group for some odd prime $p$, and let $N$ be a normal subgroup of $G$. Assume that $\chi \in \mathrm{Irr}(G)$ is a monomial character of $G$ that has $p$-power degree. Then every irreducible constituent of the restriction $\chi_N$ of $\chi$ to $N$ is monomial.*

Dade's Theorem (Theorem 1.4) is a very powerful result as it manages to handle monomial characters individually. That is, he doesn't assume that $G$ is monomial or that there are any monomial characters in $\mathrm{Irr}(G)$ except the specific prime-power degree character that is analyzed. Unfortunately, if we have characters with degrees that are divisible by more than one prime, then we can't hope to prove something similar to Theorem 1.4, as Dade has given counterexamples (in [1]) where his theorem fails when two odd primes divide $\chi(1)$.

But if we have enough monomial characters so that we have the freedom to replace, in some sense, the "bad"ones with "good" ones, we can do more. We can actually prove that a big enough section inside any normal subgroup $N$ of a monomial $p^a q^b$-group $G$ is nilpotent, provided that $p$ and $q$ are odd primes. What we show is

**Main Theorem 1.** *Let $G$ be a finite $p^a q^b$-monomial group, for some odd primes $p$ and $q$. Assume that $N$ is a normal subgroup of $G$ and that $\chi$ is an irreducible character of $N$. Then there exists a faithful linear limit $\chi^*$ of $\chi$, such that the domain $Dom(\chi^*)$ of $\chi^*$ is a nilpotent group.*

(For the definition of "faithful linear limits", see Section 10.1.)

As corollaries of the Main Theorem 1 we have

**Main Theorem 2.** *Let $G$ be an odd order monomial $p^a q^b$-group. If $N$ is a normal subgroup of $G$, then $N$ is monomial.*

and

**Main Theorem 3.** *Let $G$ be an odd order monomial $p^a q^b$-group and let $\chi \in \mathrm{Irr}(G)$. Then there exists a faithful linear limit $\chi^*$ of $\chi$ such that $\chi^*(1) = 1$, i.e., $\chi^*$ is a linear character.*

What is the desired property a "good" character has? We can control its degree. If this degree has the right properties then nilpotent subgroups appear, as the following result shows:

**Theorem 1.5.** *Assume that $G$ is a $p,q$-group, where $p$ and $q$ are distinct odd primes, and that $N, M$ are normal subgroups of $G$. Let $M = P \times S$ and $N = P \rtimes Q$, where $P$ is a $p$-group, and $S, Q$ are $q$-groups with $S \leq Q$. Assume that the center $Z(P)$ of $P$ is maximal among the abelian $G$-invariant subgroups of $P$. Let $\chi, \alpha, \beta$ and $\zeta$ be irreducible characters of $G, P, S$ and $Z(P)$ respectively that satisfy*

$$\begin{gathered}\chi \in \mathrm{Irr}(G|\alpha \times \beta) \text{ and } \alpha \in \mathrm{Irr}(P|\zeta),\\ \zeta \text{ is a faithful } G\text{-invariant character of } Z(P),\\ G(\beta) = G,\\ \chi \text{ is a monomial character of } G \text{ with } \chi(1)_q = \beta(1),\end{gathered}$$

*where $\chi(1)_q$ denotes the $q$-part of the integer $\chi(1)$. Then $Q$ centralizes $P$.*

The above theorem, that appears as Theorem 11.76 in this thesis, is heavily based on Theorem 1.3, and the work done in Dade's paper [1].

The way we use in this thesis to approach these "good" characters is by constructing a character of known degree in a subgroup of $G$ that extends to its own stabilizer in $G$. So the key tool for the proof of Theorem 1 is the following result (that appears as Theorem 4.17 in Chapter 4):

**Theorem 1.6.** *Let $P$ be a $p$-subgroup, for some prime $p$, of a finite odd order group $G$. Let $Q_1, Q$ be $q$-subgroups of $G$, for some prime $q \neq p$, with $Q_1 \leq Q$. Assume that $P$ normalizes $Q_1$, while $Q$ normalizes the product $P \cdot Q_1$. Assume further that $\beta_1$ is an irreducible character of $Q_1$. Then there exist irreducible characters $\beta_1^\nu$ of $Q_1$ and $\beta^\nu$ of $Q(\beta_1^\nu)$ such that*

$$\begin{aligned} &P(\beta_1) = P(\beta_1^\nu), \\ &Q(\beta_1) \leq Q(\beta_1^\nu) \textit{ and} \\ &\beta^\nu|_{Q_1} = \beta_1^\nu. \end{aligned}$$

*Therefore $\beta^\nu$ is an extension of $\beta_1^\nu$ to $Q(\beta_1^\nu)$.*

A possible generalization of Theorem 1.6 to the case where $Q_1$ and $Q$ are arbitrary $p'$-subgroups of $G$ would give a generalization of the Main Theorem 1. In this way the original problem (Conjecture 1.1) can be transformed to the following question:

**Question 1.7.** Does Theorem 1.6 hold if its second sentence is replaced by "Let $Q_1, Q$ be $p'$-subgroups of $G$"?

**Note:** Shortly after this thesis was submitted E. C. Dade found a counterexample to the above question. His example is not a counterexample for Conjecture 1.1. Nevertheless, it suggests strongly that a generalization of Theorem 1.6 requires some new ideas.

## 1.2 Notation

Let $G$ be a finite group with $M, N, K, K_1, \ldots, K_t$ subgroups of $G$, for some $t \geq 1$, and $g, h$ elements of $G$. Assume further that $M$ normalizes $N$. Let $\chi$ be a (complex) irreducible character of $M$, (we assume that all the characters we use in this text are over the field $\mathbb{C}$ of the complex numbers) and $\mathfrak{X}$ an irreducible $\mathbb{C}$-representation of $G$ that affords $\chi$. Assume further that $\phi_1, \ldots, \phi_t$ are irreducible characters of $K_1, \ldots, K_t$, respectively. Let $p$ be a prime number, and $\pi$ a set of primes.

The list that follows describes the notation we will be using for the rest of this thesis.

| | |
|---|---|
| $\mathbb{Z}_p$ : | the field of $p$ elements $\mathbb{Z}/p\mathbb{Z}$ |
| $Z(G)$ : | the center of $G$ |
| $\mathrm{Syl}_p(G)$ : | the set of all Sylow $p$-subgroups of $G$ |
| $\mathrm{Hall}_\pi(G)$ : | the set of all Hall $\pi$-subgroups of $G$ |
| $O_p(G)$ : | the largest normal $p$-subgroup of $G$ |
| $\Phi(G)$ : | the Frattini subgroup of $G$ |
| $g^h$ : | the $h$-conjugate $h^{-1}gh$ of $g$ |
| $K^h$ : | the $h$-conjugate group $h^{-1}Kh$ of $K$ |
| $N \leq G$ : | $N$ is a subgroup of $G$ |
| $N \trianglelefteq G$ : | $N$ is a normal subgroup of $G$ |
| $N(M \text{ in } G) = N_G(M)$ : | the normalizer of $M$ in $G$ |
| $N(K \text{ in } M) = N_M(K)$ : | the normalizer of $K$ in $M$ |
| $N(K_1, \ldots, K_t \text{ in } M)$ : | the normalizer of all $K_i$, for $i = 1, \ldots, t$ in $M$, i.e., the intersection $\bigcap_{i=1}^{t} N(K_i \text{ in } M)$ |
| $C(M \text{ in } G) = C_G(M)$ : | the centralizer of $M$ in $G$ |
| $C(K \text{ in } M) = C_M(K)$ : | the centralizer of $K$ in $M$ |
| $C(K_1, \ldots, K_t \text{ in } M)$ : | the centralizer of all $K_i$, for $i = 1, \ldots, t$ in $M$, i.e., the intersection $\bigcap_{i=1}^{t} C(K_i \text{ in } M)$ |
| $M \ltimes N$ : | the semidirect product of $M$ and $N$ when $M$ acts on $N$ |
| $K \rtimes N$ : | the semidirect product of $K$ and $N$ when $N$ acts on $K$ |
| $[M, N]$ : | the commutator subgroup of $M, N$ |
| $\mathrm{Irr}(G)$ : | the set of all complex irreducible characters of $G$ |
| $\mathrm{Lin}(G)$ : | the set of all linear complex characters of $G$ |
| $\mathrm{Ker}(\chi)$ : | the Kernel of $\chi$, i.e., $\mathrm{Ker}(\chi) = \{m \in M \mid \chi(m) = \chi(1)\}$ |
| $\chi^g$ : | the $g$-conjugate of $\chi$, i.e., $\chi^g$ is a character of $M^g$ defined as $\chi^g(m^g) := \chi(m)$ for all $m \in M$ |
| $\chi^G$ : | the induced character on $G$ |

| | |
|---|---|
| $[\chi, \psi]$ : | the inner product of $\chi, \psi \in \mathrm{Irr}(M)$, i.e. $$[\chi, \psi] = (1/\lvert G\rvert) \sum_{g \in G} \chi(g)\psi(g^{-1})$$ |
| $\mathrm{Irr}^M(N)$ : | the set of all $M$-invariant irreducible characters of $N$ |
| $\mathrm{Irr}_N^M(G)$ : | the set of all $\chi \in \mathrm{Irr}(G)$ such that $\chi$ lies above at least one character $\theta \in \mathrm{Irr}^M(N)$ |
| $\mathrm{Irr}(G\vert\chi)$ : | the set of all irreducible characters of $G$ that lie above $\chi$ |
| $G(\chi)$ : | the stabilizer of $\chi$ in $G$, i.e., the set of all elements $g$ of $G$ that satisfy $\chi^g = \chi$ |
| $G(\phi_1, \ldots, \phi_t)$ : | the stabilizer of $\phi_i$, for $i = 1, \ldots, t$, in $G$, i.e., the set $\bigcap_{i=1}^{t} G(\phi_i)$ |
| $K(\chi)$ : | the stabilizer of $\chi$ in $K$ |
| $K(\phi_1, \ldots, \phi_t)$ : | the stabilizer of $\phi_i$, for $i = 1, \ldots, t$, in $K$, i.e., the set $\bigcap_{i=1}^{t} K(\phi_i)$ |
| $\det(\mathfrak{X}(g))$ : | the determinant of the matrix $\mathfrak{X}(g)$ for some $g \in G$ |
| $\det(\chi)$ : | the linear character of $G$ defined as $(\det(\chi))(g) = \det(\mathfrak{X}(g))$ for all $g \in G$ |
| $o(\chi)$ : | the determinantal order of $\chi$, i.e., the order of the linear character $\det(\chi)$ as an element of the group $\mathrm{Lin}(G)$ |
| $[r]$ : | the integral part of a real number $r$, i.e., the largest integer $t$ such that $t \leq r$. |

Assume that $G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$ is a series of normal subgroups of $G$, for some integer $n \geq 0$. Assume further that $\chi_i$ is an irreducible character of $G_i$, for all $i = 0, 1, \ldots, n$, such that $\chi_i \in \mathrm{Irr}(G_i)$ lies above $\chi_{i-1} \in \mathrm{Irr}(G_{i-1})$, whenever $i = 1, \ldots, n$. Then we call the set $\{\chi_i\}_{i=0}^n$ *a character tower* for the series $\{G_i\}_{i=0}^n$.

If $T$ is a finite–dimensional $\mathbb{Z}_q(G)$-module that affords a $G$-invariant symplectic form $< \cdot, \cdot >$, then we will use the terminology introduced in [1]. So, if $S$ is a $\mathbb{Z}_q(G)$-submodule of $T$, then $S^\perp$ is the *perpendicular* subspace of $S$, i.e., $S^\perp := \{t \in T \mid < S, t >= 0\}$. Furthermore, $S$ is *isotropic* if $S \leq S^\perp$. $T$ is *anisotropic* if it contains no non–trivial isotropic $\mathbb{Z}_q(G)$-submodules, and is *hyperbolic* if it contains some self–perpendicular $\mathbb{Z}_q(G)$-submodule $S$, i.e., $S$ is a $\mathbb{Z}_q(G)$-submodule satisfying $S = S^\perp$.

If $U_i$, for $i = 1, \ldots, n$, are $F(G)$-submodules of an $F(G)$-module $S$, where $F$ is a field and $n$ a positive integer, we write

| | |
|---|---|
| $U_1 \dot{+} U_2$ : | for the internal direct sum of $U_1$ and $U_2$, and |
| $\dot{\sum}_{1 \leq i \leq n} U_i$ : | for the internal direct sum of the $U_i$, for all $i = 1, \ldots, n$. |

## 1.3 The general ideas in the proof

As this thesis turned out to be much longer and complicated than expected, the author would like to apologize to the reader for all the mysterious groups that appear suddenly, and for no apparent reason, in the chapters that follow. We will attempt in this section to give the main ideas of the proof, trying to avoid, as much as possible, the technical parts.

Assume that $G$ is a finite monomial group. Assume further that $G$ has order $p^a q^b$ for some distinct odd primes $p$ and $q$. Let $N$ be a normal subgroup of $G$. Fix an irreducible character $\psi$ of $N$. If $A$ is any normal subgroup of $G$ contained in $N$, and $\lambda$ is a linear character of $A$ lying under $\psi$, then Isaacs observation (Theorem 1.2), implies that we may pass from $G$ to the stabilizer $G' = G(\lambda)$ of $\lambda$ in $G$ without losing the monomiality of those irreducible characters of $G'$ that lie above $\lambda$. This way we reduce the order of the group $G$, possibly loosing some of the monomial characters, but still keeping track of those that lie above $\psi$. (This reduction procedure, that is described in Chapter 10, produces a "linear limit".) What we prove by induction is that, by applying this procedure many times, and choosing the $A$ and $\lambda$ carefully, (see Section 10.3 for an explanation of "careful"), we can reduce $G, N$ and $\psi$ to $G', N'$ and $\psi'$, respectively that satisfy

$$\begin{aligned} G' \leq G, &\text{ and } N' = N \cap G' \leq N, \text{ while } \psi' \in \operatorname{Irr}(N'), \\ N'/\operatorname{Ker}(\psi') &\text{ is nilpotent,} \\ (\psi')^N &= \psi, \\ \text{all } \chi' \in \operatorname{Irr}(G'|\psi') &\text{ are monomial.} \end{aligned} \tag{1.8}$$

This is equivalent to our Main Theorem 1. It is proved using induction on the order of $N$. Note that (1.8) easily implies that the character $\psi$ is monomial, since $\psi$ is induced from $\psi'$, and the unique character $\psi'/\operatorname{Ker}(\psi')$ of the factor group $N'/\operatorname{Ker}(\psi')$ that inflates to $\psi'$ is monomial as an irreducible character of a nilpotent group.

First notice that if $N$ is a nilpotent group, (1.8) holds trivially with $N$ in the place of $N'$. (We don't even need to apply Clifford's theorem to any normal subgroup $A$ and any linear character $\lambda \in \operatorname{Irr}(A)$.) Now suppose that $N$ is not nilpotent. Because $G$ is solvable, there exists a normal subgroup $L$ of $G$ such that $L < N$, while $N/L$ is either a $p$- or a $q$-group. We may assume that $N/L$ is a $q$-group, and that $p$ divides $|L|$. Let $\mu \in \operatorname{Irr}(L)$ be an irreducible character of $L$ lying under $\psi \in \operatorname{Irr}(N)$. We apply the reductions described above with $L$ and $\mu$ in the place of $N$ and $\psi$, i.e., we reduce $G$ using linear characters of normal subgroups of $G$ that are contained inside $L$ (again choosing these linear characters carefully). Every time $G$ gets reduced $N$ and $L$ are also reduced. Furthermore, the fixed character $\psi$ of $N$ also gets reduced, at every step, to a Clifford correspondent. So does the character $\mu \in \operatorname{Irr}(L)$. According to the inductive hypothesis applied to $L$, at some point these reductions lead us to groups $G''$ and $L''$, and to an irreducible character $\mu'' \in \operatorname{Irr}(L'')$ that induces $\mu$ while $L''/\operatorname{Ker}(\mu'')$ is nilpotent. Of course if we reduce the group $G''$ more, using normal subgroups $A$ of $G''$ contained in $L''$ and their linear characters $\lambda \in \operatorname{Lin}(A)$, the above properties remain valid. So we continue reducing until there is no normal subgroup $A \leq L''$ and linear character $\lambda \in \operatorname{Irr}(A)$ lying under $\mu''$ with $G''(\lambda) \leq G''$. At the end of this procedure we reach groups $L'$ and $G'$, and a character $\mu' \in \operatorname{Irr}(L')$, that satisfy the equivalent of (1.8) for $L$. In addition to those, the group $N$ gets reduced to $N' = G' \cap N$, and its irreducible character $\psi$ is reduced to $\psi' \in \mathrm{Irr}(N')$. Therefore we have

$$\begin{aligned} G' \le G, \text{ and } L' = L \cap G' \le L, &\text{ while } \mu' \in \mathrm{Irr}(L'), \\ N' = N \cap G' \le N &\text{ and } \psi' \in \mathrm{Irr}(N'|\mu'), \\ L'/\operatorname{Ker}(\mu') &\text{ is nilpotent}, \\ (\mu')^L &= \mu, \\ \text{all } \chi' \in \mathrm{Irr}(G'|\mu') &\text{ are monomial}, \\ \text{for any } A \trianglelefteq G' \text{ contained in } L', \text{ and any } \lambda \in \mathrm{Lin}(A) &\text{ that lies under } \mu', \text{ we get } G'(\lambda) = G'. \end{aligned} \tag{1.9}$$

It is easy to see (after all, these reductions are just repeated applications of Clifford theory) that $\psi'$ induces $\psi$, and lies above $\mu'$.

The fact $L' = L/\operatorname{Ker}(\mu')$ is nilpotent implies that the factor groups $L/\operatorname{Ker}(\mu')^g$ are all nilpotent, whenever $g \in G'$. Let $K' = \bigcap_{g \in G'} \operatorname{Ker}(\mu')^g$ and $L_1 = L'/K'$. Then $L_1$ is also nilpotent. Therefore $L_1$ splits as the direct product $L_1 = P_1 \times Q_1$ of its $p$- and $q$-Sylow subgroups $P_1$ and $Q_1$, respectively. If $\mu_1$ is the unique irreducible character of the factor group $L_1$ that inflates to $\mu' \in \mathrm{Irr}(L')$, then $\mu_1$ also decomposes as $\mu_1 = \alpha \times \beta$, where $\alpha \in \mathrm{Irr}(P_1)$ and $\beta \in \mathrm{Irr}(Q_1)$. At this point we can say more for $P_1$ and $Q_1$. If $A_1$ is any normal subgroup of $G_1 = G'/K'$ contained in $L_1$, then $A_1 = A'/K'$, where $A'$ is a normal subgroup of $G'$. If, in addition, $\lambda_1 \in \mathrm{Lin}(A_1)$ lies under $\mu_1$, then $\lambda_1$ inflates to a unique character $\lambda' \in \mathrm{Lin}(A')$ that lies under $\mu'$. As $L'$ was as reduced as possible, the character $\lambda'$ is $G'$-invariant. It also lies under $\mu'$. Hence $\operatorname{Ker}(\lambda') = \operatorname{Ker}(\mu'_{A'})$. So

$$\operatorname{Ker}(\lambda') = \bigcap_{g \in G'} \operatorname{Ker}(\lambda')^g \le \bigcap_{g \in G'} \operatorname{Ker}(\mu')^g = K'.$$

Therefore $\lambda_1$ is a faithful linear character of $A_1$, and is $G_1$-invariant. Thus every characteristic abelian subgroup of $P_1$ is cyclic and is contained in the center of $P_1$. Similarly for $Q_1$. In particular the center $Z(P_1)$ of $P_1$ is cyclic, and affords a faithful $G_1$-invariant linear character that lies under $\alpha$. Similarly the center $Z(Q_1)$ is cyclic, and affords a faithful $G_1$-invariant linear character lying under $\beta$. Furthermore, $P_1$ (and similarly $Q_1$) is of a very specific type. Either it is cyclic or it is the central product of an extra special $p$-group of exponent $p$ with the cyclic $p$-group $Z(P_1)$. Similarly for the $q$-group $Q_1$. Then clearly the characters $\alpha$ and $\beta$ are $G_1$-invariant and faithful. Hence $\mu_1 = \alpha \times \beta$ is $G_1$ invariant. We conclude that the character $\mu'$ is $G'$-invariant. So $\operatorname{Ker}(\mu') = K'$ is a normal subgroup of $G'$. Thus $G_1 = G'/\operatorname{Ker}(\mu')$. Furthermore, because $\psi' \in \mathrm{Irr}(N')$ lies above $\mu'$, the group $K'$ is contained in the kernel of $\psi' \in \mathrm{Irr}(N')$. Therefore there exists a unique irreducible character $\psi_1$ of the factor group $N_1 = N'/K'$ that inflates to $\psi' \in \mathrm{Irr}(N')$. Note that $Z(P_1)$ is maximal among the abelian normal subgroups of $G_1$ contained in $P_1$. This makes the factor group $P_1/Z(P_1)$ naturally a symplectic space (see (10.20) for the definition of the symplectic form). It is actually a symplectic $\mathbb{Z}_p(G_1/L_1)$-module. A picture of the situation is

$$\begin{array}{ccc} G_1 & & \\ | & & \\ N_1 & & \psi_1 \\ | & & | \\ L_1 = P_1 \times Q_1 & & \mu_1 = \alpha \times \beta \end{array} \tag{1.10}$$

Note that every character of $G_1$ that lies above $\mu_1$ is still monomial. Furthermore the factor group

$N_1/L_1$ is a $q$-group.

We could continue the reductions with normal subgroups of $G'$ that lie inside $N'$. But it turns out that it is not needed. The group $N'$ and the character $\psi'$ already achieved when doing the reductions for $L$, satisfy (1.8). Indeed $K' = \mathrm{Ker}(\mu') \leq \mathrm{Ker}(\psi')$, while any character of $G'$ lying above $\psi'$ is monomial, because $\psi'$ lies above $\mu'$. Therefore if we could prove that $N_1 = N'/K' = N'/\mathrm{Ker}(\mu')$ is nilpotent, then we would have that $N'/\mathrm{Ker}(\psi')$ is also nilpotent. This would prove the inductive step for (1.8). Actually, that is what we prove.

Let $Q$ be a $q$-Sylow subgroup of $N_1$. Then $N_1$ is the semidirect product $N_1 = P_1 \rtimes Q$. We prove that $Q$ centralizes $P_1$, using Theorem 1.5. Observe that the situation for the groups $G_1, N_1$ and $L_1$ looks similar to that described in Theorem 1.5. Indeed, the center $Z(P_1)$ of $P_1$ is maximal among the $G_1$-invariant abelian subgroups of $P_1$, while the characters $\alpha$ and $\beta$ are $G_1$-invariant. There is one ingredient missing from the hypothesis of the above theorem. That is, a monomial character $\chi_1$ of $G_1$ lying above $\mu_1$ and satisfying $\chi_1(1)_q = \beta(1)$. We do know that every character of $G$ above $\mu_1$ is monomial, but there is no reason for one of those monomial characters to have the desired degree. And that is the obstacle.

If the irreducible character $\beta$ of $Q_1$ extends to $G_1$, then the product of this extended character with any $p$-special character of $G_1$ lying above $\alpha$, is an irreducible character of $G_1$ that lies above $\mu_1$ and whose degree has the $q$-part equal to $\beta(1)$. Of course there is no reason for the character $\beta \in \mathrm{Irr}(Q_1)$ to extend.

A way to resolve this problem is to replace the character $\beta$ with a new character $\beta^\nu$ that extends. Actually Theorem 1.6 offers a way to replace characters. But now another problem appears. If we replace $\beta$ in $Q_1$ with a new character $\beta^\nu$ of the same group $Q_1$, then we can get an irreducible character of $G_1$ lying above $\alpha \times \beta^\nu$ with the correct degree, but which may not (and most probably is not) monomial. After all the only characters we know to be monomial in $G_1$ are those lying above $\mu_1$. Since the only source of monomial characters is back in $G$, we must change the original character $\mu \in \mathrm{Irr}(L)$.

Suppose we could find a $\mu^\nu \in \mathrm{Irr}(L)$ so that, when we reduce $G$ as above, using Isaacs observation as much as possible for subgroups of $L$, we end up with a system

$$\begin{array}{ccc} G_1^\nu & & \\ | & & \\ N_1^\nu & & \psi_1^\nu \\ | & & | \\ L_1^\nu = P_1^\nu \times Q_1^\nu & & \mu_1^\nu = \alpha_1^\nu \times \beta_1^\nu \end{array} \tag{1.11}$$

having same properties as (1.10), plus

- the character $\beta^\nu$ extends to a $q$-Sylow subgroup $\mathcal{Q}^\nu$ of $G_1^\nu$, and thus $\mu_1^\nu$ extends to $\mathcal{Q}^\nu \cdot L_1^\nu$,

- the symplectic space $P_1/Z(P_1)$ is isomorphic to $P_1^\nu/Z(P_1^\nu)$, and

- this isomorphism carries the commutator $[P_1/Z(P_1), Q]$ into $[P_1^\nu/Z(P_1^\nu), Q^\nu]$, where $Q^\nu$ is a $q$-Sylow subgroup of $N_1^\nu$.

The fact that $\beta^\nu$ extends implies, as we saw, that the $q$-Sylow subgroup $Q^\nu$ of $N_1^\nu$ centralizes $P_1^\nu$. Hence $[P_1^\nu/Z(P_1^\nu), Q^\nu] = 1$. We conclude that the group $[P_1/Z(P_1), Q]$, which is isomorphic to a subgroup of $[P_1^\nu/Z(P_1^\nu), Q^\nu]$, is also trivial. Hence $Q$ centralizes $P_1$. So the inductive step would be complete, provided that we could find such a character $\mu^\nu$.

The "miracle" is that such a character $\mu^\nu$ does exist. To find it, we had to introduce the notion of triangular sets and go through all the complicated machinery described in Chapters 5, 6 and 7. The reason is that we have a replacement theorem (Theorem 1.6) that works under special hypotheses. The character $\mu$ doesn't satisfy these hypotheses. The character $\beta$ does satisfy them, but, for the reasons explained above, if we change $\beta$ we don't get monomial characters. So instead of $\mu$ we change a character $\mu^*$ corresponding to $\mu$ in a certain subgroup $L^*$ of $L$. Both $\mu^*$ and $L^*$ satisfy the hypotheses of Theorem 1.6, so that we can replace $\mu^*$ with another $\mu^{*,\nu} \in \mathrm{Irr}(L^*)$ that extends to a $q$-Sylow subgroup of $G(\mu^*) \cdot L^*$. In addition, $\mu^*$ and $L^*$ are picked in such a way that we can get back from the new character $\mu^{*,\nu}$ of $L^*$ to a corresponding new character $\mu^\nu$ of $L$.

In Chapters 9 and 10 we show that this new character retains its extendibility properties throughout our reductions.

In Chapter 11 we put all the pieces together to prove the Main Theorem 1.

# Chapter 2

# Preliminaries

## 2.1 Glauberman correspondence

Let $A$ and $G$ be two finite groups with orders that are relatively prime. In the case that $A$ is a solvable group, Glauberman [8] constructed a 'natural' bijection between the set $\mathrm{Irr}^A(G)$ of $A$-invariant irreducible characters of $G$, and the set $\mathrm{Irr}(C(A \text{ in } G))$ of irreducible characters of the fixed points $C(A \text{ in } G)$ of $A$ in $G$. Two well known facts (see Chapter 13 (page 299) in [12]) about the Glauberman correspondence are

**Lemma 2.1.** *Suppose that $A, G$ are finite groups such that $(|A|, |G|) = 1$ and that $A$ acts on $G$. Let a group $S$ act on the semidirect product $AG$, leaving both $A$ and $G$ invariant. If $\chi$ is an $A$-invariant irreducible character of $G$, and $\chi^*$ is its $A$-Glauberman correspondent in $\mathrm{Irr}(C_G(A))$, then for any element $s \in S$, the Glauberman correspondent $(\chi^s)^*$ of $\chi^s$ equals $(\chi^*)^s$.*

Lemma 2.1 obviously implies

**Corollary 2.2.** *In the situation of Lemma 2.1, let $T$ be a subgroup of $S$. Then $\chi$ is fixed by $T$ if and only if its Glauberman correspondent $\chi^*$ is also fixed by $T$.*

## 2.2 Groups

**Proposition 2.3.** *Assume $Q, P$ are two finite groups with coprime orders, and that $Q$ acts as automorphisms of $P$. Assume further that $P$ is the product $P = P_1 \cdot P_2$, of its normal subgroup $P_1$ and some subgroup $P_2$ of $P$. If both $P_1$ and $P_2$ are $Q$-invariant, then*

$$N(Q \text{ in } P) = C(Q \text{ in } P) = C(Q \text{ in } P_1) \cdot C(Q \text{ in } P_2).$$

*Proof.* As $Q$ and $P$ have coprime orders, and $Q$ acts on $P$, we obviously have that

$$N(Q \text{ in } P) = C(Q \text{ in } P) \geq C(Q \text{ in } P_1) \cdot C(Q \text{ in } P_2).$$

Hence it remains to show that $C(Q \text{ in } P) \leq C(Q \text{ in } P_1) \cdot C(Q \text{ in } P_2)$.

As $P_1 \trianglelefteq P$, we have $C(Q \text{ in } P)/C(Q \text{ in } P_1) \cong C(Q \text{ in } P/P_1)$, by Glauberman's Lemma, 13.8 in [12]. The natural isomorphism of $P/P_1 = (P_1P_2)/P_1$ onto $P_2/(P_1 \cap P_2)$ preserves the action of $Q$. So it sends $C(Q \text{ in } P/P_1) = C(Q \text{ in } P) \cdot P_1/P_1$ onto $C(Q \text{ in } P_2/(P_1 \cap P_2)) = C(Q \text{ in } P_2) \cdot (P_1 \cap P_2)/(P_1 \cap P_2)$. So $C(Q \text{ in } P_2)$ covers $C(Q \text{ in } P)$ modulo $C(Q \text{ in } P) \cap P_1 = C(Q \text{ in } P_1)$. We conclude that $C(Q \text{ in } P) = C(Q \text{ in } P_2) \cdot C(Q \text{ in } P_1)$. Hence the proposition holds. ∎

As an easy consequence of Proposition 2.3 we have

**Corollary 2.4.** *Assume $Q, P$ are two finite groups with coprime orders, and that $Q$ acts as automorphisms of $P$. For every $i = 1, \dots, n$, let $P_i$ be a $Q$-invariant sugroup of $P$. Assume further that $P_j$ normalizes $P_i$, whenever $1 \leq i \leq j \leq n$, while $P$ is the product $P = P_1 \cdot P_2 \dots P_n$. Then*

$$N(Q \text{ in } P) = C(Q \text{ in } P) = C(Q \text{ in } P_1) \cdot C(Q \text{ in } P_2) \cdot \dots \cdot C(Q \text{ in } P_n).$$

The following theorem is a multiple application of Clifford's Theorem:

**Theorem 2.5.** *Let $G$ be a finite group. Assume that $1 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G$ is a series of normal subgroups of $G$. Assume further that we have fixed a character tower $\{\chi_i\}_{i=1}^m$ for the above series, i.e., $\chi_i \in \mathrm{Irr}(G_i)$ for $i = 1, \dots, m$, such that $\chi_i$ lies above $\chi_{i-1}$, whenever $i = 2, \dots, m$. For any $i = 1, \dots, m$ we write $G_i^m = G_i(\chi_1, \chi_2, \dots, \chi_m)$ for the stabilizer of $\chi_1, \chi_2, \dots, \chi_m$ in $G_i$, and $G^m = G(\chi_1, \dots, \chi_m)$ for the corresponding stabilizer in $G$. Then $G_1^m = G_1$ and $G_k^m = G_i^m \cap G_k = G^m \cap G_k \trianglelefteq G_i^m$, whenever $1 \leq k \leq i \leq m$. Furthermore, there exist unique characters $\chi_i^m$, for $i = 1, \dots, m$, such that*

$$\chi_1^m = \chi_1, \textit{ while } \chi_i^m \in \mathrm{Irr}(G_i^m) \textit{ lies over } \chi_1^m, \dots, \chi_{i-1}^m \\ \textit{and induces } \chi_i \in \mathrm{Irr}(G_i), \textit{ for all } i = 1, \dots, m. \tag{2.6}$$

*Furthermore, these characters satisfy*

*(1) $G^m = G(\chi_1^m, \chi_2^m, \dots, \chi_m^m)$, and*

*(2) If $G_m \leq H \leq G$ and $H^m = G^m \cap H$, then for any $\phi \in \mathrm{Irr}(H|\chi_m)$ and $\chi \in \mathrm{Irr}(G|\phi)$, there exist unique characters $\phi^m \in \mathrm{Irr}(H^m|\chi_m^m)$ and $\chi^m \in \mathrm{Irr}(G^m|\phi^m)$ that induce $\phi^m$ and $\chi$, respectively. Conversely, if $\phi^m \in \mathrm{Irr}(H^m|\chi_m^m)$ and $\chi^m \in \mathrm{Irr}(G^m|\phi^m)$, then $(\phi^m)^H \in \mathrm{Irr}(H|\chi_m)$ while $(\chi^m)^G \in \mathrm{Irr}(G|(\phi^m)^H)$.*

*Proof.* Since $G_i \trianglelefteq G_j$ and $\chi_j \in \mathrm{Irr}(G_j|\chi_i)$, whenever $1 \leq i \leq j \leq m$, we obviously have

$$G_i^m = G_i(\chi_1, \chi_2, \dots, \chi_m) = G_i(\chi_1, \chi_2, \dots, \chi_{i-1}) = G_i(\chi_1, \dots, \chi_i), \tag{2.7}$$

for any $i = 1, 2, \dots, m$. Thus $G_k^m = G_i^m \cap G_k = G^m \cap G_k$, for all $k$ with $1 \leq k \leq i$. It is also clear that $G_1^m = G_1$.

To prove the rest of the theorem we will use induction on $m$. First assume that $m = 1$. Because $G_1^1 = G_1(\chi_1) = G_1$, the only possible choice of $\chi_1^1$ satisfying (2.6) for $m = i = 1$ is

$$\chi_1^1 = \chi_1 \in \mathrm{Irr}(G_1^1). \tag{2.8}$$

Observe that, in this case, $G(\chi_1^1) = G(\chi_1) = G^1$. Furthermore, if $G_1 \leq H \leq G$, then Clifford's Theorem provides a bijection between the irreducible characters $\phi$ of $H$ lying above $\chi_1$, and the irreducible characters $\phi^1$ of $H^1 = H(\chi_1)$ that lie above $\chi_1$. Any two such characters $\phi$ and $\phi^1$ correspond if and only if $\phi^1$ induces $\phi$. Note also that, since Clifford's theory respects multiplicities, any character $\chi \in \mathrm{Irr}(G)$ that lies above $\phi$ corresponds to some $\chi^1 \in \mathrm{Irr}(G^1)$ that lies above $\phi^1 \in \mathrm{Irr}(H^1)$, and vice versa. So (1) and (2) hold for $m = 1$.

Now assume that the theorem holds for all $m$ with $m < n$, and some integer $n \geq 2$. We will prove it is also true when $m = n$. So assume that $1 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n \trianglelefteq G$ is a normal series of $G$, while $\{\chi_i\}_{i=1}^n$ is a character tower for that series. The inductive hypothesis, applied to the normal series $1 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G$ of $G$ and its character tower $\{\chi_i\}_{i=1}^{n-1}$, implies the existence of unique irreducible characters $\chi_i^{n-1}$ of $G_i^{n-1} = G(\chi_1, \ldots, \chi_{n-1})$ that satisfy the conclusions of the theorem, for $m = n-1$. Let $G_n^{n-1} = G_n(\chi_1, \ldots, \chi_{n-1}) = G^{n-1} \cap G_n$. Then we clearly have

$$G_i^n = G_i(\chi_1, \ldots, \chi_n) = G_i(\chi_1, \ldots, \chi_{n-1}) = G_i^{n-1}, \tag{2.9}$$

for all $i = 1, \ldots, n$.

The character $\chi_n \in \operatorname{Irr}(G_n)$ lies above $\chi_{n-1}$, while the series $1 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n$ is a normal series of $G_n$. Hence (2) of the theorem for $m = n-1$ implies that there exists a unique irreducible character $\chi_n^{n-1} \in \operatorname{Irr}(G_n^{n-1})$, that lies above $\chi_{n-1}^{n-1}$ and induces $\chi_n \in \operatorname{Irr}(G_n)$. We set

$$\chi_i^n := \chi_i^{n-1} \in \operatorname{Irr}(G_i^{n-1}) = \operatorname{Irr}(G_i^n), \tag{2.10}$$

for all $i = 1, \ldots, n$. Clearly the characters $\chi_i^n$ satisfy (2.6), for $m = n$ and all $i = 1, 2, \ldots, n$. (Note that $\chi_1^n = \chi_1$.) Furthermore, these characters are unique among those that satisfy (2.6) for $m = n$. Indeed, assume that $\{\psi_i^n\}_{i=1}^n$ is a character tower for $\{G_i^n\}_{i=1}^n$, so that $\psi_i^n \in \operatorname{Irr}(G_i^n)$ induces $\chi_i \in \operatorname{Irr}(G_i)$, for all $i = 1, \ldots, n$. As $G_i^n = G_i^{n-1}$ for all such $i$, the uniqueness of the characters $\{\chi_i^{n-1}\}_{i=1}^{n-1}$ achieved from the inductive argument, implies that $\psi_i^n = \chi_i^{n-1} = \chi_i^n$, whenever $i = 1, \ldots, n-1$. In addition, the character $\chi_n^n$ was picked as the unique character of $G_i^n$ that induces $\chi_n$ and lies above $\chi_{n-1}^{n-1}$. Thus $\chi_n^n = \psi_n^n$. This proves that there exist unique characters $\{\chi_i^n\}_{i=1}^n$ that satisfy (2.6), for $m = n$.

To prove that (1) also holds for the characters $\{\chi_i^n\}_{i=1}^n$, first observe that $G(\chi_1, \ldots, \chi_{n-1}) = G^{n-1} = G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1})$, by (1) for $m = n-1$. Hence

$$G^n = G(\chi_1, \ldots, \chi_{n-1}, \chi_n) = G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1}, \chi_n). \tag{2.11}$$

Furthermore, $G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1})$ normalizes both $G_{n-1}^{n-1}$ and $G_n^{n-1}$, and fixes the character $\chi_{n-1}^{n-1}$. As $\chi_n^{n-1}$ is the unique character of $G_n^{n-1}$ that lies above $\chi_{n-1}^{n-1}$ and induces $\chi_n$, we conclude that $G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1})(\chi_n) = G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1})(\chi_n^{n-1})$. This, along with (2.11) and (2.10), implies

$$G^n = G(\chi_1^{n-1}, \ldots, \chi_{n-1}^{n-1})(\chi_n^{n-1}) = G(\chi_1^n, \ldots, \chi_{n-1}^n, \chi_n^n) = G^{n-1}(\chi_n^n). \tag{2.12}$$

So (1) holds for the inductive step.

Assume now that $H$ is any subgroup of $G$ with $G_n \leq H \leq G$, while $\phi, \chi$ are irreducible characters of $H$ and $G$, respectively, so that $\chi$ lies above $\phi$ and $\phi$ above $\chi_n$. Then they both lie above $\chi_{n-1}$ as well. Hence the inductive hypothesis implies that there exist unique characters $\phi^{n-1} \in \operatorname{Irr}(H^{n-1}|\chi_{n-1}^{n-1})$ and $\chi^{n-1} \in \operatorname{Irr}(G^{n-1}|\phi^{n-1})$ that induce $\phi \in \operatorname{Irr}(H)$ and $\chi \in \operatorname{Irr}(G)$, respectively. Observe also that the inductive hypothesis for (2) guarantees that the characters $\chi^{n-1}$ and $\phi^{n-1}$ lie above $\chi_n^{n-1} \in \operatorname{Irr}(G_n^{n-1})$, since both $\chi$ and $\phi$ lie above $\chi_n$, and $\chi_n^{n-1}$ is the unique character of $G_n^{n-1}$ that induces $\chi_n$ and lies above $\chi_{n-1}^{n-1}$. Because $G_n^{n-1}$ is a normal subgroup of both $H^{n-1}$ and $G^{n-1}$, Clifford's Theorem implies the existence of unique irreducible characters $\chi^n \in \operatorname{Irr}(G^{n-1}(\chi_n^{n-1}))$ and $\phi^n \in \operatorname{Irr}(H^{n-1}(\chi_n^{n-1})$ that that lie above $\chi_n^{n-1}$ and induce $\chi^{n-1}$ and $\phi^{n-1}$, respectively. Hence $\chi^n$ induces $\chi$, and $\phi^n$ induces $\phi$. Furthermore, Clifford's Theorem implies that $\chi^n$ lies above $\phi^n$, because $\chi^{n-1}$ lies above $\phi^{n-1}$. In addition, $G^{n-1}(\chi_n^{n-1}) = G^{n-1}(\chi_n^n) = G^n$, by (2.12) and the fact that $\chi_n^n := \chi_n^{n-1}$. So $H^n = H^{n-1}(\chi_n^{n-1})$. We conclude that $\chi^n$ and $\phi^n$ satisfy (2) for the inductive step. Furthermore, they are unique with that property. Indeed, assume that $\eta \in \operatorname{Irr}(H^n|\chi_n^n)$ and $\psi \in \operatorname{Irr}(G^n|\eta)$ induce $\phi \in \operatorname{Irr}(H)$ and $\chi \in \operatorname{Irr}(G)$, respectively. Then $\eta^{H^{n-1}}$ and $\psi^{G^{n-1}}$ are irreducible characters of $H^{n-1}$ and $G^{n-1}$, (since $H^n \leq H^{n-1} \leq H$ and $G^n \leq G^{n-1} \leq G$), that induce $\phi$ and $\chi$, repsectively. Also $\psi^{G^{n-1}}$ lies above $\eta^{H^{n-1}}$, and $\eta^{H^{n-1}}$ above $\chi_{n-1}^{n-1}$, because $\chi_n^n = \chi_n^{n-1}$ lies above this last character, and $\eta$ lies above $\chi_n^n$. Hence the inductive hypothesis forces $\eta^{H^{n-1}} = \phi^{n-1}$ and $\psi^{G^{n-1}} = \chi^{n-1}$. Therefore, $\psi \in \mathrm{Irr}(G^n) = \mathrm{Irr}(G^{n-1}(\chi_n^n))$ induces $\chi^{n-1}$ and lies above $\chi_n^n = \chi_n^{n-1}$, while $\eta \in \mathrm{Irr}(H^n)$ induces $\phi^{n-1}$ and lies above $\chi_n^{n-1}$. In conclusion, $\eta, \psi$ are the $\chi_n^{n-1}$-Clifford correspondents of $\phi^{n-1}$ and $\chi^{n-1}$, respectively. So $\eta = \phi^n$ and $\psi = \chi^n$. Thus $\phi^n$ and $\psi^n$ are unique.

Conversely, if $\phi^n \in \mathrm{Irr}(H^n|\chi_n^n)$ then Clifford's Theorem implies that $(\phi^n)^{H^{n-1}}$ is an irreduible character of $H^{n-1}$. since $H^n = H^{n-1}(\chi_n^n)$. Furthermore, $(\phi^n)^{H^{n-1}}$ lies above $\chi_{n-1}^{n-1}$, as $\chi_n^n = \chi_n^{n-1}$ lies above $\chi_{n-1}^{n-1}$. This, along with the inductive argument, implies that $\phi^n$ induces an irreducible character of $H$ lying above $\chi_n$. Similarly we can work with any character $\chi^n \in \mathrm{Irr}(G^n|\phi^n)$ to show that it induces irreducibly to a character of $G$ lying above $(\phi^n)^H$. Hence (2) for the inductive step is proved. Therefore Theorem 2.5 holds. □

## 2.3 Character extensions

The following is a well known and useful result:

**Theorem 2.13.** *Assume that $N$ is a normal subgroup of $G$ such that $(|N|, |G/N|) = 1$. Let $\chi$ be a $G$-invariant irreducible character of $N$. Then there exists a unique extension $\chi^e$ of $\chi$ to $G$ such that $(o(\chi^e), |G/N|) = 1$. This is called the canonical extension of $\chi$, and has the additional property that it is the unique extension of $\chi$ to $G$ such that $o(\chi) = o(\chi^e)$.*

*Proof.* The proof follows easily from Lemma 6.24 and Theorem 11.32 of [12], as both $o(\chi)$ and $\chi(1)$ divide $|N|$, and thus are coprime to $|G : N|$. □

In the situation of the preceding theorem we can describe all the irreducible constituents of $\chi^G$, as the next result shows.

**Theorem 2.14 (Gallagher).** *Assume that $N$ is a normal subgroup of $G$. Let $\chi \in \mathrm{Irr}(N)$ be a $G$-invariant character of $N$ that extends to an irreducible character $\psi$ of $G$. Then there is a one–to–one correspondence between the irreducible characters $\gamma$ of $G$ lying above $\chi$ and the irreducible characters $\lambda$ of $G/N$. Two such $\gamma$ and $\lambda$ correspond if and only if $\gamma = \lambda \cdot \psi$. The latter product is defined as*

$$(\lambda \cdot \psi)(g) = \lambda(gN) \cdot \psi(g),$$

*for any $g \in G$.*

*Proof.* The proof follows immediately from Theorem 2.13 and Corollary 6.17 in [12]. □

Note that, under the addition hypothesis $(|N|, |G/N|) = 1$, we could have used the unique canonical extension $\chi^e$ in the place of $\psi$ in Theorem 2.14.

**Lemma 2.15.** *Assume that $G$ is a finite group, $H$ is a normal subgroup of $G$, and $N$ is a normal subgroup of $H$. Assume further that $\theta$ is an irreducible character of $N$, and that its stabilizer $H(\theta)$ in $H$ is the product $H(\theta) = N \cdot A$ of $N$ with a subgroup $A$ of $H$, such that $(|A|, |N|) = 1$.*
*Then $\theta$ has a canonical extension $\theta^e$ to $H(\theta)$. Furthermore, any irreducible character $\Psi_\gamma$ of $H(\theta)$ lying above $\theta$ is of the form $\Psi_\gamma = \gamma \cdot \theta^e$ where $\gamma \in \mathrm{Irr}(A)$ and the product is defined as $(\gamma \cdot \theta^e)(s \cdot t) = \gamma(t) \cdot \theta^e(st)$ for any $s \in N$ and any $t \in A$. As for the stabilizer $G(\gamma, \theta)$ of $\gamma$ and $\theta$ in $G$, we have:*

$$G(\gamma, \theta) = N(A \text{ in } G(\theta, \Psi_\gamma)) = N(A \text{ in } G(\theta, \Psi_\gamma^H)),$$

*where $\Psi_\gamma^H$ is the irreducible character of $H$ induced by $\Psi_\gamma$.*

*Proof.* Theorem 2.13 above implies that $\theta$ has a unique canonical extension to $H(\theta)$. This, along with Gallagher's Theorem 2.14, implies all but the last statement of the lemma.

As $G(\gamma,\theta)$ fixes $\gamma$ and $\theta$, it normalizes $A$ and $N$ respectively. Hence it normalizes the product $H(\theta) = NA$. The canonical extension, $\theta^e$, of $\theta$ to $H(\theta)$ is fixed by $G(\gamma,\theta)$, as the latter fixes $\theta$ and normalizes $H(\theta)$. Therefore $G(\gamma,\theta)$ fixes the product $\gamma \cdot \theta^e = \Psi_\gamma$. So $G(\gamma,\theta)$ is a subgroup of $N(A \text{ in } G(\theta,\Psi_\gamma))$. Because $H$ is a normal subgroup of $G$, any subgroup of $G$ that fixes $\Psi_\gamma$ also fixes the induced character $\Psi_\gamma^H$. Hence $N(A \text{ in } G(\theta,\Psi_\gamma))$ is a subgroup of $N(A \text{ in } G(\theta,\Psi_\gamma^H))$. Therefore

$$G(\gamma,\theta) \leq N(A \text{ in } G(\theta,\Psi_\gamma)) \leq N(A \text{ in } G(\theta,\Psi_\gamma^H)).$$

For the other inclusions we note that any element $g \in N(A \text{ in } G(\theta,\Psi_\gamma^H))$ fixes $\theta$ and $\Psi_\gamma^H$, normalizes $H(\theta)$, and fixes the unique Clifford correspondent $\Psi_\gamma \in \mathrm{Irr}(H(\theta)|\theta)$ of $\Psi_\gamma^H$. Furthermore, $g$ fixes $\theta^e \in \mathrm{Irr}(H(\theta))$, because it fixes $\theta$ and normalizes $H(\theta)$. As $g$ normalizes $A$, and fixes the product character $\Psi_\gamma = \gamma \cdot \theta^e$, it also fixes $\gamma$. Hence $g \in G(\gamma,\theta)$. So $N(A \text{ in } G(\theta,\Psi_\gamma^H)) \leq G(\gamma,\theta)$, and the proof of the lemma is complete. $\square$

**Proposition 2.16.** *Let $G$ be finite group of odd order such that $G = N \cdot K$, where $N$ is a normal subgroup of $G$ and $(|G/N|, |N|) = 1$. Let $H = N \cap K$ and let $\theta$ be any irreducible $K$-invariant character of $H$ that induces a $G$-invariant irreducible character $\theta^N$ of $N$. Then $\theta^N$ has a unique canonical extension, $(\theta^N)^e$, to $G$ such that $(|G/N|, o((\theta^N)^e)) = 1$, while $\theta$ has a unique canonical extension, $\theta^e$, to $K$ such that $(|K/H|, o(\theta^e)) = 1$. Furthermore, $\theta^e$ induces*

$$(\theta^e)^G = (\theta^N)^e.$$

*Proof.* Let $\pi$ be the set of primes that divide $|N|$. Then $|K/H| = |G/N|$ is a $\pi'$-number, and thus is coprime to $|H|$. As $\theta \in \mathrm{Irr}(H)$ is $K$-invariant, Theorem 2.13 implies that $\theta$ has a unique extension $\theta^e$ to $K$, with

$$o(\theta) = o(\theta^e). \tag{2.17}$$

According to Corollary 4.3 in [15], induction defines a bijection $\mathrm{Irr}(K|\theta) \to \mathrm{Irr}(G|\theta^N)$. Therefore,

$$\chi := (\theta^e)^G \in \mathrm{Irr}(G|\theta^N). \tag{2.18}$$

But $\theta^N$ extends to $G$, as it is $G$-invariant and $(|N|, |G/N|) = 1$. Let $\Psi = (\theta^N)^e \in \mathrm{Irr}(G)$ be the unique extension of $\theta^N$ such that $o(\Psi) = o(\theta^N)$ is a $\pi$-number (see Theorem 2.13). Since $\chi$ lies above $\theta^N$, Theorem 2.14 implies that

$$\chi = \mu \cdot \Psi,$$

for some $\mu \in \mathrm{Irr}(G/N)$. We compute the degree $\deg(\chi)$ in two ways. First

$$\deg(\chi) = \deg(\mu) \cdot \deg(\Psi) = \deg(\mu) \cdot \deg(\theta^N) = \deg(\mu) \cdot |N : H| \cdot \deg(\theta).$$

As $\chi = (\theta^e)^G$ we also have that

$$\deg(\chi) = |G : K| \cdot \deg(\theta^e) = |G : K| \cdot \deg(\theta) = |N : H| \cdot \deg(\theta).$$

We conclude that $\deg(\mu) = 1$. Thus $\mu \in \mathrm{Lin}(G/N)$. Therefore

$$\det(\chi) = \mu^{\Psi(1)} \det(\Psi). \tag{2.19}$$

We can now compute $o(\chi)$ in two ways. First, $o(\Psi) = o(\theta^N)$ and $\Psi(1) = \theta^N(1)$ are $\pi$-numbers. Since $\mu \in \mathrm{Irr}(G/N)$, we get that $o(\mu)$ is a $\pi'$-number. Therefore, (2.19) implies that the $\pi'$-number $o(\mu)$ divides $o(\chi)$.

On the other hand, (2.18) and Lemma 2.2 in [16] imply that

$$o(\chi) = o((\theta^e)^G) \text{ divides } 2 \cdot o(\theta^e).$$

As $G$ has odd order, we get that $o(\chi)$ divides $o(\theta^e)$. In view of (2.17), we have $o(\theta^e) = o(\theta)$, while $o(\theta) \bigm| |H|$. We conclude that $o(\chi)$ is a $\pi$-number.

Hence the only way the $\pi'$-number $o(\mu)$ can divide $o(\chi)$, is if $o(\mu) = 1$. So $\mu = 1$, and

$$(\theta^e)^G = \chi = \Psi = (\theta^N)^e,$$

as desired. ∎

**Lemma 2.20.** *Let $G$ be any finite group, and $H$ be any subgroup of $G$. If $\theta \in \mathrm{Irr}(G)$ and $\Psi \in \mathrm{Irr}(H)$ then:*

$$(\theta_H \cdot \Psi)^G = \theta \cdot (\Psi^G).$$

*Proof.* See Exercise 5.3 in [12]. ∎

As a corollary of Lemma 2.20 we can prove

**Corollary 2.21.** *Let $G = H \ltimes M$ be a finite group, and $S$ be an $H$-invariant subgroup of $M$. Let $\alpha, \theta$ be irreducible characters of $H$ and $H \ltimes S$ respectively. Then $\alpha \cdot \theta$ is a character of $HS$ defined as $(\alpha \cdot \theta)(x \cdot y) = \alpha(x) \cdot \theta(x \cdot y)$, whenever $x \in H$ and $y \in S$. Furthermore,*

$$(\alpha \cdot \theta)^G = \alpha \cdot \theta^G,$$

*where $(\alpha \cdot \theta^G)(x \cdot y) = \alpha(x) \cdot \theta^G(x \cdot y)$, whenever $x \in H$ and $y \in M$.*

*Proof.* Using the isomorphism $H \cong HS/S$, we regard $\alpha$ as a character of $HS$, defined as $\alpha(x \cdot y) = \alpha(x)$, for all $x \in H$ and $y \in S$. It is obvious that the product $\alpha \cdot \theta$ is a character of $HS$.

Furthermore, as $H \cong HM/M = G/M$, we can define an irreducible character $\alpha' \in \mathrm{Irr}(G)$ as

$$\alpha'(x \cdot m) = \alpha(x), \tag{2.22}$$

for all $x \in H$ and $m \in M$. Thus the restriction $\alpha'|_{HS}$ of $\alpha'$ to $HS$ is $\alpha \in \mathrm{Irr}(HS)$, i.e.,

$$\alpha'|_{HS} = \alpha.$$

Therefore

$$\begin{aligned}(\alpha \cdot \theta)^G &= (\alpha'|_{HS} \cdot \theta)^G \\ &= \alpha' \cdot \theta^G && \text{by Lemma 2.20} \\ &= \alpha \cdot \theta^G && \text{by (2.22).}\end{aligned}$$

This completes the proof of the corollary. ∎

# Chapter 3

# A Key Theorem

One of the main ideas of this thesis is the way an irreducible character of a finite group $G$ may correspond to an irreducible character of a subgroup of $G$. The most common such example is the Clifford correspondence. Other interesting and fruitful examples are the Glauberman correspondence and the Isaacs correspondence, that coincide when applied to groups we are dealing with in this thesis, groups of odd order. According to these correspondences, whenever an odd group $A$ acts on an odd group $G$, with $(|A|, |G|) = 1$, there is 'natural' bijection between the set $\mathrm{Irr}^A(G)$ of $A$-invariant irreducible characters of $G$, and the set $\mathrm{Irr}(C(A \text{ in } G))$ of irreducible characters of the fixed points $C(A \text{ in } G)$ of $A$ in $G$.

The Glauberman-Isaacs correspondence can be easily generalized to involve a normal series of subgroups of $G$ and not only $G$. That is, if $G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$ is a normal series of $G$, and $A$ acts coprimely on $G_i$, for all $i = 1, \ldots, n$, (both $A$ and $G$ are assumed of odd order), then we still have a bijection between the set of towers $\{\chi_i\}_{i=1}^n$ of $A$-invariant characters for the series of $G_i$, and the set of towers of irreducible characters of the normal series $C(A \text{ in } G_1) \trianglelefteq C(A \text{ in } G_2) \trianglelefteq \cdots \trianglelefteq C(A \text{ in } G_n)$ of $C(A \text{ in } G)$.

However, whether we use the Glauberman-Isaacs correspondence on a single group $G$ or on a normal series of $G$, the condition that wants the acting group $A$ to normalize every group $G_i$ involved in the series can't be avoided. The solution to this problem is given by E.C.Dade in [5]. Here we only state the results needed from that paper. The easy case, where the series is replaced by a single group, is done in Theorem 3.1, while the general case is described in Theorem 3.13.

Using the notation introduced in Section 1.2, we write $\mathrm{Irr}^A(G)$ for the $A$-invariant characters of $G$, whenever $A$ acts on $G$. In addition, if $A$ acts on a subgroup $B$ of $G$, we write $\mathrm{Irr}^A_B(G)$ for the irreducible characters of $G$ that lie above at least one $A$-invariant character of $B$. Furthermore, if $M$ is a subgroup of $G$ and $\mu \in \mathrm{Irr}(M)$ we denote by $\mathrm{Irr}(G|\mu)$ the set of irreducible characters of $G$ that lie above $\mu$, and by $\mathrm{Irr}^A_B(G|\mu)$ the intersection

$$\mathrm{Irr}^A_B(G|\mu) := \mathrm{Irr}^A_B(G) \cap \mathrm{Irr}(G|\mu).$$

If $\chi \in \mathrm{Irr}(G|\mu)$, we write $m(\mu \text{ in } \chi)$ for the multiplicity that $\mu$ appears as a constituent of the restriction $\chi|_M$ of $\chi$ in $m$, i.e., $m(\mu \text{ in } \chi) = [\chi|_M, \mu]$.

**Theorem 3.1.** *Assume that $G$ is a finite group of odd order, and that $B$ is a normal subgroup of $G$. Let $A, H$ be subgroups of $G$ such that $(|A|, |B|) = 1$, while $B$ is contained in $H$. Assume further that the subgroup $AB = A \ltimes B$ is normal in $G$. Then there is a one to one correspondence*

$$\psi \in \mathrm{Irr}^A_B(H) \leftrightarrow \psi_{(A)} \in \mathrm{Irr}(N(A \text{ in } H)),$$

*between the set* $\mathrm{Irr}_B^A(H)$ *of all characters* $\psi \in \mathrm{Irr}(H)$ *such that* $\psi$ *lies above at least one irreducible* $A$*-invariant character of* $B$*, and the set* $\mathrm{Irr}(N(A \text{ in } H))$ *of all irreducible characters of the normalizer* $N(A \text{ in } H)$ *of* $A$ *in* $H$*. We call the correspondence* $\psi \leftrightarrow \psi_{(A)}$ *an* $A$*-correspondence, and say that the characters* $\psi$ *and* $\psi_{(A)}$ *are* $A$*-correspondents of each other.*

*If* $H = B$*, then the above* $A$*-correspondence coincides with the Glauberman-Isaacs correspondence between* $\mathrm{Irr}^A(B)$ *and* $\mathrm{Irr}(C(A \text{ in } B))$*.*

*Furthermore, for any subgroup* $K$ *of* $G$ *that normalizes both* $A$ *and* $H$*, the stabilizer,* $K(\psi)$*, of any* $\psi \in \mathrm{Irr}_B^A(H)$ *in* $K$ *equals the corresponding stabilizer,* $K(\psi_{(A)})$*, of* $\psi_{(A)}$ *in* $K$*.*

*Proof.* The $A$-correspondence is done in Theorem 17.4 in [5], with $A, B, H$ here in the place of $K, L, H$ there. (Observe that, in that theorem, the $A$-correspondence is described in a more general setting.)

Theorem 17.29 in [5] implies that $A$-correspondence and Glauberman correspondence coincide when $H = B$.

The last part of the theorem follows easily from Proposition 17.10 in [5]. $\square$

Before we continue to the general case, we note that the group $H$ in Theorem 3.1 doesn't need to be normal subgroup of $G$. Furthermore, the $A$-correspondence described in the above theorem depends only on $A$ and $H$ and not on the choice of $B$ (see Proposition 17.13 in [5]).

Theorem 17.4 in [5] not only provides the character correspondence we describe in Theorem 3.1, but also gives a specific algorithm we can use to obtain this correspondence. It tells us that

**Theorem 3.2.** *Assume that* $A, B, H$ *and* $G$ *satisfy the hypotheses of Theorem 3.1. Assume further that* $\psi$ *is an irreducible character of* $H$ *that lies above at least one irreducible* $A$*-invariant character of* $B$*, i.e.,* $\psi \in \mathrm{Irr}_B^A(H)$*. Then there exists some sequence* $M_0, M_1, \dots, M_n$ *of subgroups of* $G$ *satisfying*

$$n \geq 1, \quad M_0 = B \geq M_1 \geq \cdots \geq M_n = 1, \tag{3.3a}$$

$$M_i \trianglelefteq H, \quad [M_i, A] \leq M_i, \text{ and} \tag{3.3b}$$

$$M_{i-1}/M_i \text{ is abelian,} \tag{3.3c}$$

*for all* $i = 1, 2, \dots, n$*. Any such sequence of subgroups determines a unique sequence of characters* $\theta_0, \theta_1, \dots, \theta_n$ *such that*

$$\theta_0 = \psi \in \mathrm{Irr}_B^A(H) = \mathrm{Irr}_{M_0}^A(N(AM_0 \text{ in } H)) \tag{3.4a}$$

*and*

$$\text{If } i = 1, 2, \dots, n, \text{ then } \theta_i \text{ is the unique character in } \mathrm{Irr}_{M_i}^A(N(AM_i \text{ in } H)) \text{ such that } m(\theta_i \text{ in } \theta_{i-1}) \text{ is odd.} \tag{3.4b}$$

*The character* $\theta_n \in \mathrm{Irr}_1^A(N(A \text{ in } H)) = \mathrm{Irr}(N(A \text{ in } H))$ *is independent of the choice of the sequence* $M_0, M_1, \dots, M_n$ *satisfying* (3.3)*. Furthermore,* $\theta_n$ *is the* $A$*-correspondent* $\psi_{(A)} \in \mathrm{Irr}(N(A \text{ in } H))$ *of* $\psi$*, as used in Theorem 3.1.*

It is clear from the above construction that the $A$-correspondence is preserved under epimorphic images. So we have

**Proposition 3.5.** *Assume that* $A, B, H$ *and* $G$ *satisfy the hypotheses of Theorem 3.1. Let* $\rho$ *be an epimorphism of* $G$ *onto some group* $G'$*. If* $A', B'$ *and* $H'$ *are the images under* $\rho$ *of* $A, B$ *and* $H$*,* *respectively, then $A', B', H'$ and $G'$ satisfy the hypotheses of Theorem 3.1. Furthermore, $\rho$ maps $N(A \text{ in } H)$ onto $N(A' \text{ in } H')$. Assume further that $\psi' \in \mathrm{Irr}(H')$ and $\psi = \psi' \circ \rho_H \in \mathrm{Irr}(H)$. Then $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$ if and only if $\psi \in \mathrm{Irr}_B^A(H)$. In that case $\psi_{(A)} = \psi'_{(A')} \circ \rho_{N(A \text{ in } H)} \in \mathrm{Irr}(N(A \text{ in } H))$.*

*Proof.* Clearly the groups $G', A', B'$ and $H'$ satisfy the hypotheses in Theorem 3.1.

Let $X$ be any subgroup of $G$ and $X' = \rho(X)$ be its image under $\rho$ in $G'$. Then $\rho$ restricts to an epimorphism $\rho_X$ of $X$ onto $X'$. This induces an injection $\phi' \to \phi' \circ \rho_X$ of $\mathrm{Irr}(X')$ into $\mathrm{Irr}(X)$. For any $\psi' \in \mathrm{Irr}(H')$ the above injection determines the irreducible character $\psi \in \mathrm{Irr}(H)$ satisfying $\psi = \psi' \circ \rho_H$. We remark here that any $\psi \in \mathrm{Irr}(H)$ with $\mathrm{Ker}(\rho_H) \leq \mathrm{Ker}(\psi)$ corresponds, under the above injection, to a character $\psi' \in \mathrm{Irr}(H')$, satisfying $\psi = \psi' \circ \rho_H$.

Evidently if $\psi'$ lies over a character $\phi' \in \mathrm{Irr}(B')$ then $\psi$ lies over the corresponding character $\phi = \phi' \circ \rho_B \in \mathrm{Irr}(B)$. In addition, if $A' = \rho(A)$ fixes $\phi'$, then $A$ fixes $\phi$. Hence if $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$ then $\psi \in \mathrm{Irr}_B^A(H)$. Conversely assume that $\psi \in \mathrm{Irr}_B^A(H)$ satisfies $\mathrm{Ker}(\rho_H) \leq \mathrm{Ker}(\psi)$. Then $\psi$ has a corresponding character $\psi' \in \mathrm{Irr}(H')$ satisfying $\psi = \psi' \circ \rho_H$. Because $\psi \in \mathrm{Irr}_B^A(H)$, there exists an irreducible $A$-invariant character $\phi \in \mathrm{Irr}^A(B)$ of $B$ that lies under $\psi$. Then $\mathrm{Ker}(\phi) \geq \mathrm{Ker}(\psi) \cap B \geq \mathrm{Ker}\,\rho \cap B$.. Hence $\phi$ has a corresponding character $\phi' \in \mathrm{Irr}(B')$ satisfying $\phi = \phi' \circ \rho_B$. Furthermore, $\psi'$ lies above $\phi'$, because $\psi$ lies above $\phi$. In addition, $A'$ fixes $\phi'$ because $A$ fixes $\phi$. Hence $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$. In conclusion, $\psi' \in \mathrm{Irr}_{B'}^{A'}(H)$ if and only if $\psi \in \mathrm{Irr}_B^A(H)$.

If $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$, then Theorem 3.1 applies. So $\psi'$ has an $A'$-correspondent irreducible character $\psi'_{(A')}$ in $N(A' \text{ in } H')$. To complete the proof of the proposition it suffices to show that $\rho(N(A \text{ in } H)) = N(A' \text{ in } H')$ while $\psi_{(A)} = \psi'_{(A')} \circ \rho_{N(A \text{ in } H)}$.

Let $M$ be any subgroup of $B$ such that $M \trianglelefteq H$ and $[M, A] \leq M$. Then $M' = \rho(M)$ is a subgroup of $B'$ such that $M' \trianglelefteq H'$ and $[M', A'] \leq M'$. We claim that

$$\rho(N(AM \text{ in } H)) = N(A'M' \text{ in } H'). \tag{3.6}$$

Clearly $\rho(N(AM \text{ in } H)) \leq N(A'M' \text{ in } H')$. Thus to prove (3.6) it is enough to show that any $t' \in N(A'M' \text{ in } H')$ is in the image of $N(AM \text{ in } H)$. Since $t' \in H' = \rho(H)$ there exists a $t \in H$ with $\rho(t) = t'$. If $K = \mathrm{Ker}(\rho)$, then $t$ normalizes $AMK$, since $t'$ normalizes $A'M' = \rho(AM)$. In addition, $t$ normalizes $AB \trianglelefteq G$. Hence $t$ normalizes the intersection $AB \cap AMK = A(B \cap AMK)$. The group $A$ normalizes $B \cap AMK$, since it normalizes $B, M$ and $K$. In addition, $(|A|, |B \cap AMK|) = 1$, because $(|A|, |B|) = 1$. Hence $AB \cap AMK = A(B \cap AMK) = A \ltimes (B \cap AMK)$, and the $B \cap AMK$-conjugates of $A$ are the only subgroups of order $|A|$ in $A(B \cap AMK)$. It follows that there is some element $s \in B \cap AMK$ such that $A^{ts} = A$. But $ts \in H$ normalizes $M \trianglelefteq H$. Thus $(AM)^{ts} = AM$. So $ts \in N(AM \text{ in } H)$ has image $\rho(ts) = t'\rho(s) \in N(A'M' \text{ in } H')$. In addition, the image $\rho(s)$ of $s \in AMK$ is an element of $\rho(AMK) = A'M' = \rho(AM)$ and thus lies in $\rho(N(AM \text{ in } H))$. We conclude that $t' = \rho(ts)\rho(s)^{-1}$ lies in $\rho(N(AM \text{ in } H))$. Therefore (3.6) holds.

Observe that (3.6) for $M = 1$, implies that $\rho(N(A \text{ in } H)) = N(A' \text{ in } H')$. Furthermore, if $M_0, M_1, \ldots, M_n$ are any subgroups of $G$ satisfying (3.3), their images $M'_i = \rho(M_i)$, for $i = 0, 1, \ldots, n$, satisfy the equivalent of (3.3) for $G', H', B'$ and $A'$. According to Theorem 3.2, the character $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$ determines characters $\theta'_0, \ldots, \theta'_n$ such that $\theta'_0 = \psi'$ and $\theta'_i$ , for any $i = 1, \ldots, n$, is the unique character in $\mathrm{Irr}_{M'_i}^{A'}(N(A'M'_i \text{ in } H'))$ such that $m(\theta'_i \text{ in } \theta'_{i-1})$ is odd. Since $\rho$ sends $N(AM_i \text{ in } H)$ onto $N(A'M'_i \text{ in } H')$, by (3.6), it follows that $\theta_i = \theta'_i \circ \rho_{N(AM_i \text{ in } H)}$ lies in $\mathrm{Irr}_{M_i}^A(N(AM_i \text{ in } H))$, for each $i = 0, 1, \ldots, n$. Furthermore, $\theta_0 = \psi' \circ \rho_H = \psi$ and $m(\theta_i \text{ in } \theta_{i-1}) = m(\theta'_i \text{ in } \theta'_{i-1})$ is odd, for each $i = 1, \ldots, n$. Theorem 3.2 then implies

$$\psi_{(A)} = \theta_n = \theta'_n \circ \rho_{N(AM_n \text{ in } H)} = \psi'_{(A')} \circ \rho_{N(A \text{ in } H)}.$$

Hence Proposition 3.5 holds. $\square$

Proposition (3.5) easily implies

**Corollary 3.7.** *Assume that $A, B, H$ and $G$ satisfy the hypotheses of Theorem 3.1. Let $\rho$ be an isomorphism of $G$ onto some group $G'$. If $A', B'$ and $H'$ are the isomorphic images, under $\rho$, of $A, B$ and $H$ respectively, then $A', B', H'$ and $G'$ satisfy the hypotheses of Theorem 3.1. Furthermore, $N(A' \text{ in } H')$ is the isomorphic image under $\rho$ of $N(A \text{ in } H)$. In addition, for any $\psi \in \mathrm{Irr}(H)$ there exists a $\psi' \in \mathrm{Irr}(H')$ such that $\psi = \psi' \circ \rho_H$. Then $\psi' \in \mathrm{Irr}_{B'}^{A'}(H')$ if and only if $\psi \in \mathrm{Irr}_B^A(H)$. In that case $\psi_{(A)} = \psi'_{(A')} \circ \rho_{N(A \text{ in } H)}$.*

Proposition 17.12 in [5] implies

**Proposition 3.8.** *If the group $A$ in Theorem 3.1 centralizes $B$, then $\mathrm{Irr}_B^A(H) = \mathrm{Irr}(H) = \mathrm{Irr}(N(A \text{ in } H))$ and the $A$-correspondence is the identity map of these equal sets onto themselves.*

From Proposition 17.14 in [5] we obtain

**Proposition 3.9.** *Let $A, B, H, G$ be as in Theorem 3.1. Let $A'$ be a subgroup of $A$ such that $N(A \text{ in } H) = N(A' \text{ in } H)$, and thus $N(A \text{ in } B) = N(A' \text{ in } B)$. Then $\mathrm{Irr}_B^A(H) = \mathrm{Irr}_B^{A'}(H)$ and*

$$\psi_{(A)} = \psi_{(A')},$$

*for any $\psi \in \mathrm{Irr}_B^A(H) = \mathrm{Irr}_B^{A'}(H)$.*

In the special case where the $A$-correspondence is the Glauberman correspondence (that is the case $H = B$), Proposition 3.9 translates to

**Corollary 3.10.** *Assume that $A$ acts coprimely on $B$, where $A$ and $B$ are both finite groups of odd order. If $A'$ is a subgroup of $A$ satisfying $C(A \text{ in } B) = C(A' \text{ in } B)$ then the $A$-Glauberman and the $A'$-Glauberman correspondences coincide.*

In the special case that $H = AB$, Theorem 17.36 in [5] describes clearly the $A$-correspondence of Theorem 3.1. So we get

**Theorem 3.11.** *Assume that $A, B, G$ satisfy the hypotheses of Theorem 3.1, with the additional condition that $H = AB$. Assume further that $\chi \in \mathrm{Irr}(H)$ is of the form $\chi = \alpha \cdot \beta^e$, where $\alpha \in \mathrm{Irr}(A)$ and $\beta^e$ is the canonical extension to $H$ of an irreducible $A$-invariant character $\beta \in \mathrm{Irr}^A(B)$. Then $\chi \in \mathrm{Irr}_B^A(H)$. Furthermore, $N(A \text{ in } H) = A \times C(A \text{ in } B)$, where $C(A \text{ in } B)$ is the centralizer of $A$ in $B$. In addition, the $A$-correspondent $\chi_{(A)} \in \mathrm{Irr}(N(A \text{ in } H))$ of $\chi$, is of the form*

$$\chi_{(A)} = \alpha \times \gamma,$$

*where $\gamma \in \mathrm{Irr}(C(A \text{ in } B))$ is the $A$-Glauberman correspondent of $\beta \in \mathrm{Irr}^A(B)$.*

*Proof.* See Theorem 17.36 in [5]. $\square$

The next proposition shows that the $A$-correspondence is compatible with Clifford correspondence.

**Proposition 3.12.** *Let $A, B, H$ and $G$ be as in Theorem 3.1 and let $M$ be an $A$-invariant normal subgroup of $G$ contained in $B$. Assume further that $\mu$ is an $A$-invariant irreducible character of $M$, and let $G(\mu), H(\mu)$ and $B(\mu)$ be the stabilizers of $\mu$ in $G, H$ and $B$ respectively. Let $\mu_{(A)} \in \mathrm{Irr}(N(A \text{ in } M))$ be the $A$-correspondent of $\mu$, (note this is the $A$-Glauberman correspondent of $\mu$). If $\psi \in \mathrm{Irr}_B^A(H)$ lies above $\mu$, that is, $\psi \in \mathrm{Irr}_B^A(H|\mu)$, then the $\mu$-Clifford correspondent $\psi_\mu \in \mathrm{Irr}(H(\mu))$ of $\psi$ lies in $\mathrm{Irr}_{B(\mu)}^A(H(\mu)|\mu)$. Furthermore, the $A$-correspondent $\psi_{\mu,(A)} \in \mathrm{Irr}(N(A \text{ in } H(\mu)))$ of $\psi_\mu$ is the $\mu_{(A)}$-Clifford correspondent of the $A$-correspondent $\psi_{(A)} \in \mathrm{Irr}(N(A \text{ in } H))$ of $\psi$.*

*Proof.* See Propositions 17.19 17.20, 17.22, 17.23 and Theorem 17.24 in [5]. □

We conclude with a generalization of Theorem 3.1.

**Theorem 3.13.** *Let $A, B, G$ be as in Theorem 3.1. Assume further that $G_0 = B \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n \trianglelefteq G$ is a series of normal subgroups of $G$. Then the groups $N(A \text{ in } G_0) = N(A \text{ in } B) \trianglelefteq N(A \text{ in } G_1) \trianglelefteq \cdots \trianglelefteq N(A \text{ in } G_n) \trianglelefteq N(A \text{ in } G)$, form a series of normal subgroups of $N(A \text{ in } G)$. Let $\psi_0, \psi_1, \ldots, \psi_n$ be a tower of irreducible characters for the chain $G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n$, while $\psi_0 \in \mathrm{Irr}^A(B)$. Then $\psi_i \in \mathrm{Irr}_B^A(G_i)$, for all $i = 1, 2, \ldots, n$. Let $\psi_{i,(A)} \in \mathrm{Irr}(N(A \text{ in } G_i))$ be the $A$-correspondent of $\psi_i \in \mathrm{Irr}_B^A(G_i)$, for all $i = 0, 1, \ldots, n$. Then the $\psi_{i,(A)}$ form a character tower for the chain $N(A \text{ in } G_0) = N(A \text{ in } B) \trianglelefteq N(A \text{ in } G_1) \trianglelefteq \cdots \trianglelefteq N(A \text{ in } G_n)$. This way we get a bijection between character towers $\{\psi_i\}_{i=0}^n$ for the series $\{G_i\}_{i=0}^n$ with $\psi_0 \in \mathrm{Irr}^A(B)$, and character towers $\{\psi_{i,(A)}\}_{i=0}^n$ for the series $\{N(A \text{ in } G_i)\}_{i=0}^n$. In addition, this correspondence respects restrictions and inductions, i.e.,*

*(a)* $\psi_i^{G_{i+1}} = \psi_{i+1}$ *if and only if* $\psi_{i,(A)}^{N(A \text{ in } G_{i+1})} = \psi_{i+1,(A)}$*, while*

*(b)* $\psi_{i+1}\big|_{G_i} = \psi_i$ *if and only if* $\psi_{i+1,(A)}\big|_{N(A \text{ in } G_i)} = \psi_{i,(A)}$*,*

*for any $i = 1, \ldots, n-1$.*

*Furthermore, for any subgroup $K$ of $G$ that normalizes $A$, the stabilizer, $K(\psi_i)$, of $\psi_i$ in $K$ equals the corresponding stabilizer, $K(\psi_{i,(A)})$, of $\psi_{i,(A)}$ in $K$, for all $i = 0, 1, \ldots, n$.*

*Proof.* See Theorem 17.15 and Propositions 17.16 and 17.17 in [5]. □

# Chapter 4

# Changing Characters

## 4.1 A "petite" change

Assume that $G, Q_1, Q_2$ and $P$ are finite groups that satisfy

**Hypothesis 4.1.** *$Q_1, Q_2$ are $q$-subgroups of $G$, for some odd prime $q$, with $Q_1 \trianglelefteq Q_2$. Furthermore, $P$ is a $p$-group, for some prime $p$ with $2 \neq p \neq q$, that normalizes $Q_1$, while $P \cdot Q_1$ is normalized by $Q_2$.*

Let $\beta_1$ be an irreducible character of $Q_1$. Our main goal in this section is to show how we can change the character, $\beta_1 \in \mathrm{Irr}(Q_1)$, to a new one, $\beta_1^\nu \in \mathrm{Irr}(Q_1)$, so that

(1) $P(\beta_1) = P(\beta_1^\nu)$ and $Q_2(\beta_1) \leq Q_2(\beta_1^\nu)$, while

(2) $\beta_1^\nu$ can be extended to $Q_2(\beta_1^\nu)$.

The following diagram describes that situation:

$$\begin{tikzcd}
& Q_2 \arrow[dl] \arrow[d, dash] & & \\
P \arrow[dr] & Q_2(\beta_1^\nu) \arrow[d] & & (\beta_1^\nu)^e \arrow[d, dash] \\
& Q_1 & \beta_1 \arrow[r, mapsto] & \beta_1^\nu
\end{tikzcd} \tag{4.2}$$

Most of the work towards that direction is done in

**Lemma 4.3.** *Let $Q$ be a $q$-group acting on a $p$-group $P$, with $p \neq q$ odd primes. Let $T$ be a finite-dimensional right $\mathbb{Z}_q(Q \ltimes P)$-module such that the action of $P$ on $T$ is faithful. Then there exists an element $\tau \in T$ such that its stabilizer $(QP)(\tau)$ in $Q \ltimes P$ equals $Q$.*

*Proof.* We will prove a series of claims under the

**Inductive Assumption 4.4.** *$Q, P, T$ are chosen among all the triplets satisfying the hypothesis, but not the conclusion, of Lemma 4.3, so as to minimize first the order $|QP|$ of the semidirect product $Q \ltimes P$, and then the $\mathbb{Z}_q$-dimension $\dim_{\mathbb{Z}_q} T$ of $T$.*

These claims will lead to a contradiction, thus proving the lemma.

**Claim 4.5.** *$T$ is an indecomposable $\mathbb{Z}_q(QP)$-module.*

*Proof.* Suppose not. Let $T = T_1 \dot{+} T_2$ be a direct decomposition of $T$, where $T_1, T_2$ are nontrivial $\mathbb{Z}_q(QP)$-submodules of $T$. For $i = 1, 2$ let $K_i$ be the kernel of the action of $P$ on $T_i$. Hence $T_i$ is a $\mathbb{Z}_q(Q \ltimes P/K_i)$-module such that $P/K_i$ acts faithfully on it. As $\dim_{\mathbb{Z}_q}(T_i)$ is strictly smaller than $\dim_{\mathbb{Z}_q} T$, the minimality in Inductive Assumption 4.4 provides an element $\tau_i \in T_i$ such that $(Q \ltimes P/K_i)(\tau_i) = Q$. If we take as $\tau$ the sum, $\tau = \tau_1 + \tau_2$, then $\tau$ is an element of $T$ fixed by $Q$, as $Q$ fixes each one of the $\tau_i$ for $i = 1, 2$. Furthermore for the stabilizer of $\tau$ in $P$ we have

$$P(\tau) = \cap_{i=1}^{2} P(\tau_i) = \cap_{i=1}^{2} K_i.$$

Since $P$ acts faithfully on $T$ the last intersection is trivial. Therefore $(QP)(\tau) = Q$, which contradicts Inductive Assumption 4.4. Hence $T$ is an indecomposable $\mathbb{Z}_q(QP)$-module. $\square$

**Claim 4.6.** *The restriction $T_P$ of $T$ to $P$ is a multiple of an irreducible $Q$-invariant $\mathbb{Z}_q(P)$-module.*

*Proof.* Claim 4.5 and Clifford's Theorem imply that $T_p$ can be written as a direct sum of its $\mathbb{Z}_q(P)$-homogeneous components, i.e.,

$$T_P = U_1 \dot{+} U_2 \dot{+} \cdots \dot{+} U_r,$$

where $U = U_1 \cong mV = mV^{\sigma_1}, U_2 \cong mV^{\sigma_2}, \ldots, U_r \cong mV^{\sigma_r}$. Here $V = V^{\sigma_1}, \ldots, V^{\sigma_r}$, are the distinct conjugates of a simple $\mathbb{Z}_q(P)$-submodule, $V$, of $T_p$, and $1 = \sigma_1, \ldots, \sigma_r$ are representatives for the cosets in $Q \cdot P$ of the stabilizer,$(QP)_V$, of the isomorphism class of $V$ in $QP$. We may pick $\sigma_1, \ldots, \sigma_r$ to be representatives of the cosets in $Q$ of the stabilizer, $Q_V$, of that isomorphism class in $Q$. Note that $Q_V = Q_U$ as $U \cong mV$, where $Q_U$ is the stabilizer in $Q$ of $U$ under multiplication in $T$. If $T_P$ is not homogeneous, then $r > 1$ and $Q_U = Q_V < Q$. For $i = 1, \ldots, r$ let $\widehat{K_i}$ be the kernel of the action of $P$ on $U_i$. Then for every $i = 1, \ldots, r$ the stabilizer $Q_{U_i}$ of $U_i$ in $Q$ equals the $\sigma_i$-conjugate, $Q_U^{\sigma_i}$ , of $Q_U = Q_V$. For the corresponding kernels we similarly have $\widehat{K_i} = \widehat{K_1}^{\sigma_i}$.

As $U$ is a faithful $\mathbb{Z}_q(P/\widehat{K_1})$-module and $Q_U \lneq Q$, the minimality in Inductive Assumption 4.4 implies that there exists an element $\mu \in U$ such that

$$(Q_U \ltimes (P/\widehat{K_1}))(\mu) = Q_U.$$

For every $i = 1, \ldots, r$ we can define an element $\mu_i = \mu\sigma_i$ of $U_i$. Then $Q_{U_i}$ fixes $\mu_i$ as $Q_U$ fixes $\mu$. Furthermore if $x$ is any element of $P$ fixing $\mu_i$ then $x^{\sigma_i^{-1}}$ is an element of $P$ fixing $\mu$. Therefore $x^{\sigma_i^{-1}} \in \widehat{K_1}$, which implies that $x \in \widehat{K_i}$. Thus

$$(Q_{U_i} \ltimes (P/\widehat{K_i}))(\mu_i) = Q_{U_i}$$

for every $i = 1, \ldots, r$.

Let $\tau$ be the sum of the $\mu_i$ for $i = 1, \ldots, r$. Then $\tau$ is an element of $T$ fixed by $Q$, since multiplication by any element in $Q$ permutes the $U_i$ and the $\mu_i$ among themselves. The stabilizer $P(\tau)$ of $\tau$ in $P$ equals the intersection of the stabilizers of $\mu_i$ in $P$ for $i = 1, \ldots, r$. Since $(Q_{U_i} \ltimes (P/\widehat{K_i}))(\mu_i) = Q_{U_i}$ for every such $i$, the latter equals the intersection of $\widehat{K_i}$ for $i = 1, \ldots, r$. The faithful action of $P$ on $T$ implies that

$$P(\tau) = \cap_{i=1}^{r} \widehat{K_i} = 1.$$

Hence $T$ has an element $\tau$ with $(QP)(\tau) = Q$, contradicting Inductive Assumption 4.4. This contradiction proves Claim 4.6. $\square$

**Claim 4.7.** *There are no $Q$-invariant subgroup, $H < P$, and $\mathbb{Z}_q(QH)$-submodule, $S$, of $T_{QH}$ such that $T$ is the $\mathbb{Z}_q(QP)$-module $S^{QP}$ induced from $S$, i.e.,*

$$T = \dot{\sum_{1 \le i \le n}} S\sigma_i,$$

*where the $\sigma_i$ are representatives for the cosets $H\sigma_i$ of $H$ in $P$.*

*Proof.* Suppose Claim 4.7 is false. We choose $H$ to have maximal order among all those $Q$-invariant subgroups of $P$ that contradict Claim 4.7. Hence $T_{QH}$ has a $\mathbb{Z}_q(QH)$-submodule, $S$, such that $S^{QP} = T$. If $H$ is not normal in $P$ then its normalizer, $N_P(H)$, in $P$ satisfies $H \lhd N_P(H) < P$. Since $H$ is $Q$-invariant, $N_P(H)$ is also $Q$-invariant. Hence $S^{QN_P(H)}$ is a $\mathbb{Z}_q(QN_P(H))$-submodule of $T_{QN_P(H)}$. Furthermore $S^{QN_P(H)}$ induces $T$. Thus $N_P(H)$ is among the $Q$-invariant subgroups of $P$ that contradict Claim 4.7 with $|N_P(H)| > |H|$. So the maximality of $|H|$ implies that $H$ is normal in $P$.

Let $1 = r_1, \ldots, r_k$ be coset representatives of $H$ in $P$, and let $\bar{r}_m$ denote the image of $r_m$ in $P/H$ for $m = 1, \ldots, k$. Then $\bar{1} = \bar{r_1}, \bar{r_2}, \ldots, \bar{r_k}$ are the distinct elements of $P/H$. As $Q$ acts on $P/H$, it has to divide the $\bar{r}_m$, for $m = 1, \ldots, k$, into orbits, $\overline{R_1}, \overline{R_2}, \ldots, \overline{R_l}$, for some $l \in \{1, \ldots, k\}$. We may choose $\overline{R_1}$ to be equal to $\{\bar{r}_1\} = \{1\}$. For every $i = 2, \ldots, l$, we pick some element $\bar{r}_{i,1} \in \overline{R_i}$. Then $\overline{R_i} = \{\bar{r}_{i,1}^{q_j}\}_{j=1}^{j=k_i}$ where $k_i = |\overline{R_i}|$ and $q_j$ runs over a set $Q_j$ of coset representatives of the stabilizer, $C_Q(\bar{r}_{i,1})$, in $Q$. For every $i = 2, \ldots, l$ the stabilizer $C_Q(\bar{r}_{i,1})$ acts by conjugation on $H$ and on $r_{i,1}H$, where $r_{i,1} \in P$ has image $\bar{r}_{i,1} \in P/H$. Furthermore, $H$ acts transitively by right multiplication on $r_{i,1}H$ and $(xh)^c = x^c h^c$ for all $x \in r_{i,1}H, h \in H, c \in C_Q(\bar{r}_{i,1})$. Hence Glauberman's Lemma (13.8 in [12]) provides an element $t_{i,1} \in r_{i,1}H$ that is fixed by $C_Q(\bar{r}_{i,1})$. So $C_Q(t_{i,1}) \ge C_Q(\bar{r}_{i,1})$. Furthermore, the opposite inclusion, $C_Q(t_{i,1}) \le C_Q(\bar{r}_{i,1})$, also holds as $\bar{r}_{i,1} = t_{i,1}H$. Hence,

$$C_Q(t_{i,1}) = C_Q(\bar{r}_{i,1}).$$

In this way we can pick a $t_{i,1} \in r_{i,1}H$, for every $i = 1, \ldots, l$, such that $C_Q(t_{i,1}) = C_Q(\bar{r}_{i,1})$. We can even assume that $t_{1,1} = 1$. Let $t_{i,j}$ denote the $q_j$-conjugate, $t_{i,1}^{q_j}$, of $t_{i,1}$ for every $j = 1, \ldots, k_i$. Hence the set of all $t_{i,j}$, for $i = 1, \ldots l$ and for $j = 1, \ldots, k_i$, is a complete set of coset representatives of $H$ in $P$. Furthermore the $Q$-orbit $\overline{R_i}$ corresponds to a $Q$-orbit $R_i = \{t_{i,1}, \ldots, t_{i_{k,i}}\}$, for every $i = 1, \ldots, l$.

Let $K_S$ be the kernel of the action of $H$ on $S$. As $|H/K_S| < |P|$, the minimality in Inductive Assumption 4.4 implies that there exists $\mu \in S$ such that its stabilizer, $(Q \ltimes H/K_S)(\mu)$, in $Q \ltimes H/K_S$ equals $Q$, or equivalently $(QH)(\mu) = QK_S$. We note here that $K_S < H$. Indeed, if $H$ acts trivially on $S$, then $T$ is induced from a trivial module and thus contains both trivial and non–trivial irreducible $P$-submodules, contradicting Claim 4.5. We also have that $\mu \neq 0$ since $Q = (Q \ltimes H/K_S)(\mu) < QH/K_S$. We denote by $\mu t_{i,j}$ the $t_{i,j}$-translation of $\mu$, for every $i = 1, \ldots, l$ and for every $j = 1, \ldots, k_i$. Then $\mu t_{i,j}$ is an element of $St_{i,j}$ such that

$$(Q^{t_{i,j}}H)(\mu t_{i,j}) = Q^{t_{i,j}} K_S^{t_{i,j}}$$

Since $S^{QP} = T$ we get that

$$T = S^{QP} = \dot{\sum_{1 \le i \le l}} \dot{\sum_{1 \le j \le k_i}} St_{i,j} = S \dotplus \dot{\sum_{2 \le i \le l}} \dot{\sum_{1 \le j \le k_i}} St_{i,j}. \tag{4.8a}$$

Let $\tau$ be the element of $T$ defined by

$$\tau = -\mu + \sum_{i=2}^{l}\sum_{j=1}^{k_i} \mu t_{i,j} = -\mu + \sum_{i=2}^{l}\sum_{t_{i,j}\in R_i} \mu t_{i,j}. \tag{4.8b}$$

We claim that $\tau$ satisfies the condition in Lemma 4.3, i.e., that $(QP)(\tau) = Q$. This will contradict Inductive Assumption 4.4, and thus prove Claim 4.7. Indeed, $R_i = \{t_{i,1}, \dots, t_{i,k_i}\}$ is a $Q$-orbit for every $i = 2, \dots, l$. Also $\mu$ and $-\mu$ are $Q$-invariant as $(QH)(-\mu) = (QH)(\mu) = QK_S$. Hence $\sum_{t_{i,j}\in R_i} \mu t_{i,j}$ is $Q$-invariant. Thus $\tau$ is a $Q$-invariant element of $T$.

If $x \in H(\tau)$ then, since $H \lhd P$, we get that $(\mu t_{i,j})x = \mu x^{(t_{i,j})^{-1}} t_{i,j}$ is an element of $St_{i,j}$, for all $i = 2, \dots, l$ and $j = 1, \dots, k_i$, while $(-\mu)x$ is an element of $S$. Since $\tau x = \tau$, it follows from (4.8) that $(-\mu)x = -\mu$ and $(\mu t_{i,j})x = \mu t_{i,j}$ for every $i = 2, \dots, l$ and for every $j = 1, \dots, k_i$. Hence $x$ is an element of:

$$H(\mu) \cap \bigcap_{i=2}^{l}\bigcap_{j=1}^{k_i}(P(\mu t_{i,j}) \cap H) = \bigcap_{i=1}^{l}\bigcap_{j=1}^{k_i} H(\mu t_{i,j}) = \bigcap_{i=1}^{l}\bigcap_{j=1}^{k_i} K_S^{t_{i,j}}.$$

As $H$ acts faithfully on $T$, we get that $\bigcap_{i=1}^{l}\bigcap_{j=1}^{k_i} K_S^{t_{i,j}} = 1$. Hence $H(\tau) = 1$.

Now let $x \in P\backslash H$. We claim that $\tau x \neq \tau$. Indeed any $x \in P$ permutes the $St_{i,j}$ among themselves. If $x$ fixes $\tau$, then it also permutes among themselves the summands $-\mu$ and $\mu t_{i,j}$, for $i \neq 1$, of $\tau$. Since $Sx \neq S$ we have $(-\mu)x = \mu t_{i,j}$ for some $i = 2, \dots, l$ and some $j = 1, \dots, k_i$. But as $x \in P\backslash H$ we have that $x = ht$ for some coset representative $t = t_{i_0,j_0}$ of $H$ in $P$ with $i_0 = 2, \dots, l$ and some element $h \in H$. Hence $\mu t_{i,j} = (-\mu)x = (-\mu)ht \in St$, which implies that $t_{i,j} = t$ and $(-\mu)h = \mu$. This last equation leads to a contradiction as $h$ has odd order ($|P|$ is odd) and $\mu \neq -\mu$ ( as $S \leq T$ has odd order). Therefore $\tau^x \neq \tau$ whenever $x \in P\backslash H$. Hence $P(\tau) = H(\tau) = 1$ and $(QP)(\tau) = Q$, contradicting Inductive Assumption 4.4. This contradiction proves Claim 4.7. □

**Claim 4.9.** *Every normal abelian subgroup $A$ of $QP$ contained in $P$ is cyclic.*

*Proof.* Let $A$ be a normal subgroup of $QP$ contained in $P$, and let $T_A$ be the restriction of $T_P$ to $A$. According to Claim 4.6, and Clifford's Theorem (11.1 in CR), we have that $T_A$ can be written as a direct sum of its $\mathbb{Z}_q(A)$-homogeneous components, i.e.,

$$T_A = W_1 \dotplus W_2 \dotplus \cdots \dotplus W_s.$$

Furthermore, $P$ acts transitively on the $W_i$ for all $i = 1, \dots, s$, while $Q$ permute the $W_i$ among themselves (as $T$ is a $\mathbb{Z}_q(QP)$-module). Hence Glauberman's lemma implies that $Q$ fixes some $\mathbb{Z}_q(A)$-homogeneous component, $W$, of $T_A$. Thus $W$ is a $\mathbb{Z}_q(QA)$-submodule of $T_{QA}$. Even more, Clifford's Theorem implies that $W^{QP} = T$. This, along with Claim 4.7, implies that $W = T$. Hence $T_A \cong eV$ where $V$ is an irreducible $P$-invariant $\mathbb{Z}_q(A)$-submodule of $T$. As $P$ acts faithfully on $T$, the $\mathbb{Z}_q(A)$-module $V$ is also faithful. If $A$ is abelian, the existence of a faithful irreducible $\mathbb{Z}_q(A)$-module implies that $A$ is cyclic. Therefore, the claim is proved. □

The $q$-group $Q$ acts on the $q$-group $T$, fixing the trivial element 0 of $T$. Hence the group $Q$ fixes at least $q$ elements of $T$. So $Q$ fixes some $\tau$ with

$$\tau \in T \text{ and } \tau \neq 0. \tag{4.10}$$

Hence, to complete the proof of Lemma 4.3, by contradicting the Inductive Assumption 4.4, is enough to show that $P(\tau) = 1$

By Claim 4.9 every characteristic abelian subgroup of $P$ is cyclic. Since $p$ is odd, Theorem 4.9 in section 5.4 of [9] implies that either $P$ is cyclic or $P$ is the central product $E \odot C$, of an extra–special $p$-group $E$ of exponent $p$ and a cyclic group $C$.

If $P$ is cyclic then $Z(P) = P$. According to Claim 4.6, the $\mathbb{Z}_q(P)$-module $T_P$ is a multiple of an irreducible $Q$-invariant $\mathbb{Z}_q(P)$-module $V$, i.e., $T_P = mV$. Hence $Z(P)$ acts fix point free on $T$ as it acts fix point free on $V$ (or else $V$ wouldn't be simple). This implies that no element of $P = Z(P) - \{1\}$ could fix $\tau$. Hence $P(\tau) = 1$. So $(QP)(\tau) = Q$, contradicting Inductive Assumption 4.4. Therefore, $P$ can't be cyclic.

Hence, $P = E \odot C$, where $E = \Omega_1(P)$ is an extra special of exponent $p$ and $C = Z(P)$ is cyclic. So,

$$P = E \odot C = \Omega_1(P) \odot Z(P). \tag{4.11}$$

Therefore the factor group $\overline{P} = P/Z(P)$ is an elementary abelian $p$-group. Furthermore it affords a bilinear form $c : \overline{P} \times \overline{P} \to Z(E)$ defined, for every $\bar{x}, \bar{y} \in \overline{P}$, as $c(\bar{x}, \bar{y}) = [x, y]$, where $x, y$ are any elements of $P$ having images $\bar{x}, \bar{y}$ respectively, in $\overline{P}$. With respect to that form $\overline{P}$ is a symplectic $\mathbb{Z}_p(Q)$-module.

**Claim 4.12.** *The symplectic $\mathbb{Z}_p(Q)$-module $\overline{P}$ is anisotropic.*

*Proof.* Assume not. Then there is an isotropic non–zero $\mathbb{Z}_p(Q)$-submodule $\bar{A}$ of $\overline{P}$. Hence $c(\bar{a}, \bar{b}) = 0$ for every $\bar{a}, \bar{b} \in \bar{A}$, as $\bar{A} \subset \bar{A}^{\perp}$. Therefore, by the definition of the symplectic form $c$, we get that the inverse image $A$ of $\bar{A}$ in $P$ is an abelian subgroup of $P$ containing $Z(P)$. Since $\bar{A}$ is a $\mathbb{Z}_p(Q)$-submodule of $\overline{P}$, the abelian group $A$ is a normal subgroup of $QP$ contained in $P$. Hence by Claim 4.9 , $A$ is cyclic and properly contains the $Z(P)$. Therefore there exists an element $a \in A \setminus Z(P)$ such that $a^p$ is a generator of $Z(P)$. On the other hand according to (4.11) $a = \omega \cdot c$ where $\omega \in \Omega_1(P)$ and $c \in C = Z(P)$. Hence $a^p = \omega^p \cdot c^p = c^p$. Since $a^p$ is a generator of the cyclic $p$-group $Z(P)$ and $c \in Z(P)$, this last equation leads to a contradiction. This proves the claim. ☐

Now we can complete the proof of Lemma 4.3. If $(QP)(\tau) \neq Q$ then there exists a $Q$-invariant subgroup $D = P(\tau) \neq 1$ of $P$ such that $(QP)(\tau) = QD$. Hence the center $Z(D)$ of $D$ is a non–trivial $Q$-invariant abelian subgroup of $P$. Therefore its image $\overline{Z(D)} = Z(D)Z(P)/Z(P)$ in $\overline{P}$ is an isotropic $\mathbb{Z}_p(Q)$-submodule of $\overline{P}$. Since $\overline{P}$ is anisotropic, $\overline{Z(D)} = \bar{1}$, i.e., $Z(D)$ is contained in $Z(P)$.

As we saw in the first case, $Z(P)$ acts fix point free on $T$. This implies that no element of $Z(P) - \{1\}$ could fix $\tau$. Hence $Z(D) = 1$, contradicting the fact that $Z(P) \neq 1$. So $(QP)(\tau) = Q$, contradicting Inductive Assumption 4.4. This final contradiction completes the proof of Lemma 4.3. ☐

In terms of characters, Lemma 4.3 implies

**Corollary 4.13.** *Let $Q$ be a $q$-group acting on a $p$-group $P$ with $p \neq q$ odd primes. Suppose that the semi–direct product $Q \ltimes P$ acts on a $q$-group $Q_1$ such that the action of $P$ on $Q_1$ is faithful. Then there exists a linear character $\lambda$ of $Q_1$ whose kernel* $\mathrm{Ker}(\lambda)$ *contains the Frattini subgroup $\Phi(Q_1)$ and whose stabilizer $(QP)(\lambda)$ in $Q \ltimes P$ is $Q$.*

*Proof.* Let $T$ be the factor group $T := Q_1/\Phi(Q_1)$. Then $T$ is a $\mathbb{Z}_q(QP)$-module. We write $T^*$ for its dual $\mathbb{Z}_q(QP)$-module, i.e., $T^* = \mathrm{Hom}_{\mathbb{Z}_q}(T, \mathbb{Z}_q)$. Then $P$ acts faithfully on both, $T$ and $T^*$. Furthermore, according to Lemma 4.3 there is an element $\tau \in T^*$ whose stabilizer in $QP$ equals $Q$.

Since the linear characters of $T$ can be considered as the elements of $T^*$ composed with some faithful linear character of $\mathbb{Z}_q$, we conclude that there is a linear character $\lambda^* \in \mathrm{Lin}(T)$ whose stabilizer in $QP$ is $Q$. Let $\lambda$ be the linear character of $Q_1$ to which $\lambda^*$ inflates. Then $\Phi(Q_1) \leq \mathrm{Ker}(\lambda)$. Furthermore, $(QP)(\lambda) = (QP)(\lambda^*) = Q$, and the corollary follows. $\square$

The following straightforward lemma is necessary for the rest of the chapter, and gives a stronger result than Corollary 4.13.

**Lemma 4.14.** *Let $P$ be a $p$-subgroup of a finite group $G$ and let $Q_1 \trianglelefteq Q_2$ be $q$-subgroups of $G$, for some distinct odd primes $p$ and $q$. If $P$ normalizes $Q_1$, and $Q_2$ normalizes their product $Q_1P$, then $Q_2P$ is also a subgroup of $G$ with $Q_2 \in \mathrm{Syl}_q(Q_2P), P \in \mathrm{Syl}_p(Q_2P)$ and $Q_1P \trianglelefteq Q_2P$. Furthermore, $Q_2$ is the product $Q_2 = [Q_1, P]N(P \text{ in } Q_2)$, where $[Q_1, P] \trianglelefteq Q_2P$ and $[Q_1, P] \cap N(P \text{ in } Q_2) = C(P \text{ in } [Q_1, P]) \leq \Phi([Q_1, P])$.*

*Proof.* That the product, $Q_2P = Q_2(Q_1P)$, is a subgroup of $G$ is clear as $Q_2$ normalizes the semidirect product $Q_1 \rtimes P$. That same product $Q_1P$ is a normal subgroup of $Q_2P = Q_2(Q_1P)$. We obviously have that $Q_2 \in \mathrm{Syl}_q(Q_2P)$ and $P \in \mathrm{Syl}_p(Q_2P)$.

By Frattini's argument for the Sylow $p$-subgroup $P$ of $Q_1P \trianglelefteq Q_2P$ we get

$$Q_2P = Q_1PN(P \text{ in } Q_2P). \tag{4.15a}$$

The normalizer, $N(P \text{ in } Q_2P)$, of $P$ in $Q_2P$ contains $P$. So it is equal to $PN(P \text{ in } Q_2)$. Hence (4.15a) can be written as $Q_2P = Q_1N(P \text{ in } Q_2)P$. Since $Q_1N(P \text{ in } Q_2) \leq Q_2$ and $Q_2 \cap P = 1$, we conclude that

$$Q_2 = Q_1N(P \text{ in } Q_2). \tag{4.15b}$$

Because $(|Q_1|, |P|) = 1$, and $P$ acts on $Q_1$, we can write $Q_1$ as the product $Q_1 = [Q_1, P]N(P \text{ in } Q_1)$. The commutator subgroup $[Q_1, P]$ is a characteristic subgroup of $Q_1P$ and thus is also a normal subgroup of $Q_2$, as $Q_2$ normalizes $Q_1P$. Therefore, (4.15b) implies

$$Q_2 = [Q_1, P]N(P \text{ in } Q_2).$$

That $[Q_1, P] \cap N(P \text{ in } Q_2) = C(P \text{ in } [Q_1, P])$ is obvious as $(|Q_1|, |P|) = 1$. Also the factor group $K := [Q_1, P]/\Phi([Q_1, P])$ is abelian and thus $K = [K, P] \times C(P \text{ in } K)$. As $[Q_1, P, P] = [Q_1, P]$ (by Theorem 3.6 in section 3.5 in [9]), we get that $K = [K, P]$ and $C(P \text{ in } K) = 1$. This implies that $C(P \text{ in } [Q_1, P]) \leq \Phi([Q_1, P])$. $\square$

As an easy consequence of Corollary 4.13 and Lemma 4.14 we have:

**Proposition 4.16.** *Let $Q$ be a $q$-group acting on a $p$-group $P$ with $p \neq q$ odd primes. Suppose that the semi–direct product $Q \ltimes P$ acts on a $q$-group $Q_1$ such that the action of $P$ on $Q_1$ is faithful. Then there exists a linear character $\lambda$ of $Q_1$ such that $C(P \text{ in } Q_1) \leq \mathrm{Ker}(\lambda)$ and $(QP)(\lambda) = Q$.*

*Proof.* As $P$ acts on $Q_1$ we can write $Q_1$ as the product $Q_1 = [Q_1, P] \cdot C(P \text{ in } Q_1)$. It is clear that the product $QC(P \text{ in } Q_1)$ forms a group. Furthermore, $QC(P \text{ in } Q_1)$ normalizes $P$ and the semidirect product $(QC(P \text{ in } Q_1)) \ltimes P$ acts on $[Q_1, P]$, while the action of $P$ on $[Q_1, P]$ is faithful. Then according to Corollary 4.13 there exists a linear character $\lambda_1$ of $[Q_1, P]$ such that $(QC(P \text{ in } Q_1)P)(\lambda_1) = QC(P \text{ in } Q_1)$, while $\Phi([Q_1, P]) \leq \mathrm{Ker}(\lambda_1)$.

As we have seen in Lemma 4.14

$$[Q_1, P] \cap C(P \text{ in } Q_1) = C(P \text{ in } [Q_1, P]) \leq \Phi([Q_1, P]).$$

Since $\lambda_1$ is a linear character of $[Q_1, P]$ that is trivial on $\Phi([Q_1, P])$ and $C(P \text{ in } Q_1)$-invariant, the above inclusion implies that $\lambda_1$ has a unique extension to a linear character $\lambda$ of $Q_1$ trivial on $C(P \text{ in } Q_1)$. Furthermore, $(QP)(\lambda) = (QP)(\lambda_1) = Q$, and the proposition follows. $\square$

We are now ready to show how our first change works:

**Theorem 4.17.** *Let $Q_1 \trianglelefteq Q_2 = Q \leq G$ and $P$ satisfy Hypothesis 4.1. Assume further that $\beta_1$ is an irreducible character of $Q_1$. Then there exist irreducible characters $\beta_1^\nu$ of $Q_1$ and $\beta^\nu$ of $Q_2(\beta_1^\nu)$ such that*

$$\begin{aligned} &P(\beta_1) = P(\beta_1^\nu), \\ &Q(\beta_1) \leq Q(\beta_1^\nu) \text{ and} \\ &\beta^\nu|_{Q_1} = \beta_1^\nu. \end{aligned}$$

*Therefore $\beta^\nu$ is an extension of $\beta_1^\nu$ to $Q(\beta_1^\nu)$.*

*Proof.* Let $P(\beta_1)$ be the stabilizer of $\beta_1$ in $P$ and $P_1$ be the normalizer of $P(\beta_1)$ in $P$. Let $\overline{P_1}$ denote the factor group $P_1/P(\beta_1)$. We write $C_1$ for the centralizer, $C_1 = C(P(\beta_1) \text{ in } Q_1)$, of $P(\beta_1)$ in $Q_1$. Then it is clear that $\overline{P_1}$ acts on $C_1$.

The Glauberman–Isaacs correspondence (Theorem 13.1 in [12]), applied to the groups $P(\beta_1)$ and $Q_1$, provides an irreducible character $\theta$ of $C_1$ corresponding to the irreducible character $\beta_1$ of $Q_1$. As $P_1$ normalizes both $P(\beta_1)$ and $Q_1$ we get that $(P_1)(\theta) = (P_1)(\beta_1) = P(\beta_1)$. If $(P_1)(\theta) < P(\theta)$ then $N(P(\beta_1) \text{ in } P(\theta)) = (P_1)(\theta) > (P_1)(\theta) = P(\beta_1)$. Therefore

$$P(\theta) = (P_1)(\theta) = P(\beta_1).$$

Since $P(\beta_1)$ centralizes $C_1 = C(P(\beta_1) \text{ in } Q_1)$, we have $P(\beta_1) \leq C(C_1 \text{ in } P_1) \leq (P_1)(\theta) = P(\beta_1)$. Hence $C(C_1 \text{ in } P_1) = P(\beta_1)$ and $\overline{P_1}$ acts faithfully on $C_1$.

Let $C_2 := N(P(\beta_1) \text{ in } Q)$ be the normalizer of $P(\beta_1)$ in $Q$. Then $C_1$ is a normal subgroup of $C_2$ as $Q_1 \trianglelefteq Q_2$. Furthermore, $C_2$ normalizes $N(P(\beta_1) \text{ in } PQ_1)$ as $Q_2$ normalizes the product $PQ_1$. As $P_1C_1 = N(P(\beta_1) \text{ in } PQ_1)$ we conclude that $C_2$ normalizes the product $P_1C_1$. Hence Frattini's argument implies that $C_2 = N(P_1 \text{ in } C_2)C_1$. Let $C_2' := N(P_1 \text{ in } C_2)$. Then $C_2'$ normalizes $\overline{P_1}$ and the semidirect product $C_2' \ltimes \overline{P_1}$ acts on $C_1$. Furthermore, the action of $\overline{P_1}$ on $C_1$ is faithful. By Proposition 4.16, there exists a linear character $\lambda \in \mathrm{Lin}(C_1)$ such that $C(\overline{P_1} \text{ in } C_1) \leq \mathrm{Ker}(\lambda)$ and $(C_2'\overline{P_1})(\lambda) = C_2'$.

The last equation implies that $P_1(\lambda) = P(\beta_1)$. Thus $P(\beta_1) = P_1(\lambda) \leq P(\lambda)$. We actually have that

$$P(\lambda) = P(\beta_1).$$

Indeed, if $P(\beta_1) < P(\lambda)$, then $P(\beta_1)$ would be a proper subgroup of $N(P(\beta_1) \text{ in } P(\lambda))$. Thus $P(\beta_1) < N(P(\beta_1) \text{ in } P(\lambda)) = N(P(\beta_1) \text{ in } P)(\lambda) = P_1(\lambda) = P(\beta_1)$.

Since $C_2 = C_2'C_1$ and $C_2'$ fixes $\lambda_1$, we conclude that $C_2$ also fixes $\lambda_1$. Furthermore, for the intersection $C_1 \cap C_2'$ we get

$$C_1 \cap C_2' = C_1 \cap N(P_1 \text{ in } C_2) = C(P_1 \text{ in } C_1) \leq C(\overline{P_1} \text{ in } C_1) \leq \mathrm{Ker}(\lambda).$$

Therefore $\lambda$ can be extended to $C_2$. Furthermore, according to Theorem 6.26 in [12] and the fact that $C_2P(\beta_1)$ fixes $\lambda$, we get that $\lambda$ can be extended to $C_2P(\beta_1)$.

Let $\beta_1^\nu \in \mathrm{Irr}(Q_1)$ be the Glauberman–Issacs $P(\beta_1)$-correspondent to $\lambda$. Then as $C_2P_1$ normalizes both $P(\beta_1)$ and $Q_1$, Corollary 2.2 implies that

$$(C_2P_1)(\beta_1^\nu) = (C_2P_1)(\lambda) = C_2P(\beta_1).$$

Hence $P(\beta_1^\nu) \geq P_1(\beta_1^\nu) = P(\lambda) = P(\beta_1)$. If $P(\beta_1^\nu) > P(\beta_1)$ then

$$P(\beta_1) < N(P(\beta_1) \text{ in } P(\beta_1^\nu)) = P_1(\beta_1^\nu) = P(\beta_1).$$

Thus $P(\beta_1^\nu) = P(\lambda) = P(\beta_1)$ and

$$(C_2(PQ_1))(\beta_1^\nu) = C_2P(\beta_1)Q_1.$$

Since $C_2$ fixes $\beta_1^\nu$ and normalizes $P(\beta_1)$ we have $C_2 \leq N(P(\beta_1) \text{ in } Q(\beta_1^\nu)) \leq N(P(\beta_1) \text{ in } Q) = C_2$. Hence $C_2 = N(P(\beta_1) \text{ in } Q(\beta_1^\nu))$. Furthermore, $P(\beta_1)Q_1 = (P_1Q_1)(\beta_1^\nu)$ as $P(\beta_1^\nu) = P(\beta_1)$. Hence the group $P(\beta_1)Q_1 = (PQ_1)(\beta_1^\nu)$ is a normal subgroup of $P(\beta_1)Q(\beta_1^\nu)$ as $Q$ normalizes the product $P_1Q_1$. So we can apply the Main Theorem in [17] to the groups $P(\beta_1)Q(\beta_1^\nu)$, $P(\beta_1)Q_1$ and $Q_1$. We conclude that $\beta_1^\nu$ extends to $P(\beta_1)Q(\beta_1^\nu)$ as its $P(\beta_1)$-Glauberman correspondent $\lambda$ can be extended to $P(\beta_1)C_2 = P(\beta_1)N(P(\beta_1) \text{ in } Q(\beta_1^\nu))$. We write $\beta^\nu$ for the extension of $\beta_1^\nu$ to $Q(\beta_1^\nu)$.

To complete the proof of the theorem it remains to show that $Q(\beta_1) \leq Q(\beta_1^\nu)$. The group $(PQ_1)(\beta_1) = P(\beta_1)Q_1$ is a normal subgroup of $P(\beta_1)Q(\beta_1)$, as $Q$ normalizes $P_1Q_1$. Hence Frattini's argument implies that

$$Q(\beta_1) = Q_1N(P(\beta_1) \text{ in } Q(\beta_1)).$$

Therefore $Q(\beta_1) \leq Q_1N(P(\beta_1) \text{ in } Q) = Q_1C_2$. But we have already seen that $P(\beta_1)Q_1$ is a normal subgroup of $P(\beta_1)Q(\beta_1^\nu)$. Hence the Frattini argument implies that

$$Q(\beta_1^\nu) = Q_1N(P(\beta_1) \text{ in } Q(\beta_1^\nu)) = Q_1C_2.$$

Thus $Q(\beta_1) \leq Q(\beta_1^\nu)$ and the theorem follows. $\square$

## 4.2 A "multiple" change

In the previous section we saw how we can make a change of a character whenever we have only two $q$-groups, $Q_1$ and $Q_2$ involved. The natural question that follows from that restricted case is whether or not we can prove a similar theorem when a chain of $q$-groups, $Q_1 \trianglelefteq Q_2 \trianglelefteq \cdots \trianglelefteq Q_n \trianglelefteq Q_{n+1}$, is involved. What we will show is that, eventhought we can not do a character replacement as in the two group case, we can still find new linear characters having enough of the desired properties.

So assume that, along with the series $Q_1 \trianglelefteq Q_2 \trianglelefteq \cdots \trianglelefteq Q_n \trianglelefteq Q_{n+1} = Q$ of normal subgroups of $Q_{n+1} = Q$, where $Q_{n+1}$ is a $q$-subgroup of a finite group $G$, we also have $p$-subgroups $P_1, P_2, \ldots, P_{n-1}, P_n$ of $G$, such that $P_i$ normalizes the groups $P_j$ and $Q_j$ whenever $1 \leq j \leq i \leq n$, while $Q_i$ normalizes the semidirect product $P_j \ltimes Q_j$ whenever $1 \leq j < i \leq n+1$. Assume further that $K_i = C(Q_i \text{ in } P_i)$ for all $i = 1, \ldots, n$. Then, as we will see by the end of the section, we can find a chain of linear characters, $\beta_1^\nu, \beta_2^\nu, \ldots, \beta_n^\nu$, of $Q_1, Q_2, \ldots, Q_n$ respectively, such that

(1) $P_i(\beta_i^\nu) = K_i$ and $Q_{n+1}(\beta_i^\nu) = Q_{n+1}$ while

(2) $\beta_n^\nu$ can be extended to $Q_{n+1}(\beta_n^\nu) = Q$.

The equivalent of the diagram 4.2 in this case is:

$$
\begin{tikzcd}
 & Q = Q_{n+1} = Q(\beta_n^\nu) \arrow[dl] \arrow[dd] & (\beta_n^\nu)^e \arrow[dd, dash] \\
P_n \arrow[dr] & & \\
 & Q_n \arrow[dl] \arrow[dd] & \beta_n^\nu \arrow[dd, dash] \\
P_{n-1} \arrow[dr] & & \\
 & Q_{n-1} \arrow[dl] \arrow[dd, dashed] & \beta_{n-1}^\nu \arrow[dd, dash, dashed] \\
P_2 \arrow[dr] & & \\
 & Q_2 \arrow[dl] \arrow[dd] & \beta_2^\nu \arrow[dd, dash] \\
P_1 \arrow[dr] & & \\
 & Q_1 & \beta_1^\nu
\end{tikzcd}
\tag{4.18}
$$

We will prove in this section that such a generalization is possible when the primes $p, q$ are odd. We first need some lemmas:

**Lemma 4.19.** *Assume $Q_1, Q_2$ are two $q$-subgroups of a finite group $G$, for some odd prime $q$. Assume further that $Q_1 \trianglelefteq Q_2$ and that $A$ is a normal subgroup of $Q_1$ normalized by $Q_2$ as well. Let $P$ be a $p$-subgroup of $G$, where $2 \neq p \neq q$, such that*

*(a) $P$ normalizes $A$ and $Q_1$, and*

*(b) $Q_2$ normalizes $Q_1 P$*

*If $\widetilde{P}$ denotes the centralizer, $C(Q_1/A \text{ in } P)$, of $Q_1/A$ in $P$, then $Q_2$ normalizes the semidirect product $\widetilde{P} \ltimes A$. Thus $Q_2\widetilde{P}$ is a group with $Q_1\widetilde{P}$ and $A$ as normal subgroups. The factor group $Q_2\widetilde{P}/A = Q_2(\widetilde{P} \ltimes A)/A$ equals the semidirect product $(Q_2/A) \ltimes ((\widetilde{P}A)/A)$, of its $q$-subgroup $Q_2/A$ and its normal $p$-subgroup $(\widetilde{P}A)/A \cong \widetilde{P}$.*

*Proof.* Since $Q_2$ normalizes $Q_1 P$, the product $Q_2 P = Q_2 \cdot (Q_1 P)$ is a group. Note that $A \trianglelefteq Q_2 P$. We will use bars to denote the image in $\overline{PQ_2} = (PQ_2)/A$ of any subgroup of $PQ_2$ containing $A$. Then $\widetilde{P} = C(\overline{Q}_1 \text{ in } P)$, is a normal subgroup of $P$, as the latter normalizes both $A$ and $Q_1$. Furthermore, $\overline{PQ_1} = \overline{P} \ltimes \overline{Q}_1$ and $\overline{\widetilde{P}}$ is the centralizer in $\overline{P}$ of $\overline{Q}_1$. It follows that $\overline{\widetilde{P}} = (\widetilde{P}A)/A$ is the maximal normal $p$-subgroup of $\overline{PQ_1}$. Therefore, $\overline{Q}_2$ normalizes $\overline{\widetilde{P}}$ as it normalizes $\overline{PQ_1}$. Hence $Q_2$ normalizes the inverse image $\widetilde{P}A$ of $\overline{\widetilde{P}}$ in $PQ_2$. So $\widetilde{P}Q_2 = (\widetilde{P}A)Q_2$ is a group and thus a subgroup of $PQ_2$. Furthermore, as $Q_1 \trianglelefteq Q_2$ and $Q_2$ normalizes $\widetilde{P}A$ we get that $Q_2$ normalizes their product $Q_1(\widetilde{P}A) = Q_1\widetilde{P}$. Therefore $Q_1\widetilde{P} \trianglelefteq Q_2\widetilde{P}$.

As $Q_2$ normalizes $\widetilde{P}A$ we get that $Q_2/A$ normalizes $\widetilde{P}A/A$. Therefore $(Q_2/A) \ltimes (\widetilde{P}A/A)$ is a group. Clearly $Q_2(\widetilde{P}A)/A = (Q_2/A) \ltimes (\widetilde{P}A/A)$, and the lemma follows. □

**Lemma 4.20.** *Assume that $Q_1, Q_2, G$ and $P$ satisfy Hypothesis 4.1. Let $K := C(Q_1 \text{ in } P)$ be the kernel of the $P$-action on $Q_1$. Then $Q_2 = [Q_1, P] \cdot N(P \text{ in } Q_2)$ and $Q_1 = [Q_1, P] \cdot C(P \text{ in } Q_1)$. Furthermore, there exist linear characters $\lambda_1 \in \mathrm{Lin}(Q_1)$ and $\lambda_2 \in \mathrm{Lin}(Q_2)$ that satisfy the following three conditions:*

*(1) $\lambda_2|_{Q_1} = \lambda_1$.*

*(2) $(Q_2 \cdot P)(\lambda_1) = Q_2 \cdot K$.*

*(3) $C(P \text{ in } Q_1) \leq \mathrm{Ker}(\lambda_1)$ and $N(P \text{ in } Q_2) \leq \mathrm{Ker}(\lambda_2)$.*

*Therefore, $\lambda_1(s \cdot u) = \lambda_1(s) = \lambda_1|_{[Q_1,P]}(s)$ and $\lambda_2(s \cdot t) = \lambda_1(s)$, for all $s \in [Q_1, P], u \in C(P \text{ in } Q_1)$ and $t \in N(P \text{ in } Q_2)$.*

*Proof.* Since $Q_2$ normalizes the semidirect product $Q_1 \rtimes P$ and $Q_1 \trianglelefteq Q_2$, Lemma 4.14 implies

$$Q_2 = Q_1 N(P \text{ in } Q_2) = [Q_1, P] \cdot N(P \text{ in } Q_2), \tag{4.21}$$

where $[Q_1, P] \trianglelefteq Q_2 P$,(and thus $[Q_1, P] \trianglelefteq Q_2$). Also $N(P \text{ in } Q_2) \cap Q_1 = C(P \text{ in } Q_1)$. Furthermore, as the $p$-group $P$ normalizes the $q$-group $Q_1$, we have

$$Q_1 = [Q_1, P] \cdot C(P \text{ in } Q_1). \tag{4.22}$$

Let $C_2 = N(P \text{ in } Q_2)$. Then $C_2$ normalizes $P$, while their semidirect product $C_2 \ltimes P$ normalizes $Q_1$.

*Case 1*: Assume that $K = 1$.

Since $K = 1$, the $p$-group $P$ acts faithfully on $Q_1$. Therefore, in view of Proposition 4.16, there exist a linear character $\lambda_1 \in \mathrm{Lin}(Q_1)$ suct that $(C_2 P)(\lambda_1) = C_2$, and $C(P \text{ in } Q_1) \leq \mathrm{Ker}(\lambda_1)$.

Since $Q_2 = Q_1 C_2$ (according to (4.21)) and $C_2$ fixes $\lambda_1$, we conclude that $\lambda_1$ is $Q_2$ invariant. Furthermore, the fact that $Q_1 \cap C_2 = C(P \text{ in } Q_1) \leq \mathrm{Ker}(\lambda_1)$, implies that $\lambda_1$ extends canonicaly to a linear character $\lambda_2$ of $Q_2$ such that $N(P \text{ in } Q_2) \leq \mathrm{Ker}(\lambda_2)$. This along with (4.21) and (4.22) imply that

$$\lambda_1(s \cdot u) = \lambda_1(s) = \lambda_1|_{[Q_1,P]}(s),$$

while

$$\lambda_2(s \cdot t) = \lambda_2(s) = \lambda_1(s),$$

for all $s \in [Q_1, P], u \in C(P \text{ in } Q_1)$ and $t \in N(P \text{ in } Q_2)$.

This completes the proof of Lemma 4.20 when $K = C(Q_1 \text{ in } P) = 1$.

*Case 2*: Assume that $1 < K \leq P$.

In this case we work with the group $P' = P/K$ in the place of $P$. Note that in view of (4.22), we have

$$K = C(Q_1 \text{ in } P) = C([Q_1, P] \text{ in } P).$$

As $K = O_p(PQ_1)$ and $PQ_1 \trianglelefteq PQ_2$ we have that $K$ is a normal subgroup of $PQ_2$. Hence $K$ is a $Q_2$-invariant subgroup of $P$. Therefore the hypothesis of Lemma 4.20 are satisfied for $P'$ in the place of $P$, as $P'$ normalizes $Q_1$ while $Q_2$ normalizes their product $Q_1 \rtimes P'$.
Hence the previous case provides linear characters $\lambda_1 \in \mathrm{Lin}(Q_1)$ and $\lambda_2 \in \mathrm{Lin}(Q_2)$ such that $\lambda_2$ is an extension of $\lambda_1$ to $Q_2$. Furthermore,

$$(Q_2 \cdot P')(\lambda_1) = Q_2 \tag{4.23a}$$

while

$$C(P' \text{ in } Q_1) \leq \mathrm{Ker}(\lambda_1) \text{ and } N(P' \text{ in } Q_2) \leq \mathrm{Ker}(\lambda_2). \tag{4.23b}$$

Even more, $\lambda_1(s \cdot u) = \lambda_1(s)$ and $\lambda_2(s \cdot t) = \lambda_1(s)$, for all $s \in [Q_1, P], u \in C(P' \text{ in } Q_1)$ and $t \in N(P' \text{ in } Q_2)$.

We observe that (4.23a) implies

$$(Q_2 \cdot P)(\lambda_1) = Q_2 \cdot K,$$

as $K$ centralizes $Q_1$. Furthermore, we note that $[Q_1, P'] = [Q_1, P]$ while $N(P \text{ in } Q_2) \leq N(P' \text{ in } Q_2)$ and $C(P \text{ in } Q_1) \leq C(P' \text{ in } Q_1)$. Hence in view of (4.23b) we have $N(P \text{ in } Q_2) \leq \mathrm{Ker}(\lambda_2)$ and $C(P \text{ in } Q_1) \leq \mathrm{Ker}(\lambda_1)$. Therefore the lemma follows. □

**Theorem 4.24.** *Assume $G$ is a finite group of order $p^a q^b$ for distinct odd primes $p$ and $q$, and non–negative integers $a$ and $b$. Let $Q_1 \trianglelefteq Q_2 \trianglelefteq \cdots \trianglelefteq Q_n \trianglelefteq Q_{n+1} = Q$ be a series of normal subgroups of a $q$-subgroup $Q \leq G$, and let $P_1, P_2, \ldots, P_{n-1}, P_n$ be $p$-subgroups of $G$ such that the following hold:*

*(1) $P_i$ normalizes the groups $P_j$ and $Q_j$ whenever $1 \leq j \leq i \leq n$, while*

*(2) $Q_i$ normalizes the semidirect product $P_j \ltimes Q_j$ whenever $1 \leq j < i \leq n+1$.*

*Let $K_i$ denote the kernel of the $P_i$-action on $Q_i$, i.e., $K_i = C(Q_i \text{ in } P_i)$ for every $i = 1, \ldots n$. Then there exist linear characters $\beta_i$ of $Q_i$, for all $i = 1, \ldots, n+1$, such that:*

*(a) the restriction $\beta_i|_{Q_j}$ of $\beta_i$ to $Q_j$ equals $\beta_j$ if $1 \leq j \leq i \leq n+1$, and*

*(b) the stabilizer $(QP_i)(\beta_i)$ of $\beta_i$ in $QP_i$ equals $QK_i$ if $1 \leq i \leq n$.*

*Thus $\beta_{n+1}$ is an extension to $Q = Q_{n+1}$ of $\beta_1, \beta_2, \ldots, \beta_n$.*

*Proof.* For the proof we will use induction on $n$. The case $n = 1$ is done in Lemma 4.20.

We assume that the proposition holds for all $n$ with $1 \leq n < k$ and some $k \geq 2$. We will prove it also holds for $n = k$. Since $Q_i$ normalizes $P_1 \ltimes Q_1$, for all $i = 1, \ldots, k+1$, while $Q_1 \trianglelefteq Q_i$, Lemma 4.14 implies that $Q_i$ is the product

$$Q_i = Q_1 \cdot N(P_1 \text{ in } Q_i), \tag{4.25a}$$

of its normal subgroup $Q_1$ with $N(P_1 \text{ in } Q_i)$, where

$$Q_1 \cap N(P_1 \text{ in } Q_i) \leq C(P_1 \text{ in } Q_1). \tag{4.25b}$$

As $P_i$ normalizes the groups $Q_i, Q_1$, it also normalizes the factor group $\overline{Q}_{i,1} := Q_i/Q_1$, whenever $1 \leq i \leq k$.

We define:

$$P_{i,1} = C(\overline{Q}_{i,1} \text{ in } P_i). \tag{4.26}$$

Note that $P_{1,1} = C(Q_1/Q_1 \text{ in } P_1) = P_1$.

If we apply Lemma 4.19 to the groups $Q_i = Q_1 \cdot N(P_1 \text{ in } Q_i), Q_1, Q$ and $P_i$, for some $i = 1, \ldots, k$, in the place of $Q_1, A, Q_2$ and $P$ respectively, we conclude that

$$Q \text{ normalizes the semidirect product } P_{i,1} \ltimes Q_1. \tag{4.27}$$

Furthermore, the group $P_{i,1}$ normalizes $Q_1$, as $P_i$ does. Since $P_i$ normalizes both $P_j$ and $Q_j$ for all $j = 1, \ldots, i$ we have that $P_{i,1}$ normalizes both $P_j$ and $Q_j$ as well. Therefore $P_{i,1}$ normalizes both the factor group $\overline{Q}_{j,1}$ and the centralizer $C(\overline{Q}_{j,1} \text{ in } P_j) = P_{j,1}$, for all such $j$. Hence the product

$$\mathcal{P}_1 = P_{1,1} \cdot P_{2,1} \cdots P_{n,1} = P_1 \cdot P_{2,1} \cdots P_{n,1} \tag{4.28}$$

is a $p$-subgroup of $G$ that normalizes $Q_1$. Thus $\mathcal{P}_1 \ltimes Q_1$ is a group. In view of (4.27), the group $\mathcal{P}_1 \ltimes Q_1$ is normalized by $Q = Q_{k+1}$. Let $C_1 := C(Q_1 \text{ in } \mathcal{P}_1)$ be the centralizer of $Q_1$ in $\mathcal{P}_1$. Then Lemma 4.20 implies that there exists a linear character $\mu_1 \in \mathrm{Lin}(Q_1)$ that can be extended to a linear character $\mu_1^e \in \mathrm{Lin}(Q)$, with the following properties:

$$Q(\mu_1) = Q, \tag{4.29a}$$

$$\mathcal{P}_1(\mu_1) = C_1 = C(Q_1 \text{ in } \mathcal{P}_1) \tag{4.29b}$$

and

$$C(P_1 \text{ in } Q_1) \leq \mathrm{Ker}(\mu_1). \tag{4.29c}$$

Furthermore, for the extension character $\mu_1^e$, we have:

$$\mu_1^e|_{Q_1} = \mu_1, \tag{4.30a}$$

while

$$\mu_1^e(s \cdot t) = \mu_1(s) \tag{4.30b}$$

for all $s \in Q_1$ and $t \in N(P_1 \text{ in } Q_{k+1})$. Clearly for all $i = 1, \ldots, k$ we have that $\mu_1^e|_{Q_i} \in \mathrm{Lin}(Q_i)$. Furthermore (4.30b), along with (4.25a), implies

$$\mu_1^e|_{Q_i}(s \cdot t_i) = \mu_1(s), \tag{4.30c}$$

for all $s \in Q_1$ and $t_i \in N(P_1 \text{ in } Q_i)$.

We will use our inductive argument on the groups

$$Q_2/Q_1 \trianglelefteq Q_3/Q_1 \trianglelefteq \cdots \trianglelefteq Q_{k+1}/Q_1 = Q/Q_1.$$

Note that the above groups form a series of normal subgroups of the $q$-group $Q/Q_1$, as $Q_2 \trianglelefteq Q_3 \trianglelefteq \cdots \trianglelefteq Q = Q_{k+1}$ is a normal series of $Q$. Furthermore the group $P_i$ normalizes $Q_j/Q_1$, whenever $1 \leq j \leq i \leq k$, as $P_i$ normalizes both $P_1$ and $Q_j$. Thus $P_i \ltimes Q_i/Q_1$ is a group. Also $Q_i/Q_1$ normalizes the semidirect product $P_j \ltimes (Q_j/Q_1)$, whenever $1 \leq j \leq i \leq k+1$, as $Q_i$ normalizes the semidirect product $P_j \ltimes Q_j$. Hence by induction, there exist linear characters $\lambda_2^* \in \mathrm{Irr}(Q_2/Q_1), \ldots, \lambda_k^* \in \mathrm{Irr}(Q_k/Q_1)$, and $\lambda_{k+1}^*$ in $\mathrm{Irr}(Q/Q_1)$ such that

$$\lambda_{k+1}^*|_{Q_i/Q_1} = \lambda_i^* \tag{4.31}$$

and

$$(Q/Q_1 \cdot P_i)(\lambda_i^*) = Q/Q_1 \cdot C(Q_i/Q_1 \text{ in } P_i) = Q/Q_1 \cdot P_{i,1}, \tag{4.32}$$

for all $i = 2, \ldots, k$.
Let $\lambda_i \in \mathrm{Lin}(Q_i)$ be the linear character of $Q_i$ inflated from $\lambda_i^* \in \mathrm{Lin}(Q_i/Q_1)$. Then (4.31) and (4.32) imply:

$$\lambda_{k+1}|_{Q_i} = \lambda_i, \tag{4.33a}$$

$$Q_1 \leq \mathrm{Ker}(\lambda_i), \tag{4.33b}$$

and

$$Q(\lambda_i) = Q, \tag{4.33c}$$

for all $i = 2, \ldots, k+1$. Furthermore,

$$P_i(\lambda_i) = P_{i,1} \tag{4.33d}$$

for all $i = 2, \ldots, k$.

As $\lambda_{k+1}$ is a linear character of $Q_{k+1}$ and $Q_1 \leq Q_{k+1}$, the restriction $\lambda_1 := \lambda_{k+1}|_{Q_1}$, is a linear character of $Q_1$. Furthermore, (4.33b) implies

$$\lambda_1 = 1_{Q_1} \text{ and thus } \quad Q(\lambda_1) = Q. \tag{4.34}$$

Since $Q_i = Q_1 \cdot N(P_1 \text{ in } Q_i)$ for every $i = 1, \ldots, k+1$, (see (4.25a)), equations (4.33a) and (4.34) imply

$$\lambda_i(s \cdot t_i) = \lambda_i(t_i) = \lambda_{k+1}(t_i), \tag{4.35}$$

for all $s \in Q_1$ and $t_i \in N(P_1 \text{ in } Q_i)$.

Using the equation (4.25a), we define for all $i = 1, \ldots, k+1$,

$$\beta_i(s \cdot t_i) := \mu_1(s) \cdot \lambda_i(t_i), \tag{4.36}$$

whenever $s \in Q_1$ and $t_i \in N(P_1 \text{ in } Q_i) \leq N(P_1 \text{ in } Q_{k+1})$. According to (4.30c) and (4.35), we can rewrite the characters $\beta_i$ as

$$\beta_i = \mu_1^e|_{Q_i} \cdot \lambda_i = \mu_1^e|_{Q_i} \cdot \lambda_{k+1}|Q_i = (\mu_1^e \cdot \lambda_{k+1})|_{Q_i}.$$

Hence

$$\beta_i|_{Q_j} = \beta_j, \tag{4.37}$$

whenever $1 \leq j \leq i \leq k+1$. Therefore, $\beta_{k+1}$ is an extension of $\beta_1$ to $Q$.

As $Q$ fixes $\mu_1$ by (4.29a), and fixes $\lambda_i$ by (4.33c), it also fixes $\beta_i$, in view of (4.36). Furthermore, (4.36) implies that $P_i(\beta_i) = P_i(\mu_1) \cap P_i(\lambda_i)$ for all $i = 1, \ldots, k$. In view of (4.29b) and (4.33d) we conclude that

$$P_i(\beta_i) = C(Q_1 \text{ in } P_i) \cap P_{i,1}.$$

But $P_{i,1} = C(Q_i/Q_1 \text{ in } P_i)$ by (4.26). Hence

$$P_i(\beta_i) = C(Q_1 \text{ in } P_i) \cap C(Q_i/Q_1 \text{ in } P_i) = C(Q_i \text{ in } P_i) = K_i.$$

This completes the proof of the inductive step for $n = k$. Thus Proposition 4.24 follows. □

# Chapter 5

# Triangular Sets

## 5.1 The correspondence

Assume we have the following situation:

**Hypothesis 5.1.** *Let $G$ be an odd order group and $\pi$ any set of primes. Let*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G \tag{5.2}$$

*be a series of normal subgroups of $G$, for some arbitrary integer $m > 0$, such that $G_i/G_{i-1}$, for $i = 1, 2, \ldots, m$, is a $\pi'$-group when $i$ is odd and a $\pi$-group when $i$ is even.*

Recall the definition given in Chapter 1

**Definition 5.3.** Let $\chi_i$ be an irreducible character of $G_i$, for all $i = 0, 1, \ldots, m$, such that $\chi_i$ lies over $\chi_k$ for all $k = 0, 1, \ldots, i$. Any such collection of irreducible characters $\{\chi_i\}_{i=0}^m$ is said to be a *character tower* for the series $\{G_i\}_{i=0}^m$.

Suppose further that there exist $\pi$- and $\pi'$-groups $P_{2r}$ and $Q_{2i-1}$ respectively, along with irreducible characters $\alpha_{2r}$ and $\beta_{2i-1}$, such that

$$\begin{gathered} P_0 = 1 \text{ and } \alpha_0 = 1, \\ Q_1 = G_1 \text{ and } \beta_1 = \chi_1, \\ P_{2r} \in \mathrm{Hall}_\pi(G_{2r}(\alpha_2, \ldots, \alpha_{2r-2}, \beta_1, \ldots, \beta_{2r-1})), \\ \alpha_{2r} \in \mathrm{Irr}(P_{2r}) \text{ lies above the } Q_{2r-1}\text{-Glauberman correspondent of } \alpha_{2r-2}, \\ Q_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3})), \\ \beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1}) \text{ lies above the } P_{2i-2}\text{-Glauberman correspondent of } \beta_{2i-3} \end{gathered} \tag{5.4}$$

for all odd $2i-1$ and even $2r$ with $1 < 2i-1 \leq m$ and $1 < 2r \leq m$.

Depending on the parity of $m$ the collection of groups and characters appearing in (5.4) consists of

$$\{P_0, P_2, \ldots, P_{m-1}, Q_1, Q_3, \ldots, Q_m | \alpha_0, \ldots, \alpha_{m-1}, \beta_1, \ldots, \beta_m\} \text{ if } m \text{ is odd, and}$$
$$\{P_0, P_2, \ldots, P_m, Q_1, Q_3, \ldots, Q_{m-1} | \alpha_0, \ldots, \alpha_m, \beta_1, \ldots, \beta_{m-1}\} \text{ if } m \text{ is even.}$$

By convention we will write, for both cases, the collection of groups and characters as $\{P_{2r}, Q_{2i-1}|\alpha_{2r}, \beta_{2i-1}\}$ where $1 \leq 2i-1 \leq m$ and $0 \leq 2r \leq m$. We also write $\{Q_{2i-1}|\beta_{2i-1}\}$ for the $\pi'$-subset of the above collection and similarly, $\{P_{2r}|\alpha_{2r}\}$ for the $\pi$-subset.

**Definition 5.5.** Any set of groups and characters that satisfies (5.4) will be called a *triangular set* for the normal series (5.2).

At this point it is not clear at all that such a collection of groups and characters exists. Even worse, it is not at all obvious that (5.4) is well defined. For all we know, the group $C(P_{2i} \text{ in } Q_{2i-1})$, which is the support of the $P_{2i}$-Glauberman correspondent of $\beta_{2i-1}$, need not be a subgroup of $Q_{2i+1}$. Thus, to ask for the character $\beta_{2i+1} \in \mathrm{Irr}(Q_{2i+1})$ to lie above a character of $C(P_{2i} \text{ in } Q_{2i-1})$ seems out of place. (Of course the same problem appears for the $\pi$-groups $C(Q_{2r-1} \text{ in } P_{2r-2})$ and the character $\alpha_{2r}$). But in fact these collections of groups and characters do exist, as we will see in Section 5.2. Furthermore, we will prove in the rest of this chapter not only that triangular sets exist but also that they correspond uniquely, up to conjugation, to character towers. In particular we will prove

**Theorem 5.6.** *Assume that Hypothesis 5.1 holds. Then there is a one–to–one correspondence between $G$-conjugacy classes of character towers of* (5.2) *and $G$-conjugacy classes of triangular sets for* (5.2).

## 5.2 Triangular-sets: existence and properties

Assume that a finite group $G$ and a normal series (5.2) are given so that Hypothesis 5.1 is satisfied. Recall (see Chapter 1) that for any real number $x$, we denote by $[x]$ the greatest integer $n$ such that $n \leq x$. If we write

$$l = [(m+1)/2], textand \tag{5.7a}$$
$$k = [m/2], \tag{5.7b}$$

then $2l-1$ is the greatest odd integer in the set $\{1, \dots, m\}$ while $2k$ is the greatest even integer in the same set. Furthermore,

$$k \leq l \leq k+1, \tag{5.8}$$

where for $m$ even we get $k = l$, while for $m$ odd we have $k = l-1$. Then it is easy to construct, in a recursive way, a collection of groups and characters $Q_{2i-1}, P_{2r}, \beta_{2i-1}$ and $\alpha_{2r}$ so that the following holds:

$$P_0 = 1 \text{ and } \alpha_0 = 1, \tag{5.9a}$$
$$Q_1 = G_1 \text{ and } \beta_1 = \chi_1, \tag{5.9b}$$
$$P_{2r} \in \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2, \dots, \alpha_{2r-2}, \beta_1, \dots, \beta_{2r-1})), \tag{5.9c}$$
$$\alpha_{2r} \in \mathrm{Irr}(P_{2r}), \tag{5.9d}$$
$$Q_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \dots, \alpha_{2i-2}, \beta_1, \dots, \beta_{2i-3})), \tag{5.9e}$$
$$\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1}), \tag{5.9f}$$

whenever $2 \leq i \leq l$ and $1 \leq r \leq k$.

Notice that (5.9) is a part of (5.4). So to prove existence of triangular sets we need to show that the characters $\beta_{2i-1}$ and $\alpha_{2r}$ that appear in (5.9) can be chosen to satisfy the additional conditions required in (5.4). Before we prove this, let us see what conclusions we can draw from (5.9). The first obvious remark is that, according to (5.9e) and (5.9c), we have:

$$Q_{2i-1} \text{ normalizes the groups } P_2, \ldots, P_{2i-2}, Q_1, \ldots, Q_{2i-3}, \tag{5.10a}$$

while

$$P_{2r} \text{ normalizes the groups } P_0, P_2, \ldots, P_{2r-2}, Q_1, \ldots, Q_{2r-1}, \tag{5.10b}$$

whenever $2 \leq i \leq l$ and $1 \leq r \leq k$.

Since $P_{2r}$ normalizes $Q_{2r-1}$, the group

$$Q_{2r-1,2r} := C(P_{2r} \text{ in } Q_{2r-1}) = N(P_{2r} \text{ in } Q_{2r-1}) \tag{5.11}$$

is defined whenever $1 \leq r \leq k$ (note that the group $Q_{2r-1}$ is defined for all such $r$ as $k \leq l$). Furthermore, (5.9) implies two lemata that lead to the existence of triangular sets. We start with

**Lemma 5.12.** *For every $i$ with $1 \leq i \leq l-1$ we have*

$$Q_{2i-1,2i} = Q_{2i+1} \cap G_{2i-1}.$$

*Hence $Q_{2i-1,2i}$ is a normal subgroup of $Q_{2i+1}$.*

Note that this lemma gives us no information about the group $Q_{2k-1,2k}$ in the case of an even $m = 2l = 2k$, as the group $Q_{2k+1}$ is not defined in that case.

*Proof.* The group $G_{2i-1}$ is a normal subgroup of $G_{2i+1}$ whenever $1 \leq i < l$. Hence the definition (5.9e) of $Q_{2i+1}$ implies that

$$Q_{2i+1} \cap G_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \ldots, \alpha_{2i}, \beta_1, \ldots, \beta_{2i-1})). \tag{5.13}$$

In particular, whenever $1 < i < l$ the character $\beta_{2i-3}$ is defined, and we have

$$Q_{2i+1} \cap G_{2i-1} \leq G_{2i-1}(\alpha_2, \ldots, \alpha_{2i}, \beta_1, \ldots, \beta_{2i-1}) \leq G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3}).$$

Furthermore, $Q_{2i+1}$ normalizes $Q_{2i-1}$ (according to (5.10a)). Also $Q_{2i-1}$ is a $\pi'$-Hall subgroup of the group $G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3})$. So the intersection $Q_{2i+1} \cap G_{2i-1}$ is a $\pi'$-subgroup of $G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3})$, and normalizes the $\pi'$-Hall subgroup $Q_{2i-1}$ of that group. Therefore $Q_{2i+1} \cap G_{2i-1} \leq Q_{2i-1}$ whenever $1 < i < l$. But the last inclusion is still valid when $i = 1$, as $Q_1 = G_1$ and therefore $Q_3 \cap G_1 \leq Q_1$. Hence, $Q_{2i+1} \cap G_{2i-1} \leq Q_{2i-1}$ whenever $1 \leq i < l$. As $Q_{2i+1}$ normalizes the group $P_{2i}$ by (5.10a), we conclude that $Q_{2i+1} \cap G_{2i-1}$ is a subgroup of $N(P_{2i} \text{ in } Q_{2i-1}) = Q_{2i-1,2i}$, i.e., that

$$Q_{2i+1} \cap G_{2i-1} \leq Q_{2i-1,2i}$$

whenever $1 \leq i < l$.

To prove the opposite inclusion we remark that, as $Q_{2i-1,2i}$ centralizes $P_{2i}$, it fixes the character $\alpha_{2i}$. It also fixes the character $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$, as it is a subgroup of $Q_{2i-1}$. This, along with the definition of $Q_{2i-1}$ (see (5.9e)), implies that

$$Q_{2i-1,2i} \leq G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \alpha_{2i}, \beta_1, \ldots, \beta_{2i-3}, \beta_{2i-1}).$$

But the group $Q_{2i+1} \cap G_{2i-1}$ is a $\pi'$-Hall subgroup of $G_{2i-1}(\alpha_2, \ldots, \alpha_{2i}, \beta_1, \ldots, \beta_{2i-1})$, by (5.13). Hence the fact that $Q_{2i+1} \cap G_{2i-1}$ is contained in the $\pi'$-group $Q_{2i-1,2i}$ implies that $Q_{2i+1} \cap G_{2i-1} = Q_{2i-1,2i}$. As $G_{2i-1}$ is a normal subgroup of $G_{2i+1}$ we conclude that $Q_{2i+1} \cap G_{2i-1} \trianglelefteq Q_{2i+1}$, and the lemma follows. $\square$

Note that Lemma 5.12 resolves the problem discussed in Section 5.1, at least for the $\pi'$-groups. Indeed, the character $\beta_{2r-1}$ is fixed by the $\pi$-group $P_{2r}$, by (5.9c). Thus we can define $\beta_{2r-1,2r} \in \operatorname{Irr}(Q_{2r-1,2r})$ to be the $P_{2r}$-Glauberman correspondent of $\beta_{2r-1} \in \operatorname{Irr}(Q_{2r-1})$ whenever $1 \leq r \leq k$. Hence, in view of Lemma 5.12, and starting with $\beta_1 = \chi_1$, it makes sense to pick the character $\beta_{2i-1}$ so that it lies above $\beta_{2i-3,2i-2}$ whenever $2 \leq i \leq l$.

Similarly we can work with the $\pi$-groups. So we can define the group

$$P_{2i,2i+1} = C(Q_{2i+1} \text{ in } P_{2i}) = N(Q_{2i+1} \text{ in } P_{2i}), \tag{5.14}$$

whenever $1 \leq i < l$. (Note that $Q_{2i+1}$ normalizes $P_{2i}$, by (5.10a)). Furthermore, in a symmetric way to that we used for the $\pi'$-groups we can prove

**Lemma 5.15.** *For every $r$ with $1 \leq r \leq k-1$*

$$P_{2r,2r+1} = P_{2r+2} \cap G_{2r}.$$

*Hence $P_{2r,2r+1}$ is a normal subgroup of $P_{2r+2}$.*

Note that, as in the case of the $\pi'$-groups, we get no information about the groups $P_{2l-2,2l-1} = P_{2k,2k+1}$ that appear in the case where $m = 2l-1 = 2k+1$ is odd.

As the character $\alpha_{2i}$ is fixed by $Q_{2i+1}$ (see (5.9e)), we can define $\alpha_{2i,2i+1} \in \operatorname{Irr}(P_{2i,2i+1})$ to be the $Q_{2i+1}$-Glauberman correspondent of $\alpha_{2i} \in \operatorname{Irr}(P_{2i})$ whenever $1 \leq i < l$. Hence, in view of Lemma 5.15, and starting with $\alpha_0 = 1$, we can pick the character $\alpha_{2r} \in \operatorname{Irr}(P_{2r})$ so that it lies above $\alpha_{2r-2,2r-1}$ whenever $1 \leq r \leq k$. (Observe that, since $\alpha_0 = 1$, the only requirement for the character $\alpha_2$ is to be an irreducible character of $P_2$).

This completes the proof of the existence of triangular sets, as the groups and characters we just constructed satisfy (5.4). Indeed, we have proved

**Proposition 5.16.** *Assume that Hypothesis* (5.1) *holds for the group $G$. Then there exists a triangular set $\{P_{2r}, Q_{2i+1} | \alpha_{2r}, \beta_{2i+1}\}$ for the normal series* (5.2), *so as to satisfy the following conditions, whenever $1 \leq r \leq k$ and $2 \leq i \leq l$:*

$$P_0 = 1 \text{ and } \alpha_0 = 1, \tag{5.17a}$$

$$Q_1 = G_1 \text{ and } \beta_1 := \chi_1 \in \operatorname{Irr}(Q_1) = \operatorname{Irr}(G_1), \tag{5.17b}$$

$$P_{2r} \in \operatorname{Hall}_\pi(G_{2r}(\alpha_2, \ldots, \alpha_{2r-2}, \beta_1, \ldots, \beta_{2r-1})), \tag{5.17c}$$

$$\alpha_{2r} \in \operatorname{Irr}(P_{2r} | \alpha_{2r-2,2r-1}), \tag{5.17d}$$

$$Q_{2i-1} \in \operatorname{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3})), \tag{5.17e}$$

$$\beta_{2i-1} \in \operatorname{Irr}(Q_{2i-1} | \beta_{2i-3,2i-2}), \tag{5.17f}$$

*where $\alpha_{2r-2,2r-1}$ is the $Q_{2r-1}$-Glauberman correspondent of $\alpha_{2r-2}$ and similarly $\beta_{2i-3,2i-2}$ is the $P_{2i-2}$-Glauberman correspondent of $\beta_{2i-3}$.*

From now on, and until the end of this section, we assume that the set $\{P_{2r}, Q_{2i+1} | \alpha_{2r}, \beta_{2i+1}\}$ is a triangular set for (5.2), and therefore satisfies (5.17).

An attempt to give a diagram that describes the relations in (5.17) produces the following “Double Staircases” of groups and characters:

$$
\begin{tikzcd}
(Q_{2l-1,2l} \arrow[r, dash] & )Q_{2l-1} \arrow[d, dash] & & & & & \\
& Q_{2l-3,2l-2} \arrow[r, dash] & Q_{2l-3} & & & & \\
& & \ddots & \ddots & & & \\
& & & Q_{5,6} \arrow[r, dash] & Q_5 \arrow[d, dash] & & \\
& & & & Q_{3,4} \arrow[r, dash] & Q_3 \arrow[d, dash] & \\
& & & & & Q_{1,2} \arrow[r, dash] & Q_1
\end{tikzcd}
\tag{5.18a}
$$

and

$$
\begin{tikzcd}
(\beta_{2l-1,2l} \arrow[r, leftrightarrow] & )\beta_{2l-1} \arrow[d, dash] & & & & & \\
& \beta_{2l-3,2l-2} \arrow[r, leftrightarrow, "P_{2l-2}"'] & \beta_{2l-3} & & & & \\
& & \ddots & \ddots & & & \\
& & & \beta_{5,6} \arrow[r, leftrightarrow, "P_6"'] & \beta_5 \arrow[d, dash] & & \\
& & & & \beta_{3,4} \arrow[r, leftrightarrow, "P_4"'] & \beta_3 \arrow[d, dash] & \\
& & & & & \beta_{1,2} \arrow[r, leftrightarrow, "P_2"'] & \beta_1
\end{tikzcd}
\tag{5.18b}
$$

and similarly for the $\pi$-groups and their characters

$$
\begin{tikzcd}
(P_{2k,2k+1} \arrow[r, dash] & )P_{2k} \arrow[d, dash] & & & & & \\
& P_{2k-2,2k-1} \arrow[r, dash] & P_{2k-2} & & & & \\
& & \ddots & \ddots & & & \\
& & & P_{6,7} \arrow[r, dash] & P_6 \arrow[d, dash] & & \\
& & & & P_{4,5} \arrow[r, dash] & P_4 \arrow[d, dash] & \\
& & & & & P_{2,3} \arrow[r, dash] & P_2
\end{tikzcd}
\tag{5.19a}
$$

and

$$
\begin{tikzcd}
(\alpha_{2k,2k+1} \arrow[r, leftrightarrow] & )\alpha_{2k} \arrow[d, dash] & & & & & \\
& \alpha_{2k-2,2k-1} \arrow[r, leftrightarrow, "Q_{2k-1}"'] & \alpha_{2k-2} & & & & \\
& & \ddots & \ddots & & & \\
& & & \alpha_{6,7} \arrow[r, leftrightarrow, "Q_7"'] & \alpha_6 \arrow[d, dash] & & \\
& & & & \alpha_{4,5} \arrow[r, leftrightarrow, "Q_5"'] & \alpha_4 \arrow[d, dash] & \\
& & & & & \alpha_{2,3} \arrow[r, leftrightarrow, "Q_3"'] & \alpha_2
\end{tikzcd}
\tag{5.19b}
$$

where the groups and characters in parentheses are those extra groups and characters that appear in the case of an even $m$ (for the $\pi'$-group $Q_{2l-1,2l}$ and its character $\beta_{2l-1,2l}$) or an odd $m$ (for the $\pi$-group $P_{2k,2k+1}$ and its character $\alpha_{2k,2k+1}$) respectively. Observe, that every group appearing in (5.18a) or (5.19a) is contained in all other groups that lie above or to its right. Furthermore, any character appearing in (5.18b) or (5.19b) is a Glauberman correspondent of the character that lies on its right.

We can actually expand these staircases into the following "Double Triangles" (5.20) and (5.21) of groups and characters (which is the reason behind the name *triangular*).

$$
\begin{tikzcd}
(Q_{2l-1,2l} \arrow[r, dash] & )Q_{2l-1} \arrow[d, dash] & & & & & \\
(Q_{2l-3,2l} \arrow[r, dash] & )Q_{2l-3,2l-2} \arrow[r, dash] \arrow[d, dash] & Q_{2l-3} \arrow[d, dash] & & & & \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & & & \\
(Q_{5,2l} \arrow[r, dash] & )Q_{5,2l-2} \arrow[r, dash] \arrow[d, dash] & Q_{5,2l-4} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_5 \arrow[d, dash] & & \\
(Q_{3,2l} \arrow[r, dash] & )Q_{3,2l-2} \arrow[r, dash] \arrow[d, dash] & Q_{3,2l-4} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & Q_{3,4} \arrow[r, dash] \arrow[d, dash] & Q_3 \arrow[d, dash] & \\
(Q_{1,2l} \arrow[r, dash] & )Q_{1,2l-2} \arrow[r, dash] & Q_{1,2l-4} \arrow[r, dash] & \cdots \arrow[r, dash] & Q_{1,4} \arrow[r, dash] & Q_{1,2} \arrow[r, dash] & Q_1
\end{tikzcd}
\tag{5.20a}
$$

and

$$
\begin{tikzcd}
(\beta_{2l-1,2l} \arrow[r, leftrightarrow] & )\beta_{2l-1} \arrow[d, dash] & & & & & \\
(\beta_{2l-3,2l} \arrow[r, leftrightarrow] & )\beta_{2l-3,2l-2} \arrow[r, leftrightarrow, "P_{2l-2}"'] \arrow[d, dash] & \beta_{2l-3} \arrow[d, dash] & & & & \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & & & \\
(\beta_{5,2l} \arrow[r, leftrightarrow] & )\beta_{5,2l-2} \arrow[r, leftrightarrow, "P_{2l-2}"'] \arrow[d, dash] & \beta_{5,2l-4} \arrow[r, leftrightarrow, "P_{2l-4}"'] \arrow[d, dash] & \cdots \arrow[r, leftrightarrow, "P_6"'] & \beta_5 \arrow[d, dash] & & \\
(\beta_{3,2l} \arrow[r, leftrightarrow] & )\beta_{3,2l-2} \arrow[r, leftrightarrow, "P_{2l-2}"'] \arrow[d, dash] & \beta_{3,2l-2} \arrow[r, leftrightarrow, "P_{2l-4}"'] \arrow[d, dash] & \cdots \arrow[r, leftrightarrow, "P_6"'] & \beta_{3,4} \arrow[r, leftrightarrow, "P_4"'] \arrow[d, dash] & \beta_3 \arrow[d, dash] & \\
(\beta_{1,2l} \arrow[r, leftrightarrow] & )\beta_{1,2l-2} \arrow[r, leftrightarrow, "P_{2l-2}"'] & \beta_{1,2l-4} \arrow[r, leftrightarrow, "P_{2l-4}"'] & \cdots \arrow[r, leftrightarrow, "P_6"'] & \beta_{1,4} \arrow[r, leftrightarrow, "P_4"'] & \beta_{1,2} \arrow[r, leftrightarrow, "P_2"'] & \beta_1
\end{tikzcd}
\tag{5.20b}
$$

for the $\pi'$-groups and their characters, and similarly for the $\pi$-groups:

$$
\begin{array}{ccccccccccccc}
(P_{2k,2k+1} & \text{———} & )P_{2k} & & & & & & & & & & \\
 & & | & & & & & & & & & & \\
(P_{2k-2,2k+1} & \text{—} & )P_{2k-2,2k-1} & \text{—} & P_{2k-2} & & & & & & & & \\
 & & | & & | & & & & & & & & \\
 & & \vdots & & \vdots & & & & & & & & \\
 & & | & & | & & & & & & & & \\
(P_{6,2k+1} & \text{———} & )P_{6,2k-1} & \text{—} & P_{6,2k-3} & \text{—} & \ldots & \text{—} & P_6 & & & & \\
 & & | & & | & & & & | & & & & \\
(P_{4,2k+1} & \text{———} & )P_{4,2k-1} & \text{—} & P_{4,2k-3} & \text{—} & \ldots & \text{—} & P_{4,5} & \text{—} & P_4 & & \\
 & & | & & | & & & & | & & | & & \\
(P_{2,2k+1} & \text{———} & )P_{2,2k-1} & \text{—} & P_{2,2k-3} & \text{—} & \ldots & \text{—} & P_{2,5} & \text{—} & P_{2,3} & \text{—} & P_2
\end{array}
\tag{5.21a}
$$

and

$$
\begin{array}{ccccccccccccc}
(\alpha_{2k,2k+1} & \longleftrightarrow & )\alpha_{2k} & & & & & & & & & & \\
 & & | & & & & & & & & & & \\
(\alpha_{2k-2,2k+1} & \leftrightarrow & )\alpha_{2k-2,2k-1} & \underset{Q_{2k-1}}{\longleftrightarrow} & \alpha_{2k-2} & & & & & & & & \\
 & & | & & | & & & & & & & & \\
 & & \vdots & & \vdots & & & & & & & & \\
 & & | & & | & & & & & & & & \\
(\alpha_{6,2k+1} & \longleftrightarrow & )\alpha_{6,2k-1} & \underset{Q_{2k-1}}{\longleftrightarrow} & \alpha_{6,2k-3} & \underset{Q_{2k-3}}{\longleftrightarrow} & \cdots & \underset{Q_7}{\longleftrightarrow} & \alpha_6 & & & & \\
 & & | & & | & & & & | & & & & \\
(\alpha_{4,2k+1} & \longleftrightarrow & )\alpha_{4,2k-1} & \underset{Q_{2k-1}}{\longleftrightarrow} & \alpha_{4,2k-3} & \underset{Q_{2k-3}}{\longleftrightarrow} & \cdots & \underset{Q_7}{\longleftrightarrow} & \alpha_{4,5} & \underset{Q_5}{\longleftrightarrow} & \alpha_4 & & \\
 & & | & & | & & & & | & & | & & \\
(\alpha_{2,2k+1} & \longleftrightarrow & )\alpha_{2,2k-1} & \underset{Q_{2k-1}}{\longleftrightarrow} & \alpha_{2,2k-3} & \underset{Q_{2k-3}}{\longleftrightarrow} & \cdots & \underset{Q_7}{\longleftrightarrow} & \alpha_{2,5} & \underset{Q_5}{\longleftrightarrow} & \alpha_{2,3} & \underset{Q_3}{\longleftrightarrow} & \alpha_2
\end{array}
\tag{5.21b}
$$

where, as before, $l = [(m+1)/2]$ and $k = [m/2]$. Furthermore, the $\pi'$-groups $Q_{2i-1,2l}$ and characters $\beta_{2i-1,2l}$ in parentheses exist only when $m$ is even, and the $\pi$-groups $P_{2r,2k+1}$ and characters $\alpha_{2r,2k+1}$ only when $m$ is odd.

Before we give the long list of the groups, the characters and their properties that are involved in the above diagrams, we remark again that the groups and characters that appear in the first two diagonals of the above "Double Triangle" diagrams form the "Double Staircase" diagrams. The rest of the groups that appear in the above diagrams are defined as

$$Q_{2i-1,2j} := N(P_{2i}, P_{2i+2}, \dots, P_{2j} \text{ in } Q_{2i-1}), \tag{5.22a}$$

and

$$P_{2r,2s+1} := N(Q_{2r+1}, Q_{2r+3}, \dots, Q_{2s+1} \text{ in } P_{2r}), \tag{5.22b}$$

for all $i, j, r, s$ with $1 \leq i \leq l$, $i \leq j \leq k$, $1 \leq r \leq k$ and $r \leq s \leq l-1$. Note that, in view of (5.10a) and (5.10b), the products $P_{2i} \cdot P_{2i+2} \cdots P_{2j}$ and $Q_{2r+1} \cdot Q_{2r+3} \cdots Q_{2s+1}$ form groups, for all $i, j, r, s$ as above. This, along with (5.22), implies that

$$\begin{aligned} Q_{2i-1,2j} &= N(P_{2i}, P_{2i+2}, \dots, P_{2j} \text{ in } Q_{2i-1}) \\ &= C(P_{2i}, P_{2i+2}, \dots, P_{2j} \text{ in } Q_{2i-1}) = C(P_{2i} \cdot P_{2i+2} \cdots P_{2j} \text{ in } Q_{2i-1}), \end{aligned} \tag{5.23a}$$

and

$$\begin{aligned} P_{2r,2s+1} &= N(Q_{2r+1}, Q_{2r+3}, \dots, Q_{2s+1} \text{ in } P_{2r}) \\ &= C(Q_{2r+1}, Q_{2r+3}, \dots, Q_{2s+1} \text{ in } P_{2r}) = C(Q_{2r+1} \cdot Q_{2r+3} \cdots Q_{2s+1} \text{ in } P_{2r}), \end{aligned} \tag{5.23b}$$

whenever $1 \leq i \leq l$, $i \leq j \leq k$, $1 \leq r \leq k$ and $r \leq s \leq l-1$. Furthermore, the way the groups $Q_{2i-1,2j}$ and $P_{2r,2s+1}$ are defined, along with (5.10), implies that

$$Q_{2i-1} \text{ normalizes the groups } Q_{2t-1,2j} \text{ and } P_{2t,2j+1}, \tag{5.24a}$$

whenever $1 \leq t \leq j \leq i-1 \leq l-1$. Similarly,

$$P_{2r} \text{ normalizes the groups } Q_{2t+1,2j} \text{ and } P_{2t,2j-1}, \tag{5.24b}$$

whenever $1 \leq t < j \leq r \leq k$.

Looking at the diagrams (5.20a) and (5.21a), we seee that what (5.10) and (5.24) say is that any group on the main diagonal of these diagrams, that is $Q_{2i-1}$ or $P_{2r}$, normalizes all the other groups that lie below or to its right. They also say that $Q_{2i-1}$ normalizes all the groups in (5.21a) which are below or to the right of $P_{2i-2,2i-1}$, while $P_{2r}$ normalizes all the groups in (5.20a) which are below or to the right of $Q_{2r-1,2r}$.

Furthermore, in the case that $j > i$ and $s > r$ (with $i, j, s, r$ as in (5.23)), the groups $Q_{2i-1,2j-2}$ and $P_{2r,2s-1}$ satisfy the equations (5.23a) and (5.23b), respectively. Hence

$$Q_{2i-1,2j} = N(P_{2j} \text{ in } N(P_{2i}, P_{2i+2}, \dots, P_{2j-2} \text{ in } Q_{2i-1})) = N(P_{2j} \text{ in } Q_{2i-1,2j-2}).$$

But $P_{2j}$ normalizes $Q_{2i-1,2j-2}$ by (5.24b). Therefore

$$Q_{2i-1,2j} = N(P_{2j} \text{ in } Q_{2i-1,2j-2}) = C(P_{2j} \text{ in } Q_{2i-1,2j-2}), \tag{5.25a}$$

and similarly for the $\pi$-groups

$$P_{2r,2s+1} = N(Q_{2s+1} \text{ in } P_{2r,2s-1}) = C(Q_{2s+1} \text{ in } P_{2r,2s-1}), \tag{5.25b}$$

whenever $1 \leq i \leq l$, $i < j \leq k$, $1 \leq r \leq k$, and $r < s \leq l-1$.

According to (5.17e) and (5.17c), the groups $Q_{2i-1}$ and $P_{2r}$ were chosen to be $\pi'$-Hall and $\pi$-Hall subgroups of specific "stabilizer"-subgroups of $G_{2i-1}$ and $G_{2r}$, respectively. A similar characterization for the groups $Q_{2i-1,2j}$ and $P_{2r,2s+1}$ is described and proved in

**Proposition 5.26.** *For every $i, j$ with $1 \le i \le j \le l-1$ the following holds*

$$Q_{2i-1,2j} = Q_{2j+1} \cap G_{2i-1} \text{ and therefore} \tag{5.27a}$$

$$\begin{aligned} Q_{2i-1,2j} &\in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \dots, \alpha_{2j}, \beta_1, \dots, \beta_{2j-1})) \\ &= \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \dots, \alpha_{2j}, \beta_1, \dots, \beta_{2j-1}, \beta_{2j+1})). \end{aligned} \tag{5.27b}$$

*Similarly, for all $r, s$ with $1 \le r \le s \le k-1$ we have*

$$P_{2r,2s+1} = P_{2s+2} \cap G_{2r} \text{ and therefore} \tag{5.28a}$$

$$\begin{aligned} P_{2r,2s+1} &\in \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2, \dots, \alpha_{2s}, \beta_1, \dots, \beta_{2s+1})) \\ &= \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2, \dots, \alpha_{2s}, \alpha_{2s+2}, \beta_1, \dots, \beta_{2s+1})). \end{aligned} \tag{5.28b}$$

Note that the extra groups $Q_{2i-1,2l}$ (when $m = 2l = 2k$), and $P_{2r,2k+1}$ (when $m = 2l-1 = 2k+1$) are not covered in Proposition 5.26.

*Proof.* The definition of $Q_{2j+1}$ in (5.9e), along with the fact that $G_{2i-1}$ is a normal subgroup of $G_{2j+1}$ whenever $1 \le i \le j \le l-1$, implies that

$$Q_{2j+1} \cap G_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \dots, \alpha_{2j}, \beta_1, \dots, \beta_{2j-1})). \tag{5.29}$$

But $Q_{2j+1}$ also fixes $\beta_{2j+1}$. Hence

$$Q_{2j+1} \cap G_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \dots, \alpha_{2j}, \beta_1, \dots, \beta_{2j-1}, \beta_{2j+1})). \tag{5.30}$$

In particular, $Q_{2j+1} \cap G_{2i-1}$ is a $\pi'$-subgroup of $G_{2i-1}(\alpha_2, \dots, \alpha_{2i-2}, \beta_1, \dots, \beta_{2i-3})$. Furthermore, $Q_{2j+1}$ fixes $\beta_{2i-1}$, and so normalizes $Q_{2i-1}$ But the latter is a $\pi'$-Hall subgroup of the group $G_{2i-1}(\alpha_2, \dots, \alpha_{2i-2}, \beta_1, \dots, \beta_{2i-3})$. Therefore $Q_{2j+1} \cap G_{2i-1} \le Q_{2i-1}$. As $Q_{2j+1}$ normalizes the groups $P_2, \dots, P_{2j}$, by (5.10a), we conclude that $Q_{2j+1} \cap G_{2i-1}$ is a subgroup of $Q_{2i-1,2j} = N(P_{2i}, \dots, P_{2j} \text{ in } Q_{2i-1})$. Hence for all $i, j$ with $1 \le i \le j \le l-1$ we have

$$Q_{2j+1} \cap G_{2i-1} \le Q_{2i-1,2j}. \tag{5.31}$$

To prove the opposite inclusion, and complete the proof of (5.27), we will use induction on $j$. According to Lemma 5.12 we have $Q_{2i-1,2i} = Q_{2i+1} \cap G_{2i-1}$, for all $i$ with $1 \le i \le l-1$. Hence the proposition holds in the case that $i = j$.

Suppose that, for some fixed $r = i+1, \dots, l-1$ and for all $j$ with $1 \le i \le j < r$, we have $Q_{2i-1,2j} \le Q_{2j+1} \cap G_{2i-1}$ (and thus equality as the other inclusion is proved). Then according to (5.25a), we have $Q_{2i-1,2r} = C(P_{2r} \text{ in } Q_{2i-1,2r-2})$. By our supposition $Q_{2i-1,2r-2}$ is a subgroup of $Q_{2r-1}$. Hence $Q_{2i-1,2r} \le C(P_{2r} \text{ in } Q_{2r-1})$. But $C(P_{2r} \text{ in } Q_{2r-1}) = Q_{2r-1,2r}$, by (5.23a). Therefore $Q_{2i-1,2r} \le Q_{2r-1,2r}$. Furthermore, $Q_{2r-1,2r} \le Q_{2r+1}$, by Lemma 5.12. Hence $Q_{2i-1,2r} \le Q_{2r+1}$. This proves the inductive argument in the case that $j = r$. Hence $Q_{2i-1,2j} \le Q_{2j+1} \cap G_{2i-1}$ whenever $1 \le i \le j \le l-1$. This, along with (5.29), (5.30) and (5.31), completes the proof of (5.27).

The proof for (5.28) is similar, so we omit it. As a final remark, we observe that the only tools we used for the proof of Proposition 5.26 are the definitions of the groups $Q_{2i-1}, P_{2r}$, in (5.17e) and (5.17c), and the definitions of the groups $Q_{2i-1,2j}$ and $P_{2r,2s+1}$ in (5.22). $\square$

Proposition 5.26 implies

**Corollary 5.32.** *For all $i, j, r, s$ with $1 \leq i \leq j \leq l-1$ and $1 \leq r \leq s \leq k-1$ we have*

$$Q_{2i-1} \cap Q_{2j+1} = Q_{2i-1,2j} \text{ and } P_{2r} \cap P_{2s+2} = P_{2r,2s+1}.$$

*Therefore*

$$\begin{aligned} Q_{2i-1,2j} &= N(P_{2j} \text{ in } Q_{2i-1,2j-2}) = C(P_{2j} \text{ in } Q_{2i-1,2j-2}) \\ &= N(P_{2i}, \ldots, P_{2j} \text{ in } Q_{2i-1}) = C(P_{2i}, \ldots, P_{2j} \text{ in } Q_{2i-1}) \\ &= C(P_{2i} \cdot P_{2i+2} \cdots P_{2j} \text{ in } Q_{2i-1}) = Q_{2i-1} \cap Q_{2j+1}, \end{aligned} \tag{5.33}$$

*and*

$$\begin{aligned} P_{2r,2s+1} &= N(Q_{2s+1} \text{ in } P_{2r,2s-1}) = C(Q_{2s+1} \text{ in } P_{2r,2s-1}) \\ &= N(Q_{2r+1}, \ldots, Q_{2s+1} \text{ in } P_{2r}) = C(Q_{2r+1}, \ldots, Q_{2s+1} \text{ in } P_{2r}) \\ &= C(Q_{2r+1} \cdot Q_{2r+3} \cdots Q_{2s+1} \text{ in } P_{2r}) = P_{2r} \cap P_{2s+2}, \end{aligned} \tag{5.34}$$

*where, by convention, we write $Q_{2i-1,2i-2} = Q_{2i-1}$ and $P_{2r,2r-1} = P_{2r}$. Furthermore, for any $t, t'$ with $i \leq t \leq j$ and $r \leq t' \leq s$, where $i, j, r, s$ are as above, we have*

$$Q_{2i-1,2j} \trianglelefteq Q_{2t-1,2j} \trianglelefteq Q_{2j+1} \text{ and } P_{2r,2s+1} \trianglelefteq P_{2t',2s+1} \trianglelefteq P_{2s+2}. \tag{5.35}$$

*Similarly for the extra groups $Q_{2i-1,2l}$ and $P_{2i,2k+1}$ we have*

$$\begin{aligned} &Q_{2i-1,2l} \trianglelefteq Q_{2t-1,2l} && \text{when } m = 2k \text{ and thus } k = l, \\ &P_{2r,2k+1} \trianglelefteq P_{2t',2k+1} && \text{when } m = 2l-1 \text{ and thus } k = l-1, \end{aligned} \tag{5.36}$$

*whenever $1 \leq i \leq t \leq l$ and $1 \leq r \leq t' \leq k$.*

*Proof.* The first part follows easily from Proposition 5.26 and the two sets of inclusions $Q_{2i-1,2j} \leq Q_{2i-1} \leq G_{2i-1}$ and $P_{2r,2s+1} \leq P_{2r} \leq G_{2r}$. The multiple equations (5.33) and (5.34) are a collection of (5.23a), (5.25a), (5.23b) and (5.25b). Also (5.35) follows directly from Proposition 5.26, since $G_{2i-1} \trianglelefteq G_{2t-1}$ and $G_{2r} \trianglelefteq G_{2t'}$ whenever $1 \leq i \leq t \leq l-1$ and $1 \leq r \leq t' \leq k-1$. It remains to show that (5.36) also holds for the extra groups (whenever these exist) $Q_{2i-1,2l}$ and $P_{2r,2k+1}$. Indeed, in the case that $m = 2k$ is even (and so $k = l$) the groups $Q_{2i-1,2l}$ are well defined (see (5.22a)) for all $i = 1, \ldots, l$. Furthermore, (5.25a) implies that $Q_{2i-1,2l} = N(P_{2l} \text{ in } Q_{2i-1,2l-2})$ for all $i = 1, \ldots, l-1$. Since $Q_{2i-1,2l-2} \trianglelefteq Q_{2t-1,2l-2} \trianglelefteq Q_{2l-1}$ whenever $1 \leq i \leq t \leq l-1$, we easily have that

$$Q_{2i-1,2l} = N(P_{2l} \text{ in } Q_{2i-1,2l-2}) \trianglelefteq N(P_{2l} \text{ in } Q_{2t-1,2l-2}) = Q_{2t-1,2l} \trianglelefteq N(P_{2l} \text{ in } Q_{2l-1}) = Q_{2l-1,2l},$$

for all such $i$ and $t$. This proves (5.36) for the $\pi'$-groups. The proof for the $\pi$-groups (that occurs when $m = 2l-1 = 2k+1$) is similar. So we omit it. ∎

The following proposition covers the extra groups that Proposition 5.26 left out.

**Proposition 5.37.** *For every $i, r$ with $1 \leq i \leq l$ and $1 \leq r \leq k$ we have*

$$\begin{aligned} &Q_{2i-1,2l} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \ldots, \alpha_{2l}, \beta_1, \ldots, \beta_{2l-1})) && \textit{if } m = 2k = 2l, \\ &P_{2r,2k+1} \in \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2, \ldots, \alpha_{2k}, \beta_1, \ldots, \beta_{2k+1})) && \textit{if } m = 2l-1 = 2k+1 \ . \end{aligned} \tag{5.38}$$

*Proof.* Assume that $m = 2k = 2l$ is even. Then for all $i = 1, \ldots, l$ the groups $Q_{2i-1,2l} = C(P_{2i}, \ldots, P_{2l} \text{ in } Q_{2i-1})$ are well defined (see (5.22a)). By (5.17e) we have that $Q_{2l-1}$ is a $\pi'$-Hall subgroup of $G_{2l-1}(\alpha_2, \ldots, \alpha_{2l-2}, \beta_1, \ldots, \beta_{2l-3})$. Thus $Q_{2l-1}$ is also a $\pi'$-Hall subgroup of $G_{2l-1}(\alpha_2, \ldots, \alpha_{2l-2}, \beta_1, \ldots, \beta_{2l-1})$, as $Q_{2l-1}$ fixes $\beta_{2l-1}$. Furthermore, according to (5.28) for $r = s = k-1$, we get $P_{2l-2,2l-1} = P_{2k-2,2k-1} \in \mathrm{Hall}_{\pi}(G_{2l-2}(\alpha_2, \ldots, \alpha_{2l-2}, \beta_1, \ldots, \beta_{2l-1}))$. Thus $P_{2l-2,2l-1}$ is also a $\pi$-Hall subgroup of $G_{2l-1}(\alpha_2, \ldots, \alpha_{2l-2}, \beta_1, \ldots, \beta_{2l-1})$, since $G_{2l-1}/G_{2l-2}$ is a $\pi'$-group. Since $P_{2l-2,2l-1} = C(Q_{2l-1} \text{ in } P_{2l-2})$, we have

$$G_{2l-1}(\alpha_2, \ldots, \alpha_{2l-2}, \beta_1, \ldots, \beta_{2l-1}) = Q_{2l-1} \times P_{2l-2,2l-1}.$$

This implies that $Q_{2l-1}(\alpha_{2l})$ is a $\pi'$-Hall subgroup of $G_{2l-1}(\alpha_2, \ldots, \alpha_{2l-2}, \alpha_{2l}, \beta_1, \ldots, \beta_{2l-1})$. Furthermore, $Q_{2l-1}(\alpha_{2l}) \leq N(P_{2l} \text{ in } Q_{2l-1}) = C(P_{2l} \text{ in } Q_{2l-1}) \leq Q_{2l-1}(\alpha_{2l})$. Hence $Q_{2l-1,2l} = N(P_{2l} \text{ in } Q_{2l-1})$ is a $\pi'$-Hall subgroup of $G_{2l-1}(\alpha_2, \ldots, \alpha_{2l}, \beta_1, \ldots, \beta_{2l-1})$. Thus (5.38) holds for $i = l$. Also for any $i = 1, \ldots, l-1$ we have

$$\begin{aligned} Q_{2i-1,2l} &= N(P_{2l} \text{ in } Q_{2i-1,2l-2}) && (\text{ by (5.25a)}) \\ &= N(P_{2l} \text{ in } Q_{2l-1} \cap G_{2i-1}) && (\text{ by (5.27a)}) \\ &= N(P_{2l} \text{ in } Q_{2l-1}) \cap G_{2i-1} \\ &= Q_{2l-1,2l} \cap G_{2i-1}. \end{aligned} \tag{5.39}$$

This, along with the facts that $G_{2i-1} \trianglelefteq G_{2l-1}$ and $Q_{2l-1,2l} \in \mathrm{Hall}_{\pi'}(G_{2l-1}(\alpha_2, \ldots, \alpha_{2l}, \beta_1, \ldots, \beta_{2l-1}))$, implies that

$$Q_{2i-1,2l} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2, \ldots, \alpha_{2l}, \beta_1, \ldots, \beta_{2l-1}))$$

whenever $1 \leq i \leq l$. Hence (5.38) holds in the case $m = 2k = 2l$.

Similarly we can work with the $\pi$-groups in the case of an odd $m = 2l - 1$. □

As a straight forward consequence of (5.39) and (5.27a) we have

**Remark 5.40.** For every $i = 1, \ldots, l$ and every $j, s$ with $1 \leq i \leq j \leq s \leq k$ the following holds:

$$Q_{2i-1,2s} = Q_{2j-1,2s} \cap G_{2i-1}.$$

Regarding the possible products of the groups $Q_{2i-1,2j}$ and $P_{2r,2s+1}$ we have

**Proposition 5.41.** *For every $i, r$ with $i = 2, \ldots, l$ and $r = 1, \ldots, k$, we have*

$$G_{2r}(\alpha_2, \ldots, \alpha_{2r-2}, \beta_1, \ldots, \beta_{2r-1}) = P_{2r} \ltimes Q_{2r-1}, \tag{5.42a}$$

$$G_{2r}(\alpha_2, \ldots, \alpha_{2r}, \beta_1, \ldots, \beta_{2r-1}) = P_{2r} \times Q_{2r-1,2r}, \tag{5.42b}$$

$$G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3}) = P_{2i-2} \rtimes Q_{2i-1}, \tag{5.42c}$$

$$G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-1}) = P_{2i-2,2i-1} \times Q_{2i-1}. \tag{5.42d}$$

*Furthermore*

$$G_{2r}(\alpha_2,\ldots,\alpha_{2s-2},\beta_1,\ldots,\beta_{2s-1}) = P_{2r,2s-1}\times Q_{2r-1,2s-2}, \tag{5.43a}$$
$$G_{2r}(\alpha_2,\ldots,\alpha_{2s},\beta_1,\ldots,\beta_{2s-1}) = P_{2r,2s-1}\times Q_{2r-1,2s}, \tag{5.43b}$$
$$G_{2i-1}(\alpha_2,\ldots,\alpha_{2j-2},\beta_1,\ldots,\beta_{2j-1}) = P_{2i-2,2j-1}\times Q_{2i-1,2j-2}, \tag{5.43c}$$
$$G_{2t-1}(\alpha_2,\ldots,\alpha_{2v},\beta_1,\ldots,\beta_{2v-1}) = P_{2t-2,2v-1}\times Q_{2t-1,2v}, \tag{5.43d}$$

*whenever* $1\le i<j\le l$, $\;1\le r<s\le k$ *and* $1\le t\le v\le k$.

Note that, according to (5.8), we have $k\le l\le k+1$. Hence all the above groups are well defined.

*Proof.* Clearly (5.17a) and (5.17b), along with the fact that $G_2/G_1$ is a $\pi$-group, imply

$$Q_1 = G_1 \in \mathrm{Hall}_{\pi'}(G_1)\cap \mathrm{Hall}_{\pi'}(G_2(\beta_1))\cap \mathrm{Hall}_{\pi'}(G_2(\alpha_0,\beta_1)). \tag{5.44}$$

In addition, for all $i=2,\ldots,k$ the factor group $G_{2i}/G_{2i-1}$ is a $\pi$-group. Furthermore, in view of (5.17e) the group $Q_{2i-1}$ is a $\pi'$-Hall subgroup of $G_{2i-1}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-3})$. Hence $Q_{2i-1}$ is also a $\pi'$-Hall subgroup of $G_{2i}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-3})$. As $Q_{2i-1}$ obviously fixes the character $\beta_{2i-1}\in\mathrm{Irr}(Q_{2i-1})$, we conclude that

$$\begin{aligned} Q_{2i-1} \in{}& \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-3}))\\ &\cap \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-1}))\\ &\cap \mathrm{Hall}_{\pi'}(G_{2i}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-3}))\\ &\cap \mathrm{Hall}_{\pi'}(G_{2i}(\alpha_2,\ldots,\alpha_{2i-2},\beta_1,\ldots,\beta_{2i-1})), \end{aligned} \tag{5.45a}$$

whenever $1\le i\le k$, while

$$\begin{aligned} Q_{2l-1} \in{}& \mathrm{Hall}_{\pi'}(G_{2l-1}(\alpha_2,\ldots,\alpha_{2l-2},\beta_1,\ldots,\beta_{2l-3}))\\ &\cap \mathrm{Hall}_{\pi'}(G_{2l-1}(\alpha_2,\ldots,\alpha_{2l-2},\beta_1,\ldots,\beta_{2l-1})). \end{aligned} \tag{5.45b}$$

Note that we need to include as a special case the group $Q_{2l-1}$, since it is not covered when $m=2l-1$ is odd.

Similarly for the $\pi$-groups we have

$$\begin{aligned} P_{2r} \in{}& \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2,\ldots,\alpha_{2r-2},\beta_1,\ldots,\beta_{2r-1}))\\ &\cap \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2,\ldots,\alpha_{2r},\beta_1,\ldots,\beta_{2r-1}))\\ &\cap \mathrm{Hall}_{\pi}(G_{2r+1}(\alpha_2,\ldots,\alpha_{2r-2},\beta_1,\ldots,\beta_{2r-1}))\\ &\cap \mathrm{Hall}_{\pi}(G_{2r+1}(\alpha_2,\ldots,\alpha_{2r},\beta_1,\ldots,\beta_{2r-1})), \end{aligned} \tag{5.46a}$$

whenever $1\le r\le l-1$, while

$$\begin{aligned} P_{2k} \in{}& \mathrm{Hall}_{\pi}(G_{2k}(\alpha_2,\ldots,\alpha_{2k-2},\beta_1,\ldots,\beta_{2k-1}))\\ &\cap \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2,\ldots,\alpha_{2k},\beta_1,\ldots,\beta_{2k-1})). \end{aligned} \tag{5.46b}$$

Furthermore, (5.27) and (5.38), along with the fact that $G_{2i}/G_{2i-1}$ is a $\pi$-group, imply that

$$\begin{aligned} Q_{2i-1,2j-2} \in\ &\mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2,\dots,\alpha_{2j-2},\beta_1,\dots,\beta_{2j-3})) \\ &\cap \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha_2,\dots,\alpha_{2j-2},\beta_1,\dots,\beta_{2j-1})) \\ &\cap \mathrm{Hall}_{\pi'}(G_{2i}(\alpha_2,\dots,\alpha_{2j-2},\beta_1,\dots,\beta_{2j-3})) \\ &\cap \mathrm{Hall}_{\pi'}(G_{2i}(\alpha_2,\dots,\alpha_{2j-2},\beta_1,\dots,\beta_{2j-1})), \end{aligned} \tag{5.47a}$$

whenever $1 \leq i < j \leq l$, while for all $t$ with $1 \leq t \leq l$ we have

$$\begin{aligned} Q_{2t-1,2l} \in\ &\mathrm{Hall}_{\pi'}(G_{2t-1}(\alpha_2,\dots,\alpha_{2l},\beta_1,\dots,\beta_{2l-1})) \\ &\cap \mathrm{Hall}_{\pi'}(G_{2t}(\alpha_2,\dots,\alpha_{2l},\beta_1,\dots,\beta_{2l-1})). \end{aligned} \tag{5.47b}$$

Similarly, (5.28), (5.38) and the fact that $G_{2r+1}/G_{2r}$ is a $\pi'$-group imply that

$$\begin{aligned} P_{2r,2s-1} \in\ &\mathrm{Hall}_{\pi}(G_{2r}(\alpha_2,\dots,\alpha_{2s-2},\beta_1,\dots,\beta_{2s-1})) \\ &\cap \mathrm{Hall}_{\pi}(G_{2r}(\alpha_2,\dots,\alpha_{2s},\beta_1,\dots,\beta_{2s-1})) \\ &\cap \mathrm{Hall}_{\pi}(G_{2r+1}(\alpha_2,\dots,\alpha_{2s-2},\beta_1,\dots,\beta_{2s-1})) \\ &\cap \mathrm{Hall}_{\pi}(G_{2r+1}(\alpha_2,\dots,\alpha_{2s},\beta_1,\dots,\beta_{2s-1})), \end{aligned} \tag{5.48a}$$

whenever $1 \leq r < s \leq k$, while for all $t'$ with $1 \leq t' \leq k$ we have

$$\begin{aligned} P_{2t',2k+1} \in\ &\mathrm{Hall}_{\pi}(G_{2t'}(\alpha_2,\dots,\alpha_{2k},\beta_1,\dots,\beta_{2k+1})) \\ &\cap \mathrm{Hall}_{\pi}(G_{2t'+1}(\alpha_2,\dots,\alpha_{2k},\beta_1,\dots,\beta_{2k+1})). \end{aligned} \tag{5.48b}$$

Furthermore, $P_{2r}$ normalizes $Q_{2r-1}$, while $Q_{2i-1}$ normalizes $P_{2i-2}$. Therefore (5.44), (5.45) and (5.46) imply that

$$P_{2r} \ltimes Q_{2r-1} = G_{2r}(\alpha_2,\dots,\alpha_{2r-2},\beta_1,\dots,\beta_{2r-1}),$$

and

$$P_{2i-2} \rtimes Q_{2i-1} = G_{2i-1}(\alpha_2,\dots,\alpha_{2i-2},\beta_1,\dots,\beta_{2i-3}),$$

for all $i = 2,\dots,l$ and $r = 1,\dots,k$. For the same range of $i$ and $r$ equations (5.11) and (5.14) imply that $Q_{2r-1,2r}$ centralizes $P_{2r}$ while $P_{2i-2,2i-1}$ centralizes $Q_{2i-1}$. As these groups are $\pi$-and $\pi'$-Hall subgroups of the correct groups (see (5.46), (5.45), (5.48) and (5.47)) equations (5.42b) and (5.42d) follow.

We can work similarly for the rest of the proposition. We only remark here that, whenever $1 \leq r < s \leq k$, $1 \leq i \leq j \leq l$ and $1 \leq t \leq v \leq k$, equations (5.23a) and (5.23b) imply that $Q_{2r-1,2s-2}$ and $Q_{2r-1,2s}$ centralize $P_{2r}$ (and thus $P_{2r,2s-1}$), while $P_{2i-2,2j-1}$ and $P_{2t-2,2v-1}$ centralize $Q_{2i-1}$ and $Q_{2t-1}$, respectively. This, along with (5.48) and (5.47), implies the rest of the proposition. ∎

What about the characters that appear in the diagrams (5.20b) and (5.21b)? We have already seen, in (5.17f) and (5.17d), that $\beta_{2i-1}$ and $\alpha_{2r}$ are irreducible characters of $Q_{2i-1}$ and $P_{2r}$, respectively. Furthermore, according to (5.17c), for every $i = 1,\dots,l$ the character $\beta_{2i-1}$ is fixed by the $\pi$-groups $P_{2i}, P_{2i+2},\dots,P_{2k}$, and thus is also fixed by their product (note that their product forms a group according to (5.10b)). Similarly whenever $r = 1,\dots,k$, using (5.17e), we see that the character $\alpha_{2r}$ is fixed by the groups $Q_{2r+1}, Q_{2r+3},\dots,Q_{2l-1}$, and therefore is also fixed by their product. Hence, we can naturally make the

**Definition 5.49.** (1) We write $\beta_{2i-1,2j} \in \mathrm{Irr}(Q_{2i-1,2j})$ for the $P_{2i} \cdot P_{2i+2} \cdots P_{2j}$-Glauberman correspondent of $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$ and

(2) we write $\alpha_{2r,2s+1} \in \mathrm{Irr}(P_{2r,2s+1})$ for the $Q_{2r+1} \cdot Q_{2r+3} \cdots Q_{2s+1}$-Glauberman correspondent of $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$,

whenever $1 \leq i \leq l$, $i \leq j \leq k$, $1 \leq r \leq k$ and $r \leq s \leq l-1$.

We remark that $C(P_{2i} \cdot P_{2i+2} \cdots P_{2j}$ in $Q_{2i-1}) = Q_{2i-1,2j}$, by (5.23a). Hence the $\beta_{2i-1,2j}$ are well defined irreducible characters of $Q_{2i-1,2j}$. Similarly we see that the characters $\alpha_{2r,2s+1}$ are also well defined.

As we did with the corresponding groups, starting from the above basic properties we will describe the relations these characters satisfy. Towards that direction we state and prove

**Proposition 5.50.** *The following holds:*

$$\beta_{2i-1,2j} \in \mathrm{Irr}(Q_{2i-1,2j}) \textit{ is the } P_{2j}\textit{-Glauberman correspondent of } \beta_{2i-1,2j-2} \in \mathrm{Irr}^{P_{2j}}(Q_{2i-1,2j-2}), \textit{ and lies above } \beta_{2i-3,2j}, \beta_{2i-5,2j}, \ldots, \beta_{1,2j}, \tag{5.51}$$

*whenever $1 \leq i \leq l$ and $i \leq j \leq k$. By convention we write $\beta_{2i-1,2i-2} := \beta_{2i-1}$ when $j = i$.*

*Similarly,*

$$\alpha_{2r,2s+1} \in \mathrm{Irr}(P_{2r,2s+1}) \textit{ is the } Q_{2s+1}\textit{-Glauberman correspondent of } \alpha_{2r,2s-1} \in \mathrm{Irr}^{Q_{2s+1}}(P_{2r,2s-1}), \textit{ and lies over } \alpha_{2r-2,2s+1}, \ldots, \alpha_{2,2s+1}, \tag{5.52}$$

*whenever $1 \leq r \leq k$ and $r \leq s \leq l-1$. By convention we write $\alpha_{2r,2r-1} := \alpha_{2r}$ when $r = s$.*

*Therefore,*

$$\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1} | \beta_{2i-3,2i-2}, \beta_{2i-5,2i-2}, \ldots, \beta_{1,2i-2}), \tag{5.53}$$

*and*

$$\alpha_{2r} \in \mathrm{Irr}(P_{2r} | \alpha_{2r-2,2r-1}, \alpha_{2r-4,2r-1}, \ldots, \alpha_{2,2r-1}), \tag{5.54}$$

*whenever $i = 1, \ldots, l$ and $r = 1, \ldots, k$.*

*Proof.* In view of Definition 5.49, it is easy to see that $\beta_{2i-1,2j}$ is the $P_{2j}$-Glauberman correspondent of $\beta_{2i-1,2j-2}$ (as the latter is the $P_{2i} \cdot P_{2i+2} \cdots P_{2j-2}$-Glauberman correspondent of $\beta_{2i-1}$), for all $i, j$ with $1 \leq i \leq l$ and $i < j \leq k$. We also remark that the same argument implies that $\beta_{2i-1,2j}$ is the $P_{2t} \cdot P_{2t+2} \cdots P_{2j}$-Glauberman correspondent of $\beta_{2i-1,2t-2}$, for any $t$ with $1 \leq i < t < j$. Furthermore, the same definition tells us that $\beta_{2j-1,2j}$ is the $P_{2j}$-Glauberman correspondent of $\beta_{2j-1,2j-2} = \beta_{2j-1}$, for all $j = 1, \ldots, k$.

Thus to prove (5.51) it suffices to show that $\beta_{2i-1,2j}$ lies over $\beta_{2i-3,2j}, \ldots, \beta_{1,2j}$, for all $i, j$ with $1 \leq i \leq l$ and $i \leq j \leq k$. For this we will use induction on $i$. For $i = 1$, it holds vacuously, since the character $\beta_{2i-3,2j}$ doesn't exist. The first interesting case appears when $i = 2$. According to (5.17f) the character $\beta_3$ lies above $\beta_{1,2}$. Therefore, for any $j = 2, \ldots, k$, the $P_4 \cdots P_{2j}$-Glauberman correspondent $\beta_{3,2j}$ of $\beta_3$ lies above the $P_4 \cdots P_{2j}$-Glauberman correspondent $\beta_{1,2j}$ of $\beta_{1,2}$.

For the inductive step the argument is similar. If $i \geq 3$ and $\beta_{2i-3,2j}$ lies above $\beta_{2i-5,2j}, \ldots, \beta_{1,2j}$ for all $j = i-1, \ldots, k$ then $\beta_{2i-3,2i-2}$ lies above $\beta_{2i-5,2i-2}, \ldots, \beta_{1,2i-2}$. According to (5.17f), the character $\beta_{2i-1}$ was picked to lie above $\beta_{2i-3,2i-2}$. Therefore the $P_{2i} \cdot P_{2i+2} \cdots P_{2j}$-Glauberman correspondent $\beta_{2i-1,2j}$ of $\beta_{2i-1}$ lies above the $P_{2i} \cdot P_{2i+2} \cdots P_{2j}$-Glauberman correspondent $\beta_{2i-3,2j}$ of $\beta_{2i-3,2i-2}$, for any $j$ with $j = i, \ldots, k$. Hence, $\beta_{2i-1,2j}$ lies above $\beta_{2i-3,2j}, \beta_{2i-5,2j}, \ldots, \beta_{1,2j}$ whenever $j = i, \ldots, k$. This completes the inductive argument on $i$, thus proving (5.51).

As $\beta_{2i-1}$ lies above $\beta_{2i-3,2i-2}$ (by (5.17f)), (5.53) is an immediate consequence of (5.51).

The proof of (5.52) and (5.54) is similar. □

Looking at the "character triangles " (5.21b) and (5.20b), we can translate Proposition 5.50 as follows:

Every horizontal line in the triangles (5.21b) and (5.20b) (with the characters in parenthesis included) is formed by taking a character that is a Glauberman correspondent of the previous one. Also the vertical lines in these two triangles are formed by characters that are lying one above the other. We can say even more:

**Proposition 5.55.** *For every $i, j, t$ with $1 \leq i \leq t \leq l$ and $t \leq j \leq k$, the group $Q_{2t-1,2j}$ fixes the character $\beta_{2i-1,2j}$. Hence $\beta_{2i-1,2j}$ is the unique character in $\mathrm{Irr}(Q_{2i-1,2j})$ lying under $\beta_{2t-1,2j} \in \mathrm{Irr}(Q_{2t-1,2j})$. In addition, for every $i, j$ with $1 \leq i \leq j \leq l$, the group $Q_{2j-1}$ fixes the character $\beta_{2i-1,2j-2}$. Hence $\beta_{2i-1,2j-2}$ is the unique character in $\mathrm{Irr}(Q_{2i-1,2j-2})$ lying under $\beta_{2j-1} \in \mathrm{Irr}(Q_{2j-1})$.*

*Similarly, for every $r, s, t$ with $1 \leq r \leq t \leq k$ and $t \leq s \leq l-1$, the group $P_{2t,2s+1}$ fixes the character $\alpha_{2r,2s+1}$. Therefore, $\alpha_{2r,2s+1}$ is the unique character of $P_{2r,2s+1}$ that lies under $\alpha_{2t,2s+1} \in \mathrm{Irr}(P_{2t,2s+1})$. In addition, for every $r, s$ with $1 \leq r \leq s \leq k$, the group $P_{2s}$ fixes the character $\alpha_{2r,2s-1}$. Hence $\alpha_{2r,2s-1}$ is the unique character in $\mathrm{Irr}(P_{2r,2s-1})$ lying under $\alpha_{2s} \in \mathrm{Irr}(P_{2s})$.*

*Proof.* Because of symmetry it suffices to prove the proposition for the $q$-groups $Q_{2t-1,2j}$ and the characters $\beta_{2i-1,2j}$, for fixed $i, t, j$ in the range of the proposition.

If $1 \leq i \leq t \leq l$ and $t \leq j \leq k$, then equations (5.35) and (5.36) imply that $Q_{2i-1,2j}$ is a normal subgroup of $Q_{2t-1,2j}$. Equation (5.35) also implies that $Q_{2i-1,2j} \trianglelefteq Q_{2j-1}$, whenever $1 \leq i < j \leq l$. Therefore, according to Clifford's Theorem, it is enough to prove that $Q_{2t-1,2j}(\beta_{2i-1,2j}) = Q_{2t-1,2j}$ and $Q_{2j-1}(\beta_{2i-1,2j-2}) = Q_{2j-1}$, in order to complete the proof of Proposition 5.55.

In view of (5.17e) the group $Q_{2t-1}$ fixes $\beta_{2i-1}$. According to (5.10a), the group $Q_{2t-1}$ normalizes the groups $P_{2i}, \ldots, P_{2t-2}$. Hence its subgroup $Q_{2t-1,2j} = C(P_{2t}, \ldots, P_{2j}$ in $Q_{2t-1})$ normalizes the groups $P_{2i}, \ldots, P_{2t-2}$, centralizes $P_{2t}, \ldots, P_{2j}$, and fixes $\beta_{2i-1}$. Therefore $Q_{2t-1,2j}$ fixes $\beta_{2i-1,2j}$, which is the $P_{2i}, \ldots, P_{2j}$-Glauberman correspondent of $\beta_{2i-1}$ by (5.51). So $Q_{2t-1,2j}(\beta_{2i-1,2j}) = Q_{2t-1,2j}$

Similarly (5.17e) and (5.10a) imply that $Q_{2j-1}$ fixes $\beta_{2i-1}$ and normalizes $P_{2i}, \ldots, P_{2j-2}$, whenever $1 \leq i < j \leq l$. Hence $Q_{2j-1}$ fixes $\beta_{2i-1,2j-2}$. So $Q_{2j-1}(\beta_{2i-1,2j-2}) = Q_{2j-1}$ and the proposition follows. □

## 5.3 From towers to triangles

We are now ready to prove one direction of the correspondence in Theorem 5.6. In particular, we will prove that for any character tower of (5.2) there is a corresponding $G(\chi_1, \ldots, \chi_m)$-conjugacy class of triangular sets for (5.2). The explicit relation between the character towers and the triangular sets is described in Theorem (5.88) below. Before we give the inductive proof of that theorem, we will demonstrate, for clarity, how the correspondence works in the special cases where $m = 1, 2, 3$.

We begin with a lemma that is an easy application of Theorems 3.11 and 3.13.

**Lemma 5.56.** *Let $G$ be a finite group of odd order, and $\pi$ be any set of primes. Suppose that $N, K_1, K_2, \ldots, K_r$ are normal subgroups of $G$, for some $r \geq 1$, such that $N \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_r$. Assume further that $N = A \ltimes B$, where $B$ is a normal $\pi'$-subgroup of $G$, and $A$ is any $\pi$-subgroup of $N$. Let $\chi \in \mathrm{Irr}(N)$ be a $\pi$-factorable character of $N$. Assume that $\chi = \alpha \cdot \beta^e$ is the decomposition of $\chi$ to its $\pi$- and $\pi'$-special parts respectively, where $\beta^e$ is the canonical extension to $N$ of an irreducible $A$-invariant character $\beta \in \mathrm{Irr}^A(B)$. Let $K_i(\chi)$ be the stabilizer of $\chi$ in $K_i$, for $i = 1, \ldots, r$, and $C = C(A \text{ in } B)$ be the centralizer of $A$ in $B$.*

*Then there is a one-to-one correspondence between the character towers $\{\chi, \chi_1, \ldots, \chi_r\}$ of the series $N \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_r$, starting with $\chi$, and the character towers $\{\alpha \times \gamma, \Psi_1, \ldots, \Psi_r\}$ of the series $N(A \text{ in } N) = C \times A \trianglelefteq N(A \text{ in } K_1(\chi)) \trianglelefteq N(A \text{ in } K_2(\chi)) \trianglelefteq \cdots \trianglelefteq N(A \text{ in } K_r(\chi))$, starting with $\alpha \times \gamma \in \mathrm{Irr}(C \times A)$, where $\gamma \in \mathrm{Irr}(C)$ is the $A$-Glauberman correspondent of $\beta \in \mathrm{Irr}^A(B)$. Furthermore, for any subgroup $M$ of $N(A \text{ in } G)$ we have*

$$M(\chi, \chi_1, \ldots, \chi_r) = M(\alpha \times \gamma, \Psi_1, \ldots, \Psi_r).$$

*Proof.* Let

$$\{\chi, \chi_1, \ldots, \chi_r\}, \tag{5.57}$$

be a character tower of the normal series $N \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_r$, starting with $\chi$. According to Clifford's theorem, for every $i = 1, \ldots, r$ there exists a unique irreducible character $\chi_i^* \in \mathrm{Irr}(K_i(\chi))$ that induces $\chi_i \in \mathrm{Irr}(K_i)$ and lies above $\chi \in \mathrm{Irr}(N)$. Furthermore, the characters

$$\{\chi, \chi_1^*, \ldots, \chi_r^*\}, \tag{5.58}$$

form a tower for the normal series $N \trianglelefteq K_1(\chi) \trianglelefteq \cdots \trianglelefteq K_r(\chi)$. Hence (5.57) corresponds to (5.58). Clifford's Theorem also implies that this correspondence between (5.57) and (5.58) is invariant under any subgroup of $G(\chi)$. So, in particular,

$$G(\chi, \chi_1, \ldots, \chi_r) = G(\chi, \chi_1^*, \ldots, \chi_r^*). \tag{5.59}$$

To complete the proof of the lemma we only need to observe that Theorem 3.13 can be applied to the tower (5.58) and the normal series $N = A \ltimes B \trianglelefteq K_1(\chi) \trianglelefteq \cdots \trianglelefteq K_r(\chi)$. Note also that, in view of Theorem 3.11, the $A$-correspondent of the irreducible character $\chi = \alpha \cdot \beta^e \in \mathrm{Irr}(N)$ is ofthe form $\chi_{(A)} = \alpha \times \gamma \in \mathrm{Irr}(A \times C)$, where $\gamma \in \mathrm{Irr}(C)$ is the $A$-Glauberman correspondent of $\beta \in \mathrm{Irr}^A(B)$. Hence the character tower (5.58) has a unique $A$-correspondent character tower

$$\{\alpha \times \gamma, \Psi_1, \ldots, \Psi_r\} \tag{5.60}$$

of the series

$$A \times C \trianglelefteq N(A \text{ in } K_1(\chi)) \trianglelefteq \cdots \trianglelefteq N(A \text{ in } K_r(\chi)).$$

This way we have created a correspondence, that is the combination of a Clifford and an $A$-correspondence, between the tower (5.57) and the (5.60).

Furthermore, as $\gamma$ is the $A$-Glauberman correspondent of $\beta$ and $B$ is normal in $G$, we have that $M(\beta) = M(\gamma)$, for any group $M$ with $M \leq N(A \text{ in } G)$. Also $M(\beta) = M(\beta^e)$ as $N \trianglelefteq G$. Hence we conclude that

$$M(\chi) = M(\alpha \cdot \beta^e) = M(\alpha, \beta) = M(\alpha \times \gamma). \tag{5.61}$$

Furthermore, Theorem 3.13 implies that for all $M$ with $M \leq N(A \text{ in } G)$ we have

$$M(\chi, \chi_1^*, \dots, \chi_r^*) = M(\alpha \times \gamma, \Psi_1, \dots, \Psi_r), \tag{5.62}$$

as $M$ normalizes $N, K_1, \dots, K_r$. Therefore we have

$$\begin{aligned} M(\chi, \chi_1, \dots, \chi_r) &= M(\chi, \chi_1^*, \dots, \chi_r^*) && \text{by (5.59)} \\ &= M(\chi)(\chi, \chi_1^*, \dots, \chi_r^*) \\ &= M(\chi)(\alpha \times \gamma, \Psi_1, \dots, \Psi_r) && \text{by (5.62)} \\ &= M(\alpha \times \gamma)(\alpha \times \gamma, \Psi_1, \dots, \Psi_r) && \text{by (5.61)} \\ &= M(\alpha \times \gamma, \Psi_1, \dots, \Psi_r). \end{aligned}$$

This completes the proof of the lemma. □

**Definition 5.63.** For the rest of this thesis, the correspondence between towers

$$\{\chi = \alpha \cdot \beta^e, \chi_1, \dots, \chi_r\} \leftrightarrow \{\alpha \times \gamma, \Psi_1, \dots, \Psi_r\}$$

that is described in Lemma 5.56, will be called a *cA-correspondence* (Clifford-$A$). We call the tower $\{\alpha \times \gamma, \Psi_1, \dots, \Psi_r\}$ the *cA-correspondent* of $\{\chi = \alpha \cdot \beta^e, \chi_1, \dots, \chi_r\}$. Similarly, we call $\Psi_i$ the *cA-correspondent* of $\chi_i$, for all $i = 1, \dots, r$.

We can now look at the cases $m = 1, 2, 3$. If $m = 1$ then the normal series (5.2) consists of the groups $1 = G_0 \trianglelefteq G_1 \trianglelefteq G$. So any character tower $\{1 = \chi_0, \chi_1\}$, of this series determines the triangular set $\{1 = P_0, Q_1 = G_1 | 1 = \alpha_0, \beta_1 = \chi_1\}$. Furthermore, assume that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G \tag{5.64}$$

is a normal series of $G$, for some $n \geq m = 1$, that extends the series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G$. Assume further that we have an extension of the character tower $\{1 = \chi_0, \chi_1\}$ to a character tower $\{1 = \chi_0, \chi_1, \dots, \chi_n\}$ for the series (5.64), so that Hypothesis 5.1 holds. As $\chi_1 = \beta_1$, we have that $G_i(\chi_1) = G_i(\beta_1)$ for all $i = 1, \dots, n$. Hence we can define the groups

$$G_{i,1} := G_i(\beta_1) = G_i(\chi_1) = N(Q_1 \text{ in } G_i(\chi_1)), \tag{5.65a}$$

where the last equality holds as $Q_1 = G_1 \trianglelefteq G$. By convention, whenever we have a series as in (5.64), we will write

$$G_\infty = G. \tag{5.65b}$$

With this notation, we can also write $G_{\infty,1}$ for the stabilizer $G(\chi_1) = G(\beta_1)$. Therefore the series

$$1 = G_{0,1} \trianglelefteq G_{1,1} \trianglelefteq \cdots \trianglelefteq G_{n,1} \trianglelefteq G_{\infty,1}, \tag{5.66}$$

is a normal series of $G_{\infty,1}$. Furthermore, for any $i = 1, 2, \dots, n$, Clifford's theorem applied to the groups $G_1 \trianglelefteq G_i$ and the characters $\chi_1, \chi_i$ implies the existence of a unique character $\chi_{i,1} \in \mathrm{Irr}(G_i(\chi_1))$ that lies above $\chi_1$ and induces $\chi_i$, i.e,

$$\chi_{i,1} \in \mathrm{Irr}(G_i(\chi_1)) \text{ is the } \chi_1\text{-Clifford correspondent of } \chi_i \in \mathrm{Irr}(G_i|\chi_1). \tag{5.67}$$

Note that $\chi_{1,1} = \chi_1$. We write $\chi_{0,1} = 1$. Then it is clear that $\chi_{i,1}$ lies above $\chi_{k,1}$ whenever $1 \leq k \leq i \leq n$. This way we have created a tower $\{1 = \chi_0, \chi_{1,1} = \chi_1, \chi_{2,1}, \dots, \chi_{n,1}\}$ for the series (5.66), fully determined by the character tower $\{1 = \chi_0, \chi_1, \dots, \chi_n\}$. Furthermore, Clifford's theorem implies that for any subgroup $M$ of $G = N(Q_1 \text{ in } G)$ we have

$$M(\chi_1, \chi_2, \dots, \chi_k) = M(\chi_{1,1}, \chi_{2,1}, \dots, \chi_{k,1}), \tag{5.68}$$

for any $k = 1, 2, \dots, n$. Therefore, in the case where $m = 1$, in addition to the correspondence between towers and triangular sets for (5.2), we proved that any tower of (5.64) determines a unique tower of (5.66). This is a property that, as we will see in Theorem 5.88, carries over to every $m$. By convention, we write this first correspondence as a $cQ_1$-correspondence (even though it is a Clifford correspondence). Table 5.1 describes exactly the above relations.

$$
\begin{array}{ccccc}
G = G_\infty & & & G_{\infty,1} := G(\chi_1) & \\
\Big| & & & \Big| & \\
G_n & \chi_n & & G_{n,1} := G_n(\chi_1) & \chi_{n,1} \\
\Big| & \Big| & & \Big| & \Big| \\
\vdots & \vdots & & \vdots & \vdots \\
\Big| & \Big| & & \Big| & \Big| \\
G_3 & \chi_3 & & G_{3,1} := G_3(\chi_1) & \chi_{3,1} \\
\Big| & \Big| & \overset{\longleftrightarrow}{cQ_1} & \Big| & \Big| \\
G_2 & \chi_2 & & G_{2,1} := G_2(\chi_1) & \chi_{2,1} \\
\Big| & \Big| & & \Big| & \Big| \\
G_1 = Q_1 & \chi_1 = \beta_1 & & G_{1,1} = G_1 = Q_1 & \chi_{1,1} = \chi_1 = \beta_1 \\
\Big| & \Big| & & \Big| & \Big| \\
G_0 = 1 & \chi_0 = 1 & & G_{0,1} = 1 & \chi_{0,1} = 1
\end{array}
$$

Table 5.1: The $cQ_1$-correspondence.

The first interesting case appears when $m = 2$. Here the normal series (5.2) consists of the groups $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2$. Let

$$\{1 = \chi_0, \chi_1, \chi_2\} \tag{5.69}$$

be a character tower of that series. We have already seen (from the case $m = 1$) that the subtower $\{1 = \chi_0, \chi_1\}$ of (5.69), determines the triangular set $\{P_0 = 1, Q_1 = G_1|\alpha_0 = 1, \beta_1 = \chi_1\}$. We expand this set by picking the $\pi$-group $P_2$ to be any $\pi$-Hall subgroup of $G_2(\chi_1) = G_2(\beta_1)$. (Note that to get a triangular set for the series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2$, it is enough to expand the existing set, $\{P_0 = 1, Q_1 = G_1 | \alpha_0 = 1, \beta_1 = \chi_1\}$, by a $\pi$-group $P_2$ along with its irreducible character $\alpha_2$, so that (5.17) will be valid for this new set.) Before we see how to chose the desired irreducible character $\alpha_2 \in \mathrm{Irr}(P_2)$, we observe the following: as $G_2/G_1$ is a $\pi$-group and $P_2 \in \mathrm{Hall}_\pi(G_2(\chi_1))$, we have that $P_2$ covers $G_2(\beta_1) = G_2(\chi_1)$ modulo $G_1$. Hence,

$$G_{2,1} = G_2(\chi_1) = G_2(\beta_1) = P_2 \ltimes G_1 = P_2 \ltimes Q_1. \tag{5.70}$$

In view of the work we did in the case $m = 1$, we have that, for every $n$ with $n \geq 2$, the character tower $\{\chi_0 = 1, \chi_1 = \beta_1, \chi_2, \dots, \chi_n\}$ of the normal series (5.64), extending the tower (5.69), has a unique $Q_1$-correspondent character tower $\{\chi_{0,1} = 1, \chi_{1,1} = \beta_1, \chi_{2,1}, \dots, \chi_{n,1}\}$, of the series (5.66) (see Table 5.1).

Furthermore, equation (5.70) permits us to apply Lemma 2.15 to the groups $G_{1,1} = G_1 = Q_1$, $G_{2,1} = G_2(\chi_1)$, $G_{\infty,1}$ and the character $\chi_1 = \beta_1$ in the place of the groups $N, H, G$ and the character $\theta$, respectively. (Note that in this case $H(\theta) = H$). Thus we conclude that $\chi_1 = \beta_1$ has a unique canonical extension $\beta_1^e \in \mathrm{Irr}(G_{2,1})$. As $\chi_{2,1} \in \mathrm{Irr}(G_{2,1})$ lies above $\chi_1 = \beta_1$, Lemma 2.15 also implies that there is a unique character $\alpha_2 \in \mathrm{Irr}(P_2)$ such that

$$\chi_{2,1} = \alpha_2 \cdot \beta_1^e, \tag{5.71a}$$

while

$$G_{\infty,1}(\alpha_2, \beta_1) = N(P_2 \text{ in } G_{\infty,1}(\beta_1, \chi_{2,1})). \tag{5.71b}$$

But

$$G_{\infty,1}(\beta_1) = G_{\infty,1} = G(\beta_1) = G(\chi_1). \tag{5.71c}$$

Furthermore, (5.68) for $k = 2$ implies that

$$G_\infty(\chi_1, \chi_2) = G_\infty(\chi_{1,1}, \chi_{2,1}) = G_\infty(\chi_1, \chi_{2,1}).$$

Therefore, (5.71b) and (5.71c) imply

$$\begin{aligned} G(\alpha_2, \beta_1) &= N(P_2 \text{ in } G_{\infty,1}(\chi_{2,1})) = N(P_2 \text{ in } G_{\infty,1}(\chi_2)) \\ &= N(P_2 \text{ in } G_\infty(\chi_1, \chi_{2,1})) = N(P_2 \text{ in } G_\infty(\chi_1, \chi_2)) = N(P_2 \text{ in } G(\chi_1, \chi_2)). \end{aligned} \tag{5.71d}$$

Hence, by intersecting both sides with $G_i$, we get

$$G_i(\alpha_2, \beta_1) = N(P_2 \text{ in } G_{i,1}(\chi_{2,1})) = N(P_2 \text{ in } G_{i,1}(\chi_2)) = N(P_2 \text{ in } G_i(\chi_1, \chi_2)), \tag{5.71e}$$

whenever $i = 0, 1, \dots, n, \infty$. As $P_2$ was picked to be a $\pi$-Hall subgroup of $G_2(\beta_1)$, we obviously have that the set $\{P_0 = 1, P_2, Q_1 | \alpha_0 = 1, \alpha_2, \beta_1\}$ is a triangular set for the series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2$. Hence (5.43d) for $t = u = 1$ and (5.42b) for $r = 1$ imply

$$\begin{aligned} G_2(\alpha_2, \beta_1) &= P_2 \times Q_{1,2}, \\ G_1(\alpha_2, \beta_1) &= 1 \times Q_{1,2} = Q_{1,2}. \end{aligned}$$

Furthermore, we have a correspondence similar to the one described in Table 5.1. Indeed, in view of (5.70) and (5.71a), the normal series $G_{2,1} \trianglelefteq G_{3,1} \trianglelefteq \cdots \trianglelefteq G_{n,1} \trianglelefteq G_{\infty,1}$ , along with the $\pi$-factorable character $\chi_{2,1}$, satisfies the hypotheses of Lemma 5.56. Hence, there is a $cP_2$-correspondence between the character towers of the above series and those of the series $G_{2,2} \trianglelefteq G_{3,2} \trianglelefteq \cdots \trianglelefteq G_{n,2} \trianglelefteq G_{\infty,2}$, where

$$G_{i,2} := N(P_2 \text{ in } G_{i,1}(\chi_{2,1})) \tag{5.72}$$

for all $i = 2, \ldots, n, \infty$. Thus, the tower $\{\chi_{2,1}, \chi_{3,1}, \ldots, \chi_{n,1}\}$ has a $cP_2$-correspondent tower $\{\chi_{2,2}, \chi_{3,2}, \ldots, \chi_{n,2}\}$, where $\chi_{i,2} \in \mathrm{Irr}(G_{i,2})$, for all $i = 2, \ldots, n$. Furthermore, for any $M$ with $M \leq N(P_2 \text{ in } G_{\infty,1}) = N(P_2 \text{ in } G(\chi_1))$, we have

$$M(\chi_{2,1}, \ldots, \chi_{k,1}) = M(\chi_{2,2}, \ldots, \chi_{k,2}), \tag{5.73}$$

whenever $2 \leq k \leq n$. The same lemma describes $G_{2,2}$ as well as $\chi_{2,2}$. So we get that

$$\begin{aligned} G_{2,2} &= P_2 \times Q_{1,2}, \\ \chi_{2,2} &= \alpha_2 \times \beta_{1,2}, \end{aligned} \tag{5.74}$$

where $Q_{1,2} = N(P_2 \text{ in } Q_1) = C(P_2 \text{ in } Q_1)$ (see (5.11)), and $\beta_{1,2} \in \mathrm{Irr}(Q_{1,2})$ is the $P_2$-Glauberman correspondent of $\beta_1 \in \mathrm{Irr}^{P_2}(Q_1)$ (see Definition 5.49).

We observe that the earlier definition of $G_{i,2}$ (see (5.72)), works also for $i = 1$, as $G_{1,1} \leq G_{2,1}$ fixes $\chi_{2,1}$. So, $G_{1,2} := N(P_2 \text{ in } G_{1,1}) = Q_{1,2}$ while the character $\beta_1 = \chi_{1,1} \in \mathrm{Irr}(G_1) = \mathrm{Irr}(G_{1,1})$ has as a unique $P_2$-Glauberman correspondent the character $\beta_{1,2} \in \mathrm{Irr}(Q_{1,2})$. This, combined with the former $cP_2$-correspondence, provides a correspondence (that we also write as a $cP_2$-correspondence) between the character towers of the series (5.66) and those of the series

$$G_{0,2} = 1 \trianglelefteq G_{1,2} := N(P_2 \text{ in } G_{1,1}) = Q_{1,2} \trianglelefteq G_{2,2} \trianglelefteq \cdots \trianglelefteq G_{n,2} \trianglelefteq G_{\infty,2}, \tag{5.75}$$

described in the Table 5.2. We remark here that, for every group $M$ with $M \leq N(P_2 \text{ in } G)$, the $P_2$-Glauberman correspondence between $\mathrm{Irr}^{P_2}(Q_1)$ and $\mathrm{Irr}(Q_{1,2})$, (with the character $\chi_1 = \chi_{1,1} = \beta_1$ in the former set corresponding to the character $\chi_{1,2} = \beta_{1,2}$), is $M$-invariant. Hence

$$M(\chi_1) = M(\chi_{1,2}). \tag{5.76}$$

$$
\begin{tikzcd}[column sep=small]
G = G_{\infty,1} \arrow[d, dash] & & & G_{\infty,2} := N(P_2 \text{ in } G_{\infty,1}(\chi_{2,1})) \arrow[d, dash] & \\
G_{n,1} \arrow[d, dash] & \chi_{n,1} \arrow[d, dash] & & G_{n,2} := N(P_2 \text{ in } G_{n,1}(\chi_{2,1})) \arrow[d, dash] & \chi_{n,2} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
G_{3,1} \arrow[d, dash] & \chi_{3,1} \arrow[d, dash] & & G_{3,2} := N(P_2 \text{ in } G_{3,1}(\chi_{2,1})) \arrow[d, dash] & \chi_{3,2} \arrow[d, dash] \\
G_{2,1} = P_2 \ltimes Q_1 \arrow[d, dash] & \chi_{2,1} = \alpha_2 \cdot \beta_1^e \arrow[d, dash] & \overleftrightarrow{cP_2} & G_{2,2} = P_2 \times Q_{1,2} \arrow[d, dash] & \chi_{2,2} = \alpha_2 \times \beta_{1,2} \arrow[d, dash] \\
G_{1,1} = G_1 = Q_1 \arrow[d, dash] & \chi_{1,1} = \chi_1 = \beta_1 \arrow[d, dash] & & G_{1,2} := Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] \\
G_{0,1} = 1 & \chi_{0,1} = 1 & & G_{0,2} = 1 & \chi_{0,2} = 1
\end{tikzcd}
$$

Table 5.2: The $cP_2$-correspondence

According to (5.71e), we have that

$$G_{i,2} = N(P_2 \text{ in } G_{i,1}(\chi_{2,1})) = N(P_2 \text{ in } G_i(\chi_1, \chi_2)) = G_i(\alpha_2, \beta_1), \tag{5.77}$$

whenever $i = 1, \ldots, n, \infty$. This, along with Tables 5.1 and 5.2, implies a $cQ_1, cP_2$-correspondence described in the diagram that follows:

$$
\begin{tikzcd}[row sep=small]
G = G_\infty \arrow[d, dash] & & & G_{\infty,1} = G(\beta_1) \arrow[d, dash] & & & G_{\infty,2} = G(\alpha_2, \beta_1) \arrow[d, dash] & \\
G_n \arrow[d, dash] & \chi_n \arrow[d, dash] & & G_{n,1} = G_n(\beta_1) \arrow[d, dash] & \chi_{n,1} \arrow[d, dash] & & G_{n,2} = G_n(\alpha_2, \beta_1) \arrow[d, dash] & \chi_{n,2} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
G_3 \arrow[d, dash] & \chi_3 \arrow[d, dash] & \overleftrightarrow{cQ_1} & G_{3,1} = G_3(\beta_1) \arrow[d, dash] & \chi_{3,1} \arrow[d, dash] & \overleftrightarrow{cP_2} & G_{3,2} = G_3(\alpha_2, \beta_1) \arrow[d, dash] & \chi_{3,2} \arrow[d, dash] \\
G_2 \arrow[d, dash] & \chi_2 \arrow[d, dash] & & G_{2,1} = P_2 \ltimes Q_1 \arrow[d, dash] & \chi_{2,1} = \alpha_2 \cdot \beta_1^e \arrow[d, dash] & & G_{2,2} = P_2 \times Q_{1,2} \arrow[d, dash] & \chi_{2,2} = \alpha_2 \times \beta_{1,2} \arrow[d, dash] \\
G_1 = Q_1 \arrow[d, dash] & \chi_1 \arrow[d, dash] & & G_{1,1} = Q_1 \arrow[d, dash] & \chi_{1,1} = \chi_1 = \beta_1 \arrow[d, dash] & & G_{1,2} = Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] \\
G_0 = 1 & \chi_0 = 1 & & G_{0,1} = 1 & \chi_{0,1} = 1 & & G_{0,2} = 1 & \chi_{0,2} = 1
\end{tikzcd}
$$

Table 5.3: The $cQ_1, cP_2$-correspondence

Furthermore, for any group $M$ with $M \leq N(P_2 \text{ in } G)$ we have

$$
\begin{aligned}
M(\chi_1, \chi_2, \dots, \chi_k) &= M(\chi_1, \chi_{2,1}, \dots, \chi_{k,1}) && \text{by (5.68)} \\
&= M(\chi_1)(\chi_{2,1}, \dots, \chi_{k,1}) \\
&= M(\chi_1)(\chi_{2,2}, \dots, \chi_{k,2}) && \text{by (5.73)} \\
&= M(\chi_{1,2})(\chi_{2,2}, \dots, \chi_{k,2}) && \text{by (5.76)} \\
&= M(\chi_{1,2}, \chi_{2,2}, \dots, \chi_{k,2}).
\end{aligned}
$$

Hence

$$M(\chi_1, \chi_2, \dots, \chi_k) = M(\chi_{1,1}, \chi_{2,1}, \dots, \chi_{k,1}) = M(\chi_{1,2}, \chi_{2,2}, \dots, \chi_{k,2}), \tag{5.78}$$

whenever $1 \leq k \leq n$.

The case $m = 3$ is quite similar to $m = 2$, so we will only describe the main steps. We first pick the $\pi'$-group, $Q_3$, as any

$$Q_3 \in \mathrm{Hall}_{\pi'}(G_{3,2}) = \mathrm{Hall}_{\pi'}(G_3(\alpha_2, \beta_1)). \tag{5.79}$$

Therefore we get that

$$G_{3,2} = G_3(\alpha_2, \beta_1) = Q_3 \ltimes P_2. \tag{5.80a}$$

Even more, (5.80a) and (5.71e) for $i = 3$ imply:

$$G_{3,2} = G_3(\alpha_2, \beta_1) = Q_3 \ltimes P_2 = N(P_2 \text{ in } G_3(\chi_1, \chi_2)). \tag{5.80b}$$

To pick the character $\beta_3 \in \mathrm{Irr}(Q_3)$, we follow the same steps as we did for the character $\alpha_2$. So we apply Lemma 2.15 to the groups $P_2$, $Q_3 \ltimes P_2 = G_{3,2}$, $G_{\infty,2}$ and the character $\alpha_2$ in the place of the groups $N, H, G$ and the character $\theta$ respectively. Thus we conclude that that there is a unique character $\beta_3 \in \text{Irr}(Q_3)$ such that

$$\chi_{3,2} = \alpha_2^e \cdot \beta_3, \tag{5.81a}$$

where $\alpha_2^e \in \text{Irr}(G_{3,2})$ is the canonical extension of $\alpha_2 \in \text{Irr}(P_2)$ to $Q_3 \ltimes P_2 = G_{3,2}$. We also have that

$$G_{\infty,2}(\beta_3) = N(Q_3 \text{ in } G_{\infty,2}(\chi_{3,2})), \tag{5.81b}$$

and thus, in view of (5.77),

$$\begin{aligned} G(\beta_1, \alpha_2, \beta_3) &= N(Q_3 \text{ in } G_{\infty,2}(\chi_{3,2})) \\ &= N(Q_3 \text{ in } G(\beta_1, \alpha_2)(\chi_{3,2})) = N(Q_3 \text{ in } N(P_2 \text{ in } G(\chi_1, \chi_2))(\chi_{3,2})). \end{aligned} \tag{5.81c}$$

But (5.78), applied twice (with $k = 2$ and $M = N(P_2 \text{ in } G)$ for the first equality, and $k = 3$ and $M = N(P_2 \text{ in } G)$ for the second one), implies that

$$N(P_2 \text{ in } G(\chi_1, \chi_2))(\chi_{3,2}) = N(P_2 \text{ in } G(\chi_{1,2}, \chi_{2,2}))(\chi_{3,2}) = N(P_2 \text{ in } G(\chi_1, \chi_2))(\chi_3).$$

Hence we conclude that

$$\begin{aligned} G(\beta_1, \alpha_2, \beta_3) &= N(Q_3 \text{ in } G_{\infty,2}(\chi_{3,2})) \\ &= N(Q_3 \text{ in } N(P_2 \text{ in } G(\chi_1, \chi_2, \chi_3)) = N(P_2, Q_3 \text{ in } G(\chi_1, \chi_2, \chi_3)), \end{aligned} \tag{5.82a}$$

and thus

$$G_i(\beta_1, \alpha_2, \beta_3) = N(Q_3 \text{ in } G_{i,2}(\chi_{3,2})) = N(P_2, Q_3 \text{ in } G_i(\chi_1, \chi_2, \chi_3)), \tag{5.82b}$$

for all $i = 0, 1, \ldots, n, \infty$.

Note that $\{P_0, P_2, Q_1, Q_3 | 1, \alpha_2, \beta_1, \beta_3\}$ is a triangular set. Indeed, as $\chi_{3,2} = \alpha_2^e \cdot \beta_3$ lies above $\chi_{2,2} = \alpha_2 \times \beta_{1,2}$ (see (5.74)), we conclude that $\beta_3$ lies above the $P_2$-Glauberman correspondent $\beta_{1,2}$ of $\beta_1$. This, along with (5.79) and the fact that $\{P_0, P_2, Q_1 | 1, \alpha_2, \beta_1\}$ is a triangular set, implies that $\{P_0, P_2, Q_1, Q_3 | 1, \alpha_2, \beta_1, \beta_3\}$ satisfies (5.17), and thus is a triangular set.

To expand Table 5.3 by one more step (that will be a $cQ_3$-correspondence) we will apply (as we did for the $cP_2$-correspondence), Lemma 5.56 to the last normal series of $G_{\infty,2}$ that the above table reaches. Notice that the normal series $G_{3,2} \trianglelefteq G_{4,2} \trianglelefteq \cdots \trianglelefteq G_{n,2} \trianglelefteq G_{\infty,2}$ , along with the $\pi$-factorable character $\chi_{3,2}$, satisfies the hypotheses of Lemma 5.56. Hence there is a $cQ_3$-correspondence between the character towers of the above series and those of the series $G_{3,3} \trianglelefteq G_{4,3} \trianglelefteq \cdots \trianglelefteq G_{n,3} \trianglelefteq G_{\infty,3}$, where

$$G_{i,3} := N(Q_3 \text{ in } G_{i,2}(\chi_{3,2})), \tag{5.83}$$

for all $i = 3, \ldots, n, \infty$. Assume that the tower $\{\chi_{3,3}, \chi_{4,3}, \ldots, \chi_{n,3}\}$ is the $cQ_3$-correspondent of the tower $\{\chi_{3,2}, \chi_{4,2}, \ldots, \chi_{n,2}\}$, where $\chi_{i,3} \in \text{Irr}(G_{i,3})$ for all $i = 3, \ldots, n$. Furthermore, for any $M$ with $M \leq N(Q_3 \text{ in } G_{\infty,2}) = N(Q_3 \text{ in } N(P_2 \text{ in } G(\chi_1, \chi_2)))$ we have

$$M(\chi_{3,2}, \ldots, \chi_{k,2}) = M(\chi_{3,3}, \ldots, \chi_{k,3}), \tag{5.84}$$

whenever $3 \leq k \leq n$. Furthermore,

$$\begin{aligned} G_{3,3} &= P_{2,3} \times Q_3, \\ \chi_{3,3} &= \alpha_{2,3} \times \beta_3, \end{aligned} \tag{5.85}$$

where $P_{2,3} = N(Q_3 \text{ in } P_2)$, and $\alpha_{2,3} \in \mathrm{Irr}(P_{2,3})$ is the $Q_3$-Glauberman correspondent of $\alpha_2 \in \mathrm{Irr}^{Q_3}(P_2)$.

We expand the definition of the $G_{i,3}$ to all $i = 1, \dots, n, \infty$, that is, we write

$$G_{i,3} = N(Q_3 \text{ in } G_{i,2}(\chi_{3,2}))$$

for all such $i$. Then

$$G_{2,3} = N(Q_3 \text{ in } G_{2,2}(\chi_{3,2})) = N(Q_3 \text{ in } G_{2,2}) = N(Q_3 \text{ in } P_2 \times Q_{1,2}) = P_{2,3} \times Q_{1,2},$$

where the last equation holds, as according to (5.33), we have $Q_{1,2} = Q_3 \cap Q_1$. Furthermore, the character $\chi_{2,2} = \alpha_2 \times \beta_{1,2}$ corresponds to the character $\chi_{2,3} := \alpha_{2,3} \times \beta_{1,2} \in \mathrm{Irr}(N(Q_3 \text{ in } G_{2,2}))$, through the $Q_3$-Glauberman correspondent $\alpha_2$ of $\alpha_{2,3}$.

Also,

$$G_{1,3} = N(Q_3 \text{ in } G_{1,2}) = N(Q_3 \text{ in } Q_{1,2}) = Q_{1,2} = G_{1,2},$$

and thus we take $\chi_{1,3} := \chi_{1,2} = \beta_{1,2}$.

This, combined with the former $cQ_3$-correspondence, provides a correspondence (that we also write as $cQ_3$-correspondence) between the character towers of those of the series (5.75) and the series

$$G_{0,3} = 1 \trianglelefteq G_{1,3} = Q_{1,2} \trianglelefteq G_{2,3} = P_{2,3} \times Q_{1,2} \trianglelefteq \cdots \trianglelefteq G_{n,3} \trianglelefteq G_{\infty,3}. \tag{5.86}$$

We remark here that, for every group $M$ with $M \leq N(P_2, Q_3 \text{ in } G)$, the $Q_3$-Glauberman correspondence between $\mathrm{Irr}^{Q_3}(P_2)$ and $\mathrm{Irr}(P_{2,3})$ is $M$-invariant. In particular we have $M(\alpha_2) = M(\alpha_{2,3})$, and thus

$$M(\chi_{1,2}, \chi_{2,2}) = M(\chi_{1,3}, \chi_{2,3}).$$

Therefore, in view of (5.78), we get

$$M(\chi_1, \chi_2) = M(\chi_{1,2}, \chi_{2,2}) = M(\chi_{1,3}, \chi_{2,3}), \tag{5.87}$$

So, for any $M$ with $M \leq N(P_2, Q_3 \text{ in } G)$, we have

$$\begin{aligned}
M(\chi_1, \chi_2, \dots, \chi_n) &= M(\chi_{1,1}, \chi_{2,2}, \dots, \chi_{n,2}) && \text{by (5.78)}\\
&= M(\chi_{1,1}, \chi_{2,2})(\chi_{3,2}, \dots, \chi_{n,2}) &&\\
&= M(\chi_1, \chi_2)(\chi_{3,2}, \dots, \chi_{n,2}) && \text{by (5.87)}\\
&= M(\chi_1, \chi_2)(\chi_{3,3}, \dots, \chi_{n,3}) && \text{by (5.84), since } M(\chi_1, \chi_2) \leq N(Q_3 \text{ in } G_{\infty,2}).\\
&= M(\chi_{1,3}, \chi_{2,3})(\chi_{3,3}, \dots, \chi_{n,3}) && \text{by (5.87)}\\
&= M(\chi_{1,3}, \chi_{2,3}, \dots, \chi_{n,3}).
\end{aligned}$$

The following table gives a clear picture of the situation when $m = 3$:

$$\begin{tikzcd}[column sep=small, row sep=small]
G = G_\infty \arrow[d, dash] & & & G_{\infty,1} = G(\beta_1) \arrow[d, dash] & \\
G_n \arrow[d, dash] & \chi_n \arrow[d, dash] & & G_{n,1} = G_n(\beta_1) \arrow[d, dash] & \chi_{n,1} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
G_3 \arrow[d, dash] & \chi_3 \arrow[d, dash] & \overleftrightarrow{cQ_1} & G_{3,1} = G_3(\beta_1) \arrow[d, dash] & \chi_{3,1} \arrow[d, dash] \\
G_2 \arrow[d, dash] & \chi_2 \arrow[d, dash] & & G_{2,1} = P_2 \ltimes Q_1 \arrow[d, dash] & \chi_{2,1} = \alpha_2 \cdot \beta_1^e \arrow[d, dash] \\
G_1 = Q_1 \arrow[d, dash] & \chi_1 = \beta_1 \arrow[d, dash] & & G_{1,1} = Q_1 \arrow[d, dash] & \chi_{1,1} = \beta_1 \arrow[d, dash] \\
G_0 = 1 & \chi_0 = 1 & & G_{0,1} = 1 & \chi_{0,1} = 1
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
& G_{\infty,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & & & G_{\infty,3} = G(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \\
& G_{n,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \chi_{n,2} \arrow[d, dash] & & G_{n,3} = G_n(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \chi_{n,3} \arrow[d, dash] \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
\overleftrightarrow{cP_2} & G_{3,2} = P_2 \rtimes Q_3 \arrow[d, dash] & \chi_{3,2} = \alpha_2^e \cdot \beta_3 \arrow[d, dash] & \overleftrightarrow{cQ_3} & G_{3,3} = P_{2,3} \times Q_3 \arrow[d, dash] & \chi_{3,3} = \alpha_{2,3} \times \beta_3 \arrow[d, dash] \\
& G_{2,2} = P_2 \times Q_{1,2} \arrow[d, dash] & \chi_{2,2} = \alpha_2 \times \beta_{1,2} \arrow[d, dash] & & G_{2,3} = P_{2,3} \times Q_{1,2} \arrow[d, dash] & \chi_{2,3} = \alpha_{2,3} \times \beta_{1,2} \arrow[d, dash] \\
& G_{1,2} = Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] & & G_{1,3} = Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] \\
& G_{0,2} = 1 & \chi_{0,2} = 1 & & G_{0,3} = 1 & \chi_{0,3} = 1
\end{tikzcd}$$

Table 5.4: The $cQ_1, cQ_3, cP_2$-correspondence.

We stop here with the individual cases $m = 1, 2, 3$, hoping that it has become clear how the mechanism that produces triangular sets from character towers works. We only remark that the role of the $\pi$- and $\pi'$-groups is interchanged at every step. So the $\pi$-groups play the protagonistic role when $m$ is even, and the $\pi'$-groups when $m$ is odd. This role consists of two acts:

1) to pick the group and its character (here Lemma 2.15 is used), and

2) to create the new $cP$- or $cQ$-correspondence (for this we use Lemma 5.56).

We are ready to state and prove the inductive step of the above mechanism.

**Theorem 5.88.** *Assume that Hypothesis 5.1 holds. Then every character tower* $\{\chi_i\}_{i=0}^m$ *of* (5.2) *determines a* $G_m(\chi_0, \chi_1, \ldots, \chi_m)$*-conjugacy class of triangular sets*

$$\{P_0, \ldots, P_{2k}, Q_1, \ldots, Q_{2l-1} | \alpha_0, \ldots, \alpha_{2k}, \beta_1, \ldots, \beta_{2l-1}\} \tag{5.89}$$

*for* (5.2)*, where* $k = [m/2]$ *and* $l = [(m + 1)/2]$*, such that*

*1)Any subtower* $\{1 = \chi_0, \chi_1, \ldots, \chi_s\}$ *of the original character tower, for some* $s = 1, \ldots, m$*, determines a* $G_s(\chi_0, \chi_1, \ldots, \chi_s)$*-conjugacy class of triangular sets*

$$\{P_0, \ldots, P_{2[s/2]}, Q_1, \ldots, Q_{2[(s+1)/2]-1} | \alpha_0, \ldots, \alpha_{2[s/2]}, \beta_1, \ldots, \beta_{2[(s+1)/2]-1}\},$$

*that are subsets of* (5.89)*.*

*2)For any*

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G_{m+1} \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G \tag{5.90}$$

*extension of* (5.2) *to a normal series of* $G$ *that satisfies Hypothesis 5.1, and any extension of* $\{\chi_i\}_{i=0}^m$ *to a character tower* $\{\chi_i\}_{i=0}^n$ *of this series, there is a unique* $cP_2, \ldots, cP_{2k}, cQ_1, \ldots, cQ_{2l-1}$*-correspondent character tower* $\{1 = \chi_{0,m}, \chi_{1,m}, \ldots, \chi_{m,m}, \ldots, \chi_{n,m}\}$ *of the normal series* $1 = G_{0,m} \trianglelefteq G_{1,m} \trianglelefteq \cdots \trianglelefteq G_{m,m} \trianglelefteq \cdots \trianglelefteq G_{n,m} \trianglelefteq G_{\infty,m}$*, for all* $n$ *with* $1 \leq m \leq n$*. Here*

$$\begin{aligned} G_{i,m} = G_i(\alpha_2, \ldots, \alpha_{2k}, \beta_1, \ldots, \beta_{2l-1}) \\ = N(P_0, \ldots, P_{2k}, Q_1, \ldots, Q_{2l-1} \text{ in } G_i(\chi_1, \ldots, \chi_m)), \end{aligned} \tag{5.91}$$

*where* $i = 0, 1, \ldots, n, \infty$*.*

*3)For every* $M$ *with* $M \leq N(P_0, \ldots, P_{2k}, Q_1, \ldots, Q_{2l-1} \text{ in } G)$ *we have*

$$M(\chi_1, \ldots, \chi_n) = M(\chi_{1,s}, \ldots, \chi_{n,s}).$$

*4) For every* $i = 1, 2, \ldots, m$ *and* $s = i + 1, \ldots, m$*, these groups and characters follow the rules*

$$\begin{aligned} G_{i,i-1} &= P_i \ltimes Q_{i-1} && \textit{and} & \chi_{i,i-1} &= \alpha_i \cdot \beta_{i-1}^e, \\ G_{i,i} &= P_i \times Q_{i-1,i} && \textit{and} & \chi_{i,i} &= \alpha_i \times \beta_{i-1,i}, \\ G_{i,s} &= P_{i,2[(s+1)/2]-1} \times Q_{i-1,2[s/2]} && \textit{and} & \chi_{i,s} &= \alpha_{i,2[(s+1)/2]-1} \times \beta_{i-1,2[s/2]}, \end{aligned} \tag{5.92}$$

*whenever* $i$ *is even, and*

$$\begin{aligned} G_{i,i-1} &= P_{i-1} \rtimes Q_i && \textit{and} & \chi_{i,i-1} &= \alpha_{i-1}^e \cdot \beta_i, \\ G_{i,i} &= P_{i-1,i} \times Q_i && \textit{and} & \chi_{i,i} &= \alpha_{i-1,i} \times \beta_i, \\ G_{i,s} &= P_{i-1,2[(s+1)/2]-1} \times Q_{i,2[s/2]} && \textit{and} & \chi_{i,s} &= \alpha_{i-1,2[(s+1)/2]-1} \times \beta_{i,2[s/2]}, \end{aligned} \tag{5.93}$$

*when $i$ is odd. (Here $\beta_{i-1}^e$ is the canonical extension of $\beta_{i-1} \in \mathrm{Irr}(Q_{i-1})$ to $G_{i,i-1}$ and similarly, $\alpha_{i-1}^e$ is the canonical extension of $\alpha_{i-1} \in \mathrm{Irr}(P_{i-1})$ to $G_{i,i-1}$.)*

*Proof.* We will use induction on $m$. We have already seen that the theorem holds when $m = 1$ (also when $m = 2$ and $m = 3$).

So assume that the theorem holds for all $m = 1, \ldots, t$ and some integer $t \geq 0$. We will prove it also holds when $m = t + 1$. So assume that the normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G_{t+1} \trianglelefteq G \tag{5.94}$$

is fixed. Along with that we fix a character tower

$$\{1 = \chi_0, \chi_1, \ldots, \chi_t, \chi_{t+1}\} \tag{5.95}$$

for (5.94). As the triangular sets have different form depending on whether $t$ is even or odd, we split the proof in two symmetric cases.

**Case 1: $t$ is odd** The series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G$ is also a normal series of $G$, while the irreducible characters $\{\chi_i\}_{i=0}^t$ form a character tower for this series. Hence the inductive hypothesis implies the existence of a $G_t(\chi_1, \ldots, \chi_t)$-conjugacy class of triangular sets that is determined by the last character tower. Let $\{P_0, \ldots, P_{2k}, Q_1, \ldots, Q_{2l-1} | \alpha_0, \ldots, \alpha_{2k}, \beta_1, \ldots, \beta_{2l-1}\}$ be a representative of this conjugacy class. As $t$ is odd we have that $l = [(t+1)/2] = (t+1)/2$ while $k = [t/2] = (t-1)/2$. So $2l - 1 = t$ while $2k = t - 1$. Therefore the above triangular set has the form $\{P_0, \ldots, P_{t-1}, Q_1, \ldots, Q_t | \alpha_0, \ldots, \alpha_{t-1}, \beta_1, \ldots, \beta_t\}$.

Hence, to prove that the character tower (5.95) determines a $G_{t+1}(\chi_1, \ldots, \chi_t, \chi_{t+1})$-conjugacy class of triangular sets that respect subtowers, it is enough to prove the existence of a $\pi$-group $P_{t+1}$ unique up to conjugations by any element of $G_{t+1}(\chi_1, \ldots, \chi_t, \chi_{t+1})$, and an irreducible character $\alpha_{t+1} \in \mathrm{Irr}(P_{t+1})$ such that the set

$$\{P_0, \ldots, P_{t-1}, P_{t+1}, Q_1, \ldots, Q_t | \alpha_0, \ldots, \alpha_{t-1}, \alpha_{t+1}, \beta_1, \ldots, \beta_t\}$$

is a triangular set depending on the tower (5.95).

Let

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G_{t+1} \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G, \tag{5.96}$$

be an extension of (5.94) to a normal series of $G$ so that Hypothesis 5.1 holds, for some $n \geq t + 1$. Assume further that

$$\{1 = \chi_0, \chi_1, \ldots, \chi_t, \chi_{t+1}, \ldots, \chi_n\}, \tag{5.97}$$

is a character tower for (5.96) that extends the character tower (5.95). For any $n$ with $n \geq t$, our inductive hypothesis implies that the character tower (5.97) of (5.96) has a $cQ_1, cP_2, \ldots, cP_{t-1}, cQ_t$-correspondent character tower

$$\{1 = \chi_{0,t}, \chi_{1,t}, \ldots, \chi_{t,t}, \ldots, \chi_{n,t}\}, \tag{5.98}$$

of the normal series $1 = G_{0,t} \trianglelefteq G_{1,t} \trianglelefteq \cdots \trianglelefteq G_{t,t} \trianglelefteq \cdots \trianglelefteq G_{n,t} \trianglelefteq G_{\infty,t}$, where

$$\begin{aligned} G_{i,t} &= G_i(\alpha_2, \ldots, \alpha_{t-1}, \beta_1, \ldots, \beta_t) \\ &= N(P_0, \ldots, P_{t-1}, Q_1, \ldots, Q_t \text{ in } G_i(\chi_1, \ldots, \chi_t)). \end{aligned} \tag{5.99}$$

For every $M \leq N(P_2, \ldots, P_{t-1}, Q_1, \ldots, Q_t$ in $G)$ we also have

$$M(\chi_1, \ldots, \chi_n) = M(\chi_{1,t}, \ldots, \chi_{n,t}). \tag{5.100}$$

Furthermore, (5.92) and (5.93) for $i = t$ imply that $G_{t,t} = P_{t-1,t} \times Q_t$, while $\chi_{t,t} = \alpha_{t-1,t} \times \beta_t$. (Note that $\alpha_{t-1,t} \in \mathrm{Irr}(P_{t-1,t})$ is the $Q_t$-Glauberman correspondent of $\alpha_{t-1} \in \mathrm{Irr}^{Q_t}(P_{t-1})$). The following diagram describes the situation.

$$
\begin{tikzcd}[row sep=small]
G = G_\infty \arrow[d, dash] & & & G_{\infty,1} = G(\beta_1) \arrow[d, dash] & \\
G_n \arrow[d, dash] & \chi_n \arrow[d, dash] & & G_{n,1} = G_n(\beta_1) \arrow[d, dash] & \chi_{n,1} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
G_{t+1} \arrow[d, dash] & \chi_{t+1} \arrow[d, dash] & & G_{t+1,1} = G_{t+1}(\beta_1) \arrow[d, dash] & \chi_{t+1,1} \arrow[d, dash] \\
G_t \arrow[d, dash] & \chi_t \arrow[d, dash] & & G_{t,1} = G_t(\beta_1) \arrow[d, dash] & \chi_{t,1} \arrow[d, dash] \\
G_{t-1} \arrow[d, dash] & \chi_{t-1} \arrow[d, dash] & & G_{t-1,1} = G_{t-1}(\beta_1) \arrow[d, dash] & \chi_{t-1,1} \arrow[d, dash] \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \overleftrightarrow{cQ_1} & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
G_3 \arrow[d, dash] & \chi_3 \arrow[d, dash] & & G_{3,1} = G_3(\beta_1) \arrow[d, dash] & \chi_{3,1} \arrow[d, dash] \\
G_2 \arrow[d, dash] & \chi_2 \arrow[d, dash] & & G_{2,1} = P_2 \ltimes Q_1 \arrow[d, dash] & \chi_{2,1} = \alpha_2 \cdot \beta_1^e \arrow[d, dash] \\
G_1 = Q_1 \arrow[d, dash] & \chi_1 = \beta_1 \arrow[d, dash] & & G_{1,1} = Q_1 \arrow[d, dash] & \chi_{1,1} = \beta_1 \arrow[d, dash] \\
G_0 = 1 & \chi_0 = 1 & & G_{0,1} = 1 & \chi_{0,1} = 1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
& G_{\infty,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \\
& G_{n,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \chi_{n,2} \arrow[d, dash] \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
& G_{t+1,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \chi_{t+1,2} \arrow[d, dash] \\
& G_{t,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \chi_{t,2} \arrow[d, dash] \\
& G_{t-1,2} = G(\beta_1, \alpha_2) \arrow[d, dash] & \chi_{t-1,2} \arrow[d, dash] \\
\overleftrightarrow{cP_2} & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
& G_{3,2} = P_2 \rtimes Q_3 \arrow[d, dash] & \chi_{3,2} = \alpha_2^e \cdot \beta_3 \arrow[d, dash] \\
& G_{2,2} = P_2 \times Q_{1,2} \arrow[d, dash] & \chi_{2,2} = \alpha_2 \times \beta_{1,2} \arrow[d, dash] \\
& G_{1,2} = Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] \\
& G_{0,2} = 1 & \chi_{0,2} = 1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
& G_{\infty,3} = G(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \\
& G_{n,3} = G_n(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \chi_{n,3} \arrow[d, dash] \\
& \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
& G_{t+1,3} = G_{t+1}(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \chi_{t+1,3} \arrow[d, dash] \\
& G_{t,3} = G_t(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \chi_{t,3} \arrow[d, dash] \\
& G_{t-1,3} = G_{t-1}(\beta_1, \alpha_2, \beta_3) \arrow[d, dash] & \chi_{t-1,3} \arrow[d, dash] \\
\overleftrightarrow{cQ_3} & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
& G_{3,3} = P_{2,3} \times Q_3 \arrow[d, dash] & \chi_{3,3} = \alpha_{2,3} \times \beta_3 \arrow[d, dash] \\
& G_{2,3} = P_{2,3} \times Q_{1,2} \arrow[d, dash] & \chi_{2,3} = \alpha_{2,3} \times \beta_{1,2} \arrow[d, dash] \\
& G_{1,3} = Q_{1,2} \arrow[d, dash] & \beta_{1,2} \arrow[d, dash] \\
& G_{0,3} = 1 & \chi_{0,3} = 1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & G_{\infty,t-1} = G(\alpha_2,\dots,\alpha_{t-1},\beta_1,\dots,\beta_{t-2}) \arrow[d, dash] & \\
 & G_{n,t-1} = G_n(\alpha_2,\dots,\alpha_{t-1},\beta_1,\dots,\beta_{t-2}) \arrow[d, dash] & \chi_{n,t-1} \arrow[d, dash] \\
 & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
 & G_{t+1,t-1} = G_{t+1}(\alpha_2,\dots,\alpha_{t-1},\beta_1,\dots,\beta_{t-2}) \arrow[d, dash] & \chi_{t+1,t-1} \arrow[d, dash] \\
 & G_{t,t-1} = P_{t-1} \rtimes Q_t \arrow[d, dash] & \chi_{t,t-1} = \alpha^e_{t-1} \cdot \beta_t \arrow[d, dash] \\
 & G_{t-1,t-1} = P_{t-1} \times Q_{t-2,t-1} \arrow[d, dash] & \chi_{t-1,t-1} = \alpha_{t-1} \times \beta_{t-2,t-1} \arrow[d, dash] \\
\overleftrightarrow{cP_4, cQ_5, cP_6, \dots, cP_{t-1}} & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
 & G_{3,t-1} = P_{2,t-2} \times Q_{3,t-1} \arrow[d, dash] & \chi_{3,t-1} = \alpha_{2,t-2} \times \beta_{3,t-1} \arrow[d, dash] \\
 & G_{2,t-1} = P_{2,t-2} \times Q_{1,t-1} \arrow[d, dash] & \chi_{2,t-1} = \alpha_{2,t-2} \times \beta_{1,t-1} \arrow[d, dash] \\
 & G_{1,t-1} = Q_{1,t-1} \arrow[d, dash] & \chi_{1,t-1} = \beta_{1,t-1} \arrow[d, dash] \\
 & G_{0,t-1} = 1 & \chi_{0,t-1} = 1
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & G_{\infty,t} = G(\alpha_2, \ldots, \alpha_{t-1}, \beta_1, \ldots, \beta_{t-2}, \beta_t) \arrow[d, dash] & \\
 & G_{n,t} = G_n(\alpha_2, \ldots, \alpha_{t-1}, \beta_1, \ldots, \beta_{t-2}, \beta_t) \arrow[d, dash] & \chi_{n,t} \arrow[d, dash] \\
 & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
 & G_{t+1,t} = G_{t+1}(\alpha_2, \ldots, \alpha_{t-1}, \beta_1, \ldots, \beta_{t-2}, \beta_t) \arrow[d, dash] & \chi_{t+1,t} \arrow[d, dash] \\
 & G_{t,t} = P_{t-1,t} \times Q_t \arrow[d, dash] & \chi_{t,t} = \alpha_{t-1,t} \times \beta_t \arrow[d, dash] \\
 & G_{t-1,t} = P_{t-1,t} \times Q_{t-2,t-1} \arrow[d, dash] & \chi_{t-1,t} = \alpha_{t-1,t} \times \beta_{t-2,t-1} \arrow[d, dash] \\
\overleftrightarrow{cQ_t} & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
 & G_{3,t} = P_{2,t} \times Q_{3,t-1} \arrow[d, dash] & \chi_{3,t} = \alpha_{2,t} \times \beta_{3,t-1} \arrow[d, dash] \\
 & G_{2,t} = P_{2,t} \times Q_{1,t-1} \arrow[d, dash] & \chi_{2,t} = \alpha_{2,t} \times \beta_{1,t-1} \arrow[d, dash] \\
 & G_{1,t} = Q_{1,t-1} \arrow[d, dash] & \chi_{1,t} = \beta_{1,t-1} \arrow[d, dash] \\
 & G_{0,t} = 1 & \chi_{0,t} = 1
\end{tikzcd}
$$

Table 5.5: The $cQ_1, cQ_3, \ldots, cQ_t, cP_2, cP_4, \ldots, cP_{t-1}$-correspondence

We pick $P_{t+1}$ to be any $\pi$-Hall subgroup of $G_{t+1,t} = G_{t+1}(\alpha_2, \dots, \alpha_{t-1}, \beta_1, \dots, \beta_t)$. The factor group $G_{t+1,t}/G_{t,t}$ is a $\pi$-group, while $Q_t$ is a $\pi'$-Hall subgroup of $G_{t,t}$ normalized by $G_{t+1,t}$ (see (5.99)). Thus $P_{t+1}$ also normalizes $Q_t$, which implies that

$$G_{t+1,t} = P_{t+1} \ltimes Q_t. \tag{5.101}$$

Furthermore, $Q_t$ is a normal subgroup of $G_{\infty,t}$ (as $G_{t,t} \trianglelefteq G_{\infty,t}$), while its irreducible character $\beta_t$ is $G_{\infty,t}$-invariant. Therefore we can apply Lemma 2.15 to the groups $G_{\infty,t}, G_{t+1,t}, Q_t$ and the character $\chi_{t+1,t}$ in the place of the groups $G, H, N$ and the character $\theta$, respectively. (Note that in this case $H(\theta) = H$). Hence we conclude that $\beta_t$ has a unique canonical extension $\beta_t^e \in \mathrm{Irr}(G_{t+1,t})$. Furthermore, since $\chi_{t+1,t}$ lies above $\beta_t$, the same lemma implies the existence of a unique irreducible character $\alpha_{t+1} \in \mathrm{Irr}(P_{t+1})$ such that

$$\chi_{t+1,t} = \alpha_{t+1} \cdot \beta_t^e, \tag{5.102}$$

while $G_{\infty,t}(\beta_t, \alpha_{t+1}) = N(P_{t+1} \text{ in } G_{n,t}(\chi_{t+1,t}))$. But $G_{\infty,t}$ fixes $\beta_t$. So

$$G_{\infty,t}(\alpha_{t+1}) = N(P_{t+1} \text{ in } G_{n,t}(\chi_{t+1,t})). \tag{5.103}$$

As $\chi_{t+1,t}$ lies above $\chi_{t,t} = \alpha_{t-1,t} \times \beta_t$, equation (5.102) obviously implies that $\alpha_{t+1}$ lies above $\alpha_{t-1,t}$, which is the $Q_t$-Glauberman correspondent of $\alpha_{t-1}$. This, along with the fact that $P_{t+1}$ was picked as a $\pi$-Hall subgroup of $G_{t+1}(\alpha_2, \dots, \alpha_{t-1}, \beta_1, \dots, \beta_t)$, implies that the new $\pi$-group and its character satisfy (5.17c) and (5.17d) respectively. As we already know that the set $\{P_0, \dots, P_{t-1}, Q_1, \dots, Q_t | \alpha_0, \dots, \alpha_{t-1}, \beta_1, \dots, \beta_t\}$ is a triangular set, we conclude that

$$\{P_0, \dots, P_{t-1}, P_{t+1}, Q_1, \dots, Q_t | \alpha_0, \dots, \alpha_{t-1}, \alpha_{t+1}, \beta_1, \dots, \beta_t\} \tag{5.104}$$

is a triangular set for (5.94). Furthermore, it is clear, from the way it is constructed, that it is related to the character tower (5.95) and that it respects subtowers. Note also that the only choice for $P_{t+1}$ was that of the Hall $\pi$-subgroup of $G_{t+1,t}$. Hence $P_{t+1}$ is uniquely determined up to conjugation by an element of

$$G_{t+1,t} = N(P_0, P_2, \dots, P_{t-1}, Q_1, \dots, Q_t \text{ in } G_{t+1}(\chi_1, \dots, \chi_t)).$$

So $P_{t+1}$ is uniquely determined by an element of $G_{t+1}(\chi_1, \dots, \chi_t)$. This, along with the inductive hypothesis and the fact that $G_{t+1}(\chi_1, \dots, \chi_t) \geq G_t(\chi_1, \dots, \chi_{t-1})$ implies that the triangular set (5.104) is unquely determined up to conjugation by an element of $G_{t+1}(\chi_1, \dots, \chi_t)$. Hence the first part of Theorem 5.88 is verified for the inductive step in the case where $t$ is odd. Furthermore, as (5.104) is a triangular set, Proposition 5.41 implies that

$$\begin{aligned} G_{t+1}(\alpha_2, \dots, \alpha_{t+1}, \beta_1, \dots, \beta_t) &= P_{t+1} \times Q_{t,t+1} && \text{by (5.42b)}, \\ G_j(\alpha_2, \dots, \alpha_{t+1}, \beta_1, \dots, \beta_t) &= P_{i,t} \times Q_{i-1,t+1} && \text{if } i \text{ is even, by (5.43b)}, \\ G_j(\alpha_2, \dots, \alpha_{t+1}, \beta_1, \dots, \beta_t) &= P_{i-1,t} \times Q_{i,t+1} && \text{if } i \text{ is odd, by (5.43d)}, \end{aligned} \tag{5.105}$$

for all $j = 1, \dots, t$.

To complete the proof of the theorem (at least when $t$ is odd), it is enough to show that the character tower (5.97) determines a character tower $\{1 = \chi_{0,t+1}, \dots, \chi_{t+1,t+1}, \dots, \chi_{n,t+1}\}$ for the series $1 = G_{0,t+1} \trianglelefteq G_{1,t+1} \trianglelefteq \dots \trianglelefteq G_{t+1,t+1} \trianglelefteq \dots \trianglelefteq G_{n,t+1} \trianglelefteq G_{\infty,t+1}$, where $G_{i,t+1}$ and $\chi_{i,t+1}$ satisfy (5.91), (5.92) and (5.93). In that direction we first observe that, for every

$i = 1, \ldots, n, \infty$, we have

$$G_{i,t}(\chi_{t+1,t}) = G_{i,t}(\chi_{t+1}). \tag{5.106}$$

Indeed, in view of (5.100) for $M = G_{i,t}$ and $n = t$, we get $G_{i,t}(\chi_1, \ldots, \chi_t) = G_{i,t}(\chi_{1,t}, \ldots, \chi_{t,t})$. Hence $G_{i,t} = G_{i,t}(\chi_1, \ldots, \chi_t) = G_{i,t}(\chi_{1,t}, \ldots, \chi_{t,t})$, as $G_{i,t} \leq G_i(\chi_1, \ldots, \chi_t)$ by (5.99). So, if we apply again (5.100) for $M = G_{i,t}$ and $n = t+1$, we have

$$\begin{aligned} G_{i,t}(\chi_{t+1}) = G_{i,t}(\chi_1, \ldots, \chi_t, \chi_{t+1}) &= G_{i,t}(\chi_{1,t}, \ldots, \chi_{t,t}, \chi_{t+1,t}) \\ &= G_{i,t}(\chi_{1,t}, \ldots, \chi_{t,t})(\chi_{t+1,t}) = G_{i,t}(\chi_{t+1,t}). \end{aligned}$$

Thus (5.106) holds. Hence we conclude that

$$N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1,t})) = N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1})).$$

This, along with (5.103), implies that

$$G_{i,t}(\alpha_{t+1}) = N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1})), \tag{5.107}$$

whenever $i = 1, \ldots, n, \infty$. Thus if we define $G_{i,t+1} := G_{i,t}(\alpha_{t+1})$, equations (5.99) and (5.107) imply

$$\begin{aligned} G_{i,t+1} &= G_i(\alpha_2, \ldots, \alpha_{t-1}, \alpha_{t+1}, \beta_1, \ldots, \beta_t) \\ &= G_{i,t}(\alpha_{t+1}) = N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1})) \\ &= N(P_0, P_2, \ldots, P_{t-1}, P_{t+1}, Q_1, \ldots, Q_t \text{ in } G_i(\chi_1, \ldots, \chi_{t+1})). \end{aligned} \tag{5.108}$$

We also get (using (5.108) and (5.106))

$$G_{i,t+1} = N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1})) = N(P_{t+1} \text{ in } G_{i,t}(\chi_{t+1,t})), \tag{5.109}$$

for all $i = 1, \ldots, n, \infty$. Note that (5.108) proves that (5.91) holds for the inductive step. Also (5.105), along with (5.101) and the inductive hypothesis (for those $i$ with $i = 1, \ldots, t$), implies that the groups $G_{i,s}$ satisfy (5.92) and (5.93) whenever $1 \leq i \leq t+1$ and $i < s \leq t+1$. In particular we have

$$\begin{aligned} G_{t+1,t+1} &= P_{t+1} \times Q_{t,t+1} \\ G_{j,t+1} &= P_{j,t} \times Q_{j-1,t+1} \qquad \text{if } j \text{ is even} \\ G_{j,t+1} &= P_{j-1,t} \times Q_{j,t+1} \qquad \text{if } j \text{ is odd,} \end{aligned} \tag{5.110}$$

for all $j = 1, \ldots, t$.

To get the desired character tower for the series (5.96) (the correspondent of the tower (5.97)), we first use the inductive argument to reach the character tower (5.98). So it is enough to get a tower for (5.96) that corresponds to this latter tower. For this we split (5.98) in two pieces: the tail that consists of $\chi_{i,t}$ for all $i = 1, \ldots, t$, and the top that consists of the rest, i.e., the characters $\chi_{i,t}$ where $i = t+1, \ldots, n$.

For the top part, we apply Lemma 5.56 to the normal subgroups $G_{t+1,t}, \ldots, G_{n,t}$ of $G_{\infty,t}$ and the character $\chi_{t+1,t} = \alpha_{t+1} \cdot \beta_t^e$. This way the character tower $\{\chi_{t+1,t}, \ldots, \chi_{n,t}\}$ of the normal series $G_{t+1,t} \trianglelefteq \cdots \trianglelefteq G_{n,t}$ has a unique $cP_{t+1}$-correspondent character tower of the series

$$G_{t+1,t+1} = N(P_{t+1} \text{ in } G_{t+1,t}(\chi_{t+1,t})) \trianglelefteq \cdots \trianglelefteq G_{n,t+1} = N(P_{t+1} \text{ in } G_{n,t}(\chi_{t+1,t})).$$

We write

$$\{\chi_{t+1,t+1}, \ldots, \chi_{n,t+1}\}, \tag{5.111}$$

for this $cP_{t+1}$-correspondent tower. Note that Lemma 5.56 also determines the character $\chi_{t+1,t+1}$ as

$$\chi_{t+1,t+1} = \alpha_{t+1} \times \beta_{t,t+1} \in \mathrm{Irr}(G_{t+1,t+1}), \tag{5.112}$$

where $\beta_{t,t+1} \in \mathrm{Irr}(Q_{t,t+1}) = \mathrm{Irr}(C(P_{t+1} \text{ in } Q_t))$ is the $P_{t+1}$-Glauberman correspondent of $\beta_t \in \mathrm{Irr}^{P_{t+1}}(Q_t)$. Furthermore, according to the same lemma we have that

$$S(\chi_{t+1,t}, \ldots, \chi_{n,t}) = S(\chi_{t+1,t+1}, \ldots, \chi_{n,t+1}) \tag{5.113}$$

for any subgroup $S$ of $N(P_{t+1} \text{ in } G_{\infty,t})$.

As far as the tail of (5.98) is concerned, we observe the following: In view of (5.92) and (5.93)

$$\begin{aligned}
\chi_{t,t} &= \alpha_{t-1,t} \times \beta_t \\
\chi_{j,t} &= \alpha_{j,t} \times \beta_{j-1,t-1} \quad \text{if } j \text{ is even} \\
\chi_{j,t} &= \alpha_{j-1,t} \times \beta_{j,t-1} \quad \text{if } j \text{ is odd },
\end{aligned} \tag{5.114}$$

whenever $j = 1, \ldots, t-1$. We define

$$\begin{aligned}
\chi_{t,t+1} &= \alpha_{t-1,t} \times \beta_{t,t+1} \\
\chi_{j,t+1} &= \alpha_{j,t} \times \beta_{j-1,t+1} \quad \text{if } j \text{ is even} \\
\chi_{j,t+1} &= \alpha_{j-1,t} \times \beta_{j,t+1} \quad \text{if } j \text{ is odd },
\end{aligned} \tag{5.115}$$

where $\beta_{t,t+1}, \beta_{j,t+1}$ and $\beta_{j-1,t+1}$ are the $P_{t+1}$-Glauberman correspondents of $\beta_t, \beta_{j,t-1}$ and $\beta_{j-1,t-1}$ respectively, for all $j = 1, \ldots, t-1$. Note that all these characters are well defined characters of $Q_{t,t+1}, Q_{j,t+1}$ and $Q_{j-1,t+1}$, and form a tower by Proposition 5.50 (as (5.104) is a triangular set). Furthermore, (5.110) implies that $\chi_{j,t+1}$ and $\chi_{t,t+1}$ are characters of $G_{j,t+1}$ and $G_{t,t+1}$ respectively, for all $j = 1, \ldots, t-1$. Thus $\{1 = \chi_{0,t+1}, \chi_{1,t+1}, \ldots \chi_{t,t+1}\}$ is a character tower of the normal series $G_{0,t+1} \trianglelefteq G_{1,t+1} \trianglelefteq \cdots \trianglelefteq G_{t,t+1}$. Also we pass from the $\chi_{j,t}$ to the $\chi_{j,t+1}$ through a $P_{t+1}$-Glauberman correspondence. Thus any subgroup of $G$ that normalizes the groups $G_{1,t}, \ldots, G_{t,t}$, along with the $P_{t+1}$, leaves this correspondence invariant. But any group $T$ with $T \leq N(P_2, \ldots, P_{t-1}, P_{t+1}, Q_1, \ldots, Q_t \text{ in } G)$ normalizes the former groups (as $G_{i,t}$ is a direct product (see Table 5.5) of groups that $T$ normalizes). Hence for any such group $T$ and any $j = 1, \ldots, t$ we have

$$T(\chi_{j,t}) = T(\chi_{j,t+1}). \tag{5.116}$$

Furthermore, $\chi_{t,t+1} = \alpha_{t-1,t} \times \beta_{t,t+1}$, while $\chi_{t+1,t+1} = \alpha_{t+1} \times \beta_{t,t+1}$ (by (5.112)). As $\alpha_{t+1}$ lies above $\alpha_{t-1,t}$ (by (5.17d)), we conlcude that $\chi_{t+1,t+1}$ lies above $\chi_{t,t+1}$. Hence we have formed the tower $\{1 = \chi_{0,t+1}, \ldots, \chi_{t,t+1}, \chi_{t+1,t+1}, \ldots, \chi_{n,t+1}\}$ of (5.96), that corresponds to the tower (5.98). This, along with the inductive argument that provides the $P_2, \ldots, P_{t-1}, Q_1, \ldots, Q_t$-correspondence between (5.95) and (5.98), implies the desired correspondence between (5.95) and the tower $\{1 = \chi_{0,t+1}, \ldots, \chi_{t,t+1}, \chi_{t+1,t+1}, \ldots, \chi_{n,t+1}\}$. Furthermore, (5.102), (5.112) and (5.115), along with the inductive argument, imply that (5.92) and (5.93) hold for all $i = 1, \ldots, t+1$ and $s = i+1, \ldots, t+1$.

Also for every $M$ with $M \leq N(P_2, \ldots, P_{t-1}, P_{t+1}, Q_1, \ldots, Q_t \text{ in } G)$ we get that

$$M(\chi_1, \ldots, \chi_t) \leq N(P_{t+1} \text{ in } G_{\infty,t}),$$

(see (5.99) for a characterization of $G_{\infty,t}$). Therefore, for all such $M$ we have

$$\begin{aligned}
&M(\chi_1, \ldots, \chi_n) \\
&= M(\chi_1, \ldots, \chi_t)(\chi_1, \ldots, \chi_n) \\
&= M(\chi_1, \ldots, \chi_t)(\chi_{1,t}, \ldots, \chi_{n,t}) && \text{by (5.100)} \\
&= M(\chi_1, \ldots, \chi_t)(\chi_{t+1,t}, \ldots, \chi_{n,t})(\chi_{1,t}, \ldots, \chi_{t,t}) \\
&= M(\chi_1, \ldots, \chi_t)(\chi_{t+1,t+1}, \ldots, \chi_{n,t+1})(\chi_{1,t}, \ldots, \chi_{t,t}) && \text{by (5.113) for } S = M(\chi_1, \ldots, \chi_t) \\
&= M(\chi_{1,t}, \ldots, \chi_{t,t})(\chi_{t+1,t+1}, \ldots, \chi_{n,t+1})(\chi_{1,t}, \ldots, \chi_{t,t}) && \text{by (5.100)} \\
&= M(\chi_{1,t}, \ldots, \chi_{t,t})(\chi_{t+1,t+1}, \ldots, \chi_{n,t+1}) \\
&= M(\chi_{1,t+1}, \ldots, \chi_{t,t+1})(\chi_{t+1,t+1}, \ldots, \chi_{n,t+1}) && \text{by (5.116) for } T = M \\
&= M(\chi_{1,t+1}, \ldots, \chi_{n,t+1}).
\end{aligned}$$

This implies that part 3) of the theorem also holds for $m = t+1$. Hence the inductive step for $m = t+1$ is verified in the case of an odd $t$.

**Case 2: $t$ is even** The proof is similar to that of an odd $t$. So we will skip it. We only remark that we need to interchange the role of the $\pi$-groups with that of the $\pi'$-groups. So in this case for the inductive step we pick the $\pi'$-group $Q_{t+1}$ and its character $\beta_{t+1}$, as in the previous one we were picking the $\pi$-group $P_{t+1}$ and its character $\alpha_{t+1}$. We continue similarly, proving that the inductive step holds also in the case of an even $t$.

This completes the inductive argument and thus proving Theorem 5.88. □

The following remark is a straightforward consequence of the recursive proof of Theorem 5.88

**Remark 5.117.** Let $\{\chi_{i,m}\}_{i=0}^n$ be the unique $cP_2, \ldots, cP_{2k}, cQ_1, \ldots, cQ_{2l-1}$-correspondent of the character tower (5.97). Then its subtower $\{\chi_{i,m}\}_{i=0}^t$ is the unique $cP_2, \ldots, cP_{2k}, cQ_1, \ldots, cQ_{2l-1}$-correspondent of the subtower $\{1 = \chi_0, \chi_1, \ldots, \chi_t\}$ of (5.97), whenever $t = 0, 1, \ldots, n$. Also, if $M \leq N(P_2, \ldots, P_{2k}, Q_1, \ldots, Q_{2l-1} \text{ in } G)$ we have

$$M(\chi_1, \ldots, \chi_k) = M(\chi_{1,s}, \ldots, \chi_{k,s}).$$

## 5.4 From triangles to towers

In order to complete the proof of Theorem 5.6 it suffices to prove

**Theorem 5.118.** *Assume that Hypothesis 5.1 holds. Then every triangular set for* (5.2) *determines a character tower of* (5.2), *so that the tower is related to this triangular set via Theorem* (5.88).

*Proof.* We will use induction on the lengh $m$ of the series (5.2). If $m = 1$, then the theorem obviously holds, as we take $\chi_1 = \beta_1$.

So assume that the theorem holds for $m = 1, \ldots, t$ and some integer $t \geq 0$. We will prove it also holds for $m = t+1$. Let

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G_{t+1} \trianglelefteq G = G_\infty \tag{5.119a}$$

be a fixed normal series of $G$ that satisfies Hypothesis 5.1. Assume further that

$$\{P_{2i}, Q_{2i-1} | \alpha_{2i}, \beta_{2i-1}\} \tag{5.119b}$$

is an arbitary, but fixed, triangular set for (5.119a). We split the proof in two symmetric cases, according to the type of (5.119b).

**Case 1: $t$ is odd.** In this case the triangular set (5.119b) has the form

$$\{P_0 = 1, P_2, \dots, P_{t+1}, Q_1, \dots, Q_t | \alpha_0 = 1, \alpha_2, \dots, \alpha_{t+1}, \beta_1, \dots, \beta_t\} \tag{5.120}$$

Note that its subset

$$\{P_0 = 1, P_2, \dots, P_{t-1}, Q_1, \dots, Q_t | \alpha_0 = 1, \alpha_2, \dots, \alpha_{t-1}, \beta_1, \dots, \beta_t\} \tag{5.121a}$$

is a triangular set for the series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G = G_\infty. \tag{5.121b}$$

Hence by the inductive hypothesis there exists a character tower

$$\{\chi_0, \chi_1, \dots, \chi_t\} \tag{5.121c}$$

of (5.121b) that determines and is determined by the set (5.121a). Hence, in view of part 2) of Theorem 5.88 (for $m = t$ and $n = t+1$), there is a $cQ_1, cP_2, \dots, cP_{t-1}, cQ_t$-correspondence between the character towers of the series (5.119a) and the character towers of the series

$$1 = G_{0,t}, G_{1,t}, \dots, G_{t,t} \trianglelefteq G_{t+1,t} \trianglelefteq G_{\infty,t}, \tag{5.122}$$

where $G_{i,t} = G_i(\alpha_2, \dots, \alpha_{t-1}, \beta_1, \dots, \beta_t)$ (see (5.91)), for all $i = 0, 1, \dots, t+1, \infty$.

Let $\Psi \in \mathrm{Irr}(G_{t+1}|\chi_t)$ be any irreducible character of $G_{t+1}$ lying above $\chi_t \in \mathrm{Irr}(G_t)$. Then the characters $1 = \chi_0, \chi_1, \dots, \chi_t, \Psi$ form a tower for the series (5.119a). Let

$$\chi_{0,t}, \chi_{1,t}, \dots, \chi_{t,t}, \Psi_t,$$

be its unique $cQ_1, cP_2, \dots, cP_{t-1}, cQ_t$-correspondent tower. So $\chi_{i,t} \in \mathrm{Irr}(G_{i,t})$ for all $i = 0, 1, \dots, t$ and $\Psi_t \in \mathrm{Irr}(G_{t+1,t})$.

We remark that the above is actually a $cQ_1, cP_2, \dots, cP_{t-1}, cQ_t$-correspondence between the set $\mathrm{Irr}(G_{t+1}|\chi_t)$ of irreducible characters $\Psi$ of $G_{t+1}$ lying above $\chi_t$ and the set $\mathrm{Irr}(G_{t+1,t}|\chi_{t,t})$ of irreducible characters $\Psi_t$ of $G_{t+1,t}$ lying above $\chi_{t,t}$. This is clear in view of Remark 5.117, as the tower $\{\chi_{0,t}, \chi_{1,t}, \dots, \chi_{t,t}\}$ is the unique $cQ_1, cP_2, \dots, cP_{t-1}, cQ_t$-correspondent of the tower (5.121c). So for any $\Psi_t \in \mathrm{Irr}(G_{t+1,t}|\chi_{t,t})$ the tower $\{\chi_{0,t}, \chi_{1,t}, \dots, \chi_{t,t}, \Psi_t,\}$ has as a $cQ_1, cP_2, \dots, cP_{t-1}, cQ_t$-correspondent a tower of the form $1 = \chi_0, \chi_1, \dots, \chi_t, \Psi$ for some $\Psi \in \mathrm{Irr}(G_{t+1}|\chi_t)$.

Furthermore, according to part 4) of Theorem 5.88 (for $i = t$ odd) we get that $G_{t,t} = P_{t-1,t} \times Q_t$ while $\chi_{t,t} = \alpha_{t-1,t} \times \beta_t$. Since (5.120) is a triangular set, equation (5.42a) (for $r = (t+1)/2$) implies that

$$G_{t+1,t} = G_{t+1}(\alpha_2, \dots, \alpha_{t-1}, \beta_1, \beta_t) = P_{t+1} \ltimes Q_t.$$

Even more, according to Theorem 2.13 the $P_{t+1}$-invariant irreducible character $\beta_t$ of $Q_t$ has a unique canonical extension $\beta_t^e \in \mathrm{Irr}(P_{t+1} \ltimes Q_t)$. As $\alpha_{t+1} \in \mathrm{Irr}(P_{t+1})$, the character $\alpha_{t+1} \cdot \beta_t^e$ is an irreducible character of $G_{t+1,t} = P_{t+1} \ltimes Q_t$ (see Theorem 2.14). Also, according to (5.17d) (for $r = (t+1)/2$)), the character $\alpha_{t+1}$ lies above the $\alpha_{t-1,t}$. Hence the irreducible character $\alpha_{t+1} \cdot \beta_t^e$ of $P_{t+1} \ltimes Q_t = G_{t+1,t}$ lies above the irreducible character $\alpha_{t-1,t} \times \beta_t = \chi_{t,t}$ of $P_{t-1,t} \times Q_t = G_{t,t}$. Let $\chi_{t+1} \in \mathrm{Irr}(G_{t+1}|\chi_t)$ be the unique $cQ_1, cP_2, \ldots, cP_{t-1}, cQ_t$-correspondent of $\alpha_{t+1} \cdot \beta_t^e \in \mathrm{Irr}(G_{t+1,t}|\chi_{t,t})$. So the character tower $\{\chi_{0,t}, \chi_{1,t}, \ldots, \chi_{t,t}, \alpha_{t+1} \cdot \beta_t^e\}$ of the series (5.122) has as a unique $cQ_1, cP_2, \ldots, cP_{t-1}, cQ_t$-correspondent the tower

$$\{\chi_0, \chi_1, \ldots, \chi_t, \chi_{t+1}\} \tag{5.123}$$

of the series (5.119a). Furthermore, the steps we followed to pick the character $\chi_{t+1}$ (which are exactly the opposite of what we used to pick $P_{t+1}$ at the inductive step of Theorem 5.88) make it clear that the tower (5.123) determines the triangular set (5.120) in the way described in Theorem 5.88.

This completes the proof of the inductive step in the case of an odd $t$.

**Case 2: $t$ is even.** The proof is symmetric to that of an odd $m$, so we omit it.

This completes the proof of the theorem when $m = t + 1$, thus proving Theorem 5.118. □

Furthermore, Theorems 5.88 and 5.118, along with Corollary 3.7, imply

**Remark 5.124.** Assume that the normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G$ for $G$, along with the character tower $\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^m$, is fixed. Then conjugation by any $g \in G(\chi_1, \ldots, \chi_m)$ leads to a new choice of the $P_{2i}$, $Q_{2i-1}$, $\alpha_{2i}$ and $\beta_{2i-1}$ satisfying the same conditions (for the same $G_i$ and $\chi_i$) as the original choices.

The above remark, along with Theorems 5.88 and 5.118, easily implies Theorem 5.6.

The recursive way Theorems 5.88 and 5.118 were proved easily implies

**Remark 5.125.** Assume that the normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G$ for $G$, along with the character tower $\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^m$, is fixed. Let $\{P_{2r}, Q_{2i-1}|\alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{k,l}$ be a representative of the unique $G_m(\chi_1, \ldots, \chi_m)$-conjugacy class of triangular sets that corresponds to the above character tower according to Theorem 5.6. Then $\{\chi_i\}_{i=0}^{m-1}$ is a character tower of the normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_{m-1} \trianglelefteq G$ of $G$. Furthermore, the reduced set $\{P_{2r}, Q_{2i-1}|\alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{[(m-1)/2],[m/2]}$, is a representative of the unique $G_{m-1}(\chi_1, \ldots, \chi_{m-1})$-conjugacy class of triangular sets that corresponds to the character tower $\{\chi_i\}_{i=0}^{m-1}$.

**Corollary 5.126.** *Assume that Hypothesis 5.1 holds. Let $\{1 = \chi_0, \chi_1, \ldots, \chi_m\}$ be a tower of* (5.2) *and let $\{P_{2t}, Q_{2j-1}|\alpha_{2t}, \beta_{2j-1}\}$, for $t = 0, 1, \ldots, k$ and $j = 1, \ldots, l$, be its unique (up to conjugation) correspondent triangular set. Then*

$$Q_{2j-1} \in \mathrm{Hall}_{\pi'}(N(P_2, \ldots, P_{2j-2}, Q_1, \ldots, Q_{2j-3} \text{ in } G_{2j-1}(\chi_1, \ldots, \chi_{2j-2})),$$

*for all $j = 1, \ldots, l$, while, for all $t = 1, \ldots, k$, we also get that*

$$Q_{2t-1} \in \mathrm{Hall}_{\pi'}(N(P_2, \ldots, P_{2t-2}, Q_1, \ldots, Q_{2t-1} \text{ in } G_{2t}(\chi_1, \ldots, \chi_{2t-1})).$$

*Similarly, for the $\pi$-groups we have*

$$P_{2t} \in \mathrm{Hall}_{\pi}(N(P_2, \ldots, P_{2t-2}, Q_1, \ldots, Q_{2t-1} \text{ in } G_{2t}(\chi_1, \ldots, \chi_{2t-1})),$$

*for all* $t = 0, 1, \ldots, k$, *while, for all* $j = 0, 1, \ldots, l-1$, *we also get that*

$$P_{2j} \in \mathrm{Hall}_{\pi}(N(P_2, \ldots, P_{2j}, Q_1, \ldots, Q_{2j-1} \text{ in } G_{2j+1}(\chi_1, \ldots, \chi_{2j})).$$

*Proof.* Theorem 5.88 describes completely the relations between a character tower and its corresponding triangular set. Thus in view of (5.93) (for $i = 2j-1$) we have that $Q_{2j-1}$ is a $\pi'$-Hall subgroup of $G_{2j-1,2j-2}$, whenever $j = 1, \ldots, l$. Furthermore (5.92) (for $i = 2t$) implies that $Q_{2t-1} \in \mathrm{Hall}_{\pi'}(G_{2t,2t-1})$, for all $t = 1, \ldots, k$. But, according to (5.91), for all such $j$ and $t$ we have

$$G_{2j-1,2j-2} = N(P_2, \ldots, P_{2j-2}, Q_1, \ldots, Q_{2j-3} \text{ in } G_{2j-1}(\chi_1, \ldots, \chi_{2j-2})),$$

while

$$G_{2t,2t-1} = N(P_2, \ldots, P_{2t-2}, Q_1, \ldots, Q_{2t-1} \text{ in } G_{2t}(\chi_1, \ldots, \chi_{2t-1})).$$

Hence Corollary 5.126 follows for the $\pi'$-groups. The proof for the $\pi$-groups $P_{2t}$ is similar. $\square$

## 5.5 The groups $P_{2i}^*$: something stable in all that mess

For this section we continue working under the assumptions of Hypothesis 5.1. So $G$ is a finite group, while its arbitary (but fixed) normal series (5.2) satisfies Hypothesis 5.1. We also fix a character tower

$$1 = \chi_0, \chi_1, \dots, \chi_m \tag{5.128a}$$

of that series, along with its uniquely determined (up to conjugation) triangular set

$$\{P_0, P_2, \dots, P_{2k}, Q_1, \dots, Q_{2l-1} | \alpha_0, \alpha_2, \dots, \alpha_{2k}, \beta_1, \dots, \beta_{2l-1}\}, \tag{5.128b}$$

where $k$ and $l$ are defined as in (5.7) (i.e., $2l-1$ and $2k$ are the greatest odd and even, respectively, integers in the set $\{1, \dots, m\}$). For the normal series (5.2) and its character tower (5.128a), Theorem 2.5 can be applied. So for any $i = 1, \dots, m$ we write

$$G_i^* = G_i(\chi_1, \chi_2, \dots, \chi_{i-1})$$

and

$$G_\infty^* = G^* = G(\chi_1, \dots, \chi_m) \tag{5.129a}$$

for the stabilizers of $\chi_1, \chi_2, \dots, \chi_{i-1}$ and $\chi_1, \chi_2, \dots, \chi_m$ in $G_i$ and $G$, respectively. As $G_i \leq G_j$ for all $j$ with $0 \leq i \leq j \leq m$, the group $G_i$ fixes the characters $\chi_j$ for all such $j$. Hence

$$G_i^* = G_i(\chi_1, \dots, \chi_{i-1}) = G_i(\chi_1, \dots, \chi_m). \tag{5.129b}$$

Then, in view of Theorem 2.5, we have that $G_0^* = G_0 = 1, G_1^* = G_1$ and $G_j^* = G_i^* \cap G_j \trianglelefteq G_i^*$, whenever $0 \leq j \leq i \leq m$. Furthermore, there exist unique characters $\chi_i^*$, for $i = 1, \dots, m$, such that

$$\chi_i^* \in \mathrm{Irr}(G_i^*) \text{ lies over } \chi_1^*, \dots, \chi_{i-1}^* \text{ and induces } \chi_i, \tag{5.130a}$$

$$G_i^* = G_i(\chi_1^*, \chi_2^*, \dots, \chi_{i-1}^*) = G_i(\chi_1^*, \chi_2^*, \dots, \chi_m^*). \tag{5.130b}$$

We note here that, according to the same theorem,

$$G_1^* = G_1 \text{ and } \chi_1^* = \chi_1 \in \mathrm{Irr}(G_1^*) = \mathrm{Irr}(G_1). \tag{5.130c}$$

According to (5.10b), for each $i = 1, \dots, k$ the group $P_{2i}$ normalizes all the previously chosen $\pi$-groups $P_2, P_4, \dots, P_{2i-2}$. Hence the product:

$$P_{2i}^* = P_2 \cdot P_4 \cdots P_{2i} \tag{5.131}$$

is a group. We also define $P_0^* := 1$, therefore $P_{2i}^*$ is defined for all $i = 0, 1, \dots, k$. As will become clear, these groups play the most important role in the construction we did in the previous section. The reason is that they are the only groups that remain unchanged when we change the $\pi'$-parts (groups and characters ) of the triangular set (5.128b). The way we defined the groups $P_{2i}^*$ uses the individual $P_{2t}$ for all $t$ with $1 \leq t \leq i$. But, as the proposition that follows shows, we could have picked the groups $P_{2i}^*$ using only the groups $G_{2i}^*$.

**Proposition 5.132.** *The group $P_{2i}^*$ is a $\pi$-Hall subgroup of $G_{2i}^*$ whenever $i = 0, 1, \dots, k$. It is also a $\pi$-Hall subgroup of $G_{2i+1}^*$ for all $i = 0, 1, \dots, l-1$. Furthermore $P_{2r}^* = P_{2i}^* \cap G_{2r}^*$ and thus $P_{2r}^* \trianglelefteq P_{2i}^*$, whenever $1 \leq r \leq i \leq k$.*

To prove Proposition 5.132 we need the following lemma:

**Lemma 5.133.** *If $j = 2, \dots, m, \infty$, and $s$ are such that $2 \leq 2s \leq j$, then*

$$\begin{aligned} G_j^* &= N(P_2, \dots, P_{2s-2}, Q_1, Q_3, \dots, Q_{2s-1} \text{ in } G_j^*) \cdot G_{2s-1}^* \\ &= N(P_2, \dots, P_{2s}, Q_1, Q_3, \dots, Q_{2s-1} \text{ in } G_j^*) \cdot G_{2s}^*. \end{aligned}$$

*Proof.* We will use induction on $s$. For $s = 1$ the group $N(P_2, \dots, P_{2s-2}, Q_1, \dots, Q_{2s-1} \text{ in } G_j^*)$ equals $N(Q_1 \text{ in } G_j^*) = G_j^*$, while the normalizer $N(P_2, \dots, P_{2s}, Q_1, \dots, Q_{2s-1} \text{ in } G_j^*)$ equals the group $N(P_2, Q_1 \text{ in } G_j^*) = N(P_2 \text{ in } G_j^*)$. According to (5.17b) and (5.17c), we have that $P_2 \in \mathrm{Hall}_\pi(G_2(\chi_1)) = \mathrm{Hall}_\pi(G_2^*)$. Therefore, for any $j \geq 2$, the Frattini argument implies that $G_j^* = N(P_2 \text{ in } G_j^*) \cdot G_2^*$, as $G_2^*$ is a normal subgroup of $G_j^*$. Thus Lemma 5.133 holds when $s = 1$ and $j = 2, \dots, m, \infty$.

Assume now that Lemma 5.133 holds for all $s = 1, 2, \dots, t-1$, where $2 < 2t \leq j$. We will prove that it also holds when $s = t$. By induction, for $s = t-1$, we get that $G_j^* = N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_j^*) \cdot G_{2t-2}^*$. But

$$N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_{2t-1}^*) \trianglelefteq N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_j^*).$$

Furthermore, Corollary 5.126 implies that the group $N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_{2t-1}^*)$ has $Q_{2t-1}$ as a $\pi'$-Hall subgroup. Hence, by the Frattini argument, we have

$$\begin{aligned} &N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_j^*) = \\ &N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3}, Q_{2t-1} \text{ in } G_j^*) \cdot N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_{2t-1}^*). \end{aligned}$$

Therefore,

$$\begin{aligned} G_j^* &= N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-3} \text{ in } G_j^*) \cdot G_{2t-2}^* \\ &= N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-3}, Q_{2t-1} \text{ in } G_j^*) \cdot N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-3} \text{ in } G_{2t-1}^*) \cdot G_{2t-2}^*. \end{aligned}$$

By induction, $N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3} \text{ in } G_{2t-1}^*) \cdot G_{2t-2}^* = G_{2t-1}^*$. Hence,

$$G_j^* = N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-3}, Q_{2t-1} \text{ in } G_j^*) \cdot G_{2t-1}^*. \tag{5.134}$$

This proves the first equality in Lemma 5.133 for $s = t$.

It remains to show that $G_j^* = N(P_2, \dots, P_{2t}, Q_1, Q_3, \dots Q_{2t-1} \text{ in } G_j^*) \cdot G_{2t}^*$. By Corollary 5.126, the group $P_{2t}$ is a $\pi$-Hall subgroup of $N(P_2, \dots, P_{2t-2}, Q_1, Q_3, \dots, Q_{2t-1} \text{ in } G_{2t}^*)$. Since $N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-1} \text{ in } G_{2t}^*) \trianglelefteq N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-1} \text{ in } G_j^*)$, Frattini's argument implies that

$$\begin{aligned} &N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-1} \text{ in } G_j^*) = \\ &\qquad N(P_2, \dots, P_{2t}, Q_1, \dots, Q_{2t-1} \text{ in } G_j^*) \cdot N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-1} \text{ in } G_{2t}^*). \end{aligned}$$

The above equation, along with (5.134), implies

$$\begin{aligned} &G_j^* \\ &\quad = N(P_2, \dots, P_{2t}, Q_1, \dots, Q_{2t-1} \text{ in } G_j^*) \cdot N(P_2, \dots, P_{2t-2}, Q_1, \dots, Q_{2t-1} \text{ in } G_{2t}^*) \cdot G_{2t-1}^* \\ &\quad = N(P_2, \dots, P_{2t}, Q_1, Q_3, \dots, Q_{2t-1} \text{ in } G_j^*) \cdot G_{2t}^*. \end{aligned}$$

This proves the remaining equality in Lemma 5.133 for $s = t$. Hence the inductive proof of that lemma is complete. $\square$

*Proof of Proposition 5.132.* We will use induction on $i$. For $i = 0$ it is trivially true as $P_0^* = 1 = G_0^*$, while $G_1^*$ is a $\pi'$-group. If $i = 1$, then $P_2^* = P_2$. By (5.17c) and (5.17b) the latter group is a $\pi$-Hall subgroup of $G_2(\chi_1) = G_2^*$. Furthermore, as $G_3^*/G_2^*$ is a $\pi'$-group, $P_2^*$ is also a $\pi$-Hall subgroup of $G_3^*$. The rest of the proposition holds trivially for $i = 1$.

Now we assume that Proposition 5.132 holds for all $i = 1, 2, \ldots, t-1$, where $1 < t \leq k$. We will prove that it holds for $i = t$. We have $P_{2t-2}^* \in \mathrm{Hall}_\pi(G_{2t-2}^*)$ by induction, and $P_{2t} \in \mathrm{Hall}_\pi(N(P_2, \ldots, P_{2t-2}, Q_3, \ldots, Q_{2t-1} \text{ in } G_{2t}^*))$ by Corollary 5.126. Since $P_{2t-2}^* = P_2 \cdots P_{2t-2}$, it follows that the group $P_{2t}^* = P_{2t-2}^* \cdot P_{2t}$ is a $\pi$-subgroup of $G_{2t}^*$. So there exists a $\pi$-Hall subgroup, $\mathcal{P}_{2t}$, of $G_{2t}^*$ with

$$P_{2t}^* \leq \mathcal{P}_{2t}. \tag{5.135}$$

Since $G_{2t-2}^*$ is a normal subgroup of $G_{2t}^*$, we conclude that $\mathcal{P}_{2t} \cap G_{2t-2}^*$ is a $\pi$-Hall subgroup of $G_{2t-2}^*$. But $P_{2t-2}^*$ is a $\pi$-Hall subgroup of $G_{2t-2}^*$ such that $P_{2t-2}^* \leq P_{2t}^* \cap G_{2t-2}^* \leq \mathcal{P}_{2t} \cap G_{2t-2}^*$. Hence

$$P_{2t-2}^* = P_{2t}^* \cap G_{2t-2}^* = \mathcal{P}_{2t} \cap G_{2t-2}^*.$$

Furthermore, as $G_{2t-1}^*/G_{2t-2}^*$ is a $\pi'$-group, $P_{2t-2}^*$ is not only a $\pi$-Hall subgroup of $G_{2t-2}^*$, but also of $G_{2t-1}^*$ (note that $G_{2t-1}^*$ exists for all $t = 1, \ldots, k$). Hence

$$P_{2t-2}^* = P_{2t}^* \cap G_{2t-1}^* = \mathcal{P}_{2t} \cap G_{2t-1}^* \in \mathrm{Hall}_\pi(G_{2t-1}^*). \tag{5.136}$$

Since $G_{2t}^*/G_{2t-1}^*$ is a $\pi$-group, and $\mathcal{P}_{2t}$ is a $\pi$-Hall subroup of $G_{2t}^*$ we have that $G_{2t}^* = \mathcal{P}_{2t} \cdot G_{2t-1}^*$. Furthermore, $G_{2t}^* = N(P_2, \ldots, P_{2t-2}, Q_1, \ldots, Q_{2t-1} \text{ in } G_{2t}^*) \cdot G_{2t-1}^*$ according to Lemma 5.133. Hence, the $\pi$-Hall subgroup $P_{2t}$ of $N(P_2, \ldots, P_{2t-2}, Q_1, \ldots Q_{2t-1} \text{ in } G_{2t}^*)$ also covers $G_{2t}^*/G_{2t-1}^*$. As $P_{2t-2}^* \leq G_{2t-2}^* \leq G_{2t-1}^*$, we conclude that

$$\mathcal{P}_{2t} \cdot G_{2t-1}^* = G_{2t}^* = P_{2t} \cdot G_{2t-1}^* = P_{2t}^* \cdot G_{2t-1}^*.$$

This, along with (5.136) and (5.135), implies that $P_{2t}^* = \mathcal{P}_{2t}$. Thus $P_{2t}^*$ is a $\pi$-Hall subgroup of $G_{2t}^*$. If $t \leq l - 1$ then the group $G_{2t+1}^*$ is defined and $G_{2t+1}^*/G_{2t}^*$ is a $\pi'$-group. We conclude that $P_{2t}^* \in \mathrm{Hall}_\pi(G_{2t+1}^*)$ for all such $t$. The rest of Proposition 5.132 for $i = t$ follows easily, as $G_{2r}^* \trianglelefteq G_{2t}^*$, and $P_{2r}^* \leq P_{2t}^*$, whenever $1 \leq r \leq t \leq k$. Hence, the inductive proof of the proposition is complete. $\square$

Along with the groups $P_{2i}^*$ we have irreducible characters $\alpha_{2i}^*$ that correspond uniquely to the irreducible characters $\alpha_{2i}$ of $P_{2i}$, for all $i = 1, \ldots, k$. To prove that these characters exist and show how their correspondence with the $\alpha_{2i}$ works, we will use the following lemma:

**Lemma 5.137.** *If $1 \leq i \leq t \leq k$ then*

$$N(Q_{2i-1} \text{ in } P_{2i-2} \cdot P_{2i} \cdots P_{2t}) = P_{2i} \cdots P_{2t}. \tag{5.138}$$

*Proof.* According to (5.10b), the group $P_{2j}$ normalizes $Q_{2i-1}$ for all $j \geq i$. Hence the product $P_{2i} \cdots P_{2t}$ is contained in the normalizer $N(Q_{2i-1} \text{ in } P_{2i-2} \cdots P_{2t})$. So, $N(Q_{2i-1} \text{ in } P_{2i-2} \cdots P_{2t}) = N(Q_{2i-1} \text{ in } P_{2i-2}) \cdot P_{2i} \cdots P_{2n}$. By (5.34), we have $N(Q_{2i-1} \text{ in } P_{2i-2}) = P_{2i-2,2i-1} = P_{2i-2} \cap P_{2i}$. Therefore $N(Q_{2i-1} \text{ in } P_{2i-2})$ is a subgroup of $P_{2i} \leq P_{2i} \cdots P_{2t}$. This completes the proof of Lemma 5.137. $\square$

**Proposition 5.139.** *For all $i, t$ with $1 \leq i \leq t \leq k$ we have*

$$N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } P_{2t}^*) = P_{2i} \cdots P_{2t}. \tag{5.140}$$

*Proof.* The proof is a multiple application of Lemma 5.137.

$$\begin{aligned}
&N(Q_1, Q_3, Q_5, \dots, Q_{2i-1} \text{ in } P_{2t}^*) \\
&= N(Q_3, Q_5, \dots, Q_{2i-1} \text{ in } P_{2t}^*) && \text{since } Q_1 \trianglelefteq G, \\
&= N(Q_5, \dots, Q_{2i-1} \text{ in } N(Q_3 \text{ in } P_{2t}^*)) \\
&= N(Q_5, \dots, Q_{2i-1} \text{ in } P_4 \cdots P_{2t}) && \text{in view of Lemma 5.137}, \\
&= N(Q_7, \dots, Q_{2i-1} \text{ in } N(Q_5 \text{ in } P_4 \cdots P_{2t})) \\
&\dots \\
&= N(Q_{2i-1} \text{ in } P_{2i-2} \cdots P_{2t}) \\
&= P_{2i} \cdots P_{2t} && \text{in view of Lemma 5.137.}
\end{aligned}$$

$\square$

In particular, we have a way to recover the $P_{2i}$ from the products $P_{2i}^*$ and the $q$-groups $Q_3, \dots, Q_{2i-1}$, whenever $i = 1, \dots, k$. Indeed, (5.140) implies:

$$N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } P_{2i}^*) = P_{2i}. \tag{5.141}$$

**Lemma 5.142.** *If $1 \leq j < i \leq k$, then the product $Q_{2j+1} \cdot P_{2j} \cdot P_{2j+2} \cdots P_{2i}$ is a subgroup of $G$ having $Q_{2j+1}$ as a Hall $\pi'$-subgroup, and $P_{2j} \cdot P_{2j+2} \cdots P_{2i}$ as a Hall $\pi$-subgroup. Both the $\pi$-group $P_{2j}$ and the product $Q_{2j+1} \cdot P_{2j}$ are normal subgroups of $Q_{2j+1} \cdot P_{2j} \cdot P_{2j+2} \cdots P_{2i}$. Furthermore, $N(Q_{2j+1} \text{ in } P_{2j} \cdot P_{2j+2} \cdots P_{2i}) = P_{2j+2} \cdots P_{2i}$. Hence Theorem 3.1 gives us a one to one $Q_{2j+1}$-correspondence*

$$\alpha_{2i,2j-1}^* \underset{Q_{2j+1}}{\longleftrightarrow} \alpha_{2i,2j+1}^*$$

*between all characters $\alpha_{2i,2j+1}^* \in \mathrm{Irr}(P_{2j+2} \cdots P_{2i})$ and all characters $\alpha_{2i,2j-1}^* \in \mathrm{Irr}(P_{2j} \cdot P_{2j+2} \cdots P_{2i})$ lying over some character $\alpha_{2j,2j-1}^* \in \mathrm{Irr}^{Q_{2j+1}}(P_{2j})$. This correspondence is invariant under conjugation by elements of any subgroup $K \leq G$ normalizing both $Q_{2j+1}$ and $P_{2j} \cdot P_{2j+2} \cdots P_{2i}$. Furthermore, if $\alpha_{2s,2j-1}^* \in \mathrm{Irr}(P_{2j} \cdots P_{2s})$ is any character of $P_{2j} \cdots P_{2s}$ lying under $\alpha_{2i,2j-1}^*$ and above $\alpha_{2j,2j-1}^*$, for some $s$ with $1 \leq j < s \leq i \leq k$, then its $Q_{2j+1}$-correspondent $\alpha_{2s,2j+1}^*$ lies under the $Q_{2j+1}$-correspondent $\alpha_{2i,2j+1}^*$ of $\alpha_{2i,2j-1}^*$.*

*Proof.* We only need to show that $Q_{2j+1} \cdot P_{2j} \cdot P_{2j+2} \cdots P_{2i}$ is a group having $P_{2j}$ and $Q_{2j+1} \cdot P_{2j}$ as normal subgroups, while $P_{2j} \cdots P_{2s} \trianglelefteq P_{2j} \cdots P_{2i}$ whenever $1 \leq j < s \leq i \leq k$. The rest is an easy application of Theorem 3.13 and (5.140).

By (5.10a) the group $Q_{2j+1}$ normalizes the group $P_{2j}$. Furthermore, (5.10b) implies that the groups $P_{2j+2}, P_{2j+4}, \dots, P_{2i}$ normalize both $P_{2j}$ and $Q_{2j+1}$, while $P_{2s}$ normalizes $P_{2t}$ for all $s, t$ with $j \leq t \leq s \leq i$. Hence, the products $Q_{2j+1} \cdot P_{2j}$ and $P_{2j+2} \cdots P_{2s}$ form groups. Even more, the latter group normalizes the former one for every $s = j+1, \dots, i$. Hence $Q_{2j+1} \cdot P_{2j} \cdot P_{2j+2} \cdots P_{2i}$ forms a group having $P_{2j}$ and $Q_{2j+1} \cdot P_{2j}$ as normal subgroups. It is also clear that the product group $P_{2j} \cdots P_{2s}$ is a normal subgroup of $P_{2j} \cdots P_{2i}$ whenever $j < s \leq i$. $\square$

**Theorem 5.143.** *For any $i = 2, 3, \ldots, k$ we may form a chain*

$$\alpha^*_{2i,1} \underset{Q_3}{\longleftrightarrow} \alpha^*_{2i,3} \underset{Q_5}{\longleftrightarrow} \alpha^*_{2i,5} \underset{Q_7}{\longleftrightarrow} \cdots \underset{Q_{2i-3}}{\longleftrightarrow} \alpha^*_{2i,2i-3} \underset{Q_{2i-1}}{\longleftrightarrow} \alpha^*_{2i,2i-1} \tag{5.144}$$

*of the $Q_{2j+1}$-correspondences in Lemma 5.142. The composite $Q_3, Q_5, \ldots, Q_{2i-1}$-correspondence is one to one between all characters $\alpha^*_{2i,2i-1} \in \mathrm{Irr}(P_{2i})$ and all characters $\alpha^*_{2i,1} \in \mathrm{Irr}(P^*_{2i})$ having the following property:*

**Property 5.145.** *There exist characters $\alpha^*_{2j,1} \in \mathrm{Irr}(P^*_{2j})$, for $j = 1, 2, \ldots, i-1$, such that each $\alpha^*_{2j,1}$ is $Q_{2j+1}$-invariant and lies under $\alpha^*_{2j+2,1}$.*

*This $Q_3, Q_5, \ldots, Q_{2i-1}$-correspondence is invariant under conjugation by elements of any subgroup $K \leq G$ normalizing all the subgroups $Q_3, Q_5, \ldots, Q_{2i-1}$ and $P^*_{2i}$.*

*Proof.* It follows immediately from Lemma 5.142 that the composite correspondence is one to one between all characters in $\mathrm{Irr}(P_{2i})$ and some characters in $\mathrm{Irr}(P^*_{2i})$. It is also clear that this correspondence is invariant under conjugation by elements of $N(Q_3, Q_5, \ldots, Q_{2i-1}, P^*_{2i} \text{ in } G)$. Thus it remains to show that the image of the composite correspondence is exactly the subset of all $\alpha^*_{2i,1} \in \mathrm{Irr}(P^*_{2i})$ having Property 5.145.

We will first show that any $\alpha^*_{2i,1} \in \mathrm{Irr}(P^*_{2i})$ which is the $Q_3, Q_5, \ldots, Q_{2i-1}$-correspondent of some $\alpha^*_{2i,2i-1} \in \mathrm{Irr}(P_{2i})$ must satisfy Property 5.145. This can be done by induction on $i$. Assume that $i = 2$. Let $\alpha^*_{4,1} \in \mathrm{Irr}(P^*_4)$ be the $Q_3$-correspondent of some $\alpha^*_{4,3} \in \mathrm{Irr}(P_4)$. In this case, Lemma 5.142 (for $j = 1$ and $i = 2$), describes this $Q_3$-correspondence as one between all characters $\alpha^*_{4,3} \in \mathrm{Irr}(P_4)$ and all characters $\alpha^*_{4,1} \in \mathrm{Irr}(P_4)$ lying over some character $\alpha^*_{2,1} \in \mathrm{Irr}^{Q_3}(P_2)$. As $P_2 = P^*_2$, we obviously have that $\alpha^*_{4,1}$ lies above the $Q_3$-invariant character $\alpha^*_{2,1}$ of $P^*_2$. Thus $\alpha^*_{4,1}$ satisfies Property 5.145. This implies the $i = 2$ case.

Assume that is true for all $i = 2, \ldots, t-1$, for some $t$ with $2 < t \leq k$. We will prove it also holds when $i = t$.

Let $\alpha^*_{2t,1} \in \mathrm{Irr}(P^*_{2t})$ be the $Q_3, \ldots, Q_{2t-1}$-correspondent of some $\alpha^*_{2t,2t-1} \in \mathrm{Irr}(P_{2t})$. Then, according to Lemma 5.142, the character $\alpha^*_{2t,2t-1}$ of $P_{2t}$ has as a $Q_{2t-1}$-correspondent a character $\alpha^*_{2t,2t-3}$ of $P_{2t-2} \cdot P_{2t}$, that lies above some character $\alpha^*_{2t-2,2t-3} \in \mathrm{Irr}^{Q_{2t-1}}(P_{2t-2})$. Let $\alpha^*_{2t-2,1} \in \mathrm{Irr}(P^*_{2t-2})$ be the $Q_3, Q_5, \ldots, Q_{2t-3}$-correspondent of $\alpha^*_{2t-2,2t-3} \in \mathrm{Irr}(P_{2t-2})$, that we get by multiple applications of Lemma 5.142. As $\alpha^*_{2t-2,2t-3}$ lies under $\alpha^*_{2t,2t-3}$, we get that the $Q_3, \ldots, Q_{2t-3}$-correspondent $\alpha^*_{2t-2,1}$ of $\alpha^*_{2t-2,2t-3}$ lies under the $Q_3, \ldots, Q_{2t-3}$-correspondent $\alpha^*_{2t,1}$ of $\alpha^*_{2t,2t-3}$ (see Theorem 3.13). Furthermore, $Q_{2t-1}$ fixes $\alpha^*_{2t-2,2t-3}$ and normalizes the groups $Q_3, \ldots, Q_{2t-3}$, as well as the product group $P_2 \cdots P_{2t-2}$. Hence it also fixes the $Q_3, \ldots, Q_{2t-3}$-correspondent $\alpha^*_{2t-2,1}$ of $\alpha^*_{2t-2,2t-3}$. In conclusion, the character $\alpha^*_{2t-2,1}$ is $Q_{2t-1}$-invariant and lies under $\alpha^*_{2t,1}$. Thus $\alpha^*_{2t,1}$ satisfies Property 5.145 for $j = t-1$.

As $\alpha^*_{2t-2,1}$ is the $Q_3, \ldots, Q_{2t-3}$-correspondent of $\alpha_{2t-2,2t-3}$, the inductive hypothesis applies. Hence for every $j = 1, \ldots, t-2$, there exists a character $\alpha^*_{2j,1} \in \mathrm{Irr}(P^*_{2j})$ that is $Q_{2j+1}$-invariant and lies under $\alpha^*_{2j+2,1}$. The existence of these characters $\alpha^*_{2j,1}$, along with $\alpha^*_{2t-2,1}$, implies that $\alpha^*_{2t,1}$ satisfies Property 5.145. This completes the inductive proof that an $\alpha^*_{2i,1} \in \mathrm{Irr}(P^*_{2i})$ which is the $Q_3, Q_5, \ldots, Q_{2i-1}$-correspondent of some $\alpha^*_{2i,2i-1} \in \mathrm{Irr}(P_{2i})$ must satisfy Property 5.145.

Now assume that a character $\alpha^*_{2i,1} \in \mathrm{Irr}(P^*_{2i})$ satisfying Property 5.145 is fixed, for some $i = 2, 3, \ldots, k$. We want to construct some character $\alpha^*_{2i,2i-1} \in \mathrm{Irr}(P_{2i})$ having $\alpha^*_{2i,1}$ as its $Q_3, \ldots, Q_{2i-1}$-correspondent. To do this, we will find characters $\alpha^*_{2i,2t+1}$, for every $t = 1, \ldots, i-1$, to form the chain (5.144). So it suffices to show that for every $t$ with $1 \leq t \leq i-1$ we can apply Lemma 5.142 to get a $Q_{2t+1}$-correspondent $\alpha^*_{2i,2t+1}$ of $\alpha^*_{2i,2t-1}$. This is done in a recursive way. So, for

$t = 1$ we observe that Property 5.145 implies that $\alpha^*_{2i,1}$ lies above the $Q_3$-invariant character $\alpha^*_{2,1} \in \mathrm{Irr}(P_2) = \mathrm{Irr}(P_2^*)$. Hence, according to Lemma 5.142, the character $\alpha^*_{2i,1} \in \mathrm{Irr}(P_2 \cdots P_{2i})$ has a $Q_3$-correspondent character $\alpha^*_{2i,3} \in \mathrm{Irr}(P_4 \cdots P_{2i})$. Furthermore, as $\alpha^*_{2s,1}$ lies under $\alpha^*_{2s+2,1}$ and above $\alpha^*_{2,1}$, for every $s = 2, \dots, i-1$, (by Property 5.145), the same lemma implies that the $Q_3$-correspondent $\alpha^*_{2s,3} \in \mathrm{Irr}(P_4 \cdots P_{2s})$ of $\alpha^*_{2s,1} \in \mathrm{Irr}(P_2 \cdots P_{2s})$ is defined and lies under $\alpha^*_{2s+2,3}$. Even more, the character $\alpha^*_{2s,1}$ is $Q_{2s+1}$-invariant and $Q_{2s+1}$ normalizes both $Q_3$ and the product group $P_2 \cdots P_{2s}$, for all $s = 2, \dots, i-1$. Hence the $Q_3$-correspondent character $\alpha^*_{2s,3}$ of $\alpha^*_{2s,1}$ is also $Q_{2s+1}$-invariant.

We can now do the case $t = 2$. Indeed, the previous comment for $s = 2$ implies that $\alpha^*_{2s,3} \in \mathrm{Irr}(P_4 \cdots P_{2i})$ lies above the $Q_5$-invariant character $\alpha^*_{4,3} \in \mathrm{Irr}(P_4)$. Thus we can apply Lemma 5.142 again to get a $Q_5$-correspondent character $\alpha^*_{2i,5} \in \mathrm{Irr}(P_6 \cdots P_{2i})$. Note that the $Q_5$-correspondent $\alpha^*_{2s,5} \in \mathrm{Irr}(P_6 \cdots P_{2s})$ of $\alpha^*_{2s,3} \in \mathrm{Irr}(P_4 \cdots P_{2s})$ is defined whenever $s = 3, \dots, i-1$. This correspondent lies under $\alpha^*_{2s+2,5}$, as $\alpha^*_{2s,3}$ lies under $\alpha^*_{2s+2,3}$. Furthermore $\alpha^*_{2s,3}$ is $Q_{2s+1}$-invariant, while $Q_{2s+1}$ normalizes both $Q_5$ and the product $P_4 \cdots P_{2s}$. Therefore the $Q_5$-correspondent $\alpha^*_{2s,5}$ of $\alpha^*_{2s,3}$ is also $Q_{2s+1}$-invariant. So for $s = 3$ we have that $\alpha_{2i,5} \in \mathrm{Irr}(P_6 \cdots P_{2i})$ lies over the $Q_7$-invariant character $\alpha_{6,5} \in \mathrm{Irr}(P_6)$. Hence we can apply Lemma 5.142 again and thus get the desired correspondence for $t = 3$. We continue similarly. At the $t$-step we have the character $\alpha^*_{2i,2t-1} \in \mathrm{Irr}(P_{2t} \cdots P_{2i})$ lying over the $Q_{2t+1}$-invariant character $\alpha^*_{2t,2t-1} \in \mathrm{Irr}(P_{2t})$, while for all $s = t, \dots, i-1$ the character $\alpha^*_{2s,2t-1} \in \mathrm{Irr}(P_{2t} \cdots P_{2s})$ is $Q_{2s+1}$-invariant and lies under $\alpha^*_{2s+2,2t-1}$.

At the last step for $t = i-1$ we end up with the character $\alpha^*_{2i,2i-3} \in \mathrm{Irr}(P_{2i-2} \cdot P_{2i})$ lying over the $Q_{2i-1}$-invariant character $\alpha^*_{2i-2,2i-3} \in \mathrm{Irr}(P_{2i-2})$. So the final application of Lemma 5.142 will provide a $Q_{2i-1}$-correspondent character $\alpha^*_{2i,2i-1} \in \mathrm{Irr}(P_{2i})$ of $\alpha^*_{2i,2i-3}$, that is actually a $Q_3, Q_5, \dots, Q_{2i-1}$-correspondent of $\alpha^*_{2i}$.

This completes the proof of Theorem 5.143. ☐

**Remark 5.146.** The chain $Q_3, Q_5, \dots, Q_{2i-1}$ is empty when $i = 1$. In that case we define the $Q_3, Q_5, \dots, Q_{2i-1}$-correspondence to be the identity correspondence between $\mathrm{Irr}(P_{2i}) = \mathrm{Irr}(P_2)$ and the equal set $\mathrm{Irr}(P^*_{2i}) = \mathrm{Irr}(P^*_2)$.

Now we can make the

**Definition 5.147.** For each $i = 1, 2, \dots, k$ we define $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$ to be the $Q_3, Q_5, \dots Q_{2i-1}$-correspondent of the character $\alpha_{2i} \in \mathrm{Irr}(P_{2i})$.

So

$$\alpha^*_2 = \alpha_2 \tag{5.148}$$

by convention. Furthermore Theorem 5.143 obviously implies

**Proposition 5.149.** *If a subgroup $K \leq G$ normalizes all the subgroups $Q_3, Q_5, \dots, Q_{2i-1}$ and $P^*_{2i}$, for some $i = 1, 2, \dots, k$, then $\alpha^*_{2i}$ and $\alpha_{2i}$ have the same stabilizer $K(\alpha^*_{2i}) = K(\alpha_{2i})$, in $K$.*

**Corollary 5.150.**

$$Q_{2j+1}(\alpha^*_{2i}) = Q_{2j+1}(\alpha_{2i}) = Q_{2j+1} \tag{5.151a}$$

*and*

$$P_{2s}(\alpha^*_{2i}) = P_{2s}(\alpha_{2i}) = P_{2s}. \tag{5.151b}$$

*whenever $1 \leq i \leq j < l$ and $1 \leq i \leq s \leq k$.*

*Proof.* According to (5.10), the groups $Q_{2j+1}$ and $P_{2s}$ normalize $P_2, \dots, P_{2i}$ and $Q_3, \dots Q_{2i-1}$, for all $j, s$ with $1 \leq i \leq j < l$ and $1 \leq i \leq s \leq k$. Thus they also normalize the groups $P^*_2, \dots, P^*_{2i}$.

Hence Proposition 5.149, along with the fact that $Q_{2j+1}$ and $P_{2s}$ fix $\alpha_{2i}$ (see (5.17c,e)), implies Corollary 5.150. $\square$

The next proposition shows how the characters $\alpha^*_{2i}$ are related.

**Lemma 5.152.** *If $i = 2, 3, \ldots, k$, then $\alpha^*_{2i-2}$ is the only character in $\mathrm{Irr}(P^*_{2i-2})$ lying under $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$.*

*Proof.* Let $i = 2, 3, \ldots, k$ be fixed. We first show that $\alpha^*_{2i-2}$ lies under $\alpha^*_{2i}$. By (5.17d), the character $\alpha_{2i} \in \mathrm{Irr}(P_{2i})$ lies over the character $\alpha_{2i-2,2i-1} \in \mathrm{Irr}(P_{2i-2,2i-1})$, where $P_{2i-2,2i-1} = C(Q_{2i-1} \text{ in } P_{2i-2})$ (by (5.14)) and $\alpha_{2i-2,2i-1}$ is the $Q_{2i-1}$-Glauberman correspondent of $\alpha_{2i-2}$ (see (5.49)). According to Theorem 3.13, the $Q_{2i-1}$-correspondent $\alpha^*_{2i,2i-3} \in \mathrm{Irr}(P_{2i-2} \cdot P_{2i})$ of $\alpha_{2i} = \alpha^*_{2i,2i-1}$ (see Lemma 5.142 for $j = i-1$), lies over the $Q_{2i-1}$-Glauberman correspondent $\alpha_{2i-2} \in \mathrm{Irr}(P_{2i-2})$ of $\alpha_{2i-2,2i-1}$. It follows that $\alpha^*_{2i}$, which is the $Q_3, Q_5, \ldots, Q_{2i-3}$-correspondent of $\alpha^*_{2i,2i-3}$, lies over the $Q_3, Q_5, \ldots, Q_{2i-3}$-correspondent $\alpha^*_{2i-2}$ of $\alpha_{2i-2}$ (see Theorem 3.13).

Since $P^*_{2i-2}$ is a normal subgroup of $P^*_{2i}$, Clifford's Theorem implies that we can prove the lemma by showing that $\alpha^*_{2i-2}$ is $P^*_{2i}$-invariant. But $P^*_{2i} = P^*_{2i-2} \cdot P_{2i}$, and $P^*_{2i-2}$ fixes its own character, while $P_{2i}(\alpha^*_{2i-2}) = P_{2i}$ by (5.151b). We conclude that $\alpha^*_{2i-2}$ is $P^*_{2i}$-invariant. Thus the lemma holds. $\square$

By induction the above lemma implies

**Proposition 5.153.** *If $1 \leq j \leq i \leq k$, then $\alpha^*_{2j}$ is the only character in $\mathrm{Irr}(P^*_{2j})$ lying under $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$.*

## 5.6 The groups $Q^*_{2i-1}$

We can define groups $Q^*_{2i-1}$ similar to the $P_{2i}$. Indeed, in view of (5.10a) the product

$$Q^*_{2i-1} = Q_1 \cdot Q_3 \cdot \cdots \cdot Q_{2i-1}, \tag{5.154}$$

is a group whenever $1 \leq i \leq l$.

The groups $Q^*_{2i-1}$ are defined symmetrically to the $P^*_{2i}$, and satisfy results similar to those the $P^*_{2i}$ satisfy. The following proposition is analogous to Proposition 5.132 for the groups $Q^*_{2i-1}$. Its proof is similar.

**Proposition 5.155.** *The group $Q^*_{2i-1}$ is a $\pi'$-Hall subgroup of $G^*_{2i-1}$ whenever $1 \leq i \leq l$, while for $i = 1, \ldots, k$ we have, in addition, that $Q^*_{2i-1} \in \mathrm{Hall}_{\pi'}(G^*_{2i})$. Furthermore, $Q^*_{2r-1} = Q^*_{2i-1} \cap G^*_{2r-1}$, and thus $Q^*_{2r-1} \trianglelefteq Q^*_{2i-1}$, for all $r, i$ with $1 \leq r \leq i \leq l$.*

*Proof.* The proof is done by induction, and is totally symmetric to the proof of Proposition 5.132 for the $\pi'$-groups in the place of the $\pi$-groups. So we omit it. $\square$

Of course by $\pi, \pi'$-symmetry, we can define irreducible characters $\beta^*_{2i-1}$ of $Q^*_{2i-1}$ as we did for the characters $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$. So

**Definition 5.156.** For every $i = 1, \ldots, l$, we define $\beta^*_{2i-1} \in \mathrm{Irr}(Q^*_{2i-1})$ to be the $P_2, P_4, \ldots, P_{2i-2}$-correspondent of the character $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$. By convention

$$\beta^*_1 = \beta_1.$$

Results similar to Theorem 5.143 and Propositions 5.149 and 5.153 hold for the $\beta^*$-characters. According to Propositions 5.132 and 5.155 we get

**Corollary 5.157.** *If $i = 1, \dots, k$ and $j = 1, \dots, l$ then*

$$G_{2i}^* = P_{2i}^* \cdot Q_{2i-1}^* \text{ and } G_{2j-1}^* = P_{2j-2}^* \cdot Q_{2j-1}^*,$$

*where $P_0^* = 1$ by convention. In particular*

$$G_m^* = P_{2k}^* \cdot Q_{2l-1}^*$$

Similar to the equations (5.138), (5.140) and (5.141) that the groups $P_{2i}^*$ satisfy, the following equations are satisfied by the groups $Q_{2i-1}^*$:

$$N(P_{2i} \text{ in } Q_{2i-1} \cdot Q_{2i+1} \cdot \dots \cdot Q_{2t+1}) = Q_{2i+1} \cdot \dots \cdot Q_{2t+1}, \tag{5.158a}$$

$$N(P_2, P_4, \dots, P_{2i} \text{ in } Q_{2t+1}^*) = Q_{2i+1} \cdot \dots \cdot Q_{2t+1} \tag{5.158b}$$

and

$$N(P_2, \dots, P_{2i} \text{ in } Q_{2i+1}^*) = Q_{2i+1}, \tag{5.158c}$$

for all $i, t$ with $1 \leq i \leq t < l$.

The proof of (5.158a) is similar to that of (5.138), using (5.33) in the place of (5.34). The equation (5.158b) follows by repeated applications of equation (5.158a) (as (5.140) followed from (5.138)), while (5.158c) is a special case (when $t = i$) of (5.158b).

In the proposition that follows we rewrite (5.141) and (5.158c) in a slightly different way.

**Proposition 5.159.** *For all $i = 1, \dots, k$ we have*

$$N(Q_{2i-1}^* \text{ in } P_{2i}^*) = P_{2i}. \tag{5.160}$$

*If $i = 1, 2, \dots, l-1$ then*

$$N(P_{2i}^* \text{ in } Q_{2i+1}^*) = Q_{2i+1}. \tag{5.161}$$

*Proof.* We use induction on $i$ to prove (5.160) and (5.161) simultaneously. As $Q_1^* = Q_1$ is a normal subgroup of $G$, it is clear that $N(Q_1^* \text{ in } P_2^*) = P_2^* = P_2$. Hence (5.160) is true for $i = 1$. Furthermore, (5.158c) for $i = 1$ coincides with (5.161) for $i = 1$. Thus the proposition holds for $i = 1$.

Assume the proposition is true for all $i$ with $1 \leq i < t$, for some $t = 2, \dots, l-1$, (note that either $k = l$ or $k = l-1$). We will prove it also holds for $i = t$, i.e., we will show that $N(Q_{2t-1}^* \text{ in } P_{2t}^*) = P_{2t}$ and $N(P_{2t}^* \text{ in } Q_{2t+1}^*) = Q_{2t+1}$. According to (5.141) we have that

$$P_{2t} = N(Q_1, Q_3, \dots, Q_{2t-1} \text{ in } P_{2t}^*) \leq N(Q_{2t-1}^* \text{ in } P_{2t}^*).$$

Hence the right side of (5.160) for $i = t$, is contained in the left side. For the other inclusion, we observe that, by induction, $Q_3 = N(P_2^* \text{ in } Q_3^*), Q_5 = N(P_4^* \text{ in } Q_5^*), \dots, Q_{2t-1} = N(P_{2t-2}^* \text{ in } Q_{2t-1}^*)$. In view of Propositions 5.132 and 5.155, $P_{2r}^* = P_{2t}^* \cap G_{2r}^*$ and $Q_{2r-1}^* = Q_{2t-1}^* \cap G_{2r-1}^*$. Hence $N(Q_{2t-1}^* \text{ in } P_{2t}^*)$ normalizes both $P_{2r}^*$ (as it is a subgroup of $P_{2t}^*$) and $Q_{2r-1}^*$ (as it normalizes the group $Q_{2t-1}^*$) whenever $1 \leq r \leq t$. Hence, $N(Q_{2t-1}^* \text{ in } P_{2t}^*)$ also normalizes the groups $Q_3 = N(P_2^* \text{ in } Q_3^*), \dots, Q_{2t-1} = N(P_{2t-2}^* \text{ in } Q_{2t-1}^*)$. So the left side in (5.160) for $i = t$ is contained in the right, and the inductive step for the first equation is complete.

The proof for (5.161) is similar. Indeed, according to (5.158c) we get:

$$Q_{2t+1} = N(P_2, P_4, \ldots, P_{2t} \text{ in } Q^*_{2t+1}) \leq N(P^*_{2t} \text{ in } Q^*_{2t+1}).$$

Thus the right side of (5.161) for $i = t$, is contained in the left side.

For the other inclusion, we observe that (5.160) holds for all $i$ with $1 \leq i \leq t$. Hence, $P_2 = N(Q^*_1 \text{ in } P^*_2), P_4 = N(Q^*_3 \text{ in } P^*_4), \ldots, P_{2t} = N(Q^*_{2t-1} \text{ in } P^*_{2t})$. Furthermore, Propositions 5.132 and 5.155 imply that $N(P^*_{2t} \text{ in } Q^*_{2t+1})$ normalizes the groups $P^*_{2r}$ and $Q^*_{2r-1}$ whenever $1 \leq r \leq t$. So it also normalizes the groups $P_2, P_4, \ldots, P_{2t}$. Hence $N(P^*_{2t} \text{ in } Q^*_{2t+1}) \leq N(P_2, \ldots, P_{2t} \text{ in } Q^*_{2t+1})$. This completes the proof of Proposition 5.159 for all $i$ with $1 \leq i \leq l - 1$.

It remains to show that, in the case where $k = l$, equation (5.160) holds for $i = k$. But even in this case the same argument we gave in the inductive proof of (5.160) works, as (5.161) is valid for all $i = 1, 2, \ldots, l - 1 = k - 1$. Hence Proposition 5.159 holds in all cases. □

We close this section noticing that we have some freedom in the choice of $P^*_{2i}$ and $Q^*_{2i-1}$, i.e.,

**Remark 5.162.** As we have see in Remark 5.124, conjugation by any $g \in G^*$ leads to a new choice of the $P_{2i}$, $Q_{2i-1}$, $\alpha_{2i}$ and $\beta_{2i-1}$ satisfying the same conditions (for the same $G_i$ and $\chi_i$) as the original choices. This conjugation replaces each $P^*_{2i}$ or $Q^*_{2i-1}$ by its $g$-conjugate. In particular, we can choose $g \in G^*$ so that $(P^*_{2k})^g$ and $(Q^*_{2l-1})^g$ are any given $\pi$-Hall and $\pi'$-Hall subgroups, respectively, of $G^*_m$.

## 5.7 When $\pi$-*split* groups are involved

In this section we are interested in the special case where $\pi$-split groups appear in the normal series (5.2). In particular, we will see that the triangular sets, in this case, have a very simple form. What we mean for a group to be $\pi$-split is given in

**Definition 5.163.** A finite group $H$ is called $\pi$-split if it is the direct product

$$H = H_\pi \times H_{\pi'}$$

of a $\pi$-group $H_\pi$, and a $\pi'$-group $H_{\pi'}$.

Obviously, $H_\pi$ and $H_{\pi'}$ are the unique $\pi$-and $\pi'$-Hall subgroups of $H$. Also, if $S$ is any subgroup of $H$, then $S$ is also $\pi$-split. Furthermore, if $\chi \in \mathrm{Irr}(H)$ is an irreducible character of $H$, then $\chi$ also $\pi$-splits as

$$\chi = \chi_\pi \times \chi_{\pi'},$$

where $\chi_\pi \in \mathrm{Irr}(H_\pi)$ and $\chi_{\pi'} \in \mathrm{Irr}(H_{\pi'})$ are the $\pi$-and $\pi'$-parts to which $\chi$ decomposes.

Assume now that the normal series (5.2), in addition to its usual properties described in Hypothesis 5.1, contains some $\pi$-split group $G_i$, for some $i = 1, \ldots, m$. Clearly, if $G_i$ is $\pi$-split then $G_j$ is also $\pi$-split for all $j = 1, \ldots, i$. Let $s$ be the largest integer, with $1 \leq s \leq m$, such that $G_s$ is $\pi$-split. Note that $s$ is necessarily bigger than 0 as $G_1$ is a $\pi'$-group and thus a $\pi$-split group. Let

$$\{\chi_0, \chi_1, \ldots, \chi_m\} \tag{5.164}$$

be a fixed but arbitrary character tower for (5.2). Then

$$G_i = G_{i,\pi} \times G_{i,\pi'}, \tag{5.165a}$$
$$\chi_i = \chi_{i,\pi} \times \chi_{i,\pi'}, \tag{5.165b}$$

whenever $0 \leq i \leq s$. Furthermore, the groups $G_i^*$ and $G^*$ defined in Section 5.5 (see (5.129a) and (5.129b)), and their characters $\chi_i^*$ (see (5.130a)), satisfy

$$G_i^* = G_{i,\pi}^* \times G_{i,\pi'}^* = G_{i,\pi}(\chi_1, \ldots, \chi_{i-1}) \times G_{i,\pi'}(\chi_1, \ldots, \chi_{i-1}), \tag{5.166a}$$
$$\chi_i^* = \chi_{i,\pi}^* \times \chi_{i,\pi'}^*, \tag{5.166b}$$

whenever $0 \leq i \leq s$.

Let

$$\{P_{2r}, Q_{2i-1} | \alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{k,l}, \tag{5.167}$$

be a representative of the conjugacy class of triangular sets of (5.2) that corresponds to the tower (5.164), by Theorem 5.6. All the groups, their characters, and their properties, that were introduced and proved in the previous sections with respect to a given triangular set, (like $Q_{2i-1,2r}, P_{2r,2i-1}, P_{2r}^*$ etc.), are applied to the set (5.167). Furthermore, we write

$$l_s := [(s+1)/2] \text{ and } k_s := [s/2], \tag{5.168}$$

for the greatest integers less than $(s+1)/2$ and $s/2$, respectively. (This agrees with the definition that was given in (5.7).

In the situations where (5.165) occurs, the first $n$ groups in the triangular set (5.167) are unique and satisfy

**Theorem 5.169.** *Assume that* (5.165) *holds. Then*

$$P_{2r} = P^*_{2r} = G^*_{2r,\pi} = G_{2r,\pi}(\chi_1, \dots, \chi_{2r-1}), \tag{5.170a}$$

$$Q_{2i-1} = Q^*_{2i-1} = G^*_{2i-1,\pi'} = G_{2i-1,\pi}(\chi_1, \dots, \chi_{2i-2}), \tag{5.170b}$$

$$G^*_{2r} = P_{2r} \times Q_{2r-1} = G_{2r,2r-1} = G_{2r,2r}, \tag{5.170c}$$

$$G^*_{2i-1} = P_{2i-2} \times Q_{2i-1} = G_{2i-1,2i-2} = G_{2i-1,2i-1}, \tag{5.170d}$$

$$\alpha_{2r} = \alpha^*_{2r} = \chi^*_{2r,\pi}, \tag{5.170e}$$

$$\beta_{2i-1} = \chi^*_{2i-1,\pi'}, \tag{5.170f}$$

$$\chi^*_{2r} = \chi_{2r,2r} = \alpha_{2r} \times \beta_{2r-1}, \tag{5.170g}$$

$$\chi^*_{2i-1} = \chi_{2i-1,2i-1} = \alpha_{2i-2} \times \beta_{2i-1}. \tag{5.170h}$$

*whenever* $1 \leq r \leq k_s$ *and* $1 \leq i \leq l_s$.

*Furthermore, the groups* $P_2 \trianglelefteq P_4 \trianglelefteq \cdots \trianglelefteq P_{2k_s}$ *and* $Q_1 \trianglelefteq Q_3 \trianglelefteq \cdots \trianglelefteq Q_{2l_s-1}$*, form a normal series for* $P_{2k_s}$ *and* $Q_{2l_s-1}$*, respectively. In addition,* $P_{2k_s}$ *centralizes* $Q_{2l_s-1}$*. Thus*

$$P_{2r,2t+1} = P_{2r} \textit{ and } Q_{2i-1,2j} = Q_{2i-1}, \tag{5.171}$$

$$\alpha_{2r,2t+1} = \alpha_{2r} \textit{ and } \beta_{2i-1,2j} = \beta_{2i-1}, \tag{5.172}$$

*whenever* $1 \leq r \leq t \leq l_s - 1$ *and* $1 \leq i \leq j \leq k_s$.

*Proof.* Corollary 5.157, along with (5.166a), implies that $G^*_{2r} = G^*_{2r,\pi} \times G^*_{2r,\pi'} = P^*_{2r} \cdot Q^*_{2r-1}$ and $G^*_{2i-1} = G^*_{2i-1,\pi} \times G^*_{2i-1,\pi'} = P^*_{2i-2} \cdot Q^*_{2i-1}$, whenever $1 \leq r \leq k_s$ and $1 \leq i \leq l_s$. Note that the last such group is $G^*_s$, wich satisfies

$$G^*_s = G^*_{s,\pi} \times G^*_{s,\pi'} = P^*_{2k_s} \cdot Q^*_{2l_s-1}. \tag{5.173}$$

Hence

$$P^*_{2r} = G_{2r,\pi}(\chi_1, \dots, \chi_{2r-1}) = G^*_{2r,\pi} = G^*_{2r+1,\pi}, \tag{5.174}$$

$$Q^*_{2i-1} = G_{2i-1,\pi'}(\chi_1, \dots, \chi_{2i-2}) = G^*_{2i-1,\pi'} = G^*_{2i,\pi'}, \text{while} \tag{5.175}$$

$$P^*_{2k_n} = G^*_{2k_s,\pi} \text{ and } Q^*_{2l_s-1} = G^*_{2l_s-1,\pi'}, \tag{5.176}$$

for all $r, i$ with $1 \leq r < l_s$ and $1 \leq i \leq k_s$. This, along with Proposition 5.159, implies

$$P_{2r} = N(Q^*_{2r-1} \text{ in } P^*_{2r}) = N(G^*_{2r,\pi'} \text{ in } G^*_{2r,\pi}) = G^*_{2r,\pi'} = P^*_{2r},$$
$$Q_{2i-1} = N(P^*_{2i-2} \text{ in } Q^*_{2i-1}) = N(G^*_{2i-1,\pi} \text{ in } G^*_{2i-1,\pi'}) = G^*_{2i-1,\pi'} = Q^*_{2i-1},$$

whenever $1 \leq r \leq k_s$ and $1 \leq i \leq l_s$. Hence (5.170a,b) holds.

Furthermore, the fact that the groups $P^*_2, P^*_4, \dots, P^*_{2k_s}$ form a normal series for $P^*_{2k_s}$ (see Proposition 5.132), implies that $P_2 \trianglelefteq P_4 \trianglelefteq \cdots \trianglelefteq P_{2k_s}$ is a normal series for $P_{2k_s}$. Similarly, the $\pi'$-groups $Q_{2i-1}$ form a normal series $Q_1 \trianglelefteq Q_3 \trianglelefteq \cdots \trianglelefteq Q_{2l_s-1}$ for $Q_{2l_s-1}$. According to (5.173) the $\pi$-Hall subgroup $P_{2k_s}$ of $G^*_s$ centralizes the $\pi'$-Hall subgroup $Q_{2l_s-1}$ of that same group. Hence $P_{2r}$ centralizes $Q_{2i-1}$, for all $r = 1, \dots, k_s$ and $i = 1, \dots, l_s$. Thus (see (5.33) and (5.34)), $Q_{2i-1,2j} = C(P_{2i}, \dots, P_{2j} \text{ in } Q_{2i-1}) = Q_{2i-1}$ and $P_{2r,2t+1} = C(Q_{2r+1}, \dots, Q_{2t+1} \text{ in } P_{2r}) = P_{2r}$, for all $1 \leq i \leq j \leq k_s$ and $1 \leq r \leq t \leq l_s - 1$. Furthermore, the $P_{2i}, \dots, P_{2j}$-Clifford correspondent $\beta_{2i-1,2j}$ of $\beta_{2i-1}$ coincides with $\beta_{2i-1}$. Similarly we get that $\alpha_{2r,2t+1} = \alpha_{2r}$. Hence the last part of the theorem holds.

To prove (5.170c,d) it suffices to notice that, according to (5.92) and (5.93),

$$\begin{aligned} G_{2r,2r-1} &= P_{2r} \ltimes Q_{2r-1}, \\ G_{2r,2r} &= P_{2r} \times Q_{2r-1,2r}, \\ G_{2i-1,2i-2} &= P_{2i-2} \rtimes Q_{2i-1}, \\ G_{2i-1,2i-1} &= P_{2i-2,2i-1} \times Q_{2i-1}, \end{aligned}$$

for all $r = 1, \ldots, k_s$ and $i = 1, \ldots, l_s$. But $Q_{2r-1,2r} = Q_{2r-1}$ and $P_{2i-2,2i-1} = P_{2i-2}$, by (5.171). Thus all the above products are direct, and we get

$$G_{2r,2r-1} = G_{2r,2r} = P_{2r} \times Q_{2r-1} = P^*_{2r} \times Q^*_{2r-1} = G^*_{2r}.$$

So (5.170c) holds. The proof for (5.170d) is analogous.

Notice that (5.170c,d) clearly imply that the groups $P_{2r}$ and $Q_{2i-1}$, as characteristic subgroups of $G^*_{2r}$ and $G^*_{2i-1}$, respectively, are normal subgroups of $G^* = G(\chi_1, \ldots, \chi_m)$, for all $r = 1, \ldots, k_s$ and $i = 1, \ldots, l_s$. This is actually the reason that the group $G_{2r,2r}$, defined as $G_{2r,2r} = N(P_0, \ldots, P_{2r}, Q_1, \ldots, Q_{2r-1}$ in $G_{2r}(\chi_1, \ldots, \chi_{2r}))$ in (5.91), coincides with $G_{2r}(\chi_1, \ldots, \chi_{2r}) = G^*_{2r}$, for all $r = 1, \ldots, k_s$. (Similarly we work for the group $G_{2i-1,2i-1}$). In addition, this implies that the $cP_2, \ldots, cP_{2r}, cQ_1, \ldots, cQ_{2r-1}$-correspondent $\chi_{2r,2r}$ of $\chi_{2r}$ is nothing else but a multiple Clifford correspondent, and thus coincides with $\chi^*_{2r}$, i.e., $\chi_{2r,2r} = \chi^*_{2r}$. Similarly, $\chi_{2i-1,2i-1} = \chi^*_{2i-1}$. But, according to (5.92) and (5.93), we have that $\chi_{2r,2r} = \alpha_{2r} \times \beta_{2r-1,2r}$ and $\chi_{2i-1,2i-1} = \alpha_{2i-2,2i-1} \times \beta_{2i-1}$. Hence

$$\begin{aligned} \chi^*_{2r} &= \chi_{2r,2r} = \alpha_{2r} \times \beta_{2r-1,2r} = \alpha_{2r} \times \beta_{2r-1}, \\ \chi^*_{2i-1} &= \chi_{2i-1,2i-1} = \alpha_{2i-2,2i-1} \times \beta_{2i-1} = \alpha_{2i-2} \times \beta_{2i-1}, \end{aligned}$$

whenever $1 \leq r \leq k_s$ and $1 \leq i \leq l_s$. This, along with (5.166b), implies (5.170f,g,h) and one equality in (5.170e), namely $\alpha_{2r} = \chi^*_{2r,\pi}$.

It remains to show that $\alpha^*_{2r} = \alpha_{2r}$. But $\alpha^*_{2r} \in \mathrm{Irr}(P^*_{2r})$ is the $Q_3, \ldots, Q_{2r-1}$-correspondent of $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$, while $P^*_{2r} = P_{2r}$ centralizes the $\pi'$-groups $Q_3, \ldots, Q_{2r-1}$, for all $r = 1, \ldots, k_s$. Thus this correspondence is trivial, i.e., $\alpha^*_{2r} = \alpha_{2r}$, for all such $r$. This completes the proof of the theorem. $\square$

The following is a straight forward application of Theorem 5.169.

**Corollary 5.177.** *Assume that $G_i$ and $\chi_i$ satisfy* (5.165)*, for all $i = 1, \ldots, s$. In addition, assume that $G$ fixes $\chi_i$ for all $i = 1, \ldots, s-1$. Then the triangular set* (5.167) *satisfies*

$$P_{2r} = P^*_{2r} = G_{2r,\pi}, \tag{5.178}$$
$$Q_{2i-1} = Q^*_{2i-1} = G_{2i,\pi'}, \tag{5.179}$$
$$\alpha_{2r} = \chi_{2r,\pi}, \tag{5.180}$$
$$\beta_{2i-1} = \chi_{2i-1,\pi'}, \tag{5.181}$$
$$\chi_{2r} = \alpha_{2r} \times \beta_{2r-1}, \tag{5.182}$$
$$\chi_{2i-1} = \alpha_{2i-2} \times \beta_{2i-1} \tag{5.183}$$

*whenever $1 \leq r \leq k_s$ and $1 \leq i \leq l_s$.*

# Chapter 6

# The Group $G' = G(\alpha^*_{2k})$

Assume that a finite odd group $G$ is given, along with a normal series (5.2) and a fixed triangular set for this series. We have already seen in Section 5.5 how to define the characters $\alpha^*_{2i}$ of the product groups $P^*_{2i}$ whenever $1 \leq i \leq k$. In this chapter we analyze the group $G' := G(\alpha^*_{2k})$ with ultimate purpose to reach triangular sets for this group.

## 6.1 $\pi'$-Hall subgroups of $G'$: the group $\widehat{Q}$

The following remark is an easy consequence of Proposition 5.153 and the fact that $G'$ normalizes $P^*_{2i} = P^*_{2k} \cap G_{2i}$.

**Remark 6.1.** $G' = G(\alpha^*_{2k}) = G(\alpha^*_2, \alpha^*_4, \dots, \alpha^*_{2k})$

**Proposition 6.2.** *For every $i = 1, \dots, k$ we have*

$$G'(\beta_1, \dots, \beta_{2i-1}) \leq G'(\beta_{2i-1,2k}) \text{ and} \tag{6.3a}$$

$$G'(\beta_1, \dots, \beta_{2i-1}) \leq G'(\chi_1, \dots, \chi_{2i}) \leq G'(\chi_1, \dots, \chi_{2i-1}). \tag{6.3b}$$

*In addition*

$$G'(\beta_1, \dots, \beta_{2l-1}) \leq G'(\chi_1, \dots, \chi_{2l-1}). \tag{6.3c}$$

*Proof.* Let $T_i = G'(\beta_1, \dots, \beta_{2i-1})$ for some fixed $i \in \{1, \dots, k\}$. In view of Remark 6.1 we have that $T_i$ fixes $\alpha^*_2, \dots, \alpha^*_{2k}$, and thus normalizes the groups $P^*_2, P^*_4, \dots, P^*_{2k}$. Furthermore, it normalizes the $\pi'$-groups $Q_1, Q_3, \dots, Q_{2i-1}$ and therefore also normalizes the product group $P_{2i} \cdots P_{2k} = N(Q_1, Q_3, \dots Q_{2i-1} \text{ in } P^*_{2k})$ (see (5.140)). As $T_i$ fixes $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$ and normalizes $P_{2i} \cdots P_{2k}$, it also fixes the $P_{2i} \cdots P_{2k}$-Glauberman correspondent $\beta_{2i-1,2k} \in \mathrm{Irr}(Q_{2i-1,2k})$ of $\beta_{2i-1}$ (see Definition 5.49). This implies (6.3a).

We will use induction on $i$ to prove (6.3b). If $i = 1$, then obviously $T_1 = G'(\beta_1) = G'(\chi_1)$, as $\beta_1 = \chi_1$. Also, $T_1$ normalizes $P_2 = P^*_2$ and fixes $\alpha_2 = \alpha^*_2$. Hence it also normalizes $P_2 \cdot Q_1 = G_2(\beta_1)$ (see (5.70)), and thus fixes the canonical extension $\beta^e_1$ of $\beta_1$ to $P_2 Q_1$. Therefore it fixes the character $\chi_{2,1} = \alpha_2 \cdot \beta^e_1$ (see (5.71a)). According to (5.68) we have $T_1(\chi_1, \chi_2) = T_1(\chi_{1,1}, \chi_{2,1})$. But $T_1(\chi_1) = T_1 = T_1(\chi_{2,1})$, while $\chi_{1,1} = \chi_1$. Hence $T_1(\chi_2) = T_1(\chi_1, \chi_2) = T_1(\chi_{1,1}, \chi_{2,1}) = T_1$. Therefore, $T_1$ fixes $\chi_2$ and (6.3b) is proved for $i = 1$.

Assume (6.3b) is true for $i = t-1$ and some $t = 2, 3, \dots, k$. We will prove it also holds for $i = t$. The inductive hypothesis implies that

$$T_t = G'(\beta_1, \beta_3, \dots, \beta_{2t-1}) \leq G'(\beta_1, \beta_3, \dots, \beta_{2t-3}) \leq G'(\chi_1, \chi_3, \dots, \chi_{2t-2}).$$

Thus it suffices to show that $T_t$ fixes $\chi_{2t-1}$ and $\chi_{2t}$. By (5.92) and (5.93) we have

$$\chi_{2t-1,2t-1} = \alpha_{2t-2,2t-1} \times \beta_{2t-1}$$
$$\chi_{2t,2t-1} = \alpha_{2t} \cdot \beta^{e}_{2t-1}.$$

We have already seen that $T_t$ normalizes $P^*_{2t-2}$ and $P^*_{2t}$, and fixes their characters $\alpha^*_{2t-2}$ and $\alpha^*_{2t}$. Also it normalizes $Q_1, \dots, Q_{2t-3}, Q_{2t-1}$, and thus normalizes $P_{2t-2} = N(Q_1, \dots, Q_{2t-3} \text{ in } P^*_{2t-2})$ and $P_{2t} = N(Q_1, \dots, Q_{2t-1} \text{ in } P^*_{2t})$, (see (5.141)). Hence $T_t$ also fixes the $Q_1, Q_3, \dots, Q_{2t-3}$-correspondent $\alpha_{2t-2} \in \operatorname{Irr}(P_{2t-2})$ of $\alpha^*_{2t-2}$, as well as the $Q_1, \dots, Q_{2t-1}$-correspondent $\alpha_{2t} \in \operatorname{Irr}(P_{2t})$ of $\alpha^*_{2t}$ (see Proposition 5.149). As $T_t$ also normalizes $Q_{2t-1}$, it fixes the $Q_{2t-1}$-Glauberman correspondent $\alpha_{2t-2,2t-1} \in \operatorname{Irr}(P_{2t-2,2t-1})$ of $\alpha_{2t-2} \in \operatorname{Irr}(P_{2t-2})$. So $T_t$ fixes the characters $\alpha_{2t}, \alpha_{2t-2}$ and $\alpha_{2t-2,2t-1}$.

Also $T_t$ fixes $\beta_{2t-1}$ and normalizes $P_{2t} \cdot Q_{2t-1} = G_{2t,2t-1}$ (see (5.92)). Hence it fixes the canonical extension $\beta^e_{2t-1} \in \operatorname{Irr}(G_{2t,2t-1})$ of $\beta_{2t-1}$ to $G_{2t,2t-1}$. Therefore, $T_t$ fixes $\alpha_{2t-2,2t-1}, \beta_{2t-1}, \alpha_{2t}$ and $\beta^e_{2t-1}$, and thus fixes $\chi_{2t-1,2t-1}$ and $\chi_{2t,2t-1}$. This, along with the inductive hypothesis on $T_t$, implies that

$$T_t(\chi_1, \chi_3, \dots, \chi_{2t-2}, \chi_{2t-1,2t-1}, \chi_{2t,2t-1}) = T_t. \tag{6.4}$$

We note that $T_t$ normalizes all the $\pi$-groups $P_2, P_4, \dots, P_{2t-4}, P_{2t-2}$. This is clear, as for every $j = 1, \dots, t-1$ we have $P_{2j} = N(Q_1, \dots, Q_{2j-1} \text{ in } P^*_{2j})$ (by (5.141)). Hence we conclude that $T_t \leq N(Q_1, Q_3, \dots, Q_{2t-1}, P_2, P_4, \dots, P_{2t-2} \text{ in } G)$. Therefore Theorem 5.88 (part 3 for $n = 2t$ and $n = 2t-2$ respectively) implies that $T_t(\chi_1, \dots, \chi_{2t}) = T_t(\chi_{1,2t-1}, \dots, \chi_{2t,2t-1})$ and $T_t(\chi_1, \dots, \chi_{2t-2}) = T_t(\chi_{1,2t-1}, \dots, \chi_{2t-2,2t-1})$. Hence

$$\begin{aligned}
T_t(\chi_1, \dots, \chi_{2t}) &= T_t(\chi_{1,2t-1}, \dots, \chi_{2t,2t-1}) \\
&= T_t(\chi_{1,2t-1}, \dots, \chi_{2t-2,2t-1})(\chi_{2t-1,2t-1}, \chi_{2t,2t-1}) \\
&= T_t(\chi_1, \dots, \chi_{2t-2})(\chi_{2t-1,2t-1}, \chi_{2t,2t-1}) \\
&= T_t. && \text{by (6.4)}
\end{aligned}$$

So $T_t \leq G'(\chi_1, \dots, \chi_{2t})$. This proves the inductive step for $i = t$, and thus (6.3b) for every $i = 1, \dots, k$.

It remains to show (6.3c). Observe that this additional case is not covered by (6.3b) only when $m$ is odd, since for $m$ even we have $k = l$. The arguments for this last step are similar to those we used at the inductive step. Indeed, $T_l$ fixes $\alpha^*_{2l-2}$ and thus fixes its $Q_1, \dots, Q_{2l-3}$-correspondent $\alpha_{2l-2} \in \operatorname{Irr}(P_{2l-2})$. It also fixes the $Q_{2l-1}$-Glauberman correspondent $\alpha_{2l-2,2l-1}$ of $\alpha_{2l-2}$. Hence $T_l$ fixes $\alpha_{2l-2,2l-1} \times \beta_{2l-1} = \chi_{2l-1,2l-1}$ (see (5.93)). Therefore, $T_l = T_l(\chi_1, \dots, \chi_{2k}, \chi_{2l-1,2l-1})$. Furthermore, $T_l \leq N(Q_1, \dots, Q_{2l-1}, P_2, \dots, P_{2l-2} \text{ in } G)$. Therefore, Part 3 of Theorem 5.88 implies that

$$\begin{aligned}
T_l(\chi_1, \dots, \chi_{2l-1}) &= T_l(\chi_{1,2l-1}, \dots, \chi_{2l-1,2l-1}) \\
&= T_l(\chi_{1,2l-1}, \dots, \chi_{2l-2,2l-1})(\chi_{2l-1,2l-1}) \\
&= T_l(\chi_1, \dots, \chi_{2l-2})(\chi_{2l-1,2l-1}) \\
&= T_l.
\end{aligned}$$

So $T_l \leq G'(\chi_1, \dots, \chi_{2l-1})$. This proves (6.3c) and completes the proof of the proposition. ☐

Our next goal is to show that the smallest group in (6.3b) has $\pi$-index in the largest one. The following lemma helps in this direction.

**Lemma 6.5.** *If* $T \le N(Q_1, \ldots, Q_{2i-1} \text{ in } G')$*, for some* $i$ *with* $1 \le i \le l$*, then*

$$T(\chi_1, \ldots, \chi_{2i-1}) \le T(\beta_1, \ldots, \beta_{2i-1}).$$

*Proof.* We will use induction on $i$. If $i = 1$, then the lemma is obviously true, as $\chi_1 = \beta_1$, so that $T(\chi_1) = T(\beta_1)$ for any $T \le G$. Assume that the lemma holds for all $i = 1, \ldots, t-1$, and some $t = 2, 3, \ldots, l$. We will prove it also holds for $i = t$.

Let $T$ be a subgroup of $N(Q_1, \ldots, Q_{2t-1} \text{ in } G')$. Then, according to the inductive hypothesis, $T(\chi_1, \ldots, \chi_{2t-1}) \le T(\chi_1, \ldots, \chi_{2t-3}) \le T(\beta_1, \ldots, \beta_{2t-3})$. Furthermore, in view of Remark 6.1, the group $T$ fixes the characters $\alpha_2^*, \alpha_4^*, \ldots, \alpha_{2k}^*$ and normalizes the groups $P_2^*, P_4^*, \ldots, P_{2k}^*$. Hence $T$ normalizes the groups $P_{2i} = N(Q_1, \ldots, Q_{2i-1} \text{ in } P_{2i}^*)$ (see (5.141)), as well as the product groups $P_{2i} \cdots P_{2k} = N(Q_1, \ldots, Q_{2i-1} \text{ in } P_{2k}^*)$ (see (5.140)), whenever $1 \le i \le t$. So we get that $T \le N(Q_1, \ldots, Q_{2t-1}, P_2, \ldots, P_{2t-2} \text{ in } G)$, which, in view of Theorem 5.88 (Part 3), implies that

$$T(\chi_1, \ldots, \chi_{2t-1}) = T(\chi_{1,2t-1}, \ldots, \chi_{2t-1,2t-1}).$$

Hence $T(\chi_1, \ldots, \chi_{2t-1})$ fixes $\chi_{2t-1,2t-1}$. But the last character equals $\alpha_{2t-2,2t-1} \times \beta_{2t-1}$, (see (5.93)). Hence $T(\chi_1, \ldots, \chi_{2t-1})$ fixes $\beta_{2t-1}$. Therefore $T(\chi_1, \ldots, \chi_{2t-1}) \le T(\beta_1, \ldots, \beta_{2t-3}, \beta_{2t-1})$.

This completes the proof of the inductive argument, and thus that of Lemma 6.5. □

**Theorem 6.6.** *There exists a* $\pi'$*-Hall subgroup* $\mathcal{Q}$ *of* $G'$ *such that, for every* $i = 1, \ldots, l$*, we have*

$$\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1}) \in \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2i-1})) \quad \textit{and} \tag{6.7a}$$

$$\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1}) \le \mathcal{Q}(\beta_1, \ldots, \beta_{2i-1}). \tag{6.7b}$$

*Furthermore, whenever* $1 \le i \le l-1$ *we have*

$$\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1}) \textit{ normalizes } \quad Q_{2i+1}. \tag{6.7c}$$

*Proof.* As

$$G'(\chi_1, \ldots, \chi_{2l-1}) \le G'(\chi_1, \ldots, \chi_{2l-3}) \le \cdots \le G'(\chi_1) \le G',$$

it is obvious that we can pick a $\pi'$-Hall subgroup $\mathcal{Q}$ of $G'$ that satisfies (6.7a) for all $i = 1, \ldots, l$. We will modify $\mathcal{Q}$, using induction on $i$, so that the rest of the theorem also holds.

If $i = 1$, then we obviously have that (6.7b) holds, as $\chi_1 = \beta_1$. Thus it suffices to show that we can modify $\mathcal{Q}$ so that (6.7a) holds for all $i = 1, \ldots, l$ while (6.7c) holds for $i = 1$. According to Remark 6.1 and (5.148), the group $G'$ fixes $\alpha_2^* = \alpha_2$. Hence $G'(\beta_1) \le G(\alpha_2, \beta_1)$. Therefore, $G'(\beta_1)$ normalizes $G_3(\alpha_2, \beta_1)$, and thus $\mathcal{Q}(\beta_1)$ normalizes $G_3(\alpha_2, \beta_1)$. According to (5.93), the semidirect product $Q_3 \ltimes P_2$ equals $G_3(\alpha_2, \beta_1)$. As the $\pi'$-group $\mathcal{Q}(\beta_1)$ normalizes $Q_3 \ltimes P_2$, it has to normalize a $P_2$-conjugate of $Q_3$. Hence $\mathcal{Q}(\beta_1)^{\sigma_2}$ normalizes $Q_3$, for some $\sigma_2 \in P_2$. But $P_2$ fixes $\alpha_{2k}^*$ (as $P_2 \le P_{2k}^*$) as well as $\beta_1 = \chi_1$. It also fixes the characters $\chi_2, \ldots, \chi_{2l-1}$ as $P_2 \le G_2 \le \cdots \le G_{2l-1}$. Therefore, $P_2 \le G'(\chi_1, \ldots, \chi_{2i-1})$ whenever $1 \le i \le l$. Hence $\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})^{\sigma_2} = \mathcal{Q}^{\sigma_2}(\chi_1, \ldots, \chi_{2i-1})$ and, in addition, $\mathcal{Q}^{\sigma_2}$ and $\mathcal{Q}^{\sigma_2}(\chi_1, \ldots, \chi_{2i-1})$ are $\pi'$-Hall subgroups of $G'$ and $G'(\chi_1, \ldots, \chi_{2i-1})$ respectively (as $\mathcal{Q}$ and $\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})$ are, and $P_2 \le G'(\chi_1, \ldots, \chi_{2i-1})$). Furthermore $\mathcal{Q}^{\sigma_2}(\chi_1)$ normalizes $Q_3$. Hence the group $\mathcal{Q}^{\sigma_2}$ satisfies (6.7a) for every $i = 1, \ldots, l$ as well as (6.7b) and (6.7c) for $i = 1$. So we can replace $\mathcal{Q}$ by $\mathcal{Q}^{\sigma_2}$ and assume that (6.7a) holds for every $i = 1, \ldots, l$ while (6.7b) and (6.7c) hold for $i = 1$

The same type of argument as the one we gave for $i = 1$ will make the inductive step work. So, assume that $\mathcal{Q}$ has been modified so that it satisfies (6.7a) for all $i = 1, \ldots, l$, and in addition, satisfies the rest of the theorem for all $i \leq t-1$, for some $t = 2, \ldots, l-1$. We will show that there is a $G'$-conjugate of $\mathcal{Q}$ that satisfies (6.7a) for all $i = 1, \ldots, l$ and the rest of Theorem 6.6 whenever $1 \leq i \leq t$.

According to the inductive hypothesis $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1}) \leq \mathcal{Q}(\chi_1, \ldots, \chi_{2t-3}) \leq \cdots \leq \mathcal{Q}(\chi_1)$ while, $\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})$ normalizes the group $Q_{2i+1}$ for all $i = 1, \ldots, t-1$. So $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})$ normalizes the groups $Q_1, \ldots, Q_{2t-3}, Q_{2t-1}$. Hence Lemma 6.5 implies that $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1}) \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2t-3}, \beta_{2t-1})$. This, along with the inductive hypothesis, implies that

$$\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1}) \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2i-1}), \tag{6.8}$$

whenever $1 \leq i \leq t$.

The group $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})$ fixes the characters $\alpha^*_{2i}$ and normalizes the groups $Q_{2i-1}$ and $P_{2i}$ for all $i = 1, \ldots, t$. Hence Proposition 5.149 implies that $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})$ also fixes the $Q_3, \ldots, Q_{2i-1}$-correspondent $\alpha_{2i} \in \mathrm{Irr}(P_{2i})$ of $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$ for all such $i$. This, along with (6.8), implies that $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1}) \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2t-1}, \alpha_2, \ldots, \alpha_{2t})$. Hence $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})$ normalizes the group $G_{2t+1}(\beta_1, \ldots, \beta_{2t-1}, \alpha_2, \ldots, \alpha_{2t})$, (as $G_{2t+1} \trianglelefteq G$). According to (5.93) and (5.91) the latter group equals $G_{2t+1,2t} = P_{2t} \rtimes Q_{2t+1}$. Hence the $\pi'$-group $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})$ normalizes $P_{2t} \rtimes Q_{2t+1}$, and thus normalizes a $P_{2t}$-conjugate of $Q_{2t+1}$. Thus there exists an element $\sigma \in P_{2t}$ such that $\mathcal{Q}(\chi_1, \ldots, \chi_{2t-1})^\sigma$ normalizes $Q_{2t+1}$. But $P_{2t}$ is a subgroup of $G_{2t+1,2t}$, where the latter group equals $N(Q_1, \ldots, Q_{2t-1}, P_2, \ldots, P_{2t}$ in $G_{2t+1}(\chi_1, \ldots, \chi_{2t}))$ (see (5.91)). Therefore, $P_{2t}$ fixes the characters $\chi_1, \ldots, \chi_{2t}$. Furthermore, $P_{2t} \leq G_{2t+1} \leq G_{2t+2} \leq \cdots \leq G_{2l-1}$ which implies that $P_{2t}$ also fixes the characters $\chi_{2t+1}, \chi_{2t+2}, \ldots, \chi_{2l-1}$. Hence $\mathcal{Q}^\sigma(\chi_1, \ldots, \chi_{2i-1}) = \mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})^\sigma$, whenever $1 \leq i \leq l$. As $P_{2t}$ also fixes $\alpha^*_{2k}$, we get that $P_{2t} \leq G'(\chi_1, \ldots, \chi_{2i-1})$. So, $\mathcal{Q}^\sigma$ and $\mathcal{Q}^\sigma(\chi_1, \ldots, \chi_{2i-1})$ are $\pi'$-Hall subgroups of $G'$ and $G'(\chi_1, \ldots, \chi_{2i-1})$ respectively, (as $\mathcal{Q}$ and $\mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})$ are) for all $i = 1, \ldots, l$. Hence (6.7a) holds for the group $\mathcal{Q}^\sigma$ and all $i = 1, \ldots, l$.

Furthermore, $P_{2t}$ fixes the characters $\beta_1, \ldots, \beta_{2t-1}$, by (5.17c). So for the $\sigma$-conjugate $\mathcal{Q}^\sigma$ of $\mathcal{Q}$ we get that $\mathcal{Q}^\sigma(\beta_1, \ldots, \beta_{2i-1}) = \mathcal{Q}(\beta_1, \ldots, \beta_{2i-1})^\sigma$ whenever $i = 1, \ldots, t$. Hence in view of (6.8) we get

$$\mathcal{Q}^\sigma(\chi_1, \ldots, \chi_{2i-1}) = \mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})^\sigma \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2i-1})^\sigma = \mathcal{Q}^\sigma(\beta_1, \ldots, \beta_{2i-1}),$$

whenever $1 \leq i \leq t$. Thus (6.7b) holds for $\mathcal{Q}^\sigma$ and all $i = 1, \ldots, t$. As far as (6.7c) is concerned, we note that $\sigma$ was picked so that $\mathcal{Q}^\sigma(\chi_1, \ldots, \chi_{2t-1})$ normalizes $Q_{2t+1}$. Also for every $i$ with $1 \leq i < t$, the inductive hypothesis, along with the fact that $P_{2t}$ normalizes $Q_{2i+1}$, implies that $\mathcal{Q}^\sigma(\chi_1, \ldots, \chi_{2i-1}) = \mathcal{Q}(\chi_1, \ldots, \chi_{2i-1})^\sigma$ normalizes $Q_{2i+1}$. Hence $\mathcal{Q}^\sigma$ also satisfies (6.7c) for all $i = 1, \ldots, t$. This completes the inductive proof. So there exists a $\pi'$-Hall subgroup $\mathcal{Q}$ of $G'$ that satisfies (6.7a) for $i = 1, \ldots, l$, as well as (6.7b) and (6.7c) for $i = 1, \ldots, l-1$.

To complete the proof of the theorem it suffices to show that $\mathcal{Q}$ satisfies (6.7b) for $i = l$. The group $\mathcal{Q}$ we have picked so far satisfies that extra condition. Indeed, $\mathcal{Q}(\chi_1, \ldots, \chi_{2l-3}) \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2l-3})$, by (6.7b) for $i = l-1$. So $\mathcal{Q}(\chi_1, \ldots, \chi_{2l-3})$ normalizes the groups $Q_1, \ldots, Q_{2l-3}$. It also normalizes $Q_{2l-1}$ by (6.7c) for $i = l-1$. Hence Lemma 6.5 with $T = \mathcal{Q}(\chi_1, \ldots, \chi_{2l-3})$, implies that $\mathcal{Q}(\chi_1, \ldots, \chi_{2l-1}) \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2l-1})$. This completes the proof of the theorem. $\square$

The following fact was proved in the $i = 1$ case of Theorem 6.6. We state it here separately as we will use it again later.

**Remark 6.9.** The group $G'(\beta_1)$ normalizes $G_3(\alpha_2, \beta_1)$.

As an easy consequence of Proposition 6.2 and Theorem 6.6 we get

**Corollary 6.10.** *For every $i = 1, \ldots, k$ we have*

$$\mathrm{Hall}_{\pi'}(G'(\beta_1, \ldots, \beta_{2i-1})) \subseteq \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2i})) \subseteq \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2i-1})),$$

*while*

$$\mathrm{Hall}_{\pi'}(G'(\beta_1, \ldots, \beta_{2l-1})) \subseteq \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2l-1})).$$

*Proof.* We only need to note that in view of (6.3b) we have

$$G'(\chi_1, \ldots, \chi_{2i-1}) \geq G'(\chi_1, \ldots, \chi_{2i}) \geq G'(\beta_1, \ldots, \beta_{2i-1}).$$

The rest follows from (6.7a), (6.7b) and (6.3c). □

A similar statement to that of Corollary 6.10 holds for $G'(\beta_{2i-1,2k})$ and $G'(\beta_1, \ldots, \beta_{2i-1})$. To prove it we start with the following general lemma.

**Lemma 6.11.** *Assume that $\mathcal{P}$ is a $\pi$-subgroup of a finite group $\mathcal{G}$, and that $\mathcal{S}_1, \mathcal{S}$ and $\mathcal{T}$ are $\pi'$-subgroups of $\mathcal{G}$ such that $\mathcal{T}$ normalizes $\mathcal{P}$, that $\mathcal{S}$ normalizes the semidirect product $\mathcal{T} \ltimes \mathcal{P}$, and that $\mathcal{S}_1$ is a subgroup of $\mathcal{S}$ normalizing $\mathcal{T}$. Then there exists $t \in \mathcal{P}$ so that the following three conditions are satisfied:*

*(i) $\mathcal{S}^t$ normalizes $\mathcal{T}$,*

*(ii) $\mathcal{S}_1 \leq \mathcal{S}^t$ and*

*(iii) $t$ centralizes $\mathcal{S}_1$.*

*Proof.* As the $\pi'$-group $\mathcal{S}$ normalizes the product $\mathcal{T} \ltimes \mathcal{P}$, it will normalize one of the $\pi'$-Hall subgroups of that product. Therefore there exists $s \in \mathcal{P}$ such that $\mathcal{S}^s$ normalizes $\mathcal{T}$. Thus $\mathcal{S}^s\mathcal{T}$ forms a $\pi'$-Hall subgroup of $\mathcal{S}^s\mathcal{T} \ltimes \mathcal{P}$. As $\mathcal{S}_1$ normalizes $\mathcal{T}$, the product $\mathcal{S}_1\mathcal{T}$ is a $\pi'$-subgroup of $\mathcal{S}^s\mathcal{T} \ltimes \mathcal{P} =< \mathcal{S}, \mathcal{T}, \mathcal{P} >$. So there is some $x \in \mathcal{P}$ such that the $\pi'$-Hall subgroup $(\mathcal{S}^s\mathcal{T})^x$ of $\mathcal{S}^s\mathcal{T} \ltimes \mathcal{P}$, contains $\mathcal{S}_1\mathcal{T}$. Hence $\mathcal{T} \leq \mathcal{S}_1\mathcal{T} \leq \mathcal{S}^{sx}\mathcal{T}^x$. As $\mathcal{T}$ is a $\pi'$-Hall subgroup of $\mathcal{T} \ltimes \mathcal{P}$, it follows that

$$\mathcal{T} = \mathcal{S}^{sx}\mathcal{T}^x \cap (\mathcal{T} \ltimes \mathcal{P}).$$

Now, the group $\mathcal{T}^x$ clearly normalizes the intersection $\mathcal{S}^{sx}\mathcal{T}^x \cap (\mathcal{T} \ltimes \mathcal{P}) = \mathcal{T}$. Hence

$$\mathcal{T}^x = \mathcal{T}.$$

Even more, $\mathcal{S}_1 \leq \mathcal{S}_1\mathcal{T} \leq \mathcal{S}^{sx}\mathcal{T}^x$. So

$$\mathcal{S}_1 \leq \mathcal{S}^{sx}\mathcal{T} \cap (\mathcal{S} \ltimes \mathcal{P}) = \mathcal{S}^{sx}.$$

Hence if we take $t = sx$, then (i) and (ii) are both satisfied. To see that $t$ also centralizes $\mathcal{S}_1$ we observe that $\mathcal{S}$ and its subgroup $\mathcal{S}_1$ normalize the unique $\pi$-Hall subgroup $\mathcal{P}$ of $\mathcal{T} \ltimes \mathcal{P}$. So the commutator $[t, \mathcal{S}_1]$ is contained in $\mathcal{P}$. Furthermore, if $a \in \mathcal{S}_1$ then $a^t \in \mathcal{S}^t$, as $\mathcal{S}_1 \leq \mathcal{S}$. So $a^{-1}a^t \in \mathcal{S}_1\mathcal{S}^t = \mathcal{S}^t$. But $a^{-1}a^t = [a, t] \in \mathcal{P}$. Hence $[a, t] \in \mathcal{S}^t \cap \mathcal{P} = 1$, and the lemma follows. □

We can now prove

**Theorem 6.12.** *There exists a $\pi'$-Hall subgroup $\mathcal{Q}$ of $G'$ such that for every $i = 1, \dots, k$ we have*

$$T \leq \mathcal{Q}(\beta_{2i-1,2k}) \in \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k})) \tag{6.13a}$$

$$\mathcal{Q}(\beta_{2i-1,2k}) \leq G'(\beta_1, \dots, \beta_{2i-1}), \tag{6.13b}$$

*where $T := Q_{2l-1,2k} = Q_{2k-1,2k}$ in case of an even $m$, or $T := Q_{2l-1}$ in case of an odd $m$. Furthermore, for every $i = 1, \dots, l-1$ we have*

$$\mathcal{Q}(\beta_{2i-1,2k}) \text{ normalizes } \; Q_{2i+1}. \tag{6.13c}$$

*Proof.* Note that the group $\mathcal{Q}$ in this theorem need not be the same as the group $\mathcal{Q}$ in Theorem 6.6.

Assume that $m$ is even. Then the group $Q_{2k-1,2k}$ fixes $\alpha_{2k}$ as $Q_{2k-1,2k} = C(P_{2k}$ in $Q_{2k-1})$. Thus Proposition 5.149 implies that $Q_{2k-1,2k}$ fixes $\alpha^*_{2k}$. Hence in the case of an even $m$ we have $T = Q_{2k-1,2k} \leq G'$. As we clearly have that $Q_{2k-1,2k}$ fixes $\beta_{2k-1,2k}$, we conclude that $T = Q_{2k-1,2k} \leq G'(\beta_{2k-1,2k})$. If $m$ is odd, then Corollary 5.150 implies that $Q_{2l-1}$ fixes $\alpha^*_{2k}$ as it fixes $\alpha_{2k}$ (see (5.17e)). Hence, in the case of an odd $m$, we have that $T = Q_{2l-1} \leq G'$. Furthermore, Proposition 5.55 implies that $T = Q_{2l-1}$ fixes $\beta_{2k-1,2k}$. Therefore, in the case of an odd $m$, and thus in every case, we get that $T \leq G'(\beta_{2k-1,2k})$. In view of Table 5.20a we also get that, independent of the type of $T$, we have

$$Q_{1,2k} \leq Q_{3,2k} \leq \cdots \leq Q_{2k-1,2k} \leq T. \tag{6.14}$$

Furthermore, $G'(\beta_{2i-1,2k})$ normalizes $Q_{2i-1,2k}$, and thus normalizes $Q_{2r-1,2k} = Q_{2i-1,2k} \cap G_{2r-1}$ for all $r = 1, \dots, i$ and all $i = 1, \dots, k$ (see Remark 5.40). Hence $G'(\beta_{2i-1,2k})$ fixes the unique character $\beta_{2r-1,k}$ of $Q_{2r-1,2k}$ that lies under $\beta_{2i-1,2k}$ (see Proposition 5.55). Hence $G'(\beta_{2i-1,2k}) \leq G'(\beta_{2r-1,2k})$ whenever $1 \leq r \leq i \leq k$. This implies that we have the following series of subgroups

$$T \leq G'(\beta_{2k-1,2k}) \leq G'(\beta_{2k-3,2k}) \leq \cdots \leq G'(\beta_{1,2k}) \leq G', \tag{6.15}$$

that is independent of the type of $T$. So it is clear that there exists a $\pi'$-Hall subgroup $\mathcal{Q}$ of $G'$ that satisfies (6.13a) for all $i = 1, \dots, k$. As in the proof of Theorem 6.6, we will use induction on $i$ to modify $\mathcal{Q}$ so that the rest of the theorem also holds.

For $i = 1$ we note that $G'(\beta_{1,2k})$ normalizes the groups $P^*_{2k}$ and $Q_1$. Hence it fixes the $P^*_{2k}$-Glauberman correspondent $\beta_1$ of $\beta_{1,2k}$. Thus $G'(\beta_{1,2k}) \leq G'(\beta_1)$. In addition, we have seen (see Remark 6.9), that $G'(\beta_1)$ normalizes $G_3(\alpha_2, \beta_1)$. So $\mathcal{Q}(\beta_{1,2k})$ normalizes $G_3(\alpha_2, \beta_1) = Q_3 \ltimes P_2$. Even more, $T$ is a subgroup of $\mathcal{Q}(\beta_{1,2k})$ and normalizes $Q_3$, as $Q_{2k-1}$ and $Q_{2l-1}$ do. Hence Lemma 6.11, with $\mathcal{S} = \mathcal{Q}(\beta_{1,2k})$, $\mathcal{S}_1 = T$, $\mathcal{T} = Q_3$ and $\mathcal{P} = P_2$, implies that there exists some $\sigma \in C(T$ in $P_2)$ such that $\mathcal{Q}(\beta_{1,2k})^\sigma$ normalizes $Q_3$. As $\sigma \in C(T$ in $P_2)$, we get that $\sigma$ also centralizes $Q_{2i-1,2k}$ for all $i = 1, \dots, k$ (see (6.14)). Hence $\sigma$ fixes the characters $\beta_{2i-1,2k} \in \mathrm{Irr}(Q_{2i-1,2k})$ for all such $i$.. As $\sigma \in P_2 \leq G' = G(\alpha^*_{2k})$, we get that $\sigma \in G'(\beta_{2i-1,2k})$. This implies that $\mathcal{Q}(\beta_{2i-1,2k})^\sigma = \mathcal{Q}^\sigma(\beta_{2i-1,2k})$, and that $\mathcal{Q}(\beta_{2i-1,2k})^\sigma \in \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k}))$ for all $i = 1, \dots, k$. Thus we can now work with $\mathcal{Q}^\sigma$ in the place of $\mathcal{Q}$ and conclude that this $\pi'$-Hall subgroup of $G'$ not only satisfies (6.13a) for every $i = 1, \dots, k$, but also the rest of the theorem for $i = 1$.

We will work similarly for the inductive step. So assume that $\mathcal{Q}$ has been modified so that it satisfies (6.13a) for all $i = 1, \dots, k$ and, in addition, satisfies the rest of the theorem for all $i \leq t-1$ and some $t = 1, \dots, l-1$. We will show that there is a $G'(\beta_{2k-1,2k})$-conjugate of $\mathcal{Q}$ that satisfies (6.13a) for $i = 1, \dots, k$ and the rest of Theorem 6.12 whenever $1 \leq i \leq t$. The argument here is very similar to the one we gave for the proof of Theorem 6.6. The only important difference is that $P_{2t}$ is not in general a subgroup of $G'(\beta_{2i-1,2k})$. So we can't just conjugate $\mathcal{Q}$ by an arbitary element of $P_{2t}$, or else (6.13a) need not hold for that conjugate. But Lemma 6.11 will solve this difficulty as $\sigma$ can be picked from $C(T \text{ in } P_{2t})$.

In view of (6.15) and the inductive hypothesis (as $\mathcal{Q}$ satisfies (6.13a) for all $i = 1, \ldots, k$) we have that

$$T \leq \mathcal{Q}(\beta_{2t-1,2k}) \leq \mathcal{Q}(\beta_{2t-3,2k}) \leq \cdots \leq \mathcal{Q}(\beta_{1,2k}).$$

Therefore, $\mathcal{Q}(\beta_{2t-1,2k})$ normalizes $Q_{2i+1}$ for all $i = 1, \ldots, t-1$, as $\mathcal{Q}(\beta_{2i-1,2k})$ does. In view of Remark 6.1, the group $\mathcal{Q}$ fixes the characters $\alpha_2^*, \alpha_4^*, \ldots, \alpha_{2k}^*$ and normalizes the groups $P_2^*, P_4^*, \ldots, P_{2k}^*$. Hence $\mathcal{Q}(\beta_{2t-1,2k})$ normalizes the groups $P_{2i}$ (see (5.141)) as well as the product groups $P_{2i} \cdots P_{2k}$ (see (5.140)) whenever $1 \leq i \leq t$.

Since $\mathcal{Q}(\beta_{2t-1,2k})$ fixes the characters $\beta_{2i-1,2k}$ and normalizes the groups $Q_{2i-1}$ and $P_{2i} \cdots P_{2k}$, we conclude that it also fixes the $P_{2i} \cdots P_{2k}$-Glauberman correspondent $\beta_{2i-1} \in \operatorname{Irr}(Q_{2i-1})$ of $\beta_{2i-1,2k} \in \operatorname{Irr}(Q_{2i-1,2k})$, whenever $1 \leq i \leq t$. Hence $\mathcal{Q}(\beta_{2t-1,2k}) \leq G'(\beta_1, \ldots, \beta_{2t-1})$. Therefore, in view of (6.3b), we get that

$$\mathcal{Q}(\beta_{2t-1,2k}) \leq G'(\beta_1, \ldots, \beta_{2t-1}) \leq G'(\chi_1, \ldots, \chi_{2t}). \tag{6.16}$$

This, along with the inductive hypothesis and (6.3b), implies that

$$\mathcal{Q}(\beta_{2i-1,2k}) \leq G'(\beta_1, \ldots, \beta_{2i-1}) \leq G'(\chi_1, \ldots, \chi_{2i}), \tag{6.17}$$

for all $i = 1, \ldots, t$.

If we collect all the groups that $\mathcal{Q}(\beta_{2t-1,2k})$ normalizes, and the characters it fixes, we have

$$\mathcal{Q}(\beta_{2t-1,2k}) \leq N(P_2, \ldots, P_{2t}, Q_1, \ldots, Q_{2t-1} \text{ in } G(\chi_1, \ldots, \chi_{2t})).$$

Hence $\mathcal{Q}(\beta_{2t-1,2k})$ normalizes the group $N(P_2, \ldots, P_{2t}, Q_1, \ldots, Q_{2t-1} \text{ in } G_{2t+1}(\chi_1, \ldots, \chi_{2t}))$. In view of (5.91) and (5.93), the latter group is $G_{2t+1,2t} = P_{2t} \rtimes Q_{2t+1}$. Furthermore, $T$ normalizes $Q_{2t+1}$. (Note here that, at the last case where $t = l-1$, the last $\pi'$-group is $Q_{2t+1} = Q_{2l-1}$. We still have that $T$ normalizes $Q_{2t+1}$ as, if $m$ is even, then $Q_{2t+1} = Q_{2l-1} = Q_{2k-1}$ is normalized by $Q_{2k-1,2k} = T$, while if $m$ is odd, then clearly $T = Q_{2l-1}$ normalizes $Q_{2t+1} = Q_{2l-1}$.) The inductive hypothesis implies that $T \leq \mathcal{Q}(\beta_{2t-1,2k})$. Also $\mathcal{Q}(\beta_{2t-1,2k})$ normalizes $P_{2t} \rtimes Q_{2t+1}$, while its subgroup $T$ normalizes $Q_{2t+1}$. Therefore Lemma 6.11 applies and provides an element $s \in C(T \text{ in } P_{2t})$ such that $\mathcal{Q}(\beta_{2t-1,2k})^s$ normalizes $Q_{2t+1}$. As $s \in C(T \text{ in } P_{2t})$, the inclusions (6.14) imply that $s$ centralizes $Q_{2i-1,2k}$ for all $i = 1, \ldots, k$. Hence $s \in G(\alpha_{2k}^*, \beta_{2i-1,2k})$. Thus

$$T \leq \mathcal{Q}(\beta_{2i-1,2k})^s = \mathcal{Q}^s(\beta_{2i-1,2k}) \qquad \text{and} \qquad \mathcal{Q}(\beta_{2i-1,2k})^s \in \operatorname{Hall}_{\pi'}(G'(\beta_{2i-1,2k})),$$

whenever $1 \leq i \leq k$. So $\mathcal{Q}^s$ satisfies (6.13a) for all such $i$.

Also $\mathcal{Q}^s(\beta_{2t-1,2k})$ normalizes $Q_{2t+1}$, while for all $i = 1, \ldots, t-1$ the group $P_{2t}$ normalizes $Q_{2i+1}$. So $\mathcal{Q}(\beta_{2i-1,2k})^s$ normalizes $Q_{2i+1}$ as $\mathcal{Q}(\beta_{2i-1,2k})$ does and $s \in P_{2t}$. Hence $\mathcal{Q}^s$ satisfies (6.13c) for all $i = 1, \ldots, t$.

Furthermore, as $P_{2t}$ fixes the characters $\beta_1, \ldots, \beta_{2t-1}$, we get

$$\begin{aligned} \mathcal{Q}^s(\beta_{2i-1,2k}) = \mathcal{Q}(\beta_{2i-1,2k})^s \leq \mathcal{Q}(\beta_1, \ldots, \beta_{2i-1})^s = \quad & \text{by (6.17)} \\ \mathcal{Q}^s(\beta_1, \ldots, \beta_{2i-1}), \quad & \text{as } s \in P_{2t}, \end{aligned}$$

whenever $1 \leq i \leq t$. Thus $\mathcal{Q}^s$ also satisfies (6.13b) for all such $i$. This completes the proof of the inductive step, and thus provides a $\mathcal{Q} \in \mathrm{Hall}_{\pi'}(G')$ satisfying both (6.13a) and (6.13b), for all $i = 1, \ldots, k$, along with (6.13c), for all $i = 1, \ldots, l-1$.

To complete the proof of the theorem is enough to check that (6.13b) also holds for $i = k$ (as $k \leq l \leq k+1$). The argument is exactly the same one we used at the inductive step to prove (6.16). So we omit it. $\square$

An obvious consequence of Proposition 6.2 and Theorem 6.12 is

**Corollary 6.18.** *For all $i = 1, \ldots, k$ we have*

$$\mathrm{Hall}_{\pi'}(G'(\beta_1, \ldots, \beta_{2i-1})) \subseteq \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k})).$$

We can now introduce the group $\widehat{Q}$.

**Theorem 6.19.** *There exists a $\pi'$-subgroup $\widehat{Q}$ of $G' = G(\alpha^*_{2k})$ such that*

$$\widehat{Q} \in \mathrm{Hall}_{\pi'}(G'), \tag{6.20a}$$

$$\begin{aligned}\widehat{Q}(\beta_{2i-1,2k}) \in \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k})) \cap \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2i-1})) \cap \\ \mathrm{Hall}_{\pi'}(G'(\chi_1, \ldots, \chi_{2i})) \cap \mathrm{Hall}_{\pi'}(G'(\beta_1, \ldots, \beta_{2i-1})),\end{aligned} \tag{6.20b}$$

$$\widehat{Q}(\beta_{2i-1,2k}) = \widehat{Q}(\chi_1, \ldots, \chi_{2i-1}) = \widehat{Q}(\chi_1, \ldots, \chi_{2i}) = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1}) \text{ and} \tag{6.20c}$$

$$\widehat{Q}(\chi_1, \ldots, \chi_{2i-1}) \leq \widehat{Q}(\alpha_2, \ldots, \alpha_{2i}), \tag{6.20d}$$

*for all $i = 1, \ldots, k$. In addition, for all $i$ with $1 \leq i \leq l-1$ we get*

$$\widehat{Q}(\beta_{2i-1,2k}) \text{ normalizes } Q_{2i+1}. \tag{6.21}$$

*Proof.* Let $\mathcal{Q}$ be any $\pi'$-group satisfying the conditions in Theorem 6.12. We will show that $\widehat{Q} := \mathcal{Q}$ is the desired group.

Clearly $\widehat{Q}$ satisfies (6.20a) and (6.21), for all $i = 1, \ldots, l-1$, as $\mathcal{Q}$ is a $\pi'$-Hall subgroup of $G'$ that satisfies (6.13c). Furthermore, (6.13b) and (6.3a) imply that

$$\widehat{Q}(\beta_{2i-1,2k}) \leq \widehat{Q}(\beta_1, \ldots, \beta_{2i-1}) \leq \widehat{Q}(\beta_{2i-1,2k}).$$

So $\widehat{Q}(\beta_{2i-1,2k}) = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1})$ whenever $1 \leq i \leq k$. This, along with (6.13a) and Corollary 6.18, implies that

$$\widehat{Q}(\beta_{2i-1,2k}) = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1}) \in \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k})) \cap \mathrm{Hall}_{\pi'}(G'(\beta_1, \ldots, \beta_{2i-1}))$$

for all $i = 1, \ldots, k$. Which, in view of Corollary 6.10, implies that the group $\widehat{Q}$ satisfies (6.20b).

According to (6.3b) we also get that

$$\widehat{Q}(\beta_{2i-1,2k}) = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1}) \leq \widehat{Q}(\chi_1, \ldots, \chi_{2i}) \leq \widehat{Q}(\chi_1, \ldots, \chi_{2i-1}),$$

as $\widehat{Q}$ is a subgroup of $G'$. Since $\widehat{Q}(\beta_{2i-1,2k})$ is a $\pi'$-Hall subgroup of both $G'(\chi_1, \ldots, \chi_{2i})$ and $G'(\chi_1, \ldots, \chi_{2i-1})$ we have equality everywhere, and (6.20c) follows.

It remains to show that (6.20d) also holds for $\widehat{Q}$. It follows from (6.20c) that $\widehat{Q}(\beta_{2i-1,2k})$ normalizes the groups $Q_1, \ldots, Q_{2i-1}$ and fixes the characters $\alpha^*_2, \ldots, \alpha^*_{2k}$ (by Remark 6.1), for all

$i = 1, \dots, k$. So it fixes the $Q_3, \dots, Q_{2j-1}$-correspondent $\alpha_{2j}$ of $\alpha^*_{2j}$, for all $j = 1, \dots, i$. This implies (6.20d). Thus Theorem 6.19 holds. □

**Corollary 6.22.** *Assume $\widehat{Q}$ satisfies the conditions in Theorem 6.19, and that*

$$t \in G'(\beta_{2k-1,2k}) \cap G'(\chi_1, \dots, \chi_{2k}) \cap G'(\beta_1, \dots, \beta_{2k-1}). \tag{6.23a}$$

*In addition, assume that*

$$t \in N(Q_{2k+1} \text{ in } G), \tag{6.23b}$$

*if $m = 2l - 1 = 2k + 1$ is odd. Then $\widehat{Q}^t$ also satisfies the conditions in Theorem 6.19.*

*Proof.* Obviously $\widehat{Q}^t$ is a $\pi'$-Hall subgroup of $G'$, as $t \in G'$. The fact that $\beta_{2i-1,2k}$ is the unique character of $Q_{2i-1,2k}$ that lies under $\beta_{2k-1,2k}$, implies that $t \in G'(\beta_{2k-1,2k})$ fixes $\beta_{2i-1,2k}$, for all $i = 1, \dots, k$. Hence $\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t$, for all such $i$. Furthermore, the definition of $t$ implies that

$$t \in G'(\beta_{2i-1,2k}) \cap G'(\chi_1, \dots, \chi_{2i-1}) \cap G'(\chi_1, \dots, \chi_{2i}) \cap G'(\beta_1, \dots, \beta_{2i-1}),$$

for all $i = 1, \dots, k$. As $\widehat{Q}(\beta_{2i-1,2k})$ satisfies (6.20b), we conclude that $\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t$ also satisfies (6.20b).

Even more, as $t$ fixes the various characters $\beta_{2i-1,2k}, \chi_j, \beta_{2i-1}$, for all $i = 1, \dots, k$ and $j = 1, \dots, 2k$, while (6.20c) holds for $\widehat{Q}$, we get

$$\begin{aligned}
\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t = \widehat{Q}(\chi_1, \dots, \chi_{2i-1})^t = \widehat{Q}^t(\chi_1, \dots, \chi_{2i-1}),\\
\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t = \widehat{Q}(\chi_1, \dots, \chi_{2i})^t = \widehat{Q}^t(\chi_1, \dots, \chi_{2i}),\\
\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t = \widehat{Q}(\beta_1, \dots, \beta_{2i-1})^t = \widehat{Q}^t(\beta_1, \dots, \beta_{2i}).
\end{aligned}$$

Thus $\widehat{Q}^t$ satisfies (6.20c).

Also $t$ fixes $\alpha^*_{2i}$, for all $i = 1, \dots, k$, as $t \in G' = G(\alpha^*_{2k})$. Furthermore it normalizes the groups $Q_3, \dots, Q_{2i-1}$, as it fixes $\beta_1, \dots, \beta_{2i-1}$, for all such $i$. Hence $t$ fixes the $Q_3, \dots, Q_{2i-1}$-correspondent $\alpha_{2i}$ of $\alpha^*_{2i}$. Hence

$$\widehat{Q}^t(\chi_1, \dots, \chi_{2i-1}) = \widehat{Q}(\chi_1, \dots, \chi_{2i-1})^t \leq \widehat{Q}(\alpha_2, \dots, \alpha_{2i})^t = \widehat{Q}^t(\alpha_2, \dots, \alpha_{2i}).$$

Thus (6.20d) holds for the $\widehat{Q}^t$.

By hypothesis $t$ normalizes $Q_{2k+1} = Q_{2l-1}$, in the case of an odd $m = 2l - 1$. Thus $t$ normalizes $Q_{2i+1}$, whenever $1 \leq i \leq k$. Hence (6.21), for $\widehat{Q}$, implies that $\widehat{Q}^t(\beta_{2i-1,2k}) = \widehat{Q}(\beta_{2i-1,2k})^t$ normalizes $Q^t_{2i+1} = Q_{2i+1}$, for all $i = 1, \dots, l - 1$. So $\widehat{Q}^t$ satisfies (6.21). This completes the proof of the corollary. □

From now on, we write $\widehat{Q}$ for a $\pi'$-group that satisfies all the conditions of Theorem 6.19, for a fixed system character tower–triangular set. An easy observation that follows from Theorem 6.19 is

**Corollary 6.24.** *Assume that the normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_{2k+1} \trianglelefteq G$ for $G$ is fixed (so $m = 2k + 1$ is odd). Assume also that a character tower $\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{2k+1}$ for that series and its corresponding triangular set*

$$\{P_{2i}, Q_{2i+1} | \alpha_{2i}, \beta_{2i+1}\}_{i=0}^{k} \tag{6.25a}$$

*are fixed. Then the reduced set*

$$\{P_{2i}, Q_{2i-1}, P_0 = 1 | \alpha_{2i}, \beta_{2i-1}, \alpha_0 = 1\}_{i=1}^{k} \tag{6.25b}$$

*is a representative of the unique conjugacy class of triangular sets that corresponds to the character tower* $\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{2k}$ *of the normal series* $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_{2k} \trianglelefteq G$ *of* $G$*. Furthermore, if the group* $\widehat{Q}$ *is picked so as to satisfy the conditions in Theorem 6.19 for the fixed character tower* $\{\chi_i\}_{i=0}^{2k+1}$ *and its associate triangular set, then the same group satisfies the conditions in Theorem 6.19 for the smaller character tower* $\{\chi_i\}_{i=0}^{2k}$ *and the reduced triangular set (6.25b).*

*Proof.* The first part of the corollary follows immediately from Remark 5.125. As far as the group $\widehat{Q}$ is concerned, first observe that the triangular sets in (6.25) share the group $P^*_{2k}$ and its irreducible character $\alpha^*_{2k}$. Thus they also share the group $G' = G(\alpha^*_{2k})$. Hence if $\mathcal{Q}$ was picked to satisfy the conditions in Theorem 6.19 for the set (6.25a), then $\mathcal{Q}$ satisfies both (6.20) and (6.21) for all $i = 1, \ldots, k$. But the conditions a $q$-group should satisfy to be the $\widehat{Q}$-group for the reduced set (6.25b), are (6.20) for $i = 1, \ldots, k$ and (6.21) for $i = 1, \ldots, k-1$ (since in the reduced case $l = k$). Clearly $\mathcal{Q}$ satisfies those. Hence Corollary 6.24 follows. □

The following proposition describes the relation between $\widehat{Q}(\beta_1, \cdots, \beta_{2i-1}) = \widehat{Q}(\beta_{2i-1,2k})$ and $Q_{2i+1,2k}$. Note that $Q_{2i+1,2k}$ fixes $\alpha_{2k}$, normalizes $Q_1, \ldots, Q_{2i-1}$ (see (5.17e)) and is a subgroup of $Q_{2j+1,2k} \leq Q_{2j+1}$ whenever $i \leq j \leq k-1$. Hence Proposition 5.149 implies

**Remark 6.26.**

$$Q_{1,2k} \leq Q_{2i+1,2k} \leq Q_{2i+1}(\alpha^*_{2k}) \leq G'$$

whenever $1 \leq i \leq k-1$.

We can now prove

**Proposition 6.27.** *For all* $i = 1, \ldots, k-1$ *we have*

$$\widehat{Q}(\beta_{2i-1,2k}) \cap G_{2i+1} = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1}) \cap G_{2i+1} = \widehat{Q}(\chi_1, \ldots, \chi_{2i-1}) \cap G_{2i+1} = \widehat{Q}(\chi_1, \ldots, \chi_{2i}) \cap G_{2i+1} = Q_{2i+1,2k}.$$

*Proof.* Since $G_{2i+1}$ is a normal subgroup of $G$, it follows from (6.20c) and (6.20b) that

$$\widehat{Q}(\beta_{2i-1,2k}) \cap G_{2i+1} = \widehat{Q}(\chi_1, \ldots, \chi_{2i}) \cap G_{2i+1} \in \mathrm{Hall}_{\pi'}(G_{2i+1}(\alpha^*_{2k}, \chi_1, \ldots, \chi_{2i})),$$

whenever $i = 1, \ldots, k-1$. In view of (6.20c) and (6.21) the group $\widehat{Q}(\beta_{2i-1,2k}) = \widehat{Q}(\beta_1, \ldots, \beta_{2i-1})$ normalizes the groups $Q_1, Q_3, \ldots, Q_{2i+1}$. Furthermore, as (6.20d) implies, it also normalizes the groups $P_2, P_4, \ldots, P_{2i}$. Hence

$$\widehat{Q}(\chi_1, \ldots, \chi_{2i}) \cap G_{2i+1} \leq N(P_2, \ldots, P_{2i}, Q_3, \ldots, Q_{2i+1} \text{ in } G_{2i+1}(\alpha^*_{2k}, \chi_1, \ldots, \chi_{2i})).$$

Therefore

$$\widehat{Q}(\chi_1, \ldots, \chi_{2i}) \cap G_{2i+1} \in \mathrm{Hall}_{\pi'}(N(P_2, \ldots, P_{2i}, Q_3, \ldots, Q_{2i-1} \text{ in } G_{2i+1}(\alpha^*_{2k}, \chi_1, \ldots, \chi_{2i}))).$$

According to (5.91) and (5.93) we have

$$N(P_2, \dots, P_{2i}, Q_3, \dots, Q_{2i-1} \text{ in } G_{2i+1}(\alpha^*_{2k}, \chi_1, \dots, \chi_{2i})) =$$
$$N(P_2, \dots, P_{2i}, Q_3, \dots, Q_{2i-1} \text{ in } G_{2i+1}(\chi_1, \dots, \chi_{2i}))(\alpha^*_{2k})$$
$$= G_{2i+1}(\alpha_2, \dots, \alpha_{2i}, \beta_1, \dots, \beta_{2i-1})(\alpha^*_{2k}) = (P_{2i} \rtimes Q_{2i+1})(\alpha^*_{2k}).$$

In view of (5.33) and Remark 6.26 we get

$$Q_{2i+1}(\alpha^*_{2k}) \leq N(P^*_{2k} \text{ in } Q_{2i+1}) = C(P_{2i+2} \dots P_{2k} \text{ in } Q_{2i+1}) = Q_{2i+1,2k} \leq Q_{2i+1}(\alpha^*_{2k}).$$

Also, $P_{2i}(\alpha^*_{2k}) = P_{2i}$, as $P_{2i} \leq P^*_{2k}$ and $\alpha^*_{2k} \in \mathrm{Irr}(P^*_{2k})$. Hence

$$\widehat{Q}(\chi_1, \dots, \chi_{2i}) \cap G_{2i+1} \in \mathrm{Hall}_{\pi'}(P_{2i} \rtimes Q_{2i+1,2k}).$$

Because $\widehat{Q}(\chi_1, \dots, \chi_{2i})$ normalizes both $Q_{2i+1}$ and $P^*_{2k}$, it also normalizes $N(P^*_{2k}$ in $Q_{2i+1})$. The latter equals $Q_{2i+1,2k}$ and is a $\pi'$-Hall subgroup of $P_{2i} \rtimes Q_{2i+1,2k}$. From this and the preceding statement we conclude that

$$\widehat{Q}(\chi_1, \dots, \chi_{2i-1}) \cap G_{2i+1} = \widehat{Q}(\chi_1, \dots, \chi_{2i}) \cap G_{2i+1} = Q_{2i+1,2k}.$$

This and (6.20c) imply the proposition. □

**Definition 6.28.** For every $i = 1, \dots, l$ we define

$$\widehat{Q}_{2i-1} := \widehat{Q} \cap G_{2i-1}.$$

Let $G'_s$ denote the group $G'_s = G_s(\alpha^*_{2k})$ for every $s = 0, \dots, m$. So $G'_1 \trianglelefteq G'_2 \trianglelefteq \dots \trianglelefteq G'_m$ is a normal series of $G'$, as $G_s$ is a normal subgroup of $G$. Thus Theorem 6.19 implies that

$$\widehat{Q}_{2i-1} \in \mathrm{Hall}_{\pi'}(G'_{2i-1}), \tag{6.29a}$$

$$\widehat{Q}_{2i-1}(\beta_{2j-1,2k}) \in \mathrm{Hall}_{\pi'}(G'_{2i-1}(\beta_{2j-1,2k})) \cap$$
$$\mathrm{Hall}_{\pi'}(G'_{2i-1}(\chi_1, \dots, \chi_{2j})) \cap \mathrm{Hall}_{\pi'}(G'_{2i-1}(\chi_1, \dots, \chi_{2j-1})) \cap \mathrm{Hall}_{\pi'}(G'_{2i-1}(\beta_1, \dots, \beta_{2j-1})), \tag{6.29b}$$

and $$\widehat{Q}_{2i-1}(\chi_1, \dots, \chi_{2j-1}) = \widehat{Q}_{2i-1}(\chi_1, \dots, \chi_{2j}) = \widehat{Q}_{2i-1}(\beta_1, \dots, \beta_{2j-1}) = \widehat{Q}_{2i-1}(\beta_{2j-1,2k}), \tag{6.29c}$$

whenever $1 \leq i, j \leq k$. Also, for all $i = 1, \dots, k$ and all $j = 1, \dots, l-1$ we have

$$\widehat{Q}_{2i-1}(\beta_{2j-1,2k}) \text{ normalizes } Q_{2j+1}. \tag{6.30}$$

Furthermore, for all $i = 1, \dots, k-1$, Proposition 6.27 implies that

$$\widehat{Q}_{2i+1}(\beta_{2i-1,2k}) = Q_{2i+1,2k}. \tag{6.31}$$

As $\widehat{Q}_{2i-1}(\chi_1, \dots, \chi_{2j+1}) \leq \widehat{Q}_{2i-1}(\chi_1, \dots, \chi_{2j-1})$, equation (6.29c) implies that

$$\widehat{Q}_{2i-1}(\beta_{2j+1,2k}) \leq \widehat{Q}_{2i-1}(\beta_{2j-1,2k}), \tag{6.32}$$

whenever $1 \leq i \leq k$ and $1 \leq j < k$. Furthermore, for all $i = 1, \ldots, k-1$ we have that

$$\begin{aligned} \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) &= \widehat{Q}(\beta_{2i-1,2k}) \cap G_{2i-1} \\ &= \widehat{Q}(\beta_{2i-1,2k}) \cap G_{2i+1} \cap G_{2i-1} \\ &= Q_{2i+1,2k} \cap G_{2i-1} && \text{by Proposition 6.27} \\ &= Q_{2i-1,2k}. \end{aligned}$$

In addition, (6.32) for $i = k$ and $j = k-1$ implies that $\widehat{Q}_{2k-1}(\beta_{2k-1,2k}) \leq \widehat{Q}_{2k-1}(\beta_{2k-3,2k})$. The latter group equals $Q_{2k-1,2k}$, according to (6.31) for $i = k-1$. Hence $\widehat{Q}_{2k-1}(\beta_{2k-1,2k}) \leq Q_{2k-1,2k}$. We obviously have that $Q_{2k-1,2k}$ is a subgroup of $G'_{2k-1}(\beta_{2k-1})$, (as $Q_{2k-1,2k}$ fixes $\alpha^*_{2k}$). But $\widehat{Q}_{2k-1}(\beta_{2k-1,2k})$ is a $\pi'$-Hall subgroup of $G'_{2k-1}(\beta_{2k-1})$ (see (6.29) for $i = j = k$). Therefore, $\widehat{Q}_{2k-1}(\beta_{2k-1,2k}) = Q_{2k-1,2k}$. So we conclude that

$$\widehat{Q}_{2i-1}(\beta_{2i-1,2k}) = Q_{2i-1,2k}, \tag{6.33}$$

for all $i = 1, \ldots, k$. We remark here that the the group $\widehat{Q}_1$ is an old familiar, as $\widehat{Q}_1 \in \mathrm{Hall}_{\pi'}(G'_1)$, while $G'_1 = G_1(\alpha^*_{2k}) = Q_1(\alpha^*_{2k}) = Q_{1,2k}$. Hence

$$\widehat{Q}_1 = Q_{1,2k} = G'_1. \tag{6.34}$$

We also have

**Proposition 6.35.**

$$\widehat{Q}_{2i-1} \in \mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}))) = \mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}))) \textit{ and}$$
$$\widehat{Q}_{2i-1}(\beta_{2i-1,2k}) \in \mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))) = \mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k})))$$

*for all* $i = 1, \cdots, k$.

*Proof.* Let $H$ be any subgroup of $N(P^*_{2k}$ in $G_{2i-1}(\alpha^*_{2i-2}))$. Then $H$ normalizes $P^*_{2k}$ and thus $[P^*_{2k}, H] \leq P^*_{2k}$. Also, $[P^*_{2k}, H] \leq [P^*_{2k}, G_{2i-1}] \leq G_{2i-1}$, as $P^*_{2k}$ normalizes $G_{2i-1}$ for all $i = 1, \cdots, k$. This, along with the fact that $G_{2i-1}/G_{2i-2}$ is a $\pi'$-group, implies that

$$[P^*_{2k}, H] \leq P^*_{2k} \cap G_{2i-1} = P^*_{2k} \cap G_{2i-2} = P^*_{2i-2}.$$

W conclude that $H$ centralizes the factor group $P^*_{2k}/P^*_{2i-2}$. If, in addition, $H$ is a $\pi'$-subgroup, then it fixes all the irreducible characters of $P^*_{2k}$ lying above $\alpha^*_{2i-2}$, as it fixes $\alpha^*_{2i-2}$ and centralizes $P^*_{2k}/P^*_{2i-2}$ (see Problem 13.13 in [12]). Thus $H$ fixes $\alpha^*_{2k}$, and so is contained in $G'_{2i-1} = G_{2i-1}(\alpha^*_{2k})$. Furthermore,

$$G'_{2i-1} = G_{2i-1}(\alpha^*_{2k}) = G_{2i-1}(\alpha^*_2, \ldots, \alpha^*_{2k}) \leq N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2})).$$

Applying the above argument to any $H \in \mathrm{Hall}_{\pi'}(N(P^*_{2k}$ in $G_{2i-1}(\alpha^*_{2i-2})))$, we see that $H \leq G'_{2i-1} \leq N(P^*_{2k}$ in $G_{2i-1}(\alpha^*_{2i-2}))$. So we get that

$$\mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}))) = \mathrm{Hall}_{\pi'}(G'_{2i-1}).$$

Similarly, applying the same arguments to any $H \in \mathrm{Hall}_{\pi'}(N(P^*_{2k}$ in $G_{2i-1}(\alpha^*_{2i-2}))(\beta_{2i-1,2k}))$, we see that $H \leq G'_{2i-1}(\beta_{2i-1,2k}) \leq N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}))(\beta_{2i-1,2k})$. So we have that

$$\mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}))(\beta_{2i-1,2k})) = \mathrm{Hall}_{\pi'}(G'_{2i-1}(\beta_{2i-1,2k})).$$

This, along with (6.29a) and (6.29b), implies that $\widehat{Q}_{2i-1}$ and $\widehat{Q}_{2i-1}(\beta_{2i-1,2k})$ are $\pi'$-Hall subgroups of $N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}))$ and $N(P^*_{2k} \text{ in } G_{2i-1}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$, respectively. The rest of the proposition is obvious, as $G_{2i}/G_{2i-1}$ is a $\pi$-group. □

**Lemma 6.36.** *Let $T$ be any subgroup of $N(P^*_{2k}$ in $G)$. Then $T$ normalizes $P^*_{2i}$ for all $i = 0, 1, \ldots, k$. Furthermore,*

$$N(Q_1, Q_3, \ldots, Q_{2t-1} \text{ in } TP^*_{2i}) = N(Q_1, Q_3, \ldots, Q_{2t-1} \text{ in } TP^*_{2t-2})P_{2t}P_{2t+2}\ldots P_{2i}, \qquad (6.37a)$$

*whenever $1 \leq t \leq i \leq k$, and*

$$N(Q_1, Q_3, \ldots, Q_{2t-1} \text{ in } TP^*_{2t-2}) = N(Q_{2t-1} \text{ in } N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } TP^*_{2t-4})P_{2t-2}), \quad (6.37b)$$

*for all $t = 2, 3, \ldots, k$.*

*Proof.* We first observe that, as $T$ normalizes both $P^*_{2k}$ and $G_{2i}$, it also normalizes $P^*_{2i} = P^*_{2t} \cap G^*_{2i}$ for all $i = 0, 1, \ldots, k$. Thus $TP^*_{2i}$ is a group.

Let $i = 1, \ldots, k$ be fixed. We will first prove (6.37a) using induction on $t$. In the case that $t = 1$, we clearly have that $N(Q_1 \text{ in } TP^*_{2i}) = TP^*_{2i} = N(Q_1 \text{ in } T)P^*_{2i}$, as $Q_1$ is a normal subgroup of $G$. Thus (6.37a) holds (for all $i = 1, \ldots, k$) when $t = 1$.

Now assume that (6.37a) holds for all values of $t$ with $t < s$ (for our fixed $i$), for some $s = 2, \ldots, i$. We will prove that it also holds for $t = s$. By the inductive hypothesis for $t = s - 1$ we have $N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2i}) = N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4})P_{2s-2}P_{2s}\ldots P_{2i}$. Furthermore, $N(Q_1, \ldots, Q_{2s-3}, Q_{2s-1} \text{ in } TP^*_{2i}) = N(Q_{2s-1} \text{ in } N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2i}))$. This, along with the inductive hypothesis, implies that

$$\begin{aligned} &N(Q_1, \ldots, Q_{2s-3}, Q_{2s-1} \text{ in } TP^*_{2i}) = \\ &\qquad N(Q_{2s-1} \text{ in } N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4})P_{2s-2}P_{2s}\ldots P_{2i}). \end{aligned} \qquad (6.38)$$

According to (5.10b) the groups $P_{2s}, P_{2s+2}, \ldots, P_{2i}$ normalize $Q_{2s-1}$. Furthermore, $P_{2s-2}$ normalizes the groups $Q_1, \ldots, Q_{2s-3}$. Therefore we get that $N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4})P_{2s-2} = N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4}P_{2s-2})$. Hence, in view of (6.38), we get

$$\begin{aligned} N(Q_1, \ldots, Q_{2s-3}, Q_{2s-1} \text{ in } TP^*_{2i}) &= N(Q_{2s-1} \text{ in } N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4})P_{2s-2}P_{2s}\ldots P_{2i}) = \\ &N(Q_{2s-1} \text{ in } N(Q_1, \ldots, Q_{2s-3} \text{ in } TP^*_{2s-4})P_{2s-2})P_{2s}\ldots P_{2i} = \\ &N(Q_1, \ldots, Q_{2s-3}, Q_{2s-1} \text{ in } TP^*_{2s-2})P_{2s}\ldots P_{2i}. \end{aligned}$$

This completes the proof for the inductive step, and therefore for (6.37a).

For the second equation of the lemma note that, according to (5.10b), the group $P_{2t-2}$ normalizes the $\pi'$-groups $Q_1, Q_3, \ldots, Q_{2t-3}$ whenever $t = 2, \ldots, k$. Therefore we have that

$$N(Q_1, \ldots, Q_{2t-3} \text{ in } TP^*_{2t-2}) = N(Q_1, \ldots, Q_{2t-3} \text{ in } TP^*_{2t-4})P_{2t-2}.$$

So

$$N(Q_1,\dots,Q_{2t-3},Q_{2t-1} \text{ in } TP^*_{2t-2})$$
$$= N(Q_{2t-1} \text{ in } N(Q_1,\dots,Q_{2t-3} \text{ in } TP^*_{2t-2})) = N(Q_{2t-1} \text{ in } N(Q_1,\dots,Q_{2t-3} \text{ in } TP^*_{2t-4})P_{2t-2}).$$

This completes the proof of (6.37b). Hence Lemma 6.36 is proved. ☐

If the $T$ that appears in Lemma 6.36 fixes $\beta_{2i-1,2k}$, for some $i = 1,\dots,k$, then we can prove

**Lemma 6.39.** *If $T$ is any subgroup of $N(P^*_{2k}$ in $G(\beta_{2i-1,2k}))$, for some $i = 1,\dots,k$, then $T$ normalizes $P^*_{2i-2}$, and $N(Q_1,Q_3,\dots,Q_{2t-1}$ in $TP^*_{2i-2})$ fixes $\beta_1,\beta_3,\dots,\beta_{2t-1}$, for all $t = 1,\dots,i$.*

*Proof.* We have already seen in Lemma 6.36, that $T$ normalizes $P^*_{2i-2}$. So $TP^*_{2i-2}$ is a group.

To prove the rest of the lemma, i.e., that

$$N(Q_1,Q_3,\dots,Q_{2t-1} \text{ in } TP^*_{2i-2}) \le G(\beta_1,\dots,\beta_{2t-1}), \tag{6.40}$$

for all $t = 1,\dots,i$, we will use induction on $t$.

For $t = 1$ it is enough to show that $TP^*_{2i-2} = N(Q_1$ in $TP^*_{2i-2})$ fixes $\beta_1$. According to Remark 5.55, the irreducible character $\beta_{2j-1,2k}$ is the only character of $Q_{2j-1,2k}$ lying under $\beta_{2i-1,2k}$, for all $j = 1,\dots,i$. Therefore, $T$ fixes $\beta_{2j-1,2k}$, as it fixes $\beta_{2i-1,2k}$ and normalizes $Q_{2j-1,2k} = Q_{2i-1,2k} \cap G_{2j-1}$. Hence, $T$ fixes $\beta_{1,2k}$, and normalizes $Q_1$ as well as $P^*_{2k}$. So it fixes the unique $P^*_{2k}$-Glauberman correspondent $\beta_1 \in \mathrm{Irr}(Q_1)$ of $\beta_{1,2k}$. Furthermore, $P^*_{2i-2}$ fixes $\beta_1$, according to (5.17c) and the definition (5.131) of $P^*_{2i-2}$. Hence, $TP^*_{2i-2}$ fixes $\beta_1$, and (6.40) is proved for $t = 1$.

We assume that (6.40) holds for $t = 1,\dots,s-1$, and some $s = 2,\dots,i$. We will prove it also holds for $t = s$. We need to show that $N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2i-2})$ fixes the characters $\beta_1,\beta_3,\dots,\beta_{2s-1}$. By induction for $t = s-1$ we have that

$$N(Q_1,Q_3,\dots,Q_{2s-3} \text{ in } TP^*_{2i-2}) \le G(\beta_1,\beta_3,\dots,\beta_{2s-3}).$$

As $N(Q_1,Q_3,\dots,Q_{2s-3},Q_{2s-1}$ in $TP^*_{2i-2}) \le N(Q_1,Q_3,\dots,Q_{2s-3}$ in $TP^*_{2i-2})$, we conclude that $N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2i-2})$ fixes $\beta_1,\beta_3,\dots,\beta_{2s-3}$. Hence it is enough to show it fixes $\beta_{2s-1}$. By (6.37a), we have

$$N(Q_1,Q_3,\dots,Q_{2s-1} \text{ in } TP^*_{2i-2}) = N(Q_1,Q_3,\dots,Q_{2s-1} \text{ in } TP^*_{2s-2})P_{2s}\cdots P_{2i-2},$$

where, by convention, we assume that, in the case $s = i$, we have $P_{2s}\cdots P_{2s-2} = 1$. So in that case the equation holds trivially.

According to (5.17c), the groups $P_{2s},\dots,P_{2i-2}$ fix $\beta_{2s-1}$. Hence it is enough to show that the group $N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2s-2})$ fixes $\beta_{2s-1}$.

As $N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2s-2})$ normalizes both $P^*_{2k}$ and $Q_1,\dots,Q_{2s-1}$, it normalizes the product group $P_{2s}\cdots P_{2k} = N(Q_1,\dots,Q_{2s-1}$ in $P^*_{2k})$ (see (5.140)). Therefore it also normalizes $Q_{2s-1,2k} = N(P_{2s}\cdots P_{2k}$ in $Q_{2s-1})$. Let $\sigma \in N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2s-2})$. Then $\sigma = \tau \cdot p_{2s-2}$ where $\tau \in T$ and $p_{2s-2} \in P^*_{2s-2}$. As $\tau \in T$ and $T \le G(\beta_{2i-1,2k}) \le G(\beta_{2s-1,2k})$, we have that $\tau$ normalizes $Q_{2s-1,2k}$. Since $\sigma$ also normalizes $Q_{2s-1,2k}$, so does $p_{2s-2}$. Hence $p_{2s-2} \in N(Q_{2s-1,2k}$ in $P^*_{2s-2}) = C(Q_{2s-1,2k}$ in $P^*_{2s-2})$. So $p_{2s-2}$ fixes $\beta_{2s-1,2k}$. As $T$ fixes $\beta_{2s-1,2k}$, we conclude that $\sigma$ does as well. Therefore, $N(Q_1,Q_3,\dots,Q_{2s-1}$ in $TP^*_{2s-2})$ fixes $\beta_{2s-1,2k}$, and normalizes both $Q_{2s-1}$ and $P_{2s}\cdots P_{2k}$. Hence it also fixes the $P_{2s}\cdots P_{2k}$-Glauberman correspondent $\beta_{2s-1}$ of

$\beta_{2s-1,2k} \in \mathrm{Irr}(Q_{2s-1,2k})$. So

$$N(Q_1, Q_3, \ldots, Q_{2s-1} \text{ in } TP^*_{2i-2}) \leq G_{2i}(\beta_1, \ldots, \beta_{2s-1}),$$

which completes the inductive proof of (6.40) for all $t = 1, \ldots, i$. Thus Lemma 6.39 holds. □

**Lemma 6.41.** *If $\mathcal{P}$ is any $\pi$-subgroup of $N(P^*_{2k}$ in $G(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$, for some $i = 1, \ldots, k$, then $\mathcal{P}$ normalizes $P^*_{2t}$, for all $t = 0, 1, \ldots, k$. Furthermore,*

$$N(Q_1, Q_3, \ldots, Q_{2t-1} \text{ in } \mathcal{P} \cdot P^*_{2t-2}) \cdot P_{2t-2} = N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4}) P_{2t-2}, \qquad (6.42)$$

*whenever $2 \leq t \leq i$.*

*Proof.* As $\mathcal{P}$ normalizes $P^*_{2k}$, it also normalizes $P^*_{2t} = P^*_{2k} \cap G_{2t}$ for all $t = 0, 1, \ldots, k$.

To prove (6.42) we will first do the case $t = 2$ of the equation, even though it follows from the general case, just to show the argument (which is nothing else but a Frattini argument) in its easiest form. According to Lemma 6.39, for $\mathcal{P}$ in the place of $T$, and for $t$ and $i$ there both equal to 1, he character $\beta_1$ is fixed by $\mathcal{P}$. Obviously $\mathcal{P}$ fixes $\alpha_2 = \alpha^*_2$, as $i \geq t = 2$ and $\mathcal{P}$ fixes $\alpha^*_{2i-2}$ . Therefore it normalizes $G_3(\alpha_2, \beta_1) = Q_3 \ltimes P_2$. Hence $\mathcal{P}(Q_3 \ltimes P_2)$ is a group, with $Q_3 \ltimes P_2$ as a normal subgroup. Furthermore, all the $\pi'$-Hall subgroups of $Q_3 \ltimes P_2$ are the $P_2$-conjugates of $Q_3$. So the group $\mathcal{P}P_2$ permutes these conjugates among themselves with $P_2 \leq \mathcal{P}P_2$ acting transitively. Therefore, a Frattini type argument implies that

$$\mathcal{P}P_2 = N(Q_3 \text{ in } \mathcal{P}P_2)P_2 = N(Q_1, Q_3 \text{ in } \mathcal{P}P^*_2)P_2. \qquad (6.43)$$

Clearly $N(Q_1 \text{ in } \mathcal{P}P^*_0)P_2 = \mathcal{P}P_2$, as $Q_1$ is normal in $G$ and $P^*_0 = 1$. Thus (6.42) is proved for $t = 2$.

In the general case, with $t \geq 3$, we can apply Lemma 6.39 with the present $t - 1$ as both $i$ and $t$ there. We get that $N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4})$ fixes $\beta_1, \ldots, \beta_{2t-3}$. It also fixes $\alpha^*_2, \ldots, \alpha^*_{2t-2}$, as $\mathcal{P} \leq G(\alpha^*_{2i-2})$ and $P^*_{2t-2}$ do. In view of Proposition 5.149, we conclude that $N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4})$ fixes $\alpha_2, \ldots, \alpha_{2t-2}$. Therefore $N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4})$ normalizes $G_{2t-1}(\beta_1, \ldots, \beta_{2t-3}, \alpha_2, \ldots, \alpha_{2t-2})$, which equals $Q_{2t-1} \ltimes P_{2t-2}$ by (5.91) and (5.93). Therefore, $N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4}) \cdot (Q_{2t-1} \ltimes P_{2t-2})$ is a group with $Q_{2t-1} \ltimes P_{2t-2}$ as a normal subgroup. As all the $\pi'$-Hall subgroups of $Q_{2t-1} \ltimes P_{2t-2}$ are $P_{2t-2}$-conjugates of $Q_{2t-1}$, the group $N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4}) \cdot P_{2t-2}$ permutes these conjugates among themselves, while its subgroup $P_{2t-2}$ acts transitively on them. So a Frattini type argument implies that

$$\begin{aligned} &N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4}) \cdot P_{2t-2} = \\ &\qquad\qquad N(Q_{2t-1} \text{ in } N(Q_1, Q_3, \ldots, Q_{2t-3} \text{ in } \mathcal{P} \cdot P^*_{2t-4}) \cdot P_{2t-2}) \cdot P_{2t-2}. \end{aligned}$$

According to (6.37b) this last group equals $N(Q_1, Q_3, \cdots, Q_{2t-1} \text{ in } \mathcal{P} \cdot P^*_{2t-2}) \cdot P_{2t-2}$. This completes the proof of Lemma 6.42. □

**Proposition 6.44.** *For all $i = 1, \ldots, k$, the group $P^*_{2i}(\beta_{2i-1,2k})$ is the unique normal $\pi$-Hall subgroup of the normalizer $N(P^*_{2k}$ in $G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$.*

*Proof.* Let $\mathcal{P}_{2i}$ be any $\pi$-Hall subgroup of $N(P^*_{2k}$ in $G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$, for some fixed $i = 1, \ldots, k$. $P^*_{2i}(\beta_{2i-1,2k})$ is a $\pi$-subgroup of $N(P^*_{2k}$ in $G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$, as it is contained in $G_{2i} \cap P^*_{2k}$ and fixes $\alpha^*_{2i-2}$. Furthermore, $P^*_{2i}(\beta_{2i-1,2k})$ is a normal $\pi$-subgroup of $N(P^*_{2k}$ in $G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$, and thus is contained in every $\pi$-Hall subgroup of the latter group. Hence

$$P^*_{2i}(\beta_{2i-1,2k}) \leq \mathcal{P}_{2i}. \qquad (6.45)$$

The group $P_{2i}$ fixes $\beta_{2i-1,2k} \in \mathrm{Irr}(C(P_{2i}, \dots, P_{2k} \text{ in } Q_{2i-1}))$, and is contained in $P^*_{2i}$. So we get that $P_{2i} \leq P^*_{2i}(\beta_{2i-1,2k}) \leq \mathcal{P}_{2i}$. According to (5.10b) the $\pi$-group $P_{2i}$ normalizes the $\pi'$-groups $Q_1, \dots, Q_{2i-1}$. So $P_{2i} \leq N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i}) \leq N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})$. Lemma 6.39, for $T = \mathcal{P}_{2i}$ and $t = i$, implies that

$$P_{2i} \leq N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2}) \leq G_{2i}(\beta_1, \dots, \beta_{2i-1}). \tag{6.46a}$$

Furthermore, $\mathcal{P}_{2i}P^*_{2i-2}$ fixes $\alpha^*_{2i-2}$ and normalizes $P^*_{2j}$ for all $j = 1, \cdots, k$. Hence it also fixes the unique character $\alpha^*_{2j}$ of $P^*_{2j}$ lying under $\alpha^*_{2i-2}$, whenever $1 \leq j \leq i-1$. As the group $N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})$ normalizes $Q_1, Q_3, \dots, Q_{2i-1}$, it also normalizes the groups $P_{2j} = N(Q^*_{2j-1} \text{ in } P^*_{2j})$, for all $j = 1, 2, \dots, i-1$, as well as $P_{2i}$. Hence, according to Proposition 5.149, we get $N(Q_1, Q_3, \cdots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})(\alpha_{2j}) = N(Q_1, Q_3, \cdots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})(\alpha^*_{2j})$. This implies that

$$N(Q_1, Q_3, \cdots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2}) \text{ fixes } \alpha_{2j} \tag{6.46b}$$

for all $j = 1, \dots, i-1$. Similarly we can see that for all $j$ and $t$ with $1 \leq j \leq t < i$ we have $N(Q_1, Q_3, \dots, Q_{2t-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2t-2})(\alpha_{2j}) = N(Q_1, Q_3, \dots, Q_{2t-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2t-2})(\alpha^*_{2j})$. Hence

$$N(Q_1, Q_3, \cdots, Q_{2t-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2t-2}) \text{ fixes } \alpha_{2j}, \tag{6.47}$$

whenever $1 \leq j \leq t < i$.

The inclusions (6.46a), along with (6.46b), (5.91) and (5.92), imply that

$$P_{2i} \leq N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2}) \leq G_{2i}(\alpha_2, \dots \alpha_{2i-2}, \beta_1, \dots \beta_{2i-1}) = P_{2i} \ltimes Q_{2i-1}.$$

Since $N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})$ is a $\pi$-group, and $P_{2i}$ is a $\pi$-Hall subgroup of $P_{2i} \ltimes Q_{2i-1}$, it follows that

$$P_{2i} = N(Q_1, Q_3, \cdots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2}). \tag{6.48}$$

To finish the proof of Proposition 6.44, we only need to show, according to (6.45), that $\mathcal{P}_{2i} \leq P^*_{2i}$. We actually have the stronger equality

$$P^*_{2i} = \mathcal{P}_{2i}P^*_{2i-2}. \tag{6.49}$$

To prove (6.49) we will use Lemma 6.41 with $\mathcal{P}_{2i}$ in the place of $\mathcal{P}$. Indeed,

$$\begin{aligned}
P^*_{2i} &= P_{2i}P_{2i-2}P_{2i-4}\cdots P_2 \\
&= N(Q_1, Q_3, \dots, Q_{2i-1} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-2})P_{2i-2}P_{2i-4}\cdots P_2 && \text{(by (6.48) )} \\
&= N(Q_1, Q_3, \dots, Q_{2i-3} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-4})P_{2i-2}P_{2i-4}\cdots P_2 && \text{(by (6.42) for } t = i) \\
&= N(Q_1, Q_3, \dots, Q_{2i-3} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-4})P_{2i-4}P_{2i-6}\cdots P_2P_{2i-2} \\
&= N(Q_1, Q_3, \dots, Q_{2i-5} \text{ in } \mathcal{P}_{2i} \cdot P^*_{2i-6})P_{2i-6}P_{2i-8}\cdots P_2P_{2i-2}P_{2i-4} && \text{(by (6.42) for } t = i-1) \\
&= \cdots = N(Q_1, Q_3, Q_5 \text{ in } \mathcal{P}_{2i} \cdot P^*_4)P_4P_2P_{2i-2}P_{2i-4}\cdots P_8P_6 \\
&= N(Q_1, Q_3 \text{ in } \mathcal{P}_{2i} \cdot P^*_2)P_4P_2P_{2i-2}P_{2i-4}\cdots P_8P_6 && \text{(by (6.42) for } t = 3) \\
&= N(Q_1, Q_3 \text{ in } \mathcal{P}_{2i} \cdot P^*_2)P_2P_{2i-2}P_{2i-4}\cdots P_8P_6P_4 \\
&= \mathcal{P}_{2i}P_2P_{2i-2}P_{2i-4}\cdots P_6P_4 && \text{(by (6.42) for } t = 2) \\
&= \mathcal{P}_{2i}P^*_{2i-2}.
\end{aligned}$$

Hence (6.49) holds, and Proposition 6.44 is proved. □

We finish this section with a complete characterization of $N(P^*_{2k}$ in $G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$.

**Theorem 6.50.** *For every $i = 1, \dots, k$ we have*

$$N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k})) = P^*_{2i}(\beta_{2i-1,2k}) \rtimes \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) = P^*_{2i}(\beta_{2i-1,2k}) \rtimes Q_{2i-1,2k}.$$

*Proof.* This follows easily from Proposition 6.35, Proposition 6.44 and equation (6.33). $\square$

## 6.2 The irreducible characters $\hat{\beta}_{2i-1}$ of $\widehat{Q}_{2i-1}$.

We are now able to define irreducible characters $\hat{\beta}_{2i-1}$ of $\widehat{Q}_{2i-1}$, for all $i = 1, \cdots, k$, closely related to the $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$. In fact, we will prove

**Proposition 6.51.** *For every $i = 1, \dots, k$ we write $\hat{\beta}_{2i-1}$ for the character $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}}$ of $\widehat{Q}_{2i-1}$ induced by $\beta_{2i-1,2k} \in \mathrm{Irr}(Q_{2i-1,2k})$. Then $\hat{\beta}_{2i-1}$ lies in $\mathrm{Irr}(\widehat{Q}_{2i-1} | \beta_{2i-1,2k})$, while $\hat{\beta}_1 = \beta_{1,2k}$.*

*Proof.* Let $i = 1, \dots, k$ be fixed. For any subgroup $H$ of $G$ containing $Q_{2i-1,2k}$, we write $\beta^H_{2i-1,2k}$ for the induced character $(\beta_{2i-1,2k})^H$ of $H$. We will first prove that, for all $j = 1, \dots, i$, we have

$$\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-1,2k})} \in \mathrm{Irr}(\widehat{Q}_{2i-1}(\beta_{2j-1,2k}) | \beta_{2i-1,2k}, \beta_{2i-3,2k}, \dots, \beta_{1,2k}). \tag{6.52}$$

For the proof of (6.52) we will use induction on $i - j = 0, \dots, i-1$.

We treat the case where $i = 1$ separately. Let $i = j = 1$. Then, according to (6.34), we have that $\widehat{Q}_1 = Q_{1,2k}$. Therefore

$$\hat{\beta}_1 = \beta_{1,2k} = \beta_{1,2k}^{Q_{1,2k}}, \tag{6.53}$$

is an irreducible character of $\widehat{Q}_1 = Q_{1,2k}$. So equation (6.52), as well as Proposition 6.51, holds trivially when $i = j = 1$.

For any $i > 1$ the equalities (6.31) and (6.33), imply that

$$Q_{2i-1,2k} = \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) = \widehat{Q}_{2i-1}(\beta_{2i-3,2k}). \tag{6.54}$$

This, along with the inclusion in (6.32), implies that we can form a series

$$\begin{aligned} Q_{1,2k} \leq Q_{3,2k} \leq \cdots \leq Q_{2i-3,2k} \leq Q_{2i-1,2k} = \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) = \\ \widehat{Q}_{2i-1}(\beta_{2i-3,2k}) \leq \widehat{Q}_{2i-1}(\beta_{2i-5,2k}) \leq \cdots \leq \widehat{Q}_{2i-1}(\beta_{1,2k}) \leq \widehat{Q}_{2i-1} \end{aligned} \tag{6.55}$$

of subgroups of $\widehat{Q}_{2i-1}$. Even more, (6.30), along with the fact that $\widehat{Q}_{2i-1} \leq G'$ normalizes $P^*_{2k}$, implies that

$$\widehat{Q}_{2i-1}(\beta_{2j-3,2k}) \text{ normalizes } Q_{2j-1,2k} = N(P^*_{2k} \text{ in } Q_{2j-1}), \tag{6.56}$$

for any $j = 2, \dots, i$.

If $i - j = 0$, i.e., $i = j$, then $\widehat{Q}_{2i-1}(\beta_{2j-1,2k}) = \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) = Q_{2i-1,2k}$ by (6.54). Thus $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-1,2k})} = \beta_{2i-1,2k} \in \mathrm{Irr}(Q_{2i-1,2k}) = \mathrm{Irr}(\widehat{Q}_{2i-1}(\beta_{2j-1,2k}))$. Furthermore, $\beta_{2i-1,2k}$ lies above $\beta_{2i-3,2k}, \dots, \beta_{1,2k}$, as we can see in Diagram (5.20b). Hence (6.52) holds in the case that $i - j = 0$. As $\widehat{Q}_{2i-1}(\beta_{2i-3,2k}) = Q_{2i-1,2k}$ by (6.54), we also get that $\beta_{2i-1,2k} = \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2i-3,2k})}$. Hence (6.52) also holds for $j = i - 1$.

For the inductive step it is enough to prove that, if (6.52) holds for some $j = 2, \dots, i-1$, then it also holds for $j-1$ (as we induct on $i-j$). So assume that (6.52) holds for $j$. It suffices to show that

$$\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-3,2k})} \in \mathrm{Irr}(\widehat{Q}_{2i-1}(\beta_{2j-3,2k})|\beta_{2i-1,2k}, \beta_{2i-3,2k}, \cdots, \beta_{1,2k}).$$

It follows from (6.55) that $Q_{2j-1,2k} \leq \widehat{Q}_{2i-1}(\beta_{2j-1,2k}) \leq \widehat{Q}_{2i-1}(\beta_{2j-3,2k})$, where $Q_{2j-1,2k}$ is a normal subgroup of $\widehat{Q}_{2i-1}(\beta_{2j-3,2k})$, by (6.56). We clearly have that $\widehat{Q}_{2i-1}(\beta_{2j-1,2k})$ equals the group $\widehat{Q}_{2i-1}(\beta_{2j-3,2k})(\beta_{2j-1,2k})$. Furthermore, by the inductive hypothesis we know that $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-1,2k})}$ is an irreducible character of $\widehat{Q}_{2i-1}(\beta_{2j-1,2k})$ that lies above $\beta_{2j-1,2k}$. So Clifford's theory can be applied to the normal group $Q_{2j-1,2k}$ of $\widehat{Q}_{2i-1}(\beta_{2j-3,2k})$, the character $\beta_{2j-1,2k} \in \mathrm{Irr}(Q_{2j-1,2k})$, the stabilizer $\widehat{Q}_{2i-1}(\beta_{2j-1,2k})$ of that character in $\widehat{Q}_{2i-1}(\beta_{2j-3,2k})$, and the irreducible character $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-1,2k})}$ of $\widehat{Q}_{2i-1}(\beta_{2j-1,2k})$ that lies above $\beta_{2j-1,2k}$. Therefore we conclude that $\beta_{2i-1,2k}^{\widehat{Q}_{2j-3,2k}} = (\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2j-1,2k})})^{\widehat{Q}_{2i-1}(\beta_{2j-3,2k})}$ is an irreducible character of $\widehat{Q}_{2i-1}(\beta_{2j-3,2k})$. Furthermore, it lies above $\beta_{2i-1,2k}$, and thus also lies above $\beta_{2i-3,2k}, \dots, \beta_{1,2k}$. This completes the proof of the inductive step. hence (6.52) holds for all $i = 1, \dots, k$ and $j = 1, \dots, i$.

To complete the proof of the proposition, we note that, for any fixed $i = 1, \dots, k$, equation (6.52) for $j = 1$, implies that $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{1,2k})} \in \mathrm{Irr}(\widehat{Q}_{2i-1}(\beta_{1,2k})|\beta_{1,2k})$. Furthermore, $\widehat{Q}_{2i-1}$ normalizes $Q_{1,2k} = N(P_{2k}^* \text{ in } Q_1)$. Thus Clifford's theory, applied to the groups $Q_{1,2k} \trianglelefteq \widehat{Q}_{2i-1}$, implies that $\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{1,2k})}$ induces an irreducible character of $\widehat{Q}_{2i-1}$ , i.e.,

$$\hat{\beta}_{2i-1} = \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}} = (\beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{1,2k})})^{\widehat{Q}_{2i-1}} \in \mathrm{Irr}(\widehat{Q}_{2i-1}|\beta_{2i-1,2k}).$$

Hence Proposition 6.51 is proved. □

The way $\hat{\beta}_{2i-1}$ is picked implies

**Corollary 6.57.** *If $i = 1, \dots, k$, then any subgroup of $G$ that normalizes $\widehat{Q}_{2i-1}$ and fixes $\beta_{2i-1,2k}$ also fixes $\hat{\beta}_{2i-1}$. Furthermore, any subgroup of $G$ that fixes $\hat{\beta}_{2i-1}$ and $\beta_{2i-3,2k}$ also fixes $\beta_{2i-1,2k}$.*

*Proof.* The first statement is obvious, since $\hat{\beta}_{2i-1} = \beta_{2i-1}^{\widehat{Q}_{2i-1}}$.

The character $\hat{\beta}_{2i-1}$ is obtained from $\beta_{2i-1,2k}$ using a series of characters

$$\beta_{2i-1,2k} = \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2i-1,2k})} = \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2i-3,2k})}, \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2i-5,2k})}, \\ \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{2i-7,2k})}, \dots, \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}(\beta_{1,2k})}, \beta_{2i-1,2k}^{\widehat{Q}_{2i-1}} = \hat{\beta}_{2i-1},$$

each obtained from the preceding one using Clifford theory for the characters

$$\beta_{2i-5,2k}, \beta_{2i-7,2k}, \dots, \beta_{3,2k}, \beta_{1,2k},$$

in that order. Since $G(\beta_{2i-3,2k})$ fixes the characters $\beta_{1,2k}, \beta_{3,2k}, \dots, \beta_{2i-3,2k}$, Clifford theory implies the rest of the proof. □

## 6.3 $\pi$-Hall subgroups of $N(P^*_{2k}, \widehat{Q}_{2i-1}$ in $G_{2i}(\alpha^*_{2i-2}))$: the groups $\widetilde{P}_{2i}$

The following two general lemmas, along with Lemmas 6.11 and 6.44, will help us pick "nice" $\pi$-Hall subgroups $\widetilde{P}_{2i}$ of $N(P^*_{2k}, \widehat{Q}_{2i-1}$ in $G'_{2i})$.

**Lemma 6.58.** *Assume $H$ is a finite $\pi$-separable group. Let $N = N_1 \triangleright N_2 \triangleright \cdots \triangleright N_r$ be a series of normal $\pi'$-subgroups of $H$, for some integer $r \geq 1$. Let $\theta_i$ be an irreducible character of $N_i$, for each $i = 1, \cdots, r$, such that $\theta_i \in \mathrm{Irr}(N_i|\theta_{i+1}, \theta_{i+2}, \cdots, \theta_r)$. Then $H(\theta_1) \geq H(\theta_1, \theta_2) \geq \cdots \geq H(\theta_1, \ldots, \theta_r)$ and the index $|H(\theta_1) : H(\theta_1, \ldots, \theta_r)|$ is the $\pi'$-number $|N : N(\theta_1, \ldots, \theta_r)|$. Hence any $\pi$-Hall subgroup $P$ of $H(\theta_1, \ldots, \theta_r)$ is also a $\pi$-Hall subgroup of $H(\theta_1, \ldots, \theta_i)$, for each $i = 1, \ldots, r$.*

*Proof.* For every $i = 1, \ldots, r-1$ the group $H(\theta_1, \ldots, \theta_i)$ has as normal subgroups the groups $N_i \triangleright N_{i+1}$, and fixes the character $\theta_i \in \mathrm{Irr}(N_i)$. Hence it permutes among themselves all the irreducible characters of $N_{i+1}$ that lie under $\theta_i$. But this set of irreducible characters is precisely the $N_i$-conjugacy class of $\theta_{i+1}$ by Clifford's theory. So

$$H(\theta_1, \ldots, \theta_i) = H(\theta_1, \ldots, \theta_i, \theta_{i+1})N_i.$$

Therefore,

$$|H(\theta_1, \ldots, \theta_i) : H(\theta_1, \ldots, \theta_i, \theta_{i+1})| = |N_i : N_i(\theta_{i+1})|.$$

A similar argument with $N$ in the place of $H$ shows that $|N(\theta_1, \ldots, \theta_i) : N(\theta_1, \ldots, \theta_i, \theta_{i+1})| = |N_i : N_i(\theta_{i+1})|$. Hence

$$|H(\theta_1, \ldots, \theta_i) : H(\theta_1, \ldots, \theta_i, \theta_{i+1})| = |N(\theta_1, \ldots, \theta_i) : N(\theta_1, \ldots, \theta_i, \theta_{i+1})|,$$

for all $i = 1, \ldots, r-1$. This implies that

$$|H(\theta_1) : H(\theta_1, \ldots, \theta_r)| = \prod_{i=1}^{r-1} |H(\theta_1, \ldots, \theta_i) : H(\theta_1, \ldots, \theta_i, \theta_{i+1})| =$$
$$\prod_{i=1}^{r-1} |N(\theta_1, \ldots, \theta_i) : N(\theta_1, \ldots, \theta_i, \theta_{i+1})| = |N(\theta_1) : N(\theta_1, \ldots, \theta_r)|.$$

Thus the lemma holds. $\square$

The following is similar to Lemma 6.11.

**Lemma 6.59.** *Assume a finite group $N$ is the semidirect product $N = P \ltimes H$ of its $\pi$-Hall subgroup $P$ with its normal $\pi'$-Hall subgroup $H$. Assume further that $\widetilde{P}$ is a subgroup of $P$, and that $\mathcal{P}$ is any $\pi$-group of automorphisms of $N$ that normalizes $\widetilde{P}$. Then there exists $t \in H$ such that the following conditions are satisfied:*

*(i) $\mathcal{P}$ normalizes $P^t$*

*(ii) $\widetilde{P} \leq P^t$*

*(iii) $t$ centralizes $\widetilde{P}$.*

*Proof.* Since $\mathcal{P}$ normalizes $N$, we can form the external semi–direct product product $N\mathcal{P} = N \rtimes \mathcal{P}$. Furthermore, as $\mathcal{P}$ normalizes $\widetilde{P}$, the $\pi$-group $\widetilde{P}\mathcal{P}$ is a subgroup of $N\mathcal{P}$, and thus normalizes $N$.

Hence $\widetilde{P}\mathcal{P}$ normalizes a $\pi$-Hall subgroup $P^t$ of $N$, for some $t \in H$. Therefore $\mathcal{P}$, as well as $\widetilde{P}$, normalizes $P^t$. But $\widetilde{P}$ is a subgroup of $N$. so the only way it can normalize the Hall subgroup $P^t$ is to be contained in $P^t$. Or, equivalently, $(\widetilde{P})^{t^{-1}} \leq P$. Thus $s^{-1}s^{t^{-1}} \in P$ whenever $s \in \widetilde{P}$. But $\widetilde{P} \leq P$ normalizes $H$. Hence $s^{-1}s^{t^{-1}} = [s, t^{-1}]$ is also an element of $H$. As $H \cap P = 1$, we conclude that $[s, t^{-1}] = 1$, for all $s \in \widetilde{P}$. This implies that $t^{-1}$, and thus $t$, centralizes $\widetilde{P}$. So the lemma holds. □

**Lemma 6.60.** *Let $\widetilde{P}_{2i}$ be a $\pi$-Hall subgroup of $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}))$, where $i = 1, \cdots, k$. Then*

$$N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2})) = \widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1}. \tag{6.61}$$

*Furthermore,*

$$N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \cdots \hat{\beta}_{2i-1})) = N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))), \tag{6.62}$$

*whenever $i = 1, \ldots, k$.*

*Proof.* According to Proposition 6.35 the group $\widehat{Q}_{2i-1}$ is a $\pi'$-Hall subgroup of $N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}))$. Hence

$$N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2})) = \widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1}.$$

For the second part of the lemma note that, for $i = 1$ we obviously have $N(P^*_{2k} \text{ in } G_2(\alpha^*_0, \hat{\beta}_1)) = N(\widehat{Q}_1 \text{ in } N(P^*_{2k} \text{ in } G_2(\alpha^*_0, \beta_{1,2k})))$, as $\alpha^*_0 = 1$ and $\hat{\beta}_1 = \beta_{1,2k}$, by Proposition 6.51.

For any $t, i$ with $1 \leq t \leq i$, the group $N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k})))$ normalizes $\widehat{Q}_{2t-1} = \widehat{Q}_{2i-1} \cap G_{2t-1}$ and fixes $\beta_{2t-1,2k}$, as $\beta_{2t-1,2k}$ is the unique character of $Q_{2t-1,2k} = Q_{2i-1,2k} \cap G_{2t-1}$ that lies under $\beta_{2i-1,2k}$. So $N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k})))$ fixes the characters $\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}$ by Corollary 6.57. Hence

$$N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))) \leq N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \ldots \hat{\beta}_{2i-1})).$$

For the other inclusion, we use Corollary 6.57 again, but in a recursive way. We saw above that the group $N(P^*_{2k} \text{ in } G_{2i}(\hat{\beta}_1)) = N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_0, \hat{\beta}_1))$ fixes $\beta_{1,2k}$. Hence, $N(P^*_{2k} \text{ in } G_{2i}(\hat{\beta}_1, \hat{\beta}_3))$ normalizes the group $\widehat{Q}_3(\beta_{1,2k})$. In view of (6.31) the last group equals $Q_{3,2k}$. Thus Corollary 6.57 implies that $N(P^*_{2k} \text{ in } G_2(\hat{\beta}_1, \hat{\beta}_3))$ fixes $\beta_{3,2k}$, as it fixes $\hat{\beta}_3$ and $\beta_{1,2k}$, and normalizes $Q_{3,2k}$. Similarly, we get that $N(P^*_{2k} \text{ in } G_{2i}(\hat{\beta}_1, \hat{\beta}_3, \hat{\beta}_5))$ normalizes $\widehat{Q}_5(\beta_{3,2k}) = Q_{5,2k}$, and fixes both $\beta_{3,2k}$ and $\hat{\beta}_5$. Thus it also fixes $\beta_{5,2k}$, by Corollary 6.57.

We continue in this way. So after $i-1$ steps we have that $N(P^*_{2k} \text{ in } G_{2i}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-3}))$ fixes $\beta_{2i-3,2k}$. Hence the group $N(P^*_{2k} \text{ in } G_2(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}))$ normalizes $\widehat{Q}_{2i-1}(\beta_{2i-3,2k}) = Q_{2i-3,2k}$, by (6.31), and fixes both $\beta_{2i-3,2k}$ and $\hat{\beta}_{2i-1}$. Therefore Corollary 6.57 implies that it fixes $\beta_{2i-1,2k}$, i.e.,

$$N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1})) \leq N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))).$$

This completes the proof of the lemma. □

Let $i = 1, \ldots, k$. According to Theorem 6.50 the group $N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$ equals

$P^*_{2i}(\beta_{2i-1,2k}) \rtimes \widehat{Q}_{2i-1}(\beta_{2i-1,2k})$. This, along with (6.61) and (6.62), implies that

$$\begin{aligned}
&N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \cdots, \hat{\beta}_{2i-1})) \\
&= N(\widehat{Q}_{2i-1} \text{ in } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))) && \text{by (6.62)} \\
&= N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k}) \rtimes \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) && \text{by Theorem 6.50} \\
&= N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \times \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) && \text{as } \widehat{Q}_{2i-1}(\beta_{2i-1,2k}) \leq \widehat{Q}_{2i-1} \\
&= N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \times Q_{2i-1,2k}. && \text{by (6.33).}
\end{aligned}$$

Hence

$$N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \cdots, \hat{\beta}_{2i-1})) = N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \times Q_{2i-1,2k}, \tag{6.63}$$

for all $i = 1, \ldots, k$.

The following proposition implies the existence of a "good" family of groups $\widetilde{P}_{2i}$, that permit us to use Theorem 4.24 on the groups $\widehat{Q}_{2i-1}, \widehat{Q}$ and $\widetilde{P}_{2i}$, for $i = 1, \ldots, k$.

**Proposition 6.64.** *There exist $\pi$-groups $\widetilde{P}_{2i}$, for $i = 1, \ldots, k$, such that the following conditions are satisfied:*

$$\widetilde{P}_{2i} \in \mathrm{Hall}_\pi(N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}))), \tag{6.65a}$$

$$\begin{aligned}
\widetilde{P}_{2i}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}) &= N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \\
&\in \mathrm{Hall}_\pi(N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}))),
\end{aligned} \tag{6.65b}$$

$$\textit{and } \widetilde{P}_{2i} \textit{ normalizes } \widetilde{P}_{2j}, \tag{6.65c}$$

*for all $i = 1, 2, \ldots, k$ and all $j = 1, 2, \ldots, i$. Furthermore, any such $\widetilde{P}_{2i}$ satisfy*

$$\begin{aligned}
\widetilde{P}_{2i}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}_{2j-1}, \hat{\beta}_{2j+1}, \ldots, \hat{\beta}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}_{2i-1}) \\
\in \mathrm{Hall}_\pi(N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_{2j-1}, \hat{\beta}_{2j+1}, \ldots, \hat{\beta}_{2i-1})))
\end{aligned} \tag{6.66}$$

*whenever $1 \leq j \leq i \leq k$.*

*Proof.* In view of (6.63) the only Hall $\pi$-subgroup of $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}))$ is $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k}))$. As $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}))$ is a subgroup of $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}))$, we can certainly find $\widetilde{P}_{2i}$ satisfying (6.65a,b) for $i = 1, 2, \ldots, k$. We shall modify these $\widetilde{P}_{2i}$ so as to obtain new subgroups satisfying (6.65c) as well as (6.65a,b).

We first note that, whenever $1 \leq t \leq i \leq k$, the subgroup $P^*_{2i}$ normalizes $P^*_{2t}$, while $P^*_{2i}(\beta_{2i-1,2k})$ fixes $\beta_{2t-1,2k}$ by Proposition 5.55. As $\widehat{Q}_{2t-1} = \widehat{Q}_{2i-1} \cap G_{2t-1}$, we get that $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k}))$ normalizes $N(\widehat{Q}_{2t-1} \text{ in } P^*_{2t}(\beta_{2t-1,2k}))$. By (6.65b) this is equivalent to

$$\widetilde{P}_{2i}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}) \text{ normalizes } \widetilde{P}_{2t}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2t-1}) \tag{6.67}$$

whenever $1 \leq t \leq i \leq k$.

By (6.61) we have $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2})) = \widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1}$, for each $i = 1, \ldots, k$. Also $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}))$ normalizes both $P^*_{2t} = P^*_{2k} \cap G_{2t}$ (by Proposition 5.132) and $\widehat{Q}_{2t-1} = \widehat{Q}_{2i-1} \cap G_{2t-1}$ (by Definition 6.28), whenever $1 \leq t \leq i \leq k$. So it fixes the unique character

$\alpha^*_{2t-2} \in \mathrm{Irr}(P^*_{2t-2})$ lying under $\alpha^*_{2i-2}$ (see Proposition 5.153). Hence it normalizes $G_{2t}(\alpha^*_{2t-2})$. So $N(P^*_{2k}, \widehat{Q}_{2i-1}$ in $G_{2i}(\alpha^*_{2i-2}))$ normalizes $N(P^*_{2k}, \widehat{Q}_{2t-1}$ in $G_{2t}(\alpha^*_{2t-2}))$, i.e.,

$$\widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1} \text{ normalizes } \widetilde{P}_{2t} \ltimes \widehat{Q}_{2t-1}, \tag{6.68}$$

whenever $1 \leq t \leq i \leq k$.

We are going to modify the $\hat{P}_{2i}$ so that they satisfy

$$\widetilde{P}_{2i}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2i-1}) \text{ normalizes } \widetilde{P}_{2t} \tag{6.69}$$

whenever $1 \leq t \leq i \leq k$, as well as (6.65a,b). For this we will use reverse induction on $t$, starting with $t = k$ and working down. The group $\widetilde{P}_{2k}$ requires no modification, since the only possible $i$ satisfying $k \leq i \leq k$ is $i = k$, and the subgroup $\widetilde{P}_{2k}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2k-1})$ certainly normalizes $\widetilde{P}_{2k}$. For the inductive step assume that $\widetilde{P}_{2k}, \widetilde{P}_{2k-2}, \ldots, \widetilde{P}_{2s+2}$, for some integer $s = 1, 2, \ldots, k-1$, have already been modified so that (6.69) holds whenever $s < t \leq i \leq k$, and (6.65a,b) hold for all $i = 1, 2, \ldots, k$. We want to modify $\widetilde{P}_{2s}$ so that (6.65a,b) still hold for $i = s$, while (6.69) with $t = s$ holds for all $i = s, s+1, \ldots, k$.

According to (6.67) the product

$$T_{t+1} = \widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1}) \cdot \widetilde{P}_{2k-2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3}) \cdots \widetilde{P}_{2t+2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2t+1})$$

forms a $\pi$-group, whenever $1 \leq t \leq k-1$. Each factor $\widetilde{P}_{2i}(\hat{\beta}_1, \ldots, \hat{\beta}_{2i-1})$, for $i = s+1, s+2, \ldots, k$, in this product is contained in $\widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1}$, and hence normalizes $\widetilde{P}_{2s} \ltimes \widehat{Q}_{2s-1}$ by (6.68). That factor also normalizes $\widetilde{P}_{2s}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2s-1})$ by (6.67). Thus $T_{s+1}$ acts on $\widetilde{P}_{2s} \ltimes \widehat{Q}_{2s-1}$ and normalizes the subgroup $\widetilde{P}_{2s}(\hat{\beta}_1, \hat{\beta}_3, \ldots, \hat{\beta}_{2s-1})$ of $\widetilde{P}_{2s}$. Therefore we can apply Lemma 6.59 to the groups $T_{s+1}, \widetilde{P}_{2s}(\hat{\beta}_1, \ldots, \hat{\beta}_{2s-1})$ and $\widetilde{P}_{2s} \ltimes \widehat{Q}_{2s-1}$ to get an element $t \in \widehat{Q}_{2s-1}$ such that

$$(\widetilde{P}_{2s})^t \in \mathrm{Hall}_\pi(\widetilde{P}_{2s} \ltimes \widehat{Q}_{2s-1}) \text{ is normalized by } T_{s+1} \tag{6.70a}$$

and

$$\widetilde{P}_{2s}(\hat{\beta}_1, \ldots, \hat{\beta}_{2s-1}) \leq (\widetilde{P}_{2s})^t. \tag{6.70b}$$

Obviously the group $(\widetilde{P}_{2s})^t$ satisfies (6.65a) for $i = s$, as $\widehat{Q}_{2s-1} \leq N(P^*_{2k}, \widehat{Q}_{2s-1}$ in $G_{2s}(\alpha^*_{2s-2}))$. Furthermore, as $\widetilde{P}_{2s}(\beta_1, \ldots, \beta_{2s-1})$ satisfies (6.65b) for $i = s$, the inclusion (6.70b) implies that

$$(\widetilde{P}_{2s})^t(\hat{\beta}_1, \ldots, \hat{\beta}_{2s-1}) = \widetilde{P}_{2s}(\hat{\beta}_1, \ldots, \hat{\beta}_{2s-1}). \tag{6.71}$$

Therefore, $(\widetilde{P}_{2s})^t$ satisfies (6.65a,b) for $i = s$. In addition, (6.70a) implies that $\widetilde{P}_{2i}(\hat{\beta}_1, \ldots, \hat{\beta}_{2i-1})$ normalizes $(\widetilde{P}_{2s})^t$ whenever $i = s+1, \ldots, k$. Hence we can work with $(\widetilde{P}_{2s})^t$ in the place of $\widetilde{P}_{2s}$. The new group $\widetilde{P}_{2i}$ satisfies (6.69) whenever $s \leq t \leq i \leq k$.

At this point we have shown that we can find the groups $\widetilde{P}_{2i}$ that satisfy (6.65a,b) as well as (6.69) whenever $1 \leq t \leq i \leq k$. These groups can be modified further to satisfy, in addition, (6.65c). Indeed, according to (6.68), we get that $\widetilde{P}_{2k} \leq \widetilde{P}_{2k} \ltimes \widehat{Q}_{2k-3}$ normalizes $\widetilde{P}_{2k-2} \ltimes \widehat{Q}_{2k-3}$ while $\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1})$ normalizes $\widetilde{P}_{2k-2}$ by (6.69). Therefore, Lemma 6.11, with $\pi'$ in the place of $\pi$, implies that there exists $t_{2k-3} \in \widehat{Q}_{2k-3}$ such that

$$\widetilde{P}_{2k}^{t_{2k-3}} \text{ normalizes } \widetilde{P}_{2k-2}$$

and

$$\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1}) \leq \widetilde{P}_{2k}^{t_{2k-3}}.$$

In view of (6.65b) (for $\widetilde{P}_{2k}$ ) this inclusion implies that

$$\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1}) = \widetilde{P}_{2k}^{t_{2k-3}}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1}).$$

The above equation permits us to work with $\widetilde{P}_{2k}^{t_{2k-3}}$ in the place of $\widetilde{P}_{2k}$. Then $\widetilde{P}_{2k}$ satisfies (6.65a,b) and (6.69), and normalizes $\widetilde{P}_{2k-2}$.

Now the product $\widetilde{P}_{2k}\widetilde{P}_{2k-2}$ forms a group that normalizes the unique $\pi'$-Hall subgroup $\widehat{Q}_{2k-5}$ of $\widetilde{P}_{2k-4} \ltimes \widehat{Q}_{2k-5}$, and has $T_{k-1}$ as a subgroup. Furthermore, $\widetilde{P}_{2k}\widetilde{P}_{2k-2} \ltimes \widehat{Q}_{2k-5}$ normalizes $\widetilde{P}_{2k-4} \ltimes \widehat{Q}_{2k-5}$, while $T_{k-1}$ normalizes $\widetilde{P}_{2k-4}$, as $\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1})$ and $\widetilde{P}_{2k-2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3})$ do by (6.69). Hence Lemma 6.11, with $\pi'$ in the place of $\pi$, implies that we can find an element $t_{2k-5} \in \widehat{Q}_{2k-5}$ such that

$$(\widetilde{P}_{2k}\widetilde{P}_{2k-2})^{t_{2k-5}} \text{ normalizes } \widetilde{P}_{2k-4}$$

and

$$T_{k-1} \leq \widetilde{P}_{2k}^{t_{2k-5}}\widetilde{P}_{2k-2}^{t_{2k-5}}.$$

Also $t_{2k-5}$ centralizes $T_{k-1}$, and thus centralizes both $\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1})$ and $\widetilde{P}_{2k-2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3})$. Therefore $\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1})$ and $\widetilde{P}_{2k-2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3})$ are subgroups of $\widetilde{P}_{2k}^{t_{2k-5}}$ and $\widetilde{P}_{2k-2}^{t_{2k-5}}$ respectively. So, in view of (6.65b), we get that

$$\widetilde{P}_{2k}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1}) = \widetilde{P}_{2k}^{t_{2k-5}}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-1})$$

and

$$\widetilde{P}_{2k-2}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3}) = \widetilde{P}_{2k-2}^{t_{2k-5}}(\hat{\beta}_1, \ldots, \hat{\beta}_{2k-3}).$$

Hence we can replace $\widetilde{P}_{2k}$ and $\widetilde{P}_{2k-2}$ by $\widetilde{P}_{2k}^{t_{2k-5}}$ and $\widetilde{P}_{2k-2}^{t_{2k-5}}$ respectively. Then (6.65a,b) and (6.69) are satisfied by the newly modified groups $\widetilde{P}_{2k}$ and $\widetilde{P}_{2k-2}$. Furthermore, $\widetilde{P}_{2k}$ normalizes both $\widetilde{P}_{2k-2}$ and $\widetilde{P}_{2k-4}$, while $\widetilde{P}_{2k-2}$ normalizes $\widetilde{P}_{2k-4}$.

We continue similarly. At every step the product $\widetilde{P}_{2k}\widetilde{P}_{2k-2}\cdots\widetilde{P}_{2t}$ of the modified groups $\widetilde{P}_{2i}$, for $2 \leq t \leq i \leq k$, contains $T_t$. Even more, $T_t$ is a subgroup that normalizes $\widetilde{P}_{2t-2}$ while $\widetilde{P}_{2k}\widetilde{P}_{2k-2}\ldots\widetilde{P}_{2t} \ltimes \widehat{Q}_{2t-3}$ normalizes $\widetilde{P}_{2t-2} \ltimes \widehat{Q}_{2t-3}$. So Lemma 6.11 implies the existence of an element $t_{2t-3} \in \widehat{Q}_{2t-3}$ such that, if we replace $\widetilde{P}_{2k}$ and $\widetilde{P}_{2t}$ by $\widetilde{P}_{2k}^{t_{2k-3}}$ and $\widetilde{P}_{2t}^{t_{2k-3}}$ respectively, the new groups satisfy (6.65a,b) and (6.69) for $i = k, k-1, \ldots, t-1$, while $\widetilde{P}_{2i}$ normalizes $\widetilde{P}_{2j}$ whenever $k \geq i \geq j \geq t-1$.

This process stops when we reach $t = 2$. This proves that we can pick the groups $\widetilde{P}_{2i}$ to satisfy (6.65a,b,c). The additional property (6.66) follows from (6.65b) and Lemma 6.58. This completes the proof of Proposition 6.64. $\square$

A useful property of the groups $\widetilde{P}_{2i}$ is given in

**Corollary 6.72.** *For every $i = 1, \ldots, k$ we have that*

$$\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}_1, \ldots, \hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i}) \text{ and} \tag{6.73a}$$

$$\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } P_{2i}^*). \tag{6.73b}$$

*Proof.* According to (6.66) and (6.65b) we get that

$$\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}_1, \dots, \hat{\beta}_{2i-1}) = N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})).$$

In view of Corollary 6.57, the group $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k}))$ fixes $\hat{\beta}_{2i-1}$, as it normalizes $\widehat{Q}_{2i-1}$ and fixes $\beta_{2i-1,2k}$. Thus

$$\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \leq N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1})). \tag{6.74}$$

But $\widehat{Q}_{2i-1} \leq \widehat{Q} \leq G' = G(\alpha^*_{2k})$. Thus $\widehat{Q}_{2i-1}$ normalizes $P^*_{2i}$, as $\widehat{Q}$ does. We conclude that $\widehat{Q}_{2i-1}$ normalizes $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1}))$. As the latter $p$-group also normalizes the $p'$-group $\widehat{Q}_{2i-1}$, we get that $\widehat{Q}_{2i-1}$ centralizes $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1}))$. Hence $\widehat{Q}_{2i-1}$ also centralizes $\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) \leq N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1}))$. Therefore, $\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) \leq C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i})$. As the other inclusion is obvious, we conclude that $\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i})$. Hence (6.73a) holds

Furthermore, the fact that $\widehat{Q}_{2i-1}$ normalizes $P^*_{2i}(\hat{\beta}_{2i-1})$ implies that $N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1})) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1}))$. Hence we have that

$$\begin{aligned}
\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) &= N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \\
&\leq N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1})) && \text{by (6.74)} \\
&= C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\hat{\beta}_{2i-1})) \\
&\leq C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) \\
&= C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) && \text{as } \beta_{2i-1,2k} \in \mathrm{Irr}(Q_{2i-1,2k}) \\
& && \text{and } Q_{2i-1,2k} \leq \widehat{Q}_{2i-1} \\
&\leq N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}(\beta_{2i-1,2k})) \\
&= \widetilde{P}_{2i}(\hat{\beta}_{2i-1}).
\end{aligned}$$

So $\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i})$, and (6.73b) holds. This completes the proof of the corollary. □

## 6.4 Triangular sets for $G' = G(\alpha^*_{2k})$

For the following two sections we are going to keep fixed the groups $P^*_{2i}$, along with their characters $\alpha^*_{2i}$. Even though these groups and characters come from a specific triangular set of (5.2), we will forget this triangular set and treat the $P^*_{2i}$ independently. Furthermore, we make the following assumption

$$m = 2k \text{ is even.} \tag{6.75}$$

We have already seen that the groups $G'_s$, defined as $G'_s = G_s(\alpha^*_{2k}) = G' \cap G_s$ whenever $1 \leq s \leq m = 2k$, form a series

$$1 = G'_0 \trianglelefteq G'_1 \trianglelefteq \cdots \trianglelefteq G'_{2k-1} \trianglelefteq G'_{2k} \trianglelefteq G'. \tag{6.76}$$

This is a series of normal subgroups of $G'$, as the series (5.2) is such for $G$.

### 6.4.1 From $G$ to $G'$

The goal of this section is to create a triangular set for the series (6.76), related to the triangular set $\{Q_{2i-1}, P_{2i}|\beta_{2i-1}, \alpha_{2i}\}_{i=1}^k$. We remark that the latter is a triangular set for normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{2k-1} \trianglelefteq G_{2k} \trianglelefteq G, \tag{6.77}$$

that is, (5.2) for $m = 2k$. As we will see, these two triangular sets are so close related that one determines the other uniquely, up to conjugation.

To create such a set, we first need to give groups and characters that satisfy (5.17) for the series (6.76). For the $\pi'$-groups and characters we pick the groups $Q_{2i-1,2k}$ for every $i = 1, \dots, k$, along with their irreducible characters $\beta_{2i-1,2k}$. In view of Remark 6.26, we have that $Q_{2i-1,2k}$ is a subgroup of $G'$, and thus a subgroup of $G'_{2i-1} = G' \cap G_{2i-1}$ (as $Q_{2i-1,2k} \leq G_{2i-1}$), for all $i = 1, \dots, k$. We define

$$Q'_{2i-1} := Q_{2i-1,2k} \text{ and } \beta'_{2i-1} = \beta_{2i-1,2k}, \tag{6.78a}$$

and

$$P'_0 = 1 \text{ and } P'_{2i} := P^*_{2i}(\beta_{2i-1,2k}), \tag{6.78b}$$

whenever $i = 1, \dots, k$. Note that

**Lemma 6.79.** *For every $i = 1, \dots, k$ we have*

$$\begin{aligned} P'_{2i} = P^*_{2i}(\beta'_{2i-1}) = P^*_{2i}(\beta_{2i-1,2k}) = \\ N(Q_{2i-1,2k} \text{ in } P^*_{2i}) = N(Q'_{2i-1} \text{ in } P^*_{2i}) = C(Q_{2i-1,2k} \text{ in } P^*_{2i}) = C(Q'_{2i-1} \text{ in } P^*_{2i}) = \\ C(Q_{1,2k}, Q_{3,2k}, \dots, Q_{2i-1,2k} \text{ in } P^*_{2i}). \end{aligned}$$

*Furthermore,*

$$P'_{2i} \textit{ is the unique } \pi\textit{-Hall subgroup of } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k})).$$

*Proof.* Let $i = 1, \dots, k$ be fixed. Obviously $P^*_{2i}(\beta_{2i-1,2k}) \leq N(Q_{2i-1,2k} \text{ in } P^*_{2i})$. Also $Q_{2i-1,2k} = C(P_{2i}, \dots, P_{2k} \text{ in } Q_{2i-1})$ normalizes $P_2, \dots, P_{2i-2}$, as it is a subgroup of $Q_{2i-1}$ (see (5.17e)), and centralizes $P_{2i}$. Hence the $\pi'$-group $Q_{2i-1,2k}$ normalizes the $\pi$-group $P^*_{2i}$. Therefore

$$N(Q_{2i-1,2k} \text{ in } P^*_{2i}) = C(Q_{2i-1,2k} \text{ in } P^*_{2i}) \leq P^*_{2i}(\beta_{2i-1,2k}).$$

So

$$P'_{2i} = P^*_{2i}(\beta_{2i-1,2k}) = N(Q_{2i-1,2k} \text{ in } P^*_{2i}) = C(Q_{2i-1,2k} \text{ in } P^*_{2i}).$$

The fact that $C(Q'_{2i-1} \text{ in } P^*_{2i}) = C(Q_{1,2k}, Q_{3,2k}, \dots, Q_{2i-1,2k} \text{ in } P^*_{2i})$ follows easily from the fact (see (5.40)) that $Q_{2t-1,2k}$ is a subgroup (actually normal) of $Q_{2i-1,2k}$ whenever $1 \leq t \leq i$.

The rest of the lemma follows from (6.78a) and Proposition 6.44. □

What about irreducible characters $\alpha'_{2i}$ of $P'_{2i}$? Well, there is a straightforward way to pick those characters. To see this, note that $Q'_{2i-1} = Q_{2i-1,2k}$ fixes the character $\alpha^*_{2i}$ of $P^*_{2i}$. Indeed, $Q_{2i-1,2k}$ fixes $\alpha^*_{2k}$ (as it is a subgroup of $G'$) and normalizes $P^*_{2i} = G_{2i} \cap P^*_{2k}$. Hence it fixes the unique character $\alpha^*_{2i}$ of $P^*_{2i}$ that lies under $\alpha^*_{2k}$ (see Proposition 5.153). As $P'_{2i} = C(Q_{2i-1,2k} \text{ in } P^*_{2i}) = C(Q'_{2i-1} \text{ in } P^*_{2i})$, we can make the

**Definition 6.80.** For every $i = 1, \dots, k$, the character $\alpha'_{2i} \in \operatorname{Irr}(P'_{2i})$ is the $Q'_{2i-1}$-Glauberman correspondent of $\alpha^*_{2i} \in \operatorname{Irr}(P^*_{2i})$. We also set $\alpha'_0 := 1 \in \operatorname{Irr}(P'_0)$.

Now we can show

**Theorem 6.81.** *The set*

$$\{Q'_1, \dots, Q'_{2k-1}, P'_0, P'_2, \dots, P'_{2k} | \beta'_1, \dots, \beta'_{2k-1}, \alpha'_0, \alpha'_2, \dots, \alpha'_{2k}\}, \tag{6.82}$$

*given by* (6.78) *and Definition 6.80, is a triangular set for* (6.76).

*Proof.* It is enough to check that (6.82) satisfies (5.17). It obviously satisfies (5.17a). According to (6.34) we have that $Q'_1 = Q_{1,2k} = G'_1$. Hence (5.17b) holds for the set (6.82).

By Lemma 6.79 the group $P'_{2i}$ centralizes $Q'_{2i-1} = Q_{2i-1,2k}$, for all $i = 1, \dots, k$. Therefore the group that we would write as $Q'_{2i-1,2i}$ (see (5.11)) is $Q'_{2i-1}$ itself. Furthermore, the $P'_{2i}$-Glauberman correspondent of $\beta'_{2i-1}$ is the same character $\beta'_{2i-1}$, whenever $1 \leq i \leq k$. Therefore the character that we would write as $\beta'_{2i-1,2i}$ (see Definition 5.22) is nothing else but the character $\beta'_{2i-1}$. According to (5.40) and (5.51) the groups $Q'_{2i-1} = Q_{2i-1,2k}$ and their characters $\beta'_{2i-1} = \beta_{2i-1,2k}$ line up. That is, we have a series of normal subgroups

$$Q'_1 \trianglelefteq Q'_3 \trianglelefteq \cdots \trianglelefteq Q'_{2k-3} \trianglelefteq Q'_{2k-1}$$

along with their characters

$$\beta'_1, \beta'_3, \dots, \beta'_{2k-3}, \beta'_{2k-1}$$

that lie one under the other. Actually, by Proposition 5.55 the unique character of $Q'_{2j-1}$ that lies under $\beta'_{2i-1}$ is $\beta'_{2j-1}$, whenever $1 \leq j \leq i \leq k$. Hence the characters $\beta'_{2i-1}$ satisfy (5.17f) in the definition of a triangular set.

The group $P'_{2i,2i+1}$, defined as $P'_{2i,2i+1} = C(Q'_{2i+1} \text{ in } P'_{2i})$ (see (5.14)), satisfies

$$P'_{2i,2i+1} = C(Q_{2i+1,2k} \text{ in } P'_{2i}) = C(Q_{2i+1,2k} \text{ in } C(Q_{2i-1,2k} \text{ in } P^*_{2i})) = \\ C(Q_{2i+1,2k} \text{ in } P^*_{2i}) = C(Q'_{2i+1} \text{ in } P^*_{2i}),$$

whenever $1 \leq i \leq k-1$. But $Q'_{2i+1} = Q_{2i+1,2k}$ normalizes $P^*_{2i}$, and fixes its irreducible character $\alpha^*_{2i}$, as it fixes $\alpha^*_{2k}$. Hence the $Q'_{2i+1}$-Glauberman correspondent of $\alpha'_{2i}$ is the $Q'_{2i+1}$-Glauberman correspondent of $\alpha^*_{2i}$, as $\alpha'_{2i}$ is the $Q'_{2i-1}$-Glauberman correspondent of $\alpha^*_{2i}$, and $Q'_{2i-1} \trianglelefteq Q'_{2i+1}$. This implies that the character $\alpha'_{2i,2i+1} \in \operatorname{Irr}(P'_{2i,2i+1})$, defined as the $Q'_{2i+1}$-Glauberman correspondent of $\alpha'_{2i}$ (see Definition 5.49), is the $Q'_{2i+1}$-Glauberman correspondent of $\alpha^*_{2i} \in \operatorname{Irr}(P^*_{2i})$, for all $i =$

$1, \dots, k-1$. Furthermore, $\alpha'_{2i+2}$ is the $Q'_{2i+1}$-Glauberman correspondent of $\alpha^*_{2i+2}$. As $\alpha^*_{2i+2}$ lies above $\alpha^*_{2i}$ we conclude that the same holds for their $Q'_{2i+1}$-Glauberman correspondents. Thus $\alpha'_{2i+2} \in \mathrm{Irr}(P'_{2i+2})$ lies above $\alpha'_{2i,2i+1} \in \mathrm{Irr}(P'_{2i,2i+1})$ whenever $1 \leq i \leq k-1$. We obviously have that $\alpha'_2 \in \mathrm{Irr}(P'_2)$ lies above $1 = \alpha'_{0,1} \in \mathrm{Irr}(P'_0)$. Hence the characters $\alpha'_{2i}$ satisfy (5.17d).

To complete the proof of the theorem, it remains to show that the set (6.82) also satisfies (5.17c) and (5.17e).

According to (6.29b), (6.29c) and (6.31) we get that

$$Q'_{2i-1} = Q_{2i-1,2k} = \widehat{Q}_{2i-1}(\beta_{2i-3,2k}) \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha^*_{2k}, \beta_{2i-3,2k})), \tag{6.83}$$

for all $i = 1, \dots, k$. Furthermore, as $Q'_{2i-1} = Q_{2i-1,2k}$ fixes $\alpha^*_{2k}$ it also fixes $\alpha^*_{2j}$ for all $j = 1, \dots, k$, by Remark 6.1. In view of Proposition 5.55, it also fixes $\beta'_{2j-1} = \beta_{2j-1,2k}$ for all $j = 1, \dots, i-1$. Thus $Q'_{2i-1}$ normalizes the groups $Q'_{2j-1}$ and fixes $\alpha^*_{2j}$. This implies that $Q'_{2i-1}$ also fixes the $Q'_{2j-1}$-Glauberman correspondent $\alpha'_{2j}$ of $\alpha^*_{2j}$ whenever $1 \leq j \leq i-1$. Hence $Q'_{2i-1} \leq G_{2i-1}(\alpha^*_{2k}, \beta'_1, \dots, \beta'_{2i-3}, \alpha'_2, \dots, \alpha'_{2i-2})$. (We actually have even more as $Q'_{2i-1}$ fixes $\alpha'_{2i}$ and $\beta'_{2i-1}$ for all $i = 1, \dots, k$, but we don't need it here.) This, along with (6.83) and the fact that

$$G_{2i-1}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2i-2}, \beta'_1, \dots, \beta'_{2i-3}) \leq G_{2i}(\alpha^*_{2k}, \beta'_{2i-3}) = G_{2i}(\alpha^*_{2k}, \beta_{2i-3,2k}),$$

implies that

$$\begin{aligned} Q'_{2i-1} \in \mathrm{Hall}_{\pi'}(G_{2i-1}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2i-2}, \beta'_1, \dots, \beta'_{2i-3})) = \\ \mathrm{Hall}_{\pi'}(G'_{2i-1}(\alpha'_2, \dots, \alpha'_{2i-2}, \beta'_1, \dots, \beta'_{2i-3})), \end{aligned} \tag{6.84}$$

whenever $i = 1, \dots, k$. Hence (5.17e) holds for the $\pi'$-groups $Q'_{2i-1}$.

As for the $\pi$-groups, we first note that, in view of Lemma 6.79, for every $i = 1, \dots, k$ the group $P'_{2i}$ centralizes $Q'_1 = Q_{1,2k}, \dots, Q'_{2i-1} = Q_{2i-1,2k}$, and thus fixes their characters $\beta'_1, \dots, \beta'_{2i-1}$. It also fixes the characters $\alpha^*_2, \dots, \alpha^*_{2i-2}$, as $P'_{2i} \leq P^*_{2i}$. Therefore it also fixes the $Q'_{2j-1}$-Glauberman correspondent $\alpha'_{2j}$ of $\alpha^*_{2j}$, for all $j = 1, \dots, i$. Hence

$$\begin{aligned} P'_{2i} = P^*_{2i}(\beta'_{2i-1}) \leq G_{2i}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2i-2}, \beta'_1, \dots, \beta'_{2i-1}) \leq \\ N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta'_{2i-1})). \end{aligned} \tag{6.85}$$

Proposition 6.44 implies that $P'_{2i} = P^*_{2i}(\beta_{2i-1,2k})$ is the unique $\pi$-Hall subgroup of the group $N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta_{2i-1,2k}))$. This, along with (6.85), implies that $P'_{2i}$ is a $\pi$-Hall subgroup of $G_{2i}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2i-2}, \beta'_1, \dots, \beta'_{2i-1})$, whenever $1 \leq i \leq k$. As $G_{2i}(\alpha^*_{2k}) = G'_{2i}$, we conclude that (5.17e) holds for the groups $P'_{2i}$. Hence Theorem 6.81 is proved. □

For the triangular set (6.82) we can define, as it was described in Section 5.5, the groups $(P'_{2i})^* := P'_2 \cdot P'_4 \cdots P'_{2i}$, along with their irreducible characters $(\alpha'_{2i})^*$ (see Definition 5.147), whenever $1 \leq i \leq k$. Then it is easy to show that

**Proposition 6.86.**

$$(P'_{2i})^* = P^*_{2i},$$

*for every $i = 1, \dots, k$.*

*Proof.* In view of (6.78), it is clear that $P'_{2i} \leq P^*_{2i}$ for all $i = 1, \dots, k$. As $P^*_{2j} \leq P^*_{2i}$, whenever $1 \leq j \leq i$, we conclude that $(P'_{2i})^*$ is a subgroup of $P^*_{2i}$, whenever $1 \leq i \leq k$.

For the other inclusion, note that, according to Lemma 6.79, $P'_{2i} = C(Q_{2i-1,2k} \text{ in } P^*_{2i})$. But $P_{2i}$ is a subgroup of $P^*_{2i}$ and centralizes $Q_{2i-1,2k}$ (see (5.23a)). Hence $P_{2i} \leq C(Q_{2i-1,2k} \text{ in } P^*_{2i}) = P'_{2i}$ whenever $1 \leq i \leq k$. Therefore,

$$P^*_{2i} = P_2 \cdot P_4 \cdots P_{2i} \leq P'_2 \cdot P'_4 \cdots P'_{2i} = (P'_{2i})^*.$$

Hence $P^*_{2i} = (P'_{2i})^*$. □

### 6.4.2 From $G'$ to $G$

Now assume that a triangular set for $G'$ is given. It would be nice if we could pass to a triangular set of $G$ in a "reverse " way to that described in the previous section. This would not only show a path to pass from triangular sets of $G$ to $G'$ and vice versa, but also, as Theorem 5.6 suggests, a path to pass from character towers of (6.77) to character towers of (6.76) and vice versa. We couldn't hope that this would work with every triangular set of $G'$, as, after all, the triangular set that we got in the previous section has a very specific type. That type we try to reproduce in Property 6.89 that follows. In addition, we need an extra asumption for the set of primes $\pi$. We assume that

$$\pi = \{p\} \text{ consists of one prime only.} \tag{6.87}$$

So the various $\pi$-Hall subgroups become $p$-Sylow subgroups, while the $\pi'$-Hall become $p'$-Hall.

Now assume that

$$\{Q'_{2i-1}, P'_{2i} | \beta'_{2i-1}, \alpha'_{2i}\}_{i=1}^k \tag{6.88a}$$

is a triangular set for (6.76), while $Q'$ is any $p'$-subgroup of $G'$ satisfying

$$Q'_{2i-1} \trianglelefteq Q' \leq G'(\alpha'_2, \ldots, \alpha'_{2k}, \beta'_1, \ldots, \beta'_{2k-1}), \tag{6.88b}$$

whenever $1 \leq i \leq k$. Note that $Q'_{2k-1}$ works for $Q'$. Furthermore, we assume that the set (6.88a) satisfies the following property

**Property 6.89.** *For every $i = 1, \ldots, k$ we have*

$$P'_{2i} = N(Q'_{2i-1} \text{ in } P^*_{2i}) = C(Q'_{2i-1} \text{ in } P^*_{2i}) = P^*_{2i}(\beta'_{2i-1}). \tag{6.90a}$$

*In addition,*

$$P'_{2i} \quad \textit{is the unique } p\textit{-Sylow subgroup of } N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \beta'_{2i-1})). \tag{6.90b}$$

*Furthermore,*

$$\alpha'_{2i} \in \mathrm{Irr}(P'_{2i}) \textit{ is the } Q'_{2i-1}\textit{-Glauberman correspondent of } \alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i}). \tag{6.90c}$$

We remark that, as the $p'$-group $Q'_{2i}$ normalizes the $p$-group $P^*_{2i} = P^*_{2k} \cap G_{2i}$, we necessarily have that

$$N(Q'_{2i-1} \text{ in } P^*_{2i}) = C(Q'_{2i-1} \text{ in } P^*_{2i}) = P^*_{2i}(\beta'_{2i-1}),$$

for all $i = 1, \ldots, k$. Thus equation (6.90a) is equivalent to $P'_{2i} = P^*_{2i}(\beta'_{2i-1})$.

Lets see some of the conditions Property 6.89 implies for the triangular set (6.88a). Recall that, for all $i = 1, \ldots, k$, the groups $Q'_{2i-1,2k}$ are defined as $Q'_{2i-1,2k} = C(P'_{2i}, \ldots, P'_{2k} \text{ in } Q'_{2i-1}) = C(P'_{2i} \cdots P'_{2k} \text{ in } Q'_{2i-1})$ (see (5.23a)). Furthermore, for all $i = 1, \ldots, k$, the character $\beta'_{2i-1,2k}$ is the $P'_{2i} \cdots P'_{2k}$-Glauberman correspondent of $\beta'_{2i-1,2k}$, by Definition 5.49.

**Lemma 6.91.** *For every $i = 1, \ldots, k$ we have that $Q'_{2i-1,2k} = Q'_{2i-1}$ while $\beta'_{2i-1,2k} = \beta'_{2i-1}$. Therefore we get*

$$\begin{aligned} &Q'_1 \trianglelefteq Q'_3 \trianglelefteq \cdots \trianglelefteq Q'_{2k-1} \trianglelefteq Q' \textit{ while} \\ &\quad \beta'_{2j-1} \in \operatorname{Irr}(Q'_{2j-1}|\beta'_{2j-3}) \ \textit{ whenever } 2 \le j \le k \ . \end{aligned} \tag{6.92}$$

*Furthermore, whenever $1 \le j \le i \le k$ we get*

$$Q'_{2i-1}(\beta'_{2j-1}) = Q'_{2i-1}, \tag{6.93a}$$

*while*

$$Q'_1 = G'_1 = G_1(\alpha^*_{2k}) = N(P^*_{2k} \text{ in } G_1) = C(P^*_{2k} \text{ in } G_1), \tag{6.93b}$$

$$P'_2 = P^*_2 = P_2, \tag{6.93c}$$

$$P'_{2i} = P^*_{2i}(\beta'_{2i-1}) = P^*_{2i}(\beta'_1, \ldots, \beta'_{2i-1}) = C(Q'_1, \ldots, Q'_{2i-1} \text{ in } P^*_{2i}). \tag{6.93d}$$

*Proof.* According to Property 6.89 the group $P'_{2i}$ centralizes $Q'_{2i-1}$, for each $i = 1, \ldots, k$.. As $Q'_{2i-1,2i} := C(P'_{2i} \text{ in } Q'_{2i-1})$ (see (5.23a)), we conclude that $Q'_{2i-1,2i} = Q'_{2i-1}$. But, according to (5.35), the group $Q'_{2i-1,2i}$ is a normal subgroup of $Q'_{2i+1}$ whenever $1 \le i \le k-1$. Thus $Q'_{2i-1} \trianglelefteq Q'_{2i+1}$ for all such $i$ and the first part of (6.92) is proved.

As $P'_{2i}$ centralizes $Q'_{2i-1}$ and $Q'_{2j-1} \trianglelefteq Q'_{2i-1}$, we conclude that $P'_{2i}$ centralizes $Q'_{2j-1}$ whenever $1 \le j \le i \le k$. Thus $P'_{2i} = C(Q'_{2i-1} \text{ in } P^*_{2i}) = C(Q'_1, \ldots, Q'_{2i-1} \text{ in } P^*_{2i})$. Even more, as $Q'_{2j-1,2i} = C(P'_{2j}, \ldots, P'_{2i} \text{ in } Q'_{2j-1})$ (see (5.23a)), we get that $Q'_{2j-1,2i} = Q'_{2j-1}$ whenever $1 \le j \le i \le k$. Thus $\beta'_{2j-1,2i} = \beta'_{2j-1}$. This, along with (5.51), implies that $\beta'_{2j-1}$ lies above $\beta'_{2j-3}$ whenever $2 \le j \le k$. Hence the rest of (6.92) holds.

Furthermore, as $\beta'_{2j-1,2k} = \beta'_{2j-1}$, Proposition 5.55, for the $t, j, i$ there equal to $i, k, j$ here, implies that $Q'_{2i-1,2k}(\beta'_{2j-1}) = Q'_{2i-1,2k}$. Equation (6.93a) holds, as $Q'_{2i-1,2k} = Q'_{2i-1}$.

The set (6.88a) is a triangular set of (6.76). Hence, (see (5.17b)),

$$Q'_1 = G'_1 = G_1(\alpha^*_{2k}).$$

Therefore, $Q'_1 = N(P^*_{2k} \text{ in } G_1) = C(P^*_{2k} \text{ in } G_1)$, as the $p$-group $P^*_{2k}$ normalizes the $p'$-group $G_1$. So (6.93b) holds. Furthermore, we get that $P_2 = P^*_2$ centralizes $Q'_1$. This implies that $P'_2 = C(Q'_1 \text{ in } P^*_2) = P^*_2 = P_2$. Thus (6.93c) holds.

It remains to show that (6.93d) holds. As $P'_{2i}$ centralizes $Q'_1, \ldots, Q'_{2i-1}$, and is a subgroup of $P^*_{2i}$, we obviously have that $P'_{2i} \le C(Q'_1, \ldots, Q'_{2i-1} \text{ in } P^*_{2i}) \le P^*_{2i}(\beta'_1, \ldots, \beta'_{2i-1}) \le P^*_{2i}(\beta'_{2i-1})$. But $P^*_{2i}(\beta'_{2i-1}) = P'_{2i}$. This completes the proof of the lemma. $\square$

For each $i = 1, \ldots, k$, let $(P'_{2i})^*$ be the product group $(P'_{2i})^* = P'_2 \cdots P'_{2i}$, and $(\alpha'_{2i})^*$ its irreducible character that we get (see Definition 5.147) from the triangular set (6.88a). Then

**Lemma 6.94.** *For every $i = 1, \ldots, k$,*

$$(P'_{2i})^* = P^*_{2i}.$$

*Proof.* We will use induction on $i$. Equation (6.93c) verifies the $i = 1$ case. Assume the lemma is true for all $i = 1, \ldots, n-1$, where $n = 2, 3, \ldots, k$. We will prove it also holds for $i = n$. The group $Q'_{2n-1}$ normalizes $P^*_{2i}$ for all $i = 1, \ldots, k$ (as it normalizes $P^*_{2k}$). Thus $Q'_{2n-1} \ltimes P^*_{2n}$ is a group. Furthermore, the semi–direct product $Q'_{2n-1} \ltimes P^*_{2n-2}$ is a normal subgroup of $Q'_{2n-1} \ltimes P^*_{2n}$, as $Q'_{2n-1} \ltimes P^*_{2n-2} = G_{2n-1} \cap (Q'_{2n-1} \ltimes P^*_{2n})$. Hence Frattini's argument implies that

$$P^*_{2n} = N(Q'_{2n-1} \text{ in } P^*_{2n}) \cdot P^*_{2n-2}.$$

According to the inductive hypothesis $P^*_{2n-2} = (P'_{2n-2})^*$. Even more, in view of (6.90a) we get that $P'_{2n} = N(Q'_{2n-1} \text{ in } P^*_{2n})$. So we conclude that

$$P^*_{2n} = N(Q'_{2n-1} \text{ in } P^*_{2n}) \cdot P^*_{2n-2} = P'_{2n}(P'_{2n-2})^* = (P'_{2n})^*.$$

This completes the proof of the inductive step. Thus Lemma 6.94 holds. $\square$

We can now prove

**Theorem 6.95.** *Assume that the triangular set* (6.88a) *for the series* (6.76) *satisfies Property 6.89, while* (6.88b) *holds for a subgroup $Q'$ of $G'$. Then there exists a collection of groups and characters*

$$\{P^\nu_0, Q^\nu_{2i-1}, P^\nu_{2i} | \, \alpha^\nu_0, \beta^\nu_{2i-1}, \alpha^\nu_{2i}\}_{i=1}^k, \tag{6.96}$$

*with the following properties:*

$$Q' \textit{ normalizes } Q^\nu_{2i-1}, \tag{6.97a}$$

$$\begin{aligned} Q'_{2i-1} = Q^\nu_{2i-1}(\alpha^*_{2k}) = N(N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2i-1}) = \\ C(N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2i-1}) = N(P^*_{2k} \text{ in } Q^\nu_{2i-1}), \end{aligned} \tag{6.97b}$$

$$\beta^\nu_{2i-1} \in \mathrm{Irr}(Q^\nu_{2i-1}) \quad \textit{is the } N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P^*_{2k})\textit{-Glauberman correspondent of } \beta'_{2i-1}, \tag{6.97c}$$

$$P^\nu_{2i} = N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P'_{2i}) = N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P^*_{2i}), \tag{6.97d}$$

$$\begin{aligned} \alpha^\nu_2 = \alpha^*_2 \in \mathrm{Irr}(P^\nu_2), \textit{ while for } i > 1 \\ \alpha^\nu_{2i} \in \mathrm{Irr}(P^\nu_{2i}) \textit{ is the } Q^\nu_3, \ldots, Q^\nu_{2i-1}\textit{-correspondent of } \alpha^*_{2i}, \end{aligned} \tag{6.97e}$$

$$\begin{aligned} Q^\nu_1 := G_1, \quad \textit{while for } i > 2 \\ Q^\nu_{2i-1} \in \mathrm{Hall}_{p'}(G_{2i-1}(\alpha^\nu_2, \ldots, \alpha^\nu_{2i-2}, \beta^\nu_1, \ldots, \beta^\nu_{2i-3})), \end{aligned} \tag{6.97f}$$

$$\begin{aligned} P^\nu_0 := 1 \textit{ and } \alpha^\nu_0 := 1, \textit{ while for } i \geq 1 \\ P^\nu_{2i} \in \mathrm{Syl}_p(G_{2i}(\alpha^\nu_0, \alpha^\nu_2, \ldots, \alpha^\nu_{2i-2}, \beta^\nu_1, \ldots, \beta^\nu_{2i-1})), \end{aligned} \tag{6.97g}$$

$$P^*_{2j} = P^\nu_0 \cdot P^\nu_2 \cdot P^\nu_4 \cdots P^\nu_{2i-2} \cdot N(Q^\nu_1, Q^\nu_3, \ldots, Q^\nu_{2i-1} \text{ in } P^*_{2j}), \tag{6.97h}$$

$$P^*_{2i} = P^\nu_2 \cdot P^\nu_4 \cdots P^\nu_{2i}, \tag{6.97i}$$

*whenever $1 \leq i \leq j \leq k$. In* (6.97e)*, the $Q^\nu_1, \ldots, Q^\nu_{2i-1}$-correspondent refers to the correspondence described in Lemma 5.142 and Theorem 5.143.*

To prove the above theorem we will need the following easy lemma

**Lemma 6.98.** *Assume that $T, S, T_0$ are $p$-subgroups of a finite group $H$ such that $T$ normalizes $S$ while $T_0 \leq T \cap S$. Assume further that $T_0 = N(T \text{ in } S)$. Then $T_0 = S$.*

*Proof.* The group $TS$ is a $p$-group and thus nilpotent. Hence, if $T$ is a proper subgroup of $TS$, then we should have that $N(T \text{ in } TS) > T$.

But $N(T \text{ in } TS) = T \cdot N(T \text{ in } S) = T \cdot T_0$. As $T_0 \leq T$ we conclude that $N(T \text{ in } TS) = T$. Hence $T = TS$ and thus $S \leq T$. Therefore we have that $S = N(T \text{ in } S) = T_0$. $\square$

*Proof of Theorem 6.95.* We define $P_0^\nu := 1$ and $\alpha_0^\nu := 1$, so that the trivial part of (6.97g) holds. We will prove theorem using induction on $i$. Assume that $i = 1$. For $Q_1^\nu$ we take $Q_1^\nu = G_1$. Hence the first part of (6.97f) holds. According to (6.93b) we have that

$$Q_1' = Q_1^\nu(\alpha_{2k}^*) = N(P_{2k}^* \text{ in } Q_1^\nu) = C(P_{2k}^* \text{ in } Q_1^\nu).$$

As $Q_1^\nu = G_1 \trianglelefteq G$ we get that $N(Q_1^\nu \text{ in } P_{2k}^*) = P_{2k}^*$ and that $Q'$ normalizes $Q_1^\nu$. Hence $Q_1^\nu$ satisfies (6.97a,b). Furthermore, $\beta_1'$ is fixed by $P_{2k}^*$, by (6.93d). So we take $\beta_1^\nu \in \operatorname{Irr}^{P_{2k}^*}(Q_1)$ to be the $P_{2k}^*$-Glauberman correspondent of $\beta_1' \in \operatorname{Irr}(Q_1')$. Thus $\beta_1^\nu$ satisfies (6.97c).

Let $P_2^\nu := N(Q_1^\nu \text{ in } P_2^*)$. Then, as $Q_1^\nu = G_1 \trianglelefteq G$ and $P_2^* = P_2'$ (see (6.93c)), we have

$$P_2^\nu = P_2^* = P_2'.$$

So (6.97i) holds. We also define $\alpha_2^\nu := \alpha_2^*$. Thus $P_2^\nu$ and $\alpha_2^\nu$ satisfy (6.97d,e).

Let $\mathcal{P}$ be a $p$-Sylow subgroup of $G_2(\beta_1^\nu)$. Then $G_2(\beta_1^\nu) = \mathcal{P} \ltimes G_1 = \mathcal{P} \ltimes Q_1^\nu$, as $G_2/G_1$ is a $p$-group. Furthermore, $P_{2k}^*$ normalizes $G_2(\beta_1^\nu)$, as it fixes $\beta_1^\nu$. Also, $P_2^* = G_2 \cap P_{2k}^*$ is a subgroup of $G_2(\beta_1^\nu)$. Therefore Lemma 6.59 implies that we can pick $\mathcal{P}$ so that is normalized by $P_{2k}^*$, while $P_2' = P_2^* \leq \mathcal{P}$. So $P_2' \leq \mathcal{P} \cap P_{2k}^*$. The group $N(P_{2k}^* \text{ in } \mathcal{P})$ fixes the $P_{2k}^*$-Glauberman correspondent $\beta_1^\nu$ of $\beta_1'$ (as $\mathcal{P}$ does), and normalizes $P_{2k}^*$. Thus it also fixes $\beta_1'$. Hence $P_2' \leq N(P_{2k}^* \text{ in } \mathcal{P}) \leq N(P_{2k}^* \text{ in } G_2(\beta_1'))$. According to (6.90b), the group $P_2'$ is a $p$-Sylow subgroup of $N(P_{2k}^* \text{ in } G_2(\beta_1'))$. Thus Lemma 6.98 can be applied to the groups $P_{2k}^*, \mathcal{P}$ and $P_2^* = P_2'$ in the place of $T, S$ and $T_0$, respectively. Therefore we get that $\mathcal{P} = P_2' = P_2^*$. As $P_2^\nu := P_2^*$, we conclude that $P_2^\nu \in \operatorname{Syl}_p(G_2(\beta_1^\nu))$. Hence (6.97g) holds.

We complete the proof of the $i = 1$ case by observing that $N(Q_1^\nu \text{ in } P_{2j}^*) = P_{2j}^*$ as $Q_1^\nu = G_1 \trianglelefteq G$. Thus

$$P_0^\nu \cdot N(Q_1^\nu \text{ in } P_{2j}^*) = 1 \cdot P_{2j}^* = P_{2j}^*,$$

whenever $1 \leq j \leq k$. Hence (6.97h) holds.

Now assume Theorem 6.95 holds for all $i = 1, \dots, t-1$, for some $t = 2, \dots, k$. We will prove it also holds for $i = t$. To simplify this proof, we give separately the next steps that depend heavily on the inductive hypothesis,

**Step 1.** *Assume that the set $\{Q_{2i-1}^\nu, P_{2i}^\nu | \beta_{2i-1}^\nu, \alpha_{2i}^\nu\}_{i=1}^s$, for some $s = 1, \dots, k$, satisfies* (6.97c,d,e) *for all $i = 1, \dots, s$. Let $T \leq N(P_{2k}^*, Q_1^\nu, Q_3^\nu, \dots, Q_{2s-1}^\nu \text{ in } G(\alpha_{2s}^*, \alpha_2', \dots, \alpha_{2s}', \beta_1', \dots, \beta_{2s-1}'))$. Then*

$$T \leq G(\alpha_2^\nu, \dots, \alpha_{2s}^\nu, \beta_1^\nu, \dots, \beta_{2s-1}^\nu).$$

*Proof.* The group $T$ normalizes $P_{2r}^*$ for all $r = 1, \dots, s$, as it normalizes $P_{2k}^*$. It also normalizes $Q_{2r-1}^\nu$ for all such $r$. Therefore, it normalizes $N(Q_1^\nu, \dots, Q_{2r-1}^\nu \text{ in } P_{2r}^*)$. But, according to (6.97d), this last group equals $P_{2r}^\nu$ whenever $1 \leq r \leq s$. Hence $T$ normalizes $P_{2r}^\nu$ and $Q_{2r-1}^\nu$. But $T$ also fixes $\alpha_{2r}^*$, as it fixes $\alpha_{2k}^*$ (see Remark 6.1). Therefore (6.97e), along with Proposition 5.149, implies that $T$ fixes $\alpha_{2r}^\nu$ for all $r = 1, \dots, s$.

Furthermore, $T$ fixes $\beta_{2r-1}^\nu$, as it fixes its $N(Q_1^\nu, \dots, Q_{2r-1}^\nu \text{ in } P_{2k}^*)$-Glauberman correspondent $\beta_{2r-1}'$ (see (6.97c)) and normalizes $P_{2k}^*, Q_1^\nu, \dots, Q_{2r-1}^\nu$, whenever $1 \leq r \leq s$. Therefore

$$T \leq G(\alpha_2^\nu, \dots, \alpha_{2s}^\nu, \beta_1^\nu, \dots, \beta_{2s-1}^\nu),$$

and Step 1 is complete. □

The second step is

**Step 2.** *Assume that the set* $\{Q^{\nu}_{2i-1}, P^{\nu}_{2i} | \beta^{\nu}_{2i-1}, \alpha^{\nu}_{2i}\}_{i=1}^{s}$, *for some* $s = 1, \dots, k$, *satisfies* (6.97b,c,e) *for all* $i = 1, \dots, s$. *Let* $T \leq N(P^*_{2k} \text{ in } G(\alpha^{\nu}_2, \dots, \alpha^{\nu}_{2s}, \beta^{\nu}_1, \dots, \beta^{\nu}_{2s-1}))$. *Then*

$$T \leq N(P^*_{2k} \text{ in } G(\alpha^*_{2s}, \alpha'_2, \dots, \alpha'_{2s}, \beta'_1, \dots, \beta'_{2s-1})).$$

*Proof.* The group $T$ normalizes $P^*_{2r}$ for all $r = 1, \dots, s$ (even for all $r = s+1, \dots, k$, but this we will not need). As $T$ also normalizes the groups $Q^{\nu}_{2r-1}$, and fixes the characters $\alpha^{\nu}_{2r}$, it has to fix (by Proposition 5.149) the $Q^{\nu}_3, \dots, Q^{\nu}_{2r-1}$-correspondent $\alpha^*_{2r}$ of $\alpha^{\nu}_{2r}$ (see (6.97e)) for all $r = 2, \dots, s$. It also fixes $\alpha^*_2 = \alpha^{\nu}_2$.

Furthermore, $T$ normalizes $Q'_{2r-1}$, as (6.97b) implies that $Q'_{2r-1} = N(P^*_{2k} \text{ in } Q^{\nu}_{2r-1})$, whenever $1 \leq r \leq s$. The group $P'_{2r}$ satisfies (6.90a) for $i = r$. Hence $P'_{2r} = N(Q'_{2r-1} \text{ in } P^*_{2r})$. Therefore, $T$ normalizes $P'_{2r}$, as it normalizes both $Q'_{2r-1}$ and $P^*_{2r}$. This, along with the fact that $T$ fixes $\alpha^*_{2r}$, implies that $T$ fixes the $Q'_{2r-1}$-Glauberman correspondent $\alpha'_{2r}$ (see (6.90c)) of $\alpha^*_{2r}$, for all $r = 1, \dots, s$.

Even more, as $T$ fixes $\beta^{\nu}_{2r-1}$ and normalizes $Q'_{2r-1}$, it must fix the $N(Q^{\nu}_1, \dots, Q^{\nu}_{2r-1} \text{ in } P^*_{2k})$-Glauberman correspondent $\beta'_{2r-1}$ of $\beta^{\nu}_{2r-1}$ (see (6.97c)). Hence

$$T \leq N(P^*_{2k} \text{ in } G(\alpha^*_{2s}, \alpha'_2, \dots, \alpha'_{2s}, \beta'_1, \dots, \beta'_{2s-1})).$$

□

The last step is

**Step 3.** *The group* $S := N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-3} \text{ in } P^*_{2k})$ *is a subgroup of* $G(\alpha^{\nu}_2, \dots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \dots, \beta^{\nu}_{2t-3})$.

*Proof.* For every $r = 1, \dots, t-1$, the group $S = N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-3} \text{ in } P^*_{2k})$ normalizes $P^{\nu}_{2r} = N(Q^{\nu}_1, \dots, Q^{\nu}_{2r-1} \text{ in } P^*_{2r})$. Also $S$ fixes $\alpha^*_{2r}$, as it is a subgroup of $P^*_{2k}$. Therefore $S$ fixes the $Q^{\nu}_3, \dots, Q^{\nu}_{2r-1}$-correspondent $\alpha^{\nu}_{2r} \in \operatorname{Irr}(P^{\nu}_{2r})$ of $\alpha^*_{2r}$, (see (6.97e)), for all $r = 2, \dots, t-1$, as well as $\alpha^{\nu}_2 = \alpha^*_2$.

Furthermore, (6.97c) for $i = t-1$ implies that $S$ fixes $\beta^{\nu}_{2t-3}$. Similarly, the inductive hypothesis for (6.97c) implies that $N(Q^{\nu}_1, \dots, Q^{\nu}_{2r-1} \text{ in } P^*_{2k})$ fixes $\beta^{\nu}_{2r-1}$, for all $r = 1, \dots, t-2$. But $S$ is a subgroup of $N(Q^{\nu}_1, \dots, Q^{\nu}_{2r-1} \text{ in } P^*_{2k})$ for all such $r$. Hence $S$ fixes $\beta^{\nu}_{2r-1}$ whenever $1 \leq r \leq t-1$.

Therefore $S \leq G(\alpha^{\nu}_2, \dots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \dots, \beta^{\nu}_{2t-3})$, and Step 3 is proved. □

We can now continue with the proof of the theorem. The fact that (6.88a) is a triangular set for (6.76) implies that

$$\begin{aligned} Q'_{2s-1} \leq G_{2s-1}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2t-2}, \dots, \alpha'_{2s-2}, \beta'_1, \dots, \beta'_{2t-3}, \dots, \beta'_{2s-3}) \leq \\ G_{2s-1}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2t-2}, \beta'_1, \dots, \beta'_{2t-3}), \end{aligned}$$

for all $s = t, \dots, k$. Also the inductive hypothesis, 6.97a, for $i \leq t-1$ implies that $Q'_{2s-1} \leq Q'$ normalizes the groups $Q^{\nu}_1, \dots, Q^{\nu}_{2t-3}$. Hence for all $s = t, \dots, k$ we have that

$$\begin{aligned} Q'_{2s-1} \leq N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-3} \text{ in } G_{2s-1}(\alpha^*_{2k}, \alpha'_2, \dots, \alpha'_{2t-2}, \beta'_1, \dots, \beta'_{2t-3}) \leq \\ N(P^*_{2k}, Q^{\nu}_1, \dots, Q^{\nu}_{2t-3} \text{ in } G_{2s-1}(\alpha^*_{2t-2}, \alpha'_2, \dots, \alpha'_{2t-2}, \beta'_1, \dots, \beta'_{2t-3})), \end{aligned}$$

and

$$Q' \leq N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } G(\alpha_{2k}^*, \alpha_2', \dots, \alpha_{2t-2}', \beta_1', \dots, \beta_{2t-3}') \leq \\ N(P_{2k}^*, Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } G(\alpha_{2t-2}^*, \alpha_2', \dots, \alpha_{2t-2}', \beta_1', \dots, \beta_{2t-3}')).$$

This, along with Step 1, with the present $t-1$ in the place of $s$ there, and the fact that $G_{2t-1} \trianglelefteq G_{2s-1}$, implies that both $Q_{2s-1}'$ and $Q'$ normalize the group $G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu)$ and fix the characters $\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu$. In particular, for $s = t$ we get

$$Q_{2t-1}' \leq G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu), \tag{6.99a}$$

as $Q_{2t-1}' \leq G_{2t-1}$. Furthermore,

$$Q' \text{ normalizes } G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu). \tag{6.99b}$$

Let $\mathcal{Q}$ be a $p'$-Hall subgroup of $G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu)$. Since $P_{2t-2}^\nu$ satisfies (6.97g) for $i = t-1$, and since $\alpha_{2t-2}^\nu \in \mathrm{Irr}(P_{2t-2}^\nu)$, we have that

$$P_{2t-2}^\nu \in \mathrm{Syl}_p(G_{2t-2}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu)).$$

As $G_{2t-1}/G_{2t-2}$ is a $p'$-group and $G_{2t-2}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu)$ normalizes $P_{2t-2}^\nu$, we get

$$G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu) = \mathcal{Q} \ltimes P_{2t-2}^\nu. \tag{6.100}$$

This, along with (6.99), the fact that $Q_{2t-1}' \trianglelefteq Q'$ (see (6.88b)), and Lemma 6.59, implies that we can pick a conjugate $Q_{2t-1}^\nu := \mathcal{Q}^s$ of $\mathcal{Q}$, so that

$$\begin{gathered} Q_{2t-1}^\nu \in \mathrm{Hall}_{p'}(G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu)), \\ Q' \text{ normalizes } Q_{2t-1}^\nu \text{ and} \\ Q_{2t-1}' \leq Q_{2t-1}^\nu. \end{gathered} \tag{6.101}$$

It is obvious from the definition of $Q_{2t-1}^\nu$ that it satisfies (6.97a,f) for $i = t$. Furthermore, (6.100) holds for $\mathcal{Q} = Q_{2t-1}^\nu$, i.e.,

$$G_{2t-1}(\alpha_2^\nu, \dots, \alpha_{2t-2}^\nu, \beta_1^\nu, \dots, \beta_{2t-3}^\nu) = Q_{2t-1}^\nu \ltimes P_{2t-2}^\nu. \tag{6.102}$$

This, along with Step 3, implies that $N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*)$ normalizes $Q_{2t-1}^\nu \ltimes P_{2t-2}^\nu$. Hence the product $N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*) \cdot Q_{2t-1}^\nu P_{2t-2}^\nu$ is a group having $Q_{2t-1}^\nu \ltimes P_{2t-2}^\nu$ as a normal subgroup. Furthermore, (6.97d) for $i = t-1$ implies that $P_{2t-2}^\nu \leq N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*)$. Hence $N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*)$ is a $p$-Sylow subgroup of $N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*) \cdot Q_{2t-1}^\nu P_{2t-2}^\nu$. Thus Frattini's argument for the $p'$-Hall subgroup $Q_{2t-1}^\nu$ of the normal subgroup $Q_{2t-1}^\nu \ltimes P_{2t-2}^\nu$ implies that

$$N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*) = P_{2t-2}^\nu \cdot N(Q_{2t-1}^\nu \text{ in } N(Q_1^\nu, \dots, Q_{2t-3}^\nu \text{ in } P_{2k}^*)) = \\ P_{2t-2}^\nu \cdot N(Q_1^\nu, \dots, Q_{2t-3}^\nu, Q_{2t-1}^\nu \text{ in } P_{2k}^*).$$

This, along with (6.97h) for $i = t-1$ and $j = k$, implies that

$$P^*_{2k} = P^\nu_2 \cdots P^\nu_{2t-4} \cdot N(Q^\nu_1, \ldots, Q^\nu_{2t-3} \text{ in } P^*_{2k}) = \\ P^\nu_2 \cdots P^\nu_{2t-4} \cdot P^\nu_{2t-2} \cdot N(Q^\nu_1, \ldots, Q^\nu_{2t-3}, Q^\nu_{2t-1} \text{ in } P^*_{2k}).$$

Therefore, intersecting both sides of the above equation with $G_{2j}$, we get

$$P^*_{2j} = P^\nu_2 \cdots P^\nu_{2t-2} \cdot N(Q^\nu_1, \ldots, Q^\nu_{2t-3}, Q^\nu_{2t-1} \text{ in } P^*_{2j}), \tag{6.103}$$

whenever $t \leq j \leq k$. Hence (6.97h) holds for $i = t$ and $j = i, i+1, \ldots, k$.

To prove (6.97b) for $i = t$, we first note that, according to the definition of $Q^\nu_{2t-1}$ (see (6.101)), we have $Q'_{2t-1} \leq Q^\nu_{2t-1}$. Hence $Q'_{2t-1} \leq Q^\nu_{2t-1}(\alpha^*_{2k})$, as $Q'_{2t-1} \leq G' = G(\alpha^*_{2k})$. Furthermore, $Q^\nu_{2t-1}$ normalizes $Q^\nu_1, \ldots, Q^\nu_{2t-3}, Q^\nu_{2t-1}$. Hence

$$N(P^*_{2k} \text{ in } Q^\nu_{2t-1}) \leq N(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}) = C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}),$$

where the last equality holds as the $p$-group $N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k})$ normalizes the $p'$-group $Q^\nu_{2t-1}$. Thus we have

$$Q'_{2t-1} \leq Q^\nu_{2t-1}(\alpha^*_{2k}) \leq N(P^*_{2k} \text{ in } Q^\nu_{2t-1}) \leq \\ N(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}) = \\ C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}). \tag{6.104}$$

Let $T = C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1})$. Then $T$ normalizes the groups $P^\nu_2, \ldots, P^\nu_{2t-2}$, as $Q^\nu_{2t-1}$ does (it fixes their characters $\alpha^\nu_{2i}$). Hence, $T$, in view of (6.103) for $j = k$, also normalizes $P^*_{2k}$. Therefore, $T \leq N(P^*_{2k} \text{ in } Q^\nu_{2t-1})$. This , in view of (6.101), implies that

$$T \leq N(P^*_{2k} \text{ in } G_{2t-1}(\alpha^\nu_2, \ldots, \alpha^\nu_{2t-2}, \beta^\nu_1, \ldots, \beta^\nu_{2t-3})).$$

The set $\{Q^\nu_{2i-1}, P^\nu_{2i} | \beta^\nu_{2i-1}, \alpha^\nu_{2i}\}_{i=1}^{t-1}$ satisfies (6.97b,c,e) (according to the inductive hypothesis). So Step 2 for $s = t-1$, implies that $T$ satisfies

$$T \leq N(P^*_{2k} \text{ in } G_{2t-1}(\alpha^*_{2t-2}, \alpha'_2, \ldots, \alpha'_{2t-2}, \beta'_1, \ldots, \beta'_{2t-3})). \tag{6.105}$$

Equation (6.103) for $j = k$, along with (6.97i) for $i = t-1$, implies that

$$\frac{P^*_{2k}}{P^*_{2t-2}} \cong \frac{N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k})}{P^*_{2t-2} \cap N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k})}.$$

Therefore $T$ centralizes $P^*_{2k}/P^*_{2t-2}$, as it centralizes $N(Q^\nu_1, \ldots, Q^\nu_{2t-3} \text{ in } P^*_{2k})$. Also $T$ fixes $\alpha^*_{2t-2} \in \mathrm{Irr}(P^*_{2t-2})$, and is a $p'$-group. Hence (see Exercise 13.13 in [12]), $T$ fixes every irreducible character of $P^*_{2k}$ that lies above $\alpha^*_{2t-2}$. Thus $T$ fixes $\alpha^*_{2k}$. This, along with (6.105), implies that

$$T = C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}) \leq G_{2t-1}(\alpha^*_{2k}, \alpha'_2, \ldots, \alpha'_{2t-2}, \beta'_1, \ldots, \beta'_{2t-1}).$$

But $Q'_{2t-1}$ is a $p'$-Hall subgroup of $G_{2t-1}(\alpha^*_{2k}, \alpha'_2, \ldots, \alpha'_{2t-2}, \beta'_1, \ldots, \beta'_{2t-1})$, as (6.88a) is a triangular set for (6.76). Furthermore, (6.104) implies that the $p'$-group $Q'_{2t-1}$ is contained in $C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1})$. Thus $C(N(Q^\nu_1, \ldots, Q^\nu_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^\nu_{2t-1}) = Q'_{2t-1}$. This, along with (6.104), implies that

$$\begin{aligned} Q'_{2t-1} = Q^{\nu}_{2t-1}(\alpha^*_{2k}) = N(P^*_{2k} \text{ in } Q^{\nu}_{2t-1}) = \\ N(N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^{\nu}_{2t-1}) = \\ C(N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^{\nu}_{2t-1}). \end{aligned} \tag{6.106}$$

So (6.97b) holds for $i = t$. Hence we have shown that the group $Q^{\nu}_{2t-1}$ satisfies (6.97a,b,f,h) for $i = t \leq j \leq k$.

As $Q'_{2t-1} = C(N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2k}) \text{ in } Q^{\nu}_{2t-1})$ we can define $\beta^{\nu}_{2t-1} \in \mathrm{Irr}(Q^{\nu}_{2t-1})$ to be the $N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2k})$-Glauberman correspondent of $\beta'_{2t-1} \in \mathrm{Irr}(Q'_{2t-1})$. Thus $\beta^{\nu}_{2t-1}$ satisfies (6.97c) for $i = t$.

To complete the inductive step it remains to prove that we can pick a $p$-group $P^{\nu}_{2t}$, along with its irreducible character $\alpha^{\nu}_{2t}$, so that (6.97d,e,g,i) hold for $i = t$. In view of (6.106) we have that $Q'_{2t-1} = Q^{\nu}_{2t-1}(\alpha^*_{2k})$. Hence $N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2t})$ normalizes $Q'_{2t-1}$ . This, along with the fact that $P'_{2t} = N(Q'_{2t-1} \text{ in } P^*_{2t})$ by (6.90b), implies that

$$\begin{aligned} N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2t}) = N(Q'_{2t-1} \text{ in } N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2t})) = \\ N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } N(Q'_{2t-1} \text{ in } P^*_{2t})) = N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P'_{2t}). \end{aligned}$$

Let

$$\begin{aligned} M_0 := N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P'_{2t}) = N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2t}) \text{ and} \\ M := N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1} \text{ in } P^*_{2k}). \end{aligned} \tag{6.107}$$

Note that $M \cap G_{2t} = M_0$, as $P^*_{2k} \cap G_{2t} = P^*_{2t}$.

In view of Step 3, the group $M$ fixes the characters $\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-3}$ as it is a subgroup of $N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-3} \text{ in } P^*_{2k})$.Furthermore, the definition of $\beta^{\nu}_{2t-1}$ (as the $M$-correspondent of $\beta'_{2t-1}$) implies that $M$ also fixes $\beta^{\nu}_{2t-1}$. Hence $M$ normalizes $G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-1})$, while $M_0 = M \cap G_{2t}$ is a subgroup of $G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-1})$. Let $\mathcal{P}$ be a $p$-Sylow subgroup of $G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-1})$, chosen so that $\mathcal{P}$ contains $M_0$. It is clear from the fact that $G_{2t}/G_{2t-1}$ is a $p$-group, and the definition of $Q^{\nu}_{2t-1}$ (see (6.101)), that

$$G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-1}) = \mathcal{P} \ltimes Q^{\nu}_{2t-1}. \tag{6.108}$$

Therefore Lemma 6.11 implies that there exists a $Q^{\nu}_{2t-1}$-conjugate of $\mathcal{P}$ that is normalized by $M$ and contains $M_0$. So we may replace $\mathcal{P}$ by this conjugate and assume that $M_0 \leq M \cap \mathcal{P}$.

We can show the following

**Claim 6.109.** $N(M \text{ in } \mathcal{P}) = M_0$.

*Proof.* It is obvious that $M_0 \leq N(M \text{ in } \mathcal{P})$. For the other inclusion we first note that $N(M \text{ in } \mathcal{P})$ normalizes $P^{\nu}_2, \ldots, P^{\nu}_{2t-2}$ (since $\mathcal{P}$ does) and $M$. Hence $N(M \text{ in } \mathcal{P})$ normalizes $P^*_{2k} = P^{\nu}_2 \cdots P^{\nu}_{2t-2} \cdot M$ (see (6.103)). Hence

$$\begin{aligned} N(M \text{ in } \mathcal{P}) &\leq N(P^*_{2k} \text{ in } \mathcal{P}) \\ &\leq N(P^*_{2k} \text{ in } G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-3}, \beta^{\nu}_{2t-1})) \\ &\leq N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1}, P^*_{2k} \text{ in } G_{2t}(\alpha^{\nu}_2, \ldots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \ldots, \beta^{\nu}_{2t-3}))(\beta^{\nu}_{2t-1}) \\ &\leq N(Q^{\nu}_1, \ldots, Q^{\nu}_{2t-1}, P^*_{2k} \text{ in } G_{2t}(\alpha^*_{2t-2}, \alpha'_2, \ldots, \alpha'_{2t-2}, \beta'_1, \ldots, \beta'_{2t-3}))(\beta^{\nu}_{2t-1}), \end{aligned} \tag{6.110}$$

where the last inclusion holds according to Step 2 for $s = t - 1$.

Also $N(M \text{ in } \mathcal{P})$ fixes the $M$-Glauberman correspondent $\beta'_{2t-1}$ of $\beta^{\nu}_{2t-1}$, as it fixes $\beta^{\nu}_{2t-1}$. This, along with (6.110), implies that

$$N(M \text{ in } \mathcal{P}) \leq N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-1}, P^*_{2k} \text{ in } G_{2t}(\alpha^*_{2t-2}, \beta'_{2t-1})). \tag{6.111}$$

But $P'_{2t}$ satisfies (6.90b) for $i = t$. Therefore $M_0 = N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-1} \text{ in } P'_{2t})$ is a $p$-Sylow subgroup of $N(Q^{\nu}_1, \dots, Q^{\nu}_{2t-1}, P^*_{2k} \text{ in } G_{2t}(\alpha^*_{2t-2}, \beta'_{2t-1}))$. As $M_0$ is contained in $N(M \text{ in } \mathcal{P})$, inclusion (6.111) implies that $M_0 = N(M \text{ in } \mathcal{P})$. Hence the claim follows. □

The groups $M, M_0$ and $\mathcal{P}$ satisfy the hypothesis of Lemma 6.98, in the place of $T, T_0$ and $S$ respectively. So we conclude that $M_0 = \mathcal{P}$. Therefore, $M_0$ is a $p$-Sylow subgroup of the group $G_{2t}(\alpha^{\nu}_2, \dots, \alpha^{\nu}_{2t-2}, \beta^{\nu}_1, \dots, \beta^{\nu}_{2t-1})$. If we define $P^{\nu}_{2t} := M_0$, then it is clear that $P^{\nu}_{2t}$ satisfies (6.97g) for $i = t$. It also satisfies (6.97d) for $i = t$, as (6.107) shows. Equation (6.97i) for $i = t$ follows clearly from (6.97h) for $i = j = t$, (that we have already proved in (6.103)) and (6.97d) for $i = t$.

To complete the inductive step, it remains to show that we can pick a character $\alpha^{\nu}_{2t} \in \mathrm{Irr}(P^{\nu}_{2t})$ that satisfies (6.97e) for $i = t$. That is, it suffices to show that the character $\alpha^*_{2t}$ has a $Q^{\nu}_3, \dots, Q^{\nu}_{2t-1}$-correspondent. Looking at Lemma 5.142 and Theorem 5.143, where this correspondence is described, we observe that it is enought to prove the following for every $i, j$ with $2 \leq j < i \leq t$:

[1 ] $Q^{\nu}_{2j-1} \cdot P^{\nu}_{2j-2} \cdot P^{\nu}_{2j} \cdots P^{\nu}_{2i}$ is a group containing $Q^{\nu}_{2j-1} \cdot P^{\nu}_{2j-2}$ and $P^{\nu}_{2j-2}$ as normal subgroups.

[2 ] $N(Q^{\nu}_{2j-1} \text{ in } P^{\nu}_{2j-2} \cdot P^{\nu}_{2j} \cdots P^{\nu}_{2i}) = P^{\nu}_{2j} \cdots P^{\nu}_{2i}$.

[3 ] $\alpha^*_{2t}$ satisfies Property 5.145, i.e., there exist characters $\alpha^*_{2s,1} \in \mathrm{Irr}(P^*_{2s})$, for $s = 1, \dots, t$, such that $\alpha^*_{2t,1} = \alpha^*_{2t}$, and, if $s < t$, then $\alpha^*_{2s,1}$ is $Q^{\nu}_{2s+1}$-invariant and lies under $\alpha^*_{2s+2,1}$.

Part [1] is clear as, according to (6.97e,f), for every $s = j+1, \dots, i$ the group $P^{\nu}_{2s}$ normalizes $P^{\nu}_{2j-2}, \dots, P^{\nu}_{2s-2}$ and $Q^{\nu}_{2j-1}$, while $Q^{\nu}_{2j-1}$ normalizes $P^{\nu}_{2j-2}$. This remark also implies that the product $P^{\nu}_{2j} \cdots P^{\nu}_{2i}$ normalizes $Q^{\nu}_{2j-1}$. Hence

$$N(Q^{\nu}_{2j-1} \text{ in } P^{\nu}_{2j-2} \cdot P^{\nu}_{2j} \cdots P^{\nu}_{2i}) \geq P^{\nu}_{2j} \cdots P^{\nu}_{2i}. \tag{6.112}$$

But

$$\begin{aligned}
N(Q^{\nu}_{2j-1} \text{ in } P^{\nu}_{2j-2}) &= N(Q^{\nu}_{2j-1} \text{ in } N(Q^{\nu}_1, \dots, Q^{\nu}_{2j-3} \text{ in } P^*_{2j-2})) && \text{by (6.97d) for } j-1 \text{ as } i \text{ there} \\
&= N(Q^{\nu}_1, \dots, Q^{\nu}_{2j-3}, Q^{\nu}_{2j-1} \text{ in } P^*_{2j-2}) \\
&\leq N(Q^{\nu}_1, \dots, Q^{\nu}_{2j-3}, Q^{\nu}_{2j-1} \text{ in } P^*_{2j}) \\
&= P^{\nu}_{2j} && \text{by (6.97d) for } j \text{ as } i \text{ there}
\end{aligned}$$

This, along with (6.112), implies that $N(Q^{\nu}_{2j-1} \text{ in } P^{\nu}_{2j-2} \cdot P^{\nu}_{2j} \cdots P^{\nu}_{2i}) = P^{\nu}_{2j} \cdots P^{\nu}_{2i}$. So [2] follows.

Part [3] holds if we take the characters $\alpha^*_{2s}$ in the place of $\alpha^*_{2s,1}$ for $s = 1, \dots, t$. We only need to verify that $Q^{\nu}_{2s+1}$ leaves $\alpha^*_{2s}$ invariant for every $s = 1, \dots, t-1$. This is clear as $Q^{\nu}_{2s+1}$ fixes $\alpha^{\nu}_{2s}$ and normalizes the groups $Q^{\nu}_3, \dots, Q^{\nu}_{2s-1}$ (see (6.97f) with $s-1$ as the $i$ there). So it has to fix the $Q^{\nu}_3, \dots, Q^{\nu}_{2s-1}$-correspondent $\alpha^*_{2s}$ (see (6.97e) with $s$ as the $i$ there) of $\alpha^{\nu}_{2s}$. Hence [3] follows. This proves that Lemma 5.142 and Theorem 5.143 can be applied. Therefore there exists a unique character $\alpha^{\nu}_{2t} \in \mathrm{Irr}(P^{\nu}_{2t})$ that is the $Q^{\nu}_3, \dots, Q^{\nu}_{2t-1}$-correspondent of $\alpha^*_{2t}$. Thus (6.97e) holds for $i = t$.

This completes the proof of the inductive step for $i = t$. Hence Theorem 6.95 holds. □

A useful consequence of Theorem 6.95 is

**Corollary 6.113.** *For every $i = 1, \dots, k$ we have*

$$P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k} = N(Q^{\nu}_1, Q^{\nu}_3, \dots, Q^{\nu}_{2i-1} \text{ in } P^{*}_{2k}). \tag{6.114}$$

*Therefore $\beta^{\nu}_{2i-1} \in \mathrm{Irr}(Q^{\nu}_{2i-1})$ is the $P^{\nu}_{2i} \cdots P^{\nu}_{2k}$-Glauberman correspondent of $\beta'_{2i-1} \in \mathrm{Irr}(Q'_{2i-1}) = \mathrm{Irr}(C(P^{\nu}_{2i} \cdots P^{\nu}_{2k} \text{ in } Q^{\nu}_{2i-1}))$.*

*Proof.* For every $j = i, \dots, k$, the group $P^{\nu}_{2j}$ normalizes $Q^{\nu}_1, Q^{\nu}_3, \dots, Q^{\nu}_{2i-1}$ (see (6.97d)). This, along with (6.97i) and (6.97d), implies that

$$\begin{aligned} P^{\nu}_{2i} \cdots P^{\nu}_{2k} \leq N(Q^{\nu}_1, \dots, Q^{\nu}_{2i-1} \text{ in } P^{*}_{2k}) = N(Q^{\nu}_1, \dots, Q^{\nu}_{2i-1} \text{ in } P^{*}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k}) \\ N(Q^{\nu}_1, \dots, Q^{\nu}_{2i-1} \text{ in } P^{*}_{2i}) \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k} = P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k}. \end{aligned}$$

Thus (6.114) holds. The rest of the corollary is an obvious consequence of (6.114), (6.97b) and (6.97c). ☐

We have done all the neccesary work towards the proof of the main theorem of this section which is a "mirror" of Theorem 6.81. That is

**Theorem 6.115.** *The set* (6.96)*, constructed in Theorem 6.95, forms a triangular set for* (6.77)*. Furthermore, the pair* (6.88a, 6.96) *satisfies* (6.78) *and Definition 6.80, i.e.,*

$$Q'_{2i-1} = Q^{\nu}_{2i-1,2k} \text{ and } \beta'_{2i-1} = \beta^{\nu}_{2i-1,2k}, \tag{6.116a}$$

$$P^{\nu,*}_{2i} = P^{*}_{2i} = (P'_{2i})^{*} \text{ and } P'_{2i} = P^{\nu,*}_{2i}(\beta^{\nu}_{2i-1,2k}), \tag{6.116b}$$

$$\alpha^{\nu,*}_{2i} = \alpha^{*}_{2i}, \tag{6.116c}$$

$$\alpha'_{2i} \text{ is the } Q'_{2i-1}\text{-Glauberman correspondent of } \alpha^{\nu,*}_{2i} \in \mathrm{Irr}(P^{\nu,*}_{2i}), \tag{6.116d}$$

*where $P^{\nu,*}_{2i} := (P^{\nu}_{2i})^{*} = P^{\nu}_{2} \cdots P^{\nu}_{2i}$ and $\alpha^{\nu,*}_{2i} := (\alpha^{\nu}_{2i})^{*}$, whenever $1 \leq i \leq k$.*

*Proof.* The Properties (6.97j, g, f) of that the set (6.96) imply immediately parts (5.17a,b,c,e) of the definition of a triangular set.

Let $\alpha^{\nu}_{2i-2,2i-1}$ denote the irreducible character of $P^{\nu}_{2i-2,2i-1} := C(Q^{\nu}_{2i-1} \text{ in } P^{\nu}_{2i-2})$ that is the $Q^{\nu}_{2i-1}$-Glauberman correspondent of $\alpha^{\nu}_{2i-2} \in \mathrm{Irr}^{Q^{\nu}_{2i-1}}(P^{\nu}_{2i-2})$, whenever $i = 2, \dots, k$. As $Q^{\nu}_{2i-1}$ normalizes $P^{\nu}_{2i-1}$, the $Q^{\nu}_{2i-1}$-Glauberman correspondence coincides with the $Q^{\nu}_{2i-1}$-correspondence between $\mathrm{Irr}^{Q^{\nu}_{2i-1}}(P^{\nu}_{2i-2})$ and $\mathrm{Irr}(P^{\nu}_{2i-2,2i-1}) = \mathrm{Irr}(C(Q^{\nu}_{2i-1} \text{ in } P^{\nu}_{2i-2}))$, by Theorem 3.13. This, along with (6.97e), implies that $\alpha^{\nu}_{2i-2,2i-1}$ is the $Q^{\nu}_3, \dots, Q^{\nu}_{2i-3}, Q^{\nu}_{2i-1}$-correspondent of $\alpha^{*}_{2i-2}$. (Note that in the case of $\alpha^{\nu}_{2,3}$ we only have a $Q^{\nu}_3$-correspondence.) Since $\alpha^{\nu}_{2i}$ is also the $Q^{\nu}_3, \dots, Q^{\nu}_{2i-1}$-correspondent of $\alpha^{*}_{2i}$, while $\alpha^{*}_{2i}$ lies above $\alpha^{*}_{2i-2}$, we conclude that $\alpha^{\nu}_{2i} \in \mathrm{Irr}(P^{\nu}_{2i})$ lies above $\alpha^{\nu}_{2i-2,2i-1}$, whenever $1 \leq i \leq k$. This proves that the set (6.96) satisfies (5.17d).

We will work similarly to prove (5.17f), using Corollary 6.113. For every $i = 2, \dots, k-1$, the character $\beta^{\nu}_{2i-3,2i-2} \in \mathrm{Irr}(Q^{\nu}_{2i-3,2i-2}) = \mathrm{Irr}(C(P^{\nu}_{2i-2} \text{ in } Q^{\nu}_{2i-3}))$ is defined as the $P^{\nu}_{2i-2}$-Glauberman correspondent of $\beta^{\nu}_{2i-3}$ in $\mathrm{Irr}(Q^{\nu}_{2i-3})$. The $P^{\nu}_{2i-2}, P^{\nu}_{2i}, \dots, P^{\nu}_{2k}$-Glauberman correspondent of $\beta'_{2i-3} \in \mathrm{Irr}(Q'_{2i-3}) = \mathrm{Irr}(C(P^{\nu}_{2i-2}, P^{\nu}_{2i}, \dots, P^{\nu}_{2k} \text{ in } Q^{\nu}_{2i-3}))$ is the character $\beta^{\nu}_{2i-3}$, by Corollary 6.113. Hence $\beta^{\nu}_{2i-3,2i-2}$ is the $P^{\nu}_{2i}, \dots, P^{\nu}_{2k}$-Glauberman correspondent of $\beta'_{2i-3}$. Furthermore, $\beta^{\nu}_{2i-1}$ is the $P^{\nu}_{2i}, \dots, P^{\nu}_{2k}$-Glauberman correspondent of $\beta'_{2i-1}$. As $\beta'_{2i-3}$ lies under $\beta'_{2i-1}$, by Lemma 6.91, we conclude that $\beta^{\nu}_{2i-3,2i-2}$ also lies under $\beta^{\nu}_{2i-1}$. This completes the proof of (5.17f), showing that the set (6.96) is a triangular set for (6.77). Hence all the notation and the properties described in Chapter 5 can be applied to this triangular set.

According to (5.33), the group $Q^{\nu}_{2i-1,2k}$ equals $C(P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k} \text{ in } Q^{\nu}_{2i-1})$. This, along with Corollary 6.113 and (6.97b), implies

$$Q^{\nu}_{2i-1,2k} = C(P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k} \text{ in } Q^{\nu}_{2i-1}) = C(N(Q^{\nu}_{1}, \dots, Q^{\nu}_{2i-1} \text{ in } P^{*}_{2k}) \text{ in } Q^{\nu}_{2i-1}) = Q'_{2i-1}.$$

Furthermore, according to the Definition 5.49, the character $\beta^{\nu}_{2i-1,2k} \in \mathrm{Irr}(Q^{\nu}_{2i-1,2k})$ is the $P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k}$-Glauberman correspondent of $\beta^{\nu}_{2i-1}$. Therefore it coincides with $\beta'_{2i-1}$, as the latter is also the $P^{\nu}_{2i} \cdots P^{\nu}_{2k}$-Glauberman correspondent of $\beta^{\nu}_{2i-1}$, by Corollary 6.113. Hence (6.116a) holds.

If $P^{\nu,*}_{2i}$ denotes the product of the $P^{\nu}$-groups, i.e., $P^{\nu,*}_{2i} := P^{\nu}_{2} \cdots P^{\nu}_{2i}$, then in view of (6.97i) we get that $P^{\nu,*}_{2i} = P^{*}_{2i}$, for all $i = 1, \dots, k$. This, along with Proposition 6.86, implies the first part of (6.116b). The second part follows easily from the first and the facts that $P'_{2i} = P^{*}_{2i}(\beta'_{2i-1})$ (see (6.93d)) while $\beta'_{2i-1} = \beta^{\nu}_{2i-1,2k}$ (see (6.116a)).

The character $\alpha^{\nu,*}_{2i} \in \mathrm{Irr}(P^{\nu,*}_{2i}) = \mathrm{Irr}(P^{*}_{2i})$ is constructed as the $Q^{\nu}_{3}, \dots, Q^{\nu}_{2i-1}$-correspondent of $\alpha^{\nu}_{2i}$ (see Theorem 5.143). This, along with (6.97e), implies (6.116c). The relation (6.116d) now follows, easily from (6.90c).

This completes the proof of Theorem 6.115. $\square$

# Chapter 7

# The New Characters $\chi_i^\nu$ of $G_i$

Let $G$ be a finite group satisfying

$$|G| = p^a \cdot q^b, \text{ with } p \neq q \text{ primes and } a, b \text{ non negative integers .} \tag{7.1}$$

Assume further that

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{2k} \trianglelefteq G \tag{7.2}$$

is a normal series for $G$ satisfying Hypothesis 5.1 with $\pi = \{p\}$, i.e., $G_i/G_{i-1}$ is a $p$-group if $i$ is even and a $q$-group if $i$ is odd. Note also that (7.2) plays the role of (5.2) with $m = 2k$. Let $\{1 = \chi_0, \chi_1, \cdots, \chi_{2k}\}$ be a character tower for the series (7.2). We have seen in Chapter 5, Theorem 5.6, that there exists a unique, up to conjugation, triangular set

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1, r=0}^{k,k} \tag{7.3}$$

for (7.2) that corresponds to the above character tower. Of course there is no reason for the irreducible character $\beta_{2k-1}$ to extend to its own stabilizer in $G$. In addition, we have seen how to achieve the irreducible character $\alpha_{2k}^*$ of the product group $P_{2k}^* = P_2 \dot{P}_4 \cdots P_{2k}$, from the irreducible character $\alpha_{2k} \in \mathrm{Irr}(P_{2k})$ (see Definition 5.147). We have also seen how to pick a $q$-Sylow subgroup $\widehat{Q}$ of $G(\alpha_{2k}^*)$ satisfying all the conditions in Theorem 6.19. (Observe that $\pi' = \{q\}$. )

What we will prove in this chapter is that, under the above conditions, we can find a new character tower $\{1 = \chi_0^\nu, \chi_1^\nu, \ldots, \chi_{2k}^\nu\}$ for the normal series (7.2) of $G$ so that a corresponding triangular set $\{Q_{2i-1}^\nu, P_{2i}^\nu, P_0^\nu = 1 | \beta_{2i-1}^\nu, \alpha_{2i}^\nu, \alpha_0^\nu = 1\}_{i=1}^k$ satisfies

1. $P_{2i}^* = P_{2i}^{\nu,*}$ and $\alpha_{2k}^* = \alpha_{2k}^{\nu,*}$, for all $i = 1, \ldots, k$,
2. $G(\alpha_{2k}^*) = G(\alpha_{2k}^{\nu,*})$ and $\widehat{Q} = \widehat{Q^\nu}$,
3. $\beta_{2k-1,2k}^\nu$ extends to $\widehat{Q} = \widehat{Q^\nu}$

where we keep the same notation as before with teh addition of the superscript $\nu$ to any group that refers to the new character tower and triangular set. So $P_{2i}^{\nu,*} = P_2^\nu \cdots P_{2i}^{\nu,*}$, the character $\alpha_{2k}^{\nu,*}$ is an irreducible character of $P^{\nu,*}$ that is achieved using the character $\alpha_{2k}^\nu \in \mathrm{Irr}(P_{2k}^\nu)$ via Definition 5.147 for the new characters. Furthermore, $\widehat{Q^\nu}$ is a $q$-Sylow subgroup of $G(\alpha_{2k}^{\nu,*})$ that satisfies the conditions in Theorem 6.19.

For this we will put together all the complicated machinery we developed in the previous chapters. We use the same notation for the groups and the characters that was introduced in those chapters, but applied in our specific case, i.e., where $G$ satisfies (7.1), and the series (7.2), its character tower and the corresponding triangular set (7.3) are fixed.

Thus we can use all the information about the subgroups $\widehat{Q}, \widehat{Q}_{2i-1}$ and $\widetilde{P}_{2i}$ of $G' = G(\alpha^*_{2k})$, given in Chapter 6. So we can prove

**Lemma 7.4.** *The hypotheses of Theorem 4.24 are satisfied by the present group $G$, with the integer $n$ in the theorem equal to the present $k$, the $q$-subgroup $Q_{n+1} = Q$ in the theorem equal to the present $\widehat{Q}$, the $q$-subgroup $Q_i$ in th etheorem equal to the present $\widehat{Q}_{2i-1}$, for all $i = 1, 2, \ldots, n = k$, and the $p$-subgroup $P_j$ in the theorem equal to the present $\widetilde{P}_{2j}$, for all $j = 1, \ldots, n = k$.*

*Proof.* Our present group $G$ has order $p^a q^b$, as required in Theorem 4.24. By definition $\widehat{Q}$ is an arbitary subgroup of $G$ satisfying all the conditions in Theorem 6.19. In particular it is a $q$-group, as it is a $\pi'$-Hall subgroup of of $G(\alpha^*_{2k})$ and $\pi' = q$. For each $i = 1, 2, \ldots, k$, the subgroup $\widehat{Q}_{2i-1}$ is the intersection $G_{2i-1} \cap \widehat{Q}$ by Definition 6.28. Since $G_1 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_{2k-1}$ is a series of normal subgroups of $G_{2k-1}$, this implies that $\widehat{Q}_1 \trianglelefteq \widehat{Q}_3 \trianglelefteq \cdots \trianglelefteq \widehat{Q}_{2k-1}$ is a series of normal subgroups of $\widehat{Q}_{2k-1}$, as required of $Q_1 \trianglelefteq Q_2 \trianglelefteq \cdots \trianglelefteq Q_n$ in Theorem 4.24.

For every $j = 1, \ldots, k$ the group $\widetilde{P}_{2j}$ was picked to satisfy the conditions in Proposition 6.64. Hence $\widetilde{P}$ is a $p$-group, as it is a $\pi$-group and $\pi = p$. Furthermore, according to (6.65c) the group $\widetilde{P}_{2i}$ normalizes $\widetilde{P}_{2j}$, whenever $1 \leq j \leq i \leq n = k$, as required of $P_1, P_2, \ldots, P_n$ in Theorem 4.24. In addition, (6.65a) implies that $\widetilde{P}_{2i}$ also normalizes $\widehat{Q}_{2j-1}$, whenever $1 \leq j \leq i \leq k$.

According to Lemma 6.60 and (6.65a) we get that

$$N(P^*_{2k}, \widehat{Q}_{2j-1} \text{ in } G_{2j}(\alpha^*_{2j-2})) = \widetilde{P}_{2j} \ltimes \widehat{Q}_{2j-1}, \tag{7.5}$$

for $j = 1, \ldots, k$. But $\widehat{Q}_{2i-1} \leq G'_{2i-1} = G_{2i-1}(\alpha^*_{2k})$ by (6.29a). Hence $\widehat{Q}_{2i-1}$ normalizes $P^*_{2k}$ and fixes $\alpha^*_{2j}$, for all $j = 1, \ldots, i-1$. This, along with (7.5), implies that $\widehat{Q}_{2i-1}$ normalizes the semidirect product $\widetilde{P}_{2j} \ltimes \widehat{Q}_{2j-1}$, whenever $1 \leq j \leq i \leq k$. Similarly, we use $\widehat{Q} \leq G(\alpha^*_{2k})$, to see that $\widehat{Q}$ also normalizes the above semidirect product, $\widetilde{P}_{2j} \ltimes \widehat{Q}_{2j-1}$. Therefore the groups $\widehat{Q}, \widehat{Q}_{2i-1}$ and $\widetilde{P}_{2i}$ satisfy the conditions (1) and (2) in Theorem 4.24 and the lemma follows. □

Lemma 7.4 implies

**Theorem 7.6.** *There exist linear characters $\hat{\beta}^\nu_{2i-1} \in \mathrm{Lin}(\widehat{Q}_{2i-1})$ such that the following hold:*

$$\hat{\beta}^\nu_{2i-1} \in \mathrm{Irr}(\widehat{Q}_{2i-1} | \hat{\beta}^\nu_{2i-3}, \ldots, \hat{\beta}^\nu_1), \tag{7.7a}$$

$$\widehat{Q}_{2i-1}(\hat{\beta}^\nu_{2j-1}) = \widehat{Q}_{2i-1}, \tag{7.7b}$$

$$\widetilde{P}_{2i}(\hat{\beta}^\nu_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i}) = \widetilde{P}_{2i}(\hat{\beta}_{2i-1}), \tag{7.7c}$$

*and*

$$\hat{\beta}^\nu_{2k-1} \textit{ extends to an irreducible character of } \widehat{Q}, \tag{7.7d}$$

*whenever $1 \leq j \leq i \leq k$. By convention $\hat{\beta}^\nu_{-1} := 1$.*

*Proof.* According to Lemma 7.4, the groups $\{\widehat{Q}, \widehat{Q}_{2i-1}, \widetilde{P}_{2i}\}_{i=1}^k$. satisfy the conditions in Theorem 4.24, with the series $\widehat{Q}_1 \trianglelefteq \widehat{Q}_3 \trianglelefteq \cdots \trianglelefteq \widehat{Q}_{2k-1} \trianglelefteq \widehat{Q}$ here in the place of the chain $Q_1 \trianglelefteq Q_2 \trianglelefteq \cdots \trianglelefteq Q_n \trianglelefteq Q_{n+1} = Q$ in Theorem 4.24, and the sequence $\widetilde{P}_2, \widetilde{P}_4, \ldots, \widetilde{P}_{2k}$ here, in the place of the sequence $P_1, P_2, \ldots, P_n$ there.

Note also that Corollary 6.72, and in particular (6.73a), provides the additional information that $C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i}) = \widetilde{P}_{2i}(\hat{\beta}_{2i-1})$. This, along with the conclusions (a) and (b) in Theorem 4.24, implies the theorem. $\square$

An immediate consequence of (7.7b) is

**Remark 7.8.** For every $i = 1, \ldots, k$ and $j = 1, \ldots, i$, the character $\hat{\beta}^{\nu}_{2j-1}$ is the unique character of $\widehat{Q}_{2j-1}$ that lies under $\hat{\beta}^{\nu}_{2i-1} \in \mathrm{Irr}(\widehat{Q}_{2i-1})$. Hence any subgroup that fixes $\hat{\beta}^{\nu}_{2i-1}$ also fixes $\hat{\beta}^{\nu}_{2j-1}$, as it normalizes $\widehat{Q}_{2j-1} = \widehat{Q}_{2i-1} \cap G_{2j-1}$.

In the next lemma we collect some easy remarks that follow from the properties (7.7).

**Lemma 7.9.** *For every $i = 1, \ldots, k$ we have*

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \cdots, \hat{\beta}^{\nu}_{2i-1}), \tag{7.10a}$$

*and*

$$\widetilde{P}_{2i}(\hat{\beta}_{1}, \cdots, \hat{\beta}_{2i-1}) = C(\widehat{Q} \text{ in } P^{*}_{2i}) = \widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i}) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}). \tag{7.10b}$$

*Proof.* Equation (7.10a) follows easily from Remark 7.8, and the fact that $\widetilde{P}_{2i}$ normalizes $\widehat{Q}_{2j-1}$ whenever $1 \leq j \leq i \leq k$. The equation (7.7c), along with (6.66) and (6.73b), implies (7.10b) Therefore the lemma holds. $\square$

As the next proposition shows, the group $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1})$ has properties similar to those of $\widetilde{P}_{2i}(\hat{\beta}_{2i-1})$ (see (6.63) and (6.65b)).

**Proposition 7.11.** *For all $i = 1, \ldots, k$ we have*

$$\begin{aligned} \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \cdots, \hat{\beta}^{\nu}_{2i-1}) \in \mathrm{Syl}_p(N(P^{*}_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{1}, \cdots, \hat{\beta}^{\nu}_{2i-1}))) = \\ \mathrm{Syl}_p(N(P^{*}_{2k} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{2i-1}))). \end{aligned} \tag{7.12}$$

*Even more, $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1})$ is the unique $p$-Sylow subgroup of $N(P^{*}_{2k} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{2i-1}))$, and*

$$N(P^{*}_{2k} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{2i-1})) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) \times \widehat{Q}_{2i-1} = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i}) \times \widehat{Q}_{2i-1}, \tag{7.13}$$

*whenever $1 \leq i \leq k$.*

*Proof.* Let $i = 1, \ldots, k$ be fixed. According to (6.61) we have

$$N(P^{*}_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^{*}_{2i-2})) = \widetilde{P}_{2i} \ltimes \widehat{Q}_{2i-1}.$$

Therefore, $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1})$ is a $p$-subgroup of $N(P^{*}_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}))$. Hence there exists an element $s \in \widehat{Q}_{2i-1}$ such that

$$\widetilde{P}^{s}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}) \in \mathrm{Syl}_p(N(P^{*}_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^{*}_{2i-2}, \hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}))), \text{and} \tag{7.14a}$$

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}) \leq \widetilde{P}^{s}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}). \tag{7.14b}$$

But $\widehat{Q}_{2i-1}$ fixes $\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}$, by (7.7b). Thus $\widetilde{P}^{s}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}) = (\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}))^{s}$. A cardinality argument, along with (7.14b), implies that

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}) = \widetilde{P}^{s}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}) = (\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{1}, \ldots, \hat{\beta}^{\nu}_{2i-1}))^{s}.$$

This, along with (7.14a) and (7.10a), implies that

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1}) \in \mathrm{Syl}_p(N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1}))).$$

In view of Remark 7.8, every subgroup of $G$ that fixes $\hat{\beta}^{\nu}_{2i-1}$ fixes $\hat{\beta}^{\nu}_{2j-1} \in \mathrm{Irr}(\widehat{Q}_{2j-1})$, for all $1 \leq j \leq i$. Hence $N(P^*_{2k}, \widehat{Q}_{2i-1} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1})) = N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_{2i-1}))$. Therefore (7.12) holds.

According to Proposition 6.35, we have $\widehat{Q}_{2i-1} \in \mathrm{Syl}_q(N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2})))$ . Hence $\widehat{Q}_{2i-1} \in \mathrm{Syl}_q(N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_{2i-1})))$. This, along with (7.12), implies that

$$N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_{2i-1})) = \widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) \ltimes \widehat{Q}_{2i-1}. \tag{7.15}$$

But according to (7.7c) we have $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i})$. This, along with (7.15), implies (7.13). Hence Proposition 7.11 holds. $\square$

**Definition 7.16.** For every $i = 1, \dots, k$ we define $\hat{\alpha}_{2i} \in \mathrm{Irr}(\widetilde{P}_{2i}(\hat{\beta}_{2i-1})$ to be the $\widehat{Q}_{2i-1}$-Glauberman correspondent of $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$.

Note that the $\hat{\alpha}_{2i}$ are well defined, as $\alpha^*_{2i} \in \mathrm{Irr}(P^*_{2i})$ is $\widehat{Q}_{2i-1}$-invariant (since $\widehat{Q}_{2i-1} \leq G(\alpha^*_{2k})$) and $\widetilde{P}_{2i}(\hat{\beta}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i})$ (see (7.10b)).

We can now prove the main theorem of this section.

**Theorem 7.17.** *The set*

$$\{\widehat{Q}_{2i-1}, \widetilde{P}_{2i}(\hat{\beta}_{2i-1}) | \hat{\beta}^{\nu}_{2i-1}, \hat{\alpha}_{2i}\}_{i=1}^{k} \tag{7.18}$$

*is a triangular set for the series $1 = G'_0 \trianglelefteq G'_1 \trianglelefteq \cdots \trianglelefteq G'_{2k} \trianglelefteq G'$ in* (6.76)*. Furthermore, it satisfies Property 6.89, while* (6.88b) *holds for this triangular set, with $Q' = \widehat{Q}$.*

*Proof.* We will first prove that (7.18) is a triangular set for the above series, i.e., we will verify the properties (5.17) for that set and series. Assume that $i = 1, \dots, k$ is fixed.

The equations in (5.17a) hold trivially. As $\widehat{Q}_1 = Q_{1,2k} = G'_1$ (see (6.34)) and $\hat{\beta}^{\nu}_1 \in \mathrm{Irr}(\widehat{Q}_1)$, (5.17b) holds, while (5.17e) is trivially true for $i = 1$.

Assume that $i \geq 2$. According to (6.29a) the group $\widehat{Q}_{2i-1}$ is a $q$-Sylow subgroup of $G'_{2i-1}$. Furthermore, $\widehat{Q}_{2i-1}$ normalizes $\widehat{Q}_{2j-1} = \widehat{Q}_{2i-1} \cap G_{2j-1}$, and fixes $\hat{\beta}^{\nu}_{2j-1} \in \mathrm{Irr}(\widehat{Q}_{2j-1})$, by (7.7b), for all $j = 1, \dots, i-1$. The group $\widehat{Q}_{2i-1}$ also fixes $\alpha^*_{2j}$, as it fixes $\alpha^*_{2k}$, for all such $j$. Therefore, it fixes the $\widehat{Q}_{2j-1}$-Glauberman correspondent $\hat{\alpha}_{2j} \in \mathrm{Irr}(\widetilde{P}_{2j}(\hat{\beta}^{\nu}_{2j-1})) = \mathrm{Irr}(C(\widehat{Q}_{2j-1} \text{ in } P^*_{2j})$ of $\alpha^*_{2j}$in Definition 7.16, for all $j = 1, \dots, i-1$. Hence

$$\widehat{Q}_{2i-1} \leq G'_{2i-1}(\hat{\alpha}_2, \dots, \hat{\alpha}_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-3}) \leq G'_{2i-1}.$$

As $\widehat{Q}_{2i-1} \in \mathrm{Syl}_q(G'_{2i-1})$ we get that

$$\widehat{Q}_{2i-1} \in \mathrm{Syl}_q(G'_{2i-1}(\hat{\alpha}_2, \dots, \hat{\alpha}_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-3})).$$

Hence (5.17e) holds.

As $\widetilde{P}_{2i-2}(\hat{\beta}^{\nu}_{2i-3}) = C(\widehat{Q}_{2i-3} \text{ in } \widetilde{P}_{2i-2})$ by (7.7c), we have that $C(\widetilde{P}_{2i-2}(\hat{\beta}^{\nu}_{2i-3}) \text{ in } \widehat{Q}_{2i-3}) = \widehat{Q}_{2i-3}$. Therefore, the character $\hat{\beta}^{\nu}_{2i-3,2i-2} \in \mathrm{Irr}(C(\widetilde{P}_{2i-2}(\hat{\beta}^{\nu}_{2i-3}) \text{ in } \widehat{Q}_{2i-3}))$ concides with $\hat{\beta}^{\nu}_{2i-3}$. This, along with (7.7a), makes the condition (5.17f) valid.

For the $p$-groups and characters we have that $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = C(\widehat{Q}_{2i-1} \text{ in } \widetilde{P}_{2i})$ fixes $\alpha^*_{2j}$, for all $j = 1, \dots, k$, as it is a subgroup of $G' = G(\alpha^*_{2k})$. It also centralizes $\widehat{Q}_{2i-1}$, and thus centralizes $\widehat{Q}_{2j-1} \le \widehat{Q}_{2i-1}$ for all $j = 1, \dots, i$. Therefore $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1})$ fixes the $\widehat{Q}_{2j-1}$-Glauberman correspondent $\hat{\alpha}_{2j}$ of $\alpha^*_{2j}$, for all $j = 1, \dots, i$. In view of Remark 7.8, we also have that $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1})$ fixes $\hat{\beta}^{\nu}_{2j-1}$ for all such $j$. Hence

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) \le G_{2i}(\alpha^*_{2k}, \hat{\alpha}_2, \dots, \hat{\alpha}_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1}) \le N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_{2i-1})). \tag{7.19}$$

According to Proposition 7.11, the unique $p$-Sylow subgroup of $N(P^*_{2k} \text{ in } G_{2i}(\alpha^*_{2i-2}, \hat{\beta}^{\nu}_{2i-1}))$ is the group $\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1})$. This, along with (7.19) and the fact that $G'_{2i} = G_{2i}(\alpha^*_{2k})$, implies that

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) \in \mathrm{Syl}_p(G_{2i}(\alpha^*_{2k}, \hat{\alpha}_2, \dots, \hat{\alpha}_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1})) = \mathrm{Syl}_p(G'_{2i}(\hat{\alpha}_2, \dots, \hat{\alpha}_{2i-2}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2i-1})).$$

Hence (5.17c) holds.

To prove (5.17d), we first observe that $\widehat{Q}_{2i-1}$ normalizes both $\widehat{Q}_{2i-3} = G_{2i-3} \cap \widehat{Q}_{2i-1}$ and $P^*_{2i-2} = P^*_{2k} \cap G_{2i-2}$, since $\widehat{Q}2i-1 \le G' = G(\alpha^*_{2k})$ normalizes $P^*_{2k}$. Hence the $q$-group $\widehat{Q}_{2i-1}$ normalizes the $p$-group $C(\widehat{Q}_{2i-3} \text{ in } P^*_{2i-2})$ . Hence

$$C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i-2}) = C(\widehat{Q}_{2i-1} \text{ in } C(\widehat{Q}_{2i-3} \text{ in } P^*_{2i-2})).$$

By convention we set $\widehat{Q}_{-1} := 1$, so that the above equation holds trivially for $i = 1$. Let $\hat{\alpha}_{2i-2,2i-1} \in \mathrm{Irr}(C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i-2}))$ denote the $\widehat{Q}_{2i-1}$-Glauberman correspondent of the irreducible character $\hat{\alpha}_{2i-2} \in \mathrm{Irr}(C(\widehat{Q}_{2i-3} \text{ in } P^*_{2i-2}))$. (Note that since $\widehat{Q}_{2i-1}$ is a subgroup of $G' = G(\alpha^*_{2k})$ it fixes $\alpha^*_{2i-2} \in \mathrm{Irr}(P^*_{2i-2})$ and normalizes $\widehat{Q}_{2i-3}$, so it fixes the $\widehat{Q}_{2i-3}$-Glauberman correspondent $\hat{\alpha}_{2i-2}$ of $\alpha^*_{2i-2}$.) Then $\hat{\alpha}_{2i-2,2i-1}$ is the $\widehat{Q}_{2i-1}$-Glauberman correspondent of $\alpha^*_{2i-2}$. Hence $\hat{\alpha}_{2i-2,2i-1}$ lies under the $\widehat{Q}_{2i-1}$-Glauberman correspondent $\hat{\alpha}_{2i}$ of $\alpha^*_{2i}$, as $\alpha^*_{2i-2}$ lies under $\alpha^*_{2i}$. Therefore $\hat{\alpha}_{2i} \in \mathrm{Irr}(\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) | \hat{\alpha}_{2i-2,2i-1})$. So (5.17d) is satisfied. This completes the proof of (5.17). Hence (7.18) is a triangular set for (6.76).

The group $\widehat{Q}_{2i-1} = \widehat{Q} \cap G_{2i-1}$ is clearly a normal subgroup of $\widehat{Q}$, for all $i = 1, \dots, k$. Furthermore, $\widehat{Q}$ fixes the character $\hat{\beta}^{\nu}_{2k-1}$, by (7.7d). Hence, Remark 7.8 implies that $\widehat{Q}$ fixes $\hat{\beta}^{\nu}_{2i-1}$, for all $i = 1, \dots, k$. As $\widehat{Q}$ is a subgroup of $G' = G(\alpha^*_{2k})$, it fixes $\alpha^*_{2i}$, for all $i = 1, \dots, k$. Since $\widehat{Q}$ normalizes $\widehat{Q}_{2i-1}$, it fixes the $\widehat{Q}_{2i-1}$-Glauberman correspondent $\hat{\alpha}_{2i}$ of $\alpha^*_{2i}$, for all such $i$. Thus $\widehat{Q} \le G'(\hat{\alpha}_2, \dots, \hat{\alpha}_{2k}, \hat{\beta}^{\nu}_1, \dots, \hat{\beta}^{\nu}_{2k-1})$. So $\widehat{Q}$ satisfies (6.88b).

Definition 7.16 implies that the triangular set (7.18) satisfies (6.90c). It also satisfies (6.90b), according to Proposition 7.11. As we have already seen, the subgroup $\widehat{Q}_{2i-1}$ of $G'$ normalizes $P^*_{2i}$. Hence

$$P^*_{2i}(\hat{\beta}^{\nu}_{2i-1}) \le N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) \le P^*_{2i}(\hat{\beta}^{\nu}_{2i-1}).$$

So $P^*_{2i}(\hat{\beta}^{\nu}_{2i-1}) = N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i})$. This, along with (7.7c), implies that

$$\widetilde{P}_{2i}(\hat{\beta}^{\nu}_{2i-1}) = N(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) = C(\widehat{Q}_{2i-1} \text{ in } P^*_{2i}) = P^*_{2i}(\hat{\beta}^{\nu}_{2i-1}).$$

Thus (6.90a) also holds. Hence the set (7.18) satisfies Property 6.89. This completes the proof of the theorem. $\square$

All the work in Chapters 4-6 was done to prove the following theorem

**Theorem 7.20.** *Let $\{1 = \chi_0, \chi_1, \dots, \chi_{2k}\}$ be a character tower for the series (7.2), and let (7.3) be its unique, up to conjugation, corresponding triangular set. Then there exists a character tower $\{1 = \chi_0^\nu, \chi_1^\nu, \dots, \chi_{2k}^\nu\}$ for the series (7.2), with corresponding triangular set $\{Q_{2i-1}^\nu, P_{2i}^\nu, P_0^\nu = 1 | \beta_{2i-1}^\nu, \alpha_{2i}^\nu, \alpha_0^\nu = 1\}_{i=1}^k$, so that the following hold*

$$\widehat{Q}_{2i-1} = Q_{2i-1,2k}^\nu \text{ and } \hat{\beta}_{2i-1}^\nu = \beta_{2i-1,2k}^\nu, \tag{7.21a}$$

$$P_{2i}^* = P_{2i}^{\nu,*} \text{ and } \alpha_{2i}^* = \alpha_{2i}^{\nu,*}, \tag{7.21b}$$

$$\beta_{2i-1}^\nu \text{ is the } P_{2i}^\nu \cdot P_{2i+2}^\nu \cdots P_{2k}^\nu\text{-Glauberman correspondent of } \hat{\beta}_{2i-1}^\nu, \tag{7.21c}$$

$$\alpha_{2i}^\nu \text{ is the } Q_3^\nu, \dots, Q_{2i-1}^\nu\text{-correspondent of } \alpha_{2i}^*, \tag{7.21d}$$

$$Q_{2i-1}^\nu \geq Q_{2i-1,2k}, \tag{7.21e}$$

$$\widehat{Q} \text{ normalizes } Q_{2i-1}^\nu, \tag{7.21f}$$

$$\beta_{2k-1,2k}^\nu \text{ extends to } \widehat{Q}, \tag{7.21g}$$

*whenever $1 \leq i \leq k$.*

*Proof.* For the fixed triangular set (7.3) of the character tower $\{1 = \chi_0, \chi_1, \dots, \chi_{2k}\}$ we saw in Chapter 6 how to pick groups $\widehat{Q}_{2i-1}$ and $\widetilde{P}_{2i}$, along with characters $\hat{\beta}_{2i-1} \in \operatorname{Irr}(\widehat{Q}_{2i-1})$ satisfying all the conditions in Theorem 6.19 Proposition 6.64 and Proposition 6.51. Furthermore, we proved at the begining of this chapter that we can replace the characters $\hat{\beta}_{2i-1}$ with new characters $\hat{\beta}_{2i-1}^\nu \in \operatorname{Irr}(\widehat{Q}_{2i-1})$ that satisfy (7.7). Even more, as Theorem 7.17 shows, the set (7.18) is a triangular set for (6.76) that satisfies Property 6.89, while (6.88b) holds for this triangular set, with $\widehat{Q}$ in the place of $Q'$. According to Theorems 6.95 and 6.115, the set (7.18) determines a triangular set

$$\{Q_{2i-1}^\nu, P_{2i}^\nu, P_0^\nu = 1 | \beta_{2i-1}^\nu, \alpha_{2i}^\nu, \alpha_0^\nu = 1\}_{i=1}^k \tag{7.22}$$

for the series (7.2), such that (6.97) and (6.116) hold with $\widehat{Q}, \widehat{Q}_{2i-1}$ and $\widetilde{P}_{2i}(\hat{\beta}_{2i-1})$ in the place of $Q', Q'_{2i-1}$ and $P'_{2i}$, respectively. In view of Theorem 5.6, the triangular set (7.22) corresponds to a unique, up to conjugation, character tower $\{1 = \chi_0^\nu, \chi_1^\nu, \dots, \chi_{2k}^\nu\}$ for the series (7.2).

To complete the proof of the theorem it suffices to show that the set (7.22) has the properties (7.21). As it satisfies (6.116), the equations in (7.21b) follow trivially from the first part of (6.116b), and (6.116c). It also satisfies (7.21a) and (7.21d) as it satisfies (6.116a) and (6.97e) respectively. In addition (7.21c) holds, since the set (7.22) satisfies (6.97c) and Corollary 6.113. Furthermore, according to (6.97b), the group $Q_{2i-1}^\nu$ contains $\widehat{Q}_{2i-1}$, as $Q_{2i-1}^\nu(\alpha_{2k}^*) = \widehat{Q}_{2i-1}$. In addition, (6.33) implies that $\widehat{Q}_{2i-1} \geq Q_{2i-1,2k}$. We conclude that $Q_{2i-1}^\nu \geq Q_{2i-1,2k}$ and (7.21e) follows.

Clearly (6.97a) implies that $\widehat{Q}$ normalizes $Q_{2i-1}^\nu$, for all $i = 1, \dots, k$. Thus (7.21f) holds.

The last part, (7.21g), follows easily from (7.7d), as $\hat{\beta}_{2k-1}^\nu = \beta_{2k-1,2k}^\nu$ by (7.21a). □

Furthermore, the new triangular set shares one more group with the old one, as the next theorem shows.

**Theorem 7.23.** *The group $\widehat{Q}$ satisfies the conditions in Theorem 6.19 for the new groups. Hence we may assume that $\widehat{Q^\nu} = \widehat{Q}$. Then*

$$\widehat{Q^\nu} = \widehat{Q} = \widehat{Q}(\beta_{2k-1,2k}^\nu) = \widehat{Q^\nu}(\beta_{2k-1,2k}^\nu). \tag{7.24}$$

*Proof.* For the proof we need to show that $\widehat{Q}$ satisfies (6.20) and (6.21) for the new groups. Clearly, $\widehat{Q}$ is a $\pi'$-Hall subgroup of $G(\alpha_{2k}^{\nu,*})$, as $\alpha_{2k}^{\nu,*} = \alpha_{2k}^*$, by (7.21b), and $\widehat{Q}$ is a $\pi'$-Hall subgroup of $G(\alpha_{2k}^*)$.

Thus (6.20a) holds.

In view of(7.7d) the character $\hat{\beta}^{\nu}_{2k-1}$ is fixed by $\widehat{Q}$. Thus, Remark 7.8 implies that $\widehat{Q}$ fixes $\hat{\beta}^{\nu}_{2i-1}$, for all $i = 1, \dots, k$. Hence $\widehat{Q}$ fixes $\beta^{\nu}_{2i-1,2k} = \hat{\beta}^{\nu}_{2i-1}$, for all such $i$. Furthermore, $\widehat{Q}$ normalizes the groups $Q^{\nu}_{2i-1}$, by (7.21f), and $P^{\nu,*}_{2i} = P^*_{2i} = P^*_{2k} \cap G_{2i}$, as it fixes $\alpha^*_{2k} \in \mathrm{Irr}(P^*_{2k})$, whenever $1 \leq i \leq k$. But (5.141), applied to the new groups, implies that $P^{\nu}_{2i} = N(Q^{\nu}_1, \dots, Q^{\nu}_{2i-1} \text{ in } P^{\nu,*}_{2i})$, for all such $i$ We conclude that $\widehat{Q}$ normalizes $P^{\nu}_{2i}$, for all $i = 1, \dots, k$. Hence $\widehat{Q}$ fixes the $P^{\nu}_{2i} \cdot P^{\nu}_{2i+2} \cdots P^{\nu}_{2k}$-Glauberman correspondent $\beta^{\nu}_{2i-1} \in \mathrm{Irr}(Q^{\nu}_{2i-1})$ of $\hat{\beta}^{\nu}_{2i-1}$ (see (7.21c)), as it fixes $\hat{\beta}^{\nu}_{2i-1}$. So

$$\widehat{Q} = \widehat{Q}(\beta^{\nu}_{2i-1,2k}) = \widehat{Q}(\beta^{\nu}_{2i-1}),$$

whenever $1 \leq i \leq k$.

Even more, $\widehat{Q}$ fixes $\alpha^*_{2i}$, as it fixes $\alpha^*_{2k}$. We saw above that it and normalizes $Q^{\nu}_{2i-1}$ and $P^{\nu}_{2i}$, for all $i = 1, \dots, k$. Thus $\widehat{Q}$ fixes the $Q^{\nu}_3, \dots, Q^{\nu}_{2i-1}$-correspondent $\alpha^{\nu}_{2i}$ of $\alpha^*_{2i}$ in (7.21d). Hence

$$\widehat{Q}(\alpha^{\nu}_{2i}) = \widehat{Q},$$

whenever $1 \leq i \leq k$.

According to Theorem 5.88, the $cP^{\nu}_2, \dots, cP^{\nu}_{2i}, cQ^{\nu}_1, \dots, cQ^{\nu}_{2i-1}$-correspondent of $\chi^{\nu}_{2i} \in \mathrm{Irr}(G_{2i})$ is the character $\chi^{\nu}_{2i,2i} = \alpha^{\nu}_{2i} \times \beta^{\nu}_{2i-1,2i}$, for all $i = 1, \dots, k$. As we have already seen, the group $\widehat{Q}$ fixes the characters $\alpha^{\nu}_{2i}$ and $\beta^{\nu}_{2i-1}$, and normalizes the groups $Q^{\nu}_{2j-1}$ and $P^{\nu}_{2j}$ for $j = 1, \dots, i$. Thus it also fixes both the $P^{\nu}_{2i}$-Glauberman correspondent $\beta^{\nu}_{2i-1,2i}$ of $\beta^{\nu}_{2i-1}$, and the direct product $\chi^{\nu}_{2i,2i} = \alpha^{\nu}_{2i} \times \beta^{\nu}_{2i-1,2i}$. Therefore, $\widehat{Q}$ also fixes the $cP^{\nu}_2, \dots, cP^{\nu}_{2i}, cQ^{\nu}_1, \dots, cQ^{\nu}_{2i-1}$-correspondent $\chi^{\nu}_{2i}$ of $\chi^{\nu}_{2i,2i}$, for all $i = 1, \dots, k$ (see Diagram 5.5 applied to the new characters). Similarly, we can see that $\widehat{Q}$ fixes $\chi^{\nu}_{2i-1,2i-1} = \alpha^{\nu}_{2i-2,2i-1} \times \beta^{\nu}_{2i-1}$, as well as the $cP^{\nu}_2, \dots, cP^{\nu}_{2i-2}, cQ^{\nu}_1, \dots, cQ^{\nu}_{2i-1}$-correspondent $\chi^{\nu}_{2i-1}$ of $\chi^{\nu}_{2i-1,2i-1}$.

In conclusion,

$$\widehat{Q} = \widehat{Q}(\beta^{\nu}_{2i-1,2k}) = \widehat{Q}(\beta^{\nu}_{2i-1}) = \widehat{Q}(\alpha^{\nu}_{2i}) = \widehat{Q}(\chi^{\nu}_{2i}) = \widehat{Q}(\chi^{\nu}_{2i-1}), \tag{7.25}$$

whenever $1 \leq i \leq k$.

It is clear that (6.20c) and (6.20d) hold, with the new characters $\beta^{\nu}_{2i-1}, \beta^{\nu}_{2i-1,2k}, \alpha^{\nu}_{2i}, \chi^{\nu}_{2i}, \chi^{\nu}_{2i-1}$ and $\alpha^{\nu}_{2i}$ in the place of the analogous original characters. (Actually, in (6.20d) we have equality.) Furthermore, (7.25) also implies that the group $\widehat{Q} = \widehat{Q}(\beta^{\nu}_{2i-1,2k})$ is contained in $G'(\beta^{\nu}_{2i-1,2k}) \cap G'(\chi^{\nu}_1, \dots, \chi^{\nu}_{2i-1}) \cap G'(\chi^{\nu}_1, \dots, \chi^{\nu}_{2i}) \cap G'(\beta^{\nu}_1, \dots, \beta^{\nu}_{2i-1})$, Thus it is a $\pi'$-Hall subgroup of each group in this intersection, as it is a $\pi'$-Hall subgroup of $G'$. Hence $\widehat{Q}$ satisfies (6.20b, c, d) for the new groups.

It remains to show (6.21). But, as (7.21f) implies, $\widehat{Q} = \widehat{Q}(\beta^{\nu}_{2i-1,2k})$ normalizes $Q^{\nu}_{2i+1}$, for all $i = 1, \dots, k-1$. Thus (6.21) holds. This completes the proof of the theorem. □

An easy consequence is

**Corollary 7.26.** *Let $\{1 = \chi_0, \chi_1, \dots, \chi_{2k}\}$ be a character tower for the series (7.2), and let (7.3) be its unique, up to conjugation, corresponding triangular set. Assume further that $\widehat{Q}$ satisfies the conditions in Theorem 6.19 for this set and tower. Then there exist a character tower $\{1 = \chi^{\nu}_0, \chi^{\nu}_1, \dots, \chi^{\nu}_{2k}\}$ for the series (7.2), with corresponding triangular set $\{Q^{\nu}_{2i-1}, P^{\nu}_{2i}, P^{\nu}_0 =$*

$1|\beta^{\nu}_{2i-1}, \alpha^{\nu}_{2i}, \alpha^{\nu}_0 = 1\}_{i=1}^{k}$, *and a group* $\widehat{Q^{\nu}}$ *that satisfy*

$$P^*_{2k} = P^{\nu,*}_{2k} \text{ and } \alpha^*_{2k} = \alpha^{\nu,*}_{2k}, \tag{7.27a}$$

$$\widehat{Q} = \widehat{Q^{\nu}}, \tag{7.27b}$$

$$\widehat{Q^{\nu}} \text{ fixes the characters } \alpha^{\nu}_{2i}, \beta^{\nu}_{2i-1}, \chi^{\nu}_j, \tag{7.27c}$$

$$\beta^{\nu}_{2i-1,2k} \text{ extends to } \widehat{Q} = \widehat{Q^{\nu}}, \tag{7.27d}$$

*for all* $i = 1, \ldots, k$ *and* $j = 1, \ldots, 2k$.

*Proof.* Follows immediately from Theorems 7.20 and 7.23. □

If instead of the series (7.2), we consider the bigger series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{2k} \trianglelefteq G_{2k+1} \trianglelefteq G, \tag{7.28}$$

then the conclusions of Corollary 7.26 still hold, i.e.,

**Corollary 7.29.** *Let* $\{1 = \chi_0, \chi_1, \ldots, \chi_{2k+1}\}$ *be a character tower for the series* (7.28)*, and let* $\{Q_{2i+1}, P_{2i}|\beta_{2i+1}, \alpha_{2i}\}_{i=0}^{k}$ *be its unique, up to conjugation, corresponding triangular set. Assume further that* $\widehat{Q}$ *is picked to satisfy the conditions in Theorem 6.19 for this set and tower. Then there exist a character tower* $\{1 = \chi^{\nu}_0, \chi^{\nu}_1, \ldots, \chi^{\nu}_{2k}\}$ *for the series* (7.2)*, with corresponding triangular set* $\{Q^{\nu}_{2i-1}, P^{\nu}_{2i}, P^{\nu}_0 = 1|\beta^{\nu}_{2i-1}, \alpha^{\nu}_{2i}, \alpha^{\nu}_0 = 1\}_{i=1}^{k}$, *and a group* $\widehat{Q^{\nu}}$ *that satisfy*

$$P^*_{2k} = P^{\nu,*}_{2k} \text{ and } \alpha^*_{2k} = \alpha^{\nu,*}_{2k}, \tag{7.30a}$$

$$\widehat{Q} = \widehat{Q^{\nu}}, \tag{7.30b}$$

$$\widehat{Q^{\nu}} \text{ fixes the characters } \alpha^{\nu}_{2i}, \beta^{\nu}_{2i-1}, \chi^{\nu}_j, \tag{7.30c}$$

$$\beta^{\nu}_{2i-1,2k} \text{ extends to } \widehat{Q} = \widehat{Q^{\nu}}, \tag{7.30d}$$

*for all* $i = 1, \ldots, k$ *and* $j = 1, \ldots, 2k$.

*Proof.* Follows easily from Corollary 6.24 and Corollary 7.26. □

# Chapter 8

# The $\pi, \pi'$ Symmetry and the Hall System $\{\mathbf{A}, \mathbf{B}\}$

## 8.1 The group $\widehat{P}$

Let $G$ be a finite group of odd order. As we saw in Chapter 6, whenever we fix a normal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G$ of $G$ that satisfies Hypothesis 5.1, a character tower $\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^m$ for this series and its corresponding triangular set, then we can get a $\pi$-Hall subgroup $\widehat{Q}$ of $G(\alpha^*_{2k})$ with the properties described in Theorem 6.19. We also saw how to get the $\pi$-groups $\widetilde{P}$. Furthermore, we used $\widehat{Q}$ and $\widetilde{P}$ in Chapter 7, to replace the given character tower with another one having the properties described in Corollary 7.26.

Of course the $\pi - \pi'$ symmetry in the construction of the triangular sets implies that results similar to those for the $\pi'$-groups also hold for the $\pi$-groups. That is, whenever the above series, the character tower and its triangular set are fixed, we can find a $\pi$-Hall subgroup $\widehat{P}$ of $G'' = G(\beta^*_{2l-1})$ that satisfies a modification of Theorem 6.19, that is,

$$\widehat{P} \in \mathrm{Hall}_\pi(G(\beta^*_{2l-1})), \tag{8.1a}$$

$$\widehat{P}(\alpha_{2i,2l-1}) \in \mathrm{Hall}_\pi(G''(\alpha_{2i,2l-1})) \cap \mathrm{Hall}_\pi(G''(\chi_1, \ldots, \chi_{2i})) \cap \mathrm{Hall}_\pi(G''(\chi_1, \ldots, \chi_{2i+1})) \cap \mathrm{Hall}_\pi(G''(\alpha_2, \ldots, \alpha_{2i})), \tag{8.1b}$$

$$\widehat{P}(\alpha_{2i,2l-1}) = \widehat{P}(\chi_1, \ldots, \chi_{2i}) = \widehat{P}(\chi_1, \ldots, \chi_{2i+1}) = \widehat{P}(\alpha_2, \ldots, \alpha_{2i}) \text{ and} \tag{8.1c}$$

$$\widehat{P}(\chi_1, \ldots, \chi_{2i}) \leq \widehat{P}(\beta_1, \ldots, \beta_{2i+1}), \tag{8.1d}$$

for all $i = 1, \ldots, l-1$. Furthermore,

$$\widehat{P}(\alpha_{2i,2l-1}) \text{ normalizes } P_{2i+2}, \tag{8.1e}$$

for all $i = 0, 1, \ldots, k-1$.

In the particular case of a $p^a q^b$-group $G$ (where $p \neq q$ are odd primes), we get the analogue of Corollary 7.26 for the $\pi$-groups, interchanging the roles of $p$ and $q$, that is,

**Theorem 8.2.** *Let $\{1 = \chi_0, \chi_1, \ldots, \chi_{2k}\}$ be a character tower for the series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{2k}$, and let $\{Q_{2i-1}, P_{2i}, P_0 = 1 | \beta_{2i-1}, \alpha_{2i}, \alpha_0 = 1\}_{i=1}^k$ be its unique, up to conjugation, corresponding triangular set. Then there exist a character tower $\{1 = \chi^\nu_0, \chi^\nu_1, \ldots, \chi^\nu_{2k-1}\}$ for the series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{2k-1}$, a corresponding triangular set $\{Q^\nu_{2i-1}, P^\nu_{2i-2} | \beta^\nu_{2i-1}, \alpha^\nu_{2i-2}\}_{i=1}^{l=k}$, and a $p$-group*

$\widehat{P^\nu}$, *that satisfy*

$$Q^*_{2l-1} = Q^{\nu,*}_{2l-1} \text{ and } \beta^*_{2l-1} = \beta^{\nu,*}_{2l-1},$$
$$\widehat{P} = \widehat{P^\nu},$$
$$\widehat{P^\nu} \text{ fixes the characters } \alpha^\nu_{2i}, \beta^\nu_{2i-1}, \chi^\nu_j, \text{ and}$$
$$\alpha^\nu_{2i,2l-1} \text{ extends to } \widehat{P} = \widehat{P^\nu},$$

*for all* $i = 1, \ldots, l-1$ *and* $j = 1, \ldots, 2k-1$.

## 8.2 The Hall system $\{\mathbf{A}, \mathbf{B}\}$ of $G$

Let $G$ be any finite group of odd order, and $\pi$ any set of primes. If $\mathbf{A} \in \mathrm{Hall}_\pi(G)$ and $\mathbf{B} \in \mathrm{Hall}_{\pi'}(G)$, then we call the set $\{\mathbf{A}, \mathbf{B}\}$ a *Hall* $\pi, \pi'$*-system* for $G$, or, more shortly, a *Hall system* for $G$. Note that $G$ has a single conjugacy class of such Hall systems, because it is solvable. Furthermore, if $H$ is a subgroup of $G$, we say that the Hall system $\mathbf{A}, \mathbf{B}$ of $G$ *reduces* into $H$, if $\mathbf{A} \cap H, \mathbf{B} \cap H$ form a Hall system for $H$.

We start with a finite odd order group $G$, and we fix an increasing chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{8.3a}$$

of normal subgroups $G_i$ of $G$, that satisfy Hypothesis 5.1 with $n > 0$ in the place of $m$, i.e., $G_i/G_{i-1}$ is a $\pi$-group if $i$ is even, and a $\pi'$-group if $i$ is odd, for each $i = 1, 2, \ldots, n$. We also fix a character tower

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^n \tag{8.3b}$$

for the above series.

We denote by $k'$ and $l'$ the integers

$$k' = [n/2] \text{ and } l' = [(n+1)/2]$$

corresponding to $k$ and $l$ in (5.7), with $n$ in place of $m$. So $2k'$ and $2l'-1$ are the greatest even and odd integers, respectively, in the set $\{1, 2, \ldots, n\}$. As in Section 5.3, we construct a triangular set

$$\{P_{2r}, Q_{2i-1} | \alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{k',l'} \tag{8.3c}$$

corresponding to the chain (8.3a) and tower (8.3b). It, in turn, determines the groups $P^*_{2r}$ and $Q^*_{2i-1}$, for $r = 1, 2, \ldots, k'$ and $i = 1, 2, \ldots, l'$. We know by Corollary 5.157 that $P^*_{2k}$ and $Q^*_{2l-1}$ form a Hall system for $G_m(\chi_1, \chi_2, \ldots, \chi_m)$, whenever $m = 1, 2, \ldots, n$ and $k$, $l$ are related to $m$ by the usual equations in (5.7). In particular, $P^*_{2k'}$ and $Q^*_{2l'-1}$ form a Hall system for $G(\chi_1, \chi_2, \ldots, \chi_n) = G_n(\chi_1, \chi_2, \ldots, \chi_n)$. Furthermore, the groups $G(\chi_1, \ldots, \chi_m)$, for $m = 1, \ldots, n$, form a decreasing chain, i.e.,

$$G \geq G(\chi_1) \geq G(\chi_1, \chi_2) \geq \cdots \geq G(\chi_1, \ldots, \chi_n).$$

So we may choose $\mathbf{A}$ and $\mathbf{B}$ satisfying

$$\mathbf{A} \in \mathrm{Hall}_\pi(G), \mathbf{B} \in \mathrm{Hall}_{\pi'}(G), \tag{8.4a}$$

$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_h) \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_h) \text{ form a Hall system for } G(\chi_1, \chi_2, \ldots, \chi_h), \tag{8.4b}$$

$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_n) = P^*_{2k'} \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_n) = Q^*_{2l'-1}, \tag{8.4c}$$

for all $h = 1, \ldots, n$. So (8.4b) says that $\mathbf{A}, \mathbf{B}$ reduces into $G(\chi_1, \chi_2, \ldots, \chi_h)$, for each $h = 1, 2, \ldots, n$, while (8.4c) says that $\mathbf{A}, \mathbf{B}$ reduces to the Hall system $P^*_{2k'}, Q^*_{2l'-1}$ for $G(\chi_1, \chi_2, \ldots, \chi_n)$.

We fix an integer $m = 1, \ldots, n$ and we consider the normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G. \tag{8.5a}$$

The sub tower

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^m \tag{8.5b}$$

of (8.3a) is a character tower of the above series. If $k$ and $l$ are defined as in (5.7) for $m$, then Remark 5.125 implies that the subset

$$\{P_{2r}, Q_{2i-1} | \alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{k,l} \tag{8.5c}$$

of (8.3c) is a triangular set corresponding to the chain (8.5a) and tower (8.5b).

As in (5.129a), we set $G^* := G(\chi_1, \chi_2, \ldots, \chi_m)$. So we can define the intersection groups

$$\begin{aligned} \mathbf{A}^* &:= \mathbf{A} \cap G^* = \mathbf{A}(\chi_1, \ldots, \chi_m), \\ \mathbf{B}^* &:= \mathbf{B} \cap G^* = \mathbf{B}(\chi_1, \ldots, \chi_m). \end{aligned} \tag{8.6}$$

Note that these definitions depend heavily on $m$. Then we can prove

**Proposition 8.7.** *Let $m = 1, \ldots, n$ be fixed, and $k, l$ be its associate, via* (5.7)*, integers. Then*

$$P^*_{2k} \trianglelefteq \mathbf{A}^* \leq \mathbf{A} \textit{ and } Q^*_{2l-1} \trianglelefteq \mathbf{B}^* \leq \mathbf{B}, \tag{8.8a}$$

$$N(P^*_{2k} \text{ in } \mathbf{B}^*) \in \mathrm{Hall}_{\pi'}(G(\alpha^*_{2k}, \chi_1, \ldots, \chi_m)), \tag{8.8b}$$

$$N(Q^*_{2l-1} \text{ in } \mathbf{A}^*) \in \mathrm{Hall}_{\pi}(G(\beta^*_{2l-1}, \chi_1, \ldots, \chi_m)). \tag{8.8c}$$

*Proof.* We fix the integers $m, k$ and $l$, and the triangular set (8.5c) corresponding to the tower (8.5b). Then Corollary 5.157 implies that $P^*_{2k}$ and $Q^*_{2l-1}$ form a Hall system for $G^*_m$. According to (8.4b) the groups $\mathbf{A}^* = \mathbf{A} \cap G^*$ and $\mathbf{B}^* = \mathbf{B} \cap G^*$ form a Hall system for $G^*$. In view of (8.4c) the group $\mathbf{A}^*$ contains $P^*_{2k'}$, and hence contains $P^*_{2k} \leq P^*_{2k'}$. Similarly, $\mathbf{B}^*$ contains $Q^*_{2l-1}$. Since $P^*_{2k}$ and $Q^*_{2l-1}$ form a Hall system for $G^*_m$, it follows that

$$P^*_{2k} = \mathbf{A}^* \cap G^*_m \trianglelefteq \mathbf{A}^*, \tag{8.9a}$$

$$Q^*_{2l-1} = \mathbf{B}^* \cap G^*_m \trianglelefteq \mathbf{B}^*. \tag{8.9b}$$

The subgroup $G^*_m = G^* \cap G_m$ is normal in $G^*$. Hence conjugation by elements of $\mathbf{B}^*$ permutes among themselves the Hall $\pi$-subgroups of $G^*_m$. One of those Hall $\pi$-subgroups is $P^*_{2k}$. Since $G^*_m = P^*_{2k} Q^*_{2l-1}$, the normal subgroup $Q^*_{2l-1}$ of $\mathbf{B}^*$ acts transitively on those Hall $\pi$-subgroups. It follows that

$$\mathbf{B}^* = N(P^*_{2k} \text{ in } \mathbf{B}^*) \cdot Q^*_{2l-1}. \tag{8.10}$$

This implies that $N(P^*_{2k} \text{ in } \mathbf{B}^*) \cdot G^*_m / G^*_m$ is a Hall $\pi'$-subgroup of $G^*/G^*_m$. Since $N(P^*_{2k} \text{ in } \mathbf{B}^*) \cap$

$G^*_m = N(P^*_{2k} \text{ in } Q^*_{2l-1})$ is a Hall $\pi'$-subgroup of $N(P^*_{2k} \text{ in } G^*_m)$, we conclude that

$$N(P^*_{2k} \text{ in } \mathbf{B}^*) \in \mathrm{Hall}_{\pi'}(N(P^*_{2k} \text{ in } G^*)). \tag{8.11}$$

Now the group $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ normalizes $P^*_{2k}$, and thus normalizes $P^*_{2i} = P^*_{2k} \cap G_{2i}$ for all $i = 1, 2, \ldots, k$. As $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ is a subgroup of $\mathbf{B}^*$, it normalizes $Q^*_{2l-1} = \mathbf{B}^* \cap G_m$. So it normalizes $Q^*_{2j-1} = Q^*_{2l-1} \cap G_{2j-1}$ for each $j = 1, 2, \ldots, l$. Since it normalizes both $P^*_{2i}$ and $Q^*_{2i-1}$, it normalizes $P_{2i} = N(Q^*_{2i-1} \text{ in } P^*_{2i})$ (see (5.160)), for each $i = 1, 2, \ldots, k$. Similarly, it normalizes $Q_{2j-1} = N(P^*_{2j-2} \text{ in } Q^*_{2j-1})$ (see (5.161)), for each $j = 2, 3, \ldots, l$. It also normalizes $Q_1 = G_1$. The definitions of $Q_{2i-1,2j}$ and $P_{2r,2s-1}$ in (5.22a) and (5.22b) show that they, too, are normalized by $N(P^*_{2k} \text{ in } \mathbf{B}^*)$. Thus $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ normalizes every subgroup appearing in the triangles displayed as (5.20a) and (5.21a) in Chapter 5.

The group $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ also fixes all the characters $\chi_1, \chi_2, \ldots, \chi_m$, since $\mathbf{B}^*$ does. Because it also normalizes $Q_1$, $P_2$, $Q_3$, ..., it leaves invariant the $cQ_1$- , $cP_2$- , $cQ_3$-, ... correspondences in Table 5.5. Hence it fixes all the characters in that table. In particular, it fixes $\alpha_{2i}$, for $i = 1, 2, \ldots, k$ and $\beta_{2j-1}$, for $j = 1, 2, \ldots, l$. It also fixes all the characters $\alpha_{2i,2j-1}$ and $\beta_{2r-1,2s}$ in the displayed triangles (5.20b) and (5.21b). Because it fixes all the groups and characters entering into the definition of $\alpha^*_{2i}$, it also fixes that character for each $i = 1, 2, \ldots, k$. Similarly, it fixes $\beta^*_{2j-1}$ for $j = 1, 2, \ldots, l$.

At this point we know that $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ is a Hall $\pi'$-subgroup of $N(P^*_{2k} \text{ in } G^*)$ fixing $\alpha^*_{2k}$. Hence it is a Hall $\pi'$-subgroup of $G^*(\alpha^*_{2k})$. Since $G^* = G(\chi_1, \chi_2, \ldots, \chi_m)$, we get that (8.8b) follows immediately.

The proof of (8.8c) is similar, with the roles of $\pi$ and $\pi'$ interchanged. So we omit it. □

The proof of Proposition 8.7 implies

**Corollary 8.12.** *Both $N(P^*_{2k} \text{ in } \mathbf{B}^*)$ and $N(Q^*_{2l-1} \text{ in } \mathbf{A}^*)$ fix the characters $\alpha_{2i}$, for $i = 1, \ldots, k$, and $\beta_{2j-1}$, for $j = 1, \ldots, l$. They also fix $\alpha_{2l-2,2l-1}$ and $\beta_{2k-1,2k}$.*

With the above notation, we can now prove

**Theorem 8.13.** *Assume the series* (8.3a)*, the tower* (8.3b) *and the triangular set* (8.3c) *are fixed. Assume further, that $m$ is any integer with $1 \le m \le n$, and consider the series, tower and triangular set appearing in* (8.5) *for that $m$. Then we can choose a $\pi'$-Hall subgroup $\widehat{Q}$ of $G' = G(\alpha^*_{2k})$, to satisfy the conditions* (6.20) *and* (6.21) *in Theorem 6.19 for the set* (8.5c) *and the tower* (8.5b)*, along with the property*

$$N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \ldots, \chi_{2k})) = \widehat{Q}(\beta_{2k-1,2k}).$$

*Hence*

$$\widehat{Q}(\beta_{2k-1,2k}) \cdot Q^*_{2l-1} \le \mathbf{B}(\chi_1, \ldots, \chi_{2k}) \le \mathbf{B}. \tag{8.14}$$

*Proof.* Suppose first that $m = 2l-1$ is odd with $l \le l'$, so $2k = 2l-2$. Then (6.20b) tells us that the groups $G'(\beta_{2k-1,2k})$, $G'(\chi_1, \ldots, \chi_{2k})$, $G'(\chi_1, \ldots, \chi_{2k-1})$ and $G'(\beta_1, \beta_3, \ldots, \beta_{2k-1})$ have a common Hall $\pi'$-subgroup, (where $G' = G(\alpha^*_{2k})$). Proposition 8.7, and in particular (8.8b), with $2k = 2l-2$ in the place of $m$ there, implies that the $\pi'$-group $N(P^*_{2k} \text{ in } \mathbf{B}^*) = N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \ldots, \chi_{2k}))$ is a Hall $\pi'$-subgroup of the second group on this list. By Corollary 8.12 the character $\beta_{2k-1,2k}$ is fixed by $N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \ldots, \chi_{2k}))$. Hence the latter is a subgroup of $G'(\beta_{2k-1,2k})$. So it must be a Hall $\pi'$-subgroup of that group, because of its order. Similarly, it is contained in both $G'(\chi_1, \ldots, \chi_{2k-1})$ and $G'(\beta_1, \ldots, \beta_{2k-1})$. Hence it is a Hall $\pi'$-subgroup of those groups, too. So it satisfies all the conditions for $\widehat{Q}(\beta_{2k-1,2k})$ in (6.20b). Clearly it also satisfies the equations (6.20c), as these follow from (6.20b). Furthermore, $N(P^*_{2k}$ in $\mathbf{B}(\chi_1,\dots,\chi_{2k}))$ fixes $\alpha_2,\alpha_4,\dots,\alpha_{2k}$, by Corollary 8.12, and thus satisfies equation (6.20d).

According to (8.8a) (for $m=2l-1$), the group $Q_{2l-1}$ is a subgroup of $\mathbf{B}(\chi_1,\dots,\chi_{2k},\chi_{2l-1}) \leq \mathbf{B}(\chi_1,\dots,\chi_{2k})$. Furthermore, $Q_{2l-1}$ normalizes $P^*_{2k}=P^*_{2l-2}$, by (5.10a). Thus $Q_{2l-1}$ is a subgroup of $N(P^*_{2k}$ in $\mathbf{B}(\chi_1,\dots,\chi_{2k}))$. By Corollary 8.12, the latter normalizes $G(\alpha_2,\dots,\alpha_{2k},\beta_1,\dots,\beta_{2k-1})$. Hence it normalizes $G_{2l-1}(\alpha_2,\dots,\alpha_{2k},\beta_1,\dots,\beta_{2k-1}) = Q_{2l-1}\rtimes P_{2k}$, where the equality follows from (5.42c) as $2k=2l-2$. Hence $N(P^*_{2k}$ in $\mathbf{B}(\chi_1,\dots,\chi_{2k}))$ normalizes $Q_{2l-1}\rtimes P_{2k}$ and contains $Q_{2l-1}$. Therefore, it normalizes $Q_{2l-1}$. Thus it satisfies (6.21). Evidently we can choose $\widehat{Q}\in \mathrm{Hall}_q(G')$ so that $N(P^*_{2k}$ in $\mathbf{B}(\chi_1,\dots,\chi_{2k}))=\widehat{Q}(\beta_{2k-1,2k})$.

So $\widehat{Q}(\beta_{2k-1,2k}) \leq \mathbf{B}(\chi_1,\dots,\chi_{2k})$. But $Q^*_{2l-1}$ is contained in $\mathbf{B}(\chi_1,\dots,\chi_{2l-1})$, by (8.8a)and (8.6), as $m=2l-1$. As $2k=2l-2<2l-1$ in the odd case, $\mathbf{B}(\chi_1,\dots,\chi_{2l-1})\leq \mathbf{B}(\chi_1,\dots,\chi_{2k})$. Hence Theorem 8.13 follows for any odd $m$.

If $m=2k$ is even and strictly smaller than $n$, then we can still form the $2k+1$ series, by adding the group $G_{2k+1}$ and its character $\chi_{2k+1}$. Then, according to Corollary 6.24, the even system, (where $m=2k$), with the odd, (where $m=2k+1$), share the group $\widehat{Q}$. This, along the already proved odd case of Theorem 8.13, implies the first part of Theorem 8.13 when $m=2k<n$.

If $m=2k=n$, we can't form a bigger odd system, but we know exactly what group $\widehat{Q}(\beta_{2k-1,2k})$ is. Indeed, as $G_{2k}/G_{2k-1}$ is a $\pi$-group, and $\widehat{Q}(\beta_{2k-1,2k})$ is a $\pi'$-Hall subgroup of $G'(\beta_{2k-1,2k})$, by (6.20b), it must be a $\pi'$-Hall subgroup of $G'_{2k-1}(\beta_{2k-1,2k})$. This, along with (6.29b), implies

$$\widehat{Q}(\beta_{2k-1,2k}) = \widehat{Q}(\beta_{2k-1,2k})\cap G_{2k-1} = \widehat{Q}_{2k-1}(\beta_{2k-1,2k}).$$

By (6.33) this gives

$$\widehat{Q}(\beta_{2k-1,2k}) = Q_{2k-1,2k}.$$

On the other hand, in the case $m=2k=n$ we have $2l'-1=2k-1$. Thus (8.4c) implies $\mathbf{B}(\chi_1,\dots,\chi_n)=Q^*_{2k-1}$. Hence

$$N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1,\dots,\chi_{2k})) = N(P^*_{2k} \text{ in } Q^*_{2k-1}) = Q_{2k-1,2k}.$$

So the first part of Theorem 8.13 holds in the case $m=2k=n$.

Furthermore, when $m=2k$ is even, (8.8a) and (8.6) imply that $Q^*_{2l-1}\leq \mathbf{B}(\chi_1,\dots,\chi_{2k})$. Thus (8.14) follows for the even case. This completes the proof of Theorem 8.13 □

Of course, a similar result holds by $p,q$-symmetry for $\widehat{P}(\alpha_{2l-2,2l-1})\cdot P^*_{2l-2}$, whenever $l=2,3,\dots,l'$.

**Theorem 8.15.** *Assume the series* (8.3a) *the tower* (8.3b) *and the triangular set* (8.3c) *are fixed. Assume further that $m$ is any integer with $1\leq m\leq n$ and consider the series of subgroups, the character tower, and the triangular set appearing in* (8.5) *for that $m$. Then we can choose a $\pi$-Hall subgroup $\widehat{P}$ of $G(\beta^*_{2l-1})$, to satisfy* (8.1) *for the set* (8.5c) *and the tower* (8.5b), *along with the property*

$$N(Q^*_{2l-1} \text{ in } \mathbf{A}(\chi_1,\dots,\chi_{2l-1})) = \widehat{P}(\alpha_{2l-2,2l-1}).$$

*Hence*

$$\widehat{P}(\alpha_{2l-2,2l-1})\cdot P^*_{2k} \leq \mathbf{A}(\chi_1,\dots,\chi_{2l-1}) \leq \mathbf{A}. \tag{8.16}$$

## 8.3 "Shifting " properties

We assume that the normal series

$$G_0 = 1 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{8.17}$$

is fixed for some $n \geq 2$, and satisfies

$$G_2 = G_{2,\pi} \times G_{2,\pi'} \text{ is a } \pi\text{-split group (see Definition 5.163)} \quad \text{and} \tag{8.18a}$$
$$G \text{ fixes the character } \chi_1. \tag{8.18b}$$

Assume also that the character tower

$$\{1 = \chi_0, \chi_1, \ldots, \chi_n\} \tag{8.18c}$$

is fixed, while the set

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l',k'} \tag{8.18d}$$

is a representative of the conjugacy class of triangular sets that corresponds to (8.18c). Furthermore, we fix a Hall system $\{\mathbf{A}, \mathbf{B}\}$ of $G$ that satisfies (8.4), that is

$$\mathbf{A} \in \mathrm{Hall}_{\pi}(G), \mathbf{B} \in \mathrm{Hall}_{\pi'}(G), \tag{8.18e}$$
$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_h) \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_h) \text{ form a Hall system for } G(\chi_1, \chi_2, \ldots, \chi_h), \tag{8.18f}$$
$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_n) = P^*_{2k'} \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_n) = Q^*_{2l'-1}, \tag{8.18g}$$

for any $h = 1, 2, \ldots, n$. According to Corollary 5.177 this set satisfies

$$G_2 = P_2 \times Q_1, \tag{8.19a}$$
$$\chi_2 = \chi_{2,\pi} \times \chi_{2,\pi'} = \alpha_2 \times \beta_1, \tag{8.19b}$$

where $P_2$ and $Q_1 = G_1$ are the $\pi$-and $\pi'$-Hall subgroups respectively, of $G_2$.

We replace the first $\pi'$-group $G_1 = Q_1$ appearing in (8.17), by the trivial group and consider the series

$$1 \trianglelefteq G^s_1 := 1 \trianglelefteq G^s_2 := P_2 \trianglelefteq G^s_3 := G_3 \trianglelefteq G^s_4 := G_4 \trianglelefteq \cdots \trianglelefteq G^s_n := G_n = G. \tag{8.20a}$$

We call the series (8.20a) a shifting of the series (8.17). Note that (8.20a) is a normal series of $G$, that satisfies Hypothesis 5.1 with $G^s_1 = 1$. The characters

$$1, \chi^s_1 := 1, \chi^s_2 := \alpha_2, \chi^s_3 := \chi_3, \chi^s_4 := \chi_4, \ldots, \chi^s_n := \chi_n, \tag{8.20b}$$

form a character tower for the series (8.20a). In addition, the set

$$\{Q^s_1 = 1 = P^s_0, Q^s_{2i-1} = Q_{2i-1}, P^s_{2r} = P_{2r} | \beta^s_1 = 1 = \alpha^s_0, \beta^s_{2i-1} = \beta_{2i-1}, \alpha^s_{2r} = \alpha_{2r}\}_{i=2,r=1}^{l',k'} \tag{8.20c}$$

is a triangular set for (8.20a), corresponding to the character tower (8.20b) (this can be very easily verified using the fact that (8.18d) is a triangular set for (8.17) corresponding to (8.18c)). Note that the groups $Q^s_{2i-1,2j}, P^s_{2r,2t+1}$ and their characters $\beta^s_{2i-1,2j}$ and $\alpha^s_{2r,2t+1}$, respectively, remain the same as those of (8.18), whenever $2 \leq i \leq j \leq k'$ and $2 \leq r \leq t \leq l' - 1$, i.e.,

$$Q^s_{2i-1,2j} = Q_{2i-1,2j} \text{ and } P^s_{2r,2t+1} = P_{2r,2t+1}, \tag{8.21}$$

$$\beta^s_{2i-1,2j} = \beta_{2i-1,2j} \text{ and } \alpha^s_{2r,2t+1} = \alpha_{2r,2t+1}. \tag{8.22}$$

Also, the product groups $P^{*,s}_{2k} = P^s_2 \cdots P^s_{2k}$, remain unchanged, for every $k = 1, \dots, k'$, because $P^s_{2r} = P_{2r}$ whenever $1 \leq r \leq k'$. In addition, for any such $k$, the irreducible character $\alpha^*_{2k} \in \mathrm{Irr}(P^*_{2k})$ was chosen as the $Q_3, \dots, Q_{2k-1}$-correspondent of $\alpha_{2k}$ (see Definition 5.147). As neither of the above groups nor the character $\alpha_{2k}$ changes when passing to the shifted system (8.20), we conclude that also the character $\alpha^{*,s}_{2k} \in \mathrm{Irr}(P^{*,s}_{2k})$ remains unchanged, that is,

$$\alpha^{*,s}_{2k} = \alpha^*_{2k}, \tag{8.23}$$

for all $k = 1, \dots, k'$.

Furthermore, the group $Q_3$ contains $Q_1$, as $Q_3 \geq Q_{1,2} = C(P_2 \text{ in } Q_1) = Q_1$, by (8.19b). Thus

$$Q^*_{2l-1} = Q_3 \cdot Q_5 \cdots Q_{2l-1}, \tag{8.24}$$

whenever $1 < l \leq l'$. This implies that

$$Q^*_{2l-1} = Q^s_3 \cdot Q^s_5 \cdots Q^s_{2l-1} = Q^{*,s}_{2l-1}, \tag{8.25}$$

for all such $l$. Furthermore, the fact that $P_2$ centralizes both $Q_1$ and $N(P_2 \text{ in } Q^*_{2l-1}) = Q^*_{2l-1}$ implies that the $P_2, P_4, \dots, P_{2l-2}$-correspondent $\beta^*_{2l-1} \in \mathrm{Irr}(Q^*_{2l-1})$ of $\beta_{2l-1} \in \mathrm{Irr}(Q_{2l-1})$, is actually the $P_4, \dots, P_{2l-2}$-correspondent of $\beta_{2l-1}$, whenever $1 < l \leq l'$. Clearly, for all such $l$, we get $\beta^{*,s}_{2l-1} = \beta^*_{2l-1}$, because $Q^{*,s}_{2l-1} = Q^*_{2l-1}, P^s_2 = P_2, \dots, P^s_{2l-2} = P_{2l-2}$ and $\beta^s_{2l-1} = \beta_{2l-1}$.

As $G$ fixes $\chi_1$ by (8.18b), while $\chi_2 = \alpha_2 \times \beta_1$ by (8.19d), we get that $G = G(\chi_1) = G(\beta_1)$. We conclude that

$$G(\chi^s_1, \chi^s_2) = G(\alpha_2) = G(\chi_1, \chi_2), \tag{8.26a}$$

and thus

$$G(\chi^s_1, \chi^s_2, \chi^s_3, \dots, \chi^s_h) = G(\alpha_2, \chi_3, \dots, \chi_h) = G(\chi_1, \chi_2, \chi_3, \chi_4, \dots, \chi_h), \tag{8.26b}$$

for all $h = 3, \dots, n$. For any subgroup $H$ of $G$, similar equations, with $H$ in place of $G$, hold. In particular, for the Hall system $\{\mathbf{A}, \mathbf{B}\}$ we get

$$\begin{aligned}
\mathbf{A}(\chi^s_1, \chi^s_2) = \mathbf{A}(\alpha_2) = \mathbf{A}(\chi_1, \chi_2) &\text{ and } \mathbf{B}(\chi^s_1, \chi^s_2) = \mathbf{B}(\alpha_2) = \mathbf{B}(\chi_1, \chi_2),\\
\mathbf{A}(\chi^s_1, \chi^s_2, \dots, \chi^s_h) &= \mathbf{A}(\alpha_2, \chi_3, \dots, \chi_h) = \mathbf{A}(\chi_1, \chi_2, \dots, \chi_h) \text{ and}\\
\mathbf{B}(\chi^s_1, \chi^s_2, \dots, \chi^s_h) &= \mathbf{B}(\alpha_2, \chi_3, \dots, \chi_h) = \mathbf{B}(\chi_1, \chi_2, \dots, \chi_h),
\end{aligned}$$

for all $h = 3, \dots, n$. This, along with teh conditions (8.18e,f,g) which $\mathbf{A}$ and $\mathbf{B}$ satisfy, implies

$$\begin{gathered}
\mathbf{A}(\alpha_2) = \mathbf{A}(\chi^s_1, \chi^s_2) \text{ and } \mathbf{B}(\alpha_2) = \mathbf{B}(\chi^s_1, \chi^s_2) \text{ form a Hall system for } G(\alpha_2) = G(\chi^s_1, \chi^s_2)\\
\mathbf{A}(\chi^s_1, \chi^s_2, \dots, \chi^s_h) \text{ and } \mathbf{B}(\chi^s_1, \chi^s_2, \dots, \chi^s_h) \text{ form a Hall system for } G(\chi^s_1, \chi^s_2, \dots, \chi^s_h),
\end{gathered}$$

for any $h = 3, \dots, n$. Furthermore, (8.18g), along with (8.24), implies

$$\mathbf{A}(\chi^s_1, \chi^s_2, \dots, \chi^s_n) == P^*_{2k'} \text{ and } \mathbf{B}(\chi^s_1, \chi^s_2, \dots, \chi^s_n) = Q_3 \cdots Q_{2l'-1}. \tag{8.27}$$

Therefore, the groups $\mathbf{A}, \mathbf{B}$ satisfy the equivalent of (8.18e,f,g), for the shifted system (8.20).

The other groups of interest that doesn't change, when we work in the shifted case (8.20), are $\widehat{Q}$ and $\widehat{P}$, as these are defined for every fixed, but arbitrary, smaller system

$$\begin{gathered} 1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G, \\ 1, \chi_1, \chi_2, \ldots, \chi_m, \\ \{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l,k} \end{gathered} \tag{8.28a}$$

where $m = 3, \ldots, n$, and $k, l$ are related to $m$ via (5.7). Of course when we shift the original system to get (8.20), we also get the smaller shifted system

$$\begin{gathered} 1 \trianglelefteq G_1^s = 1 \trianglelefteq G_2^s = P_2 \trianglelefteq G_3^s = G_3 \trianglelefteq \cdots \trianglelefteq G_m^s = G_m \trianglelefteq G, \\ 1, \chi_1^s, \chi_2^s, \ldots, \chi_m^s, \\ \{Q_{2i-1}^s, P_{2r}^s | \beta_{2i-1}^s, \alpha_{2r}^s\}_{i=1,r=0}^{l,k} \end{gathered} \tag{8.28b}$$

Indeed, it is easy to see that the same group $\widehat{Q}$, which was picked among the $\pi'$-Hall subgroups of $G(\alpha_{2k}^*)$ to satisfy the conditions (6.20) and (6.21) in Theorem 6.19 for the system (8.28), satisfies the same conditions for the shifted system (8.28). First note that $G' = G(\alpha_{2k}^*) = G(\alpha_{2k}^{*,s})$. Hence $\widehat{Q}$ is also a $\pi'$-Hall subgroup of $G(\alpha_{2k}^{*,s})$. Furthermore, the group $\widehat{Q}$ fixes $\beta_1$, as the latter is $G$-invariant. This forces $\widehat{Q}$ to fix the $P_{2k}^*$-Glauberman correspondent $\beta_{1,2k}$ of $\beta_1$, as $\widehat{Q}$ normalizes $P_{2k}^*$. Hence $\widehat{Q}(\beta_{1,2k}) = \widehat{Q}$. For the shifted system (8.28) we have $\beta_1^s = 1$. Thus the $P_{2k}^{*,s} = P_{2k}^*$-Glauberman correspondent $\beta_{1,2k}^s$ of $\beta_1^s$ is also trivial. Hence $\widehat{Q}(\beta_{1,2k}^s) = \widehat{Q} = \widehat{Q}(\beta_{1,2k})$. In addition, $\beta_{2i-1,2k}^s = \beta_{2i-1,2k}$, for any $i = 2, \ldots, l$ We conclude that

$$\begin{aligned} &\widehat{Q}(\beta_{2i-1,2k}^s) \in \mathrm{Hall}_{\pi'}(G'(\beta_{2i-1,2k}^s)) \cap \mathrm{Hall}_{\pi'}(G'(\chi_1^s, \chi_2^s, \chi_3^s, \ldots, \chi_{2i-1}^s)) \cap \\ &\qquad\qquad \mathrm{Hall}_{\pi'}(G'(\chi_1^s, \chi_2^s, \chi_3^s, \ldots, \chi_{2i}^s)) \cap \mathrm{Hall}_{\pi'}(G'(\beta_1^s, \beta_3^s, \ldots, \beta_{2i-1}^s)), \\ &\widehat{Q}(\beta_{2i-1,2k}^s) = \widehat{Q}(\chi_1^s, \chi_2^s, \chi_3^s, \ldots, \chi_{2i-1}^s) = \widehat{Q}(\chi_1^s, \chi_2^s, \chi_3^s, \ldots, \chi_{2i}^s) = \widehat{Q}(\beta_1^s, \beta_3^s, \beta_5^s, \ldots, \beta_{2i-1}^s) \text{ and} \\ &\qquad \widehat{Q}(\chi_1^s, \chi_2^s, \chi_3^s, \ldots, \chi_{2i-1}^s) \leq \widehat{Q}(1, \alpha_2^s, \ldots, \alpha_{2i}^s) \end{aligned}$$

for all $i = 1, 2, \ldots, k$. In addition, for all $i$ with $1 \leq i \leq l-1$ we get

$$\widehat{Q}(\beta_{2i-1,2k}^s) \text{ normalizes } Q_{2i+1}^s = Q_{2i+1}.$$

So $\widehat{Q}$ remains unchanged in the shifted case, as does $\widehat{Q}(\beta_{2k-1,2k})$. Therefore the image $I$ of $\widehat{Q}(\beta_{2k-1})$ in $\mathrm{Aut}(P_{2k}^*)$ remains unchanged.

Similarly, we can show that the group $\widehat{P}$ remains unchanged in the shifted case, as does $\widehat{P}(\alpha_{2l-2,2l-1})$. So the image of the latter group in $\mathrm{Aut}(Q_{2l-1}^*)$ remains unchanged.

It is also clear that if the characters $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ and $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extend to $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$, respectively, then the same property passes to the shifted case, provided that $k \geq 2$, as none of these groups and characters really changes. In conclusion we have

**Theorem 8.29.** *Assume that the normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G$ satisfies* (8.18). *Let* (8.18c) *be a character tower for the series,* (8.18d) *the corresponding triangular set and $\{\mathbf{A}, \mathbf{B}\}$ a Hall system for $G$ that satisfies* (8.18e). *Replacing the group $G_1 = Q_1$ with the trivial group we obtain a normal series* (8.20a) *for $G$. Then* (8.20b) *is a character tower for that series, and* (8.20c) *its corresponding triangular set. Furthermore, $\{\mathbf{A}, \mathbf{B}\}$ remains a Hall system for $G$ satis-*

*fying the equivalent of (8.18e,f,g) for the series* (8.20a) *and tower* (8.20b)*. Even more, the groups* $P^*_{2k}, Q^*_{2l-1}, \widehat{Q}, \widehat{P}$*, as well as* $\widehat{Q}(\beta_{2k-1,2k})$ *and* $\widehat{P}(\alpha_{2l-2,2l-1})$ *satisfy the same conditions for the smaller system* (8.28) *and the shifted one* (8.28)*, whenever* $m = 3, \ldots, n$*.*

The above theorem makes clear that, whenever the series (8.17) satisfies (8.18a,b), we can replace the group $G_1$ with a trivial group without affecting any other group or character involved in our constructions. From now on, for simplicity, whenever such a shifting is performed, we will be writing the produced series, tower and triangular set of (8.20) as

$$1 \trianglelefteq P_2 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{8.30a}$$

$$\{1, \alpha_2, \chi_3, \ldots, \chi_n\} \tag{8.30b}$$

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=2,r=1}^{l',k'} \tag{8.30c}$$

Note that the trivial groups $G_1^s = Q_1^s = 1 = P_0^s$ and their characters, have been dropped.

# Chapter 9

# Normal Subgroups

As in Chapter 8, we fix a normal chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{9.1a}$$

for an odd order group $G$, such that Hypothesis 5.1 holds with $n$ in the place of $m$, i.e., $n > 0$ and $G_i/G_{i-1}$ is a $\pi$-group if $i$ is even, and a $\pi'$-group if $i$ is odd, for each $i = 1, 2, \dots, n$. We also fix a character tower

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{n} \tag{9.1b}$$

for the above series and a corresponding triangular set

$$\{P_{2r}, Q_{2i-1} | \alpha_{2r}, \beta_{2i-1}\}_{r=0,i=1}^{k',l'} \tag{9.1c}$$

where $k' = [n/2]$ and $l' = [(n+1)/2]$. Along with that we fix a Hall system $\mathbf{A}, \mathbf{B}$ of $G$ that satisfies (8.4), that is,

$$\mathbf{A} \in \mathrm{Hall}_{\pi}(G), \mathbf{B} \in \mathrm{Hall}_{\pi'}(G), \tag{9.2a}$$

$$\mathbf{A}(\chi_1, \chi_2, \dots, \chi_h) \text{ and } \mathbf{B}(\chi_1, \chi_2, \dots, \chi_h) \text{ form a Hall system for } G(\chi_1, \chi_2, \dots, \chi_h), \tag{9.2b}$$

$$\mathbf{A}(\chi_1, \chi_2, \dots, \chi_n) = P^*_{2k'} \text{ and } \mathbf{B}(\chi_1, \chi_2, \dots, \chi_n) = Q^*_{2l'-1}, \tag{9.2c}$$

whenever $h = 1, \dots, n$.

## 9.1 Normal $\pi'$-subgroups inside $Q_1$

We fix an integer $m = 1, \dots, n$ and we consider the normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G. \tag{9.3a}$$

The sub tower

$$\{1 = \chi_0, \chi_1, \dots, \chi_m\} \tag{9.3b}$$

of (9.1b) is a character tower for (9.3b), and the subset

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l,\,k} \tag{9.3c}$$

of (9.1c) is a representative of the conjugacy class of triangular sets that corresponds uniquely to (9.3b). (As usual, the integers $k$ and $l$ are related to $m$ via (5.7).) Thus all the groups, the characters and their properties that were defined and proved in Chapter 5 are valid for (9.3). In particular we can define groups $G_{i,s}$ and their characters $\chi_{i,s}$ (see Theorem 5.88 for their properties). Even more, we can pick the groups $\widehat{Q}$ and $\widehat{P}$ to satisfy the conditions in Theorems 8.13 and 8.15 respectively. Hence if we write

$$\mathcal{P} := \widehat{P}(\alpha_{2l-2,2l-1}) \cdot P^*_{2k}, \tag{9.4a}$$

$$\mathcal{Q} := \widehat{Q}(\beta_{2k-1,2k}) \cdot Q^*_{2l-1}, \tag{9.4b}$$

then Theorems 8.13 and 8.15, and in particular (8.14) and (8.16), imply

$$\mathcal{P} \leq \mathbf{A}(\chi_1, \ldots, \chi_{2l-1}) \leq \mathbf{A} \tag{9.4c}$$

$$\mathcal{Q} \leq \mathbf{B}(\chi_1, \ldots, \chi_{2k}) \leq \mathbf{B}. \tag{9.4d}$$

We remark that the group $\mathcal{Q}$ is well defined, as $\widehat{Q}(\beta_{2k-1,2k}) \leq \widehat{Q}(\beta_{2i-1,2k})$ normalizes the group $Q_{2i+1}$, for all $i = 1, \ldots, l-1$, by (6.21). It also normalizes $Q_1$. Thus it normalizes their product $Q_1 \cdot Q_3 \cdots Q_{2l-1} = Q^*_{2l-1}$. Similarly we can show that $\mathcal{P}$ is well defined.

For the rest of this section we assume that $S$ is a subgroup of $G_1$, and $\zeta$ is a character of $S$, satisfying

$$S \trianglelefteq G \text{ and } S \leq G_1, \tag{9.5a}$$

$$\zeta \in \mathrm{Irr}(S) \text{ is } G\text{-invariant and lies under } \beta_1. \tag{9.5b}$$

Either $S$ or $\zeta$ may be trivial. We also assume that $E$ is a normal subgroup of $G$ with

$$S \leq E \leq Q_1 = G_1. \tag{9.5c}$$

Then

**Lemma 9.6.** *There is an irreducible character $\lambda \in \mathrm{Irr}(E)$ such that $\lambda$ is $\mathbf{A}(\chi_1)$-invariant and lies under every $\chi_i$, for $i = 1, \ldots, n$. Any such $\lambda$ lies above $\zeta$.*

*Proof.* Let $\lambda_1$ be an irreducible character of $E$ lying under $\chi_1$, and thus under $\chi_i$ for any $i = 1, \ldots, n$. Then Clifford's Theorem implies that $\chi_1$ lies above the $G_1$-conjugacy class of $\lambda_1$. The $\pi$-group $\mathbf{A}(\chi_1)$ fixes $\chi_1$, and normalizes $E$, as the latter is normal in $G$. Hence $\mathbf{A}(\chi_1)$ permutes among themselves the $G_1$-conjugates of $\lambda_1$. As $(|\mathbf{A}(\chi_1)|, |G_1|) = 1$, Glauberman's Lemma (Lemma 13.8 in [12]) implies that $\mathbf{A}(\chi_1)$ fixes at least one, $\lambda$, of the $G_1$-conjugates of $\lambda_1$.

As $\zeta \in \mathrm{Irr}(S)$ is $G$-invariant and lies under $\chi_1$, Clifford's theorem implies that any irreducible character of $E$ lying under $\chi_1$ also lies above $\zeta$. Thus the character $\lambda$ satisfies all the conditions in the lemma. ∎

Note that the proof of Lemma 9.6 also shows

**Remark 9.7.** Assume that $\lambda_1 \in \mathrm{Lin}(E)$ is a linear character of $E$ lying under $\chi_1$. Then there exists a $G_1$-conjugate $\lambda \in \mathrm{Lin}(E)$ of $\lambda_1$, such that $\lambda$ is $A(\chi_1)$-invariant, and lies under $\chi_1$ and above $\zeta$.

**Remark 9.8.** The $\pi$-group $P^*_{2k'} = P_2 \cdots P_{2k'}$ fixes $\chi_1$, as every one of its factors $P_{2i}$ fixes $\chi_1 = \beta_1$, for all $i = 1, \ldots, k'$. Hence it is a subgroup of $\mathbf{A}(\chi_1)$. So $P^*_{2i}$ fixes $\lambda$, for all $i = 1, \ldots, k'$.

In the same spirit is

**Remark 9.9.** The $\pi$-group $\mathcal{P}$ fixes $\lambda$, as it is a subgroup of $\mathbf{A}(\chi_1)$, by (9.4c).

From now on we fix a character $\lambda \in \mathrm{Irr}(E)$ satisfying the conditions in Lemma 9.6.

The $\pi$-group $P_{2i}^*$ acts on the $\pi'$-group $E$, and fixes $\lambda$ by Remark 9.8, whenever $1 \leq i \leq k'$. So we can define

$$E_{2i} = C(P_{2i}^* \text{ in } E) \text{ and} \tag{9.10a}$$

$$\lambda_{2i} \in \mathrm{Irr}(E_{2i}) \text{ is the } P_{2i}^*\text{-Glauberman correspondent of } \lambda, \tag{9.10b}$$

for all $i = 1, \ldots, k'$. We obviously have that

$$E_{2i} = Q_{1,2i} \cap E = C(P_{2i}^* \text{ in } Q_1) \cap E, \tag{9.11}$$

for each such $i$. Furthermore,

$$\lambda_{2i} \text{ lies under } \beta_{1,2i}, \tag{9.12}$$

as $\beta_{1,2i}$ is the $P_{2i}^*$-Glauberman correspondent of $\beta_1$, and $\beta_1 = \chi_1$ lies over $\lambda$. It follows from the definition (9.10b) of $\lambda_{2i}$ that

$$N(\lambda) = N(\lambda_{2i}), \tag{9.13}$$

for every group $N$ with $N \leq N(P_{2i}^* \text{ in } G)$.

We also define

$$G_\lambda := G(\lambda), \tag{9.14a}$$

$$G_{i,\lambda} := G_i(\lambda) = G_\lambda \cap G_i, \tag{9.14b}$$

whenever $0 \leq i \leq n$. This way we can form the series

$$G_{0,\lambda} = G_0 = 1 \trianglelefteq G_{1,\lambda} \trianglelefteq G_{2,\lambda} \trianglelefteq \cdots \trianglelefteq G_{n,\lambda} = G_\lambda \tag{9.15}$$

of normal subgroups of the stabilizer $G_\lambda$ of $\lambda$ in $G$. Its is clear that this series satisfies Hypothesis 5.1 with $n$ in the place of $m$, as the series (9.1a) does. Furthermore, Clifford's Theorem provides unique irreducible characters $\chi_{i,\lambda} \in \mathrm{Irr}(G_{i,\lambda})$ lying above $\lambda$ and inducing $\chi_i$, whenever $i = 0, 1, \ldots n$, i.e.,

$$\chi_{i,\lambda} \in \mathrm{Irr}(G_{i,\lambda}|\lambda) \text{ and } (\chi_{i,\lambda})^{G_i} = \chi_i. \tag{9.16a}$$

Clearly we get that

$$\chi_{0,\lambda} = \chi_0 = 1. \tag{9.16b}$$

As the $\chi_i$ lie above each other, the same holds for the characters $\chi_{i,\lambda}$, i.e., $\chi_{i,\lambda}$ lies above $\chi_{j,\lambda}$ whenever $0 \leq j \leq i \leq n$. This way we have formed a character tower

$$\{1 = \chi_{0,\lambda}, \chi_{1,\lambda}, \ldots, \chi_{n,\lambda}\} \tag{9.17a}$$

for the series (9.15). Hence Theorem 5.6, applied to the tower (9.17a), implies the existence of a unique $G_\lambda$-conjugacy class of triangular sets for (9.15) that correspond to the tower (9.17a). Let

$$\{Q_{2i-1,\lambda}, P_{2r,\lambda} | \beta_{2i-1,\lambda}, \alpha_{2r,\lambda}\}_{i=1,r=0}^{l',\,k'} \tag{9.17b}$$

be a representative of this class. All the groups, the characters and their properties that were described in Chapter 5 are valid for the $\lambda$-situation. We follow the same notation as in Chapter 5, with the addition of an extra $\lambda$ in the subscripts, to refer to this $\lambda$-situation. As a small sample we give the following list:

$$
\begin{gathered}
Q_{2r-1,2r,\lambda} = C(P_{2r,\lambda} \text{ in } Q_{2r-1,\lambda}), \text{ see (5.11)}, \\
\beta_{2r-1,2r,\lambda} \in \mathrm{Irr}(Q_{2r-1,2r,\lambda}) \text{ is the } P_{2r,\lambda}\text{-Glauberman correspondent of } \beta_{2r-1,\lambda} \in \mathrm{Irr}(Q_{2r-1,\lambda}) , \\
P_{2i,2i+1,\lambda} = C(Q_{2i+1,\lambda} \text{ in } P_{2i,\lambda}), \text{ by (5.14)}, \\
\alpha_{2i,2i+1,\lambda} \in \mathrm{Irr}(P_{2i,2i+1,\lambda}) \text{ is the } Q_{2i+1,\lambda}\text{-Glauberman correspondent of } \alpha_{2i,\lambda} \in \mathrm{Irr}(P_{2i,\lambda}) , \\
G_{2i,2i-1,\lambda} = N(P_{2,\lambda}, \dots, P_{2i-2,\lambda}, Q_{1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } G_{2i,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i-1,\lambda})), \text{ see (5.91)}, \\
\chi_{2i,2i-1,\lambda} \text{ is the } cP_{2,\lambda}, \dots, cP_{2i-2,\lambda}, cQ_{1,\lambda}, \dots, cQ_{2i-1,\lambda}\text{-correspondent of } \chi_{2i,\lambda}, \text{ see} \\
\text{Theorem 5.88 and the Definition 5.63}
\end{gathered}
$$

for all $r = 1, \dots, k'$ and $i = 1, \dots, l' - 1$.

In this section we will describe the relations between the sets (9.1c) and (9.17b). We start with the groups $G^*_{i,\lambda}$ defined as

$$
\begin{aligned}
G^*_{0,\lambda} &:= G_{0,\lambda} = 1, \\
G^*_{i,\lambda} &:= G_{i,\lambda}(\chi_{1,\lambda}, \dots \chi_{i-1,\lambda}) = G_{i,\lambda}(\chi_{1,\lambda}, \dots, \chi_{n,\lambda}), \\
G^*_{\lambda} &:= G_{\lambda}(\chi_{1,\lambda}, \dots, \chi_{n,\lambda}),
\end{aligned} \tag{9.18}
$$

whenever $i = 1, \dots, n$. Note that this is equivalent to the definitions of $G^*_i$ and $G^*$ that were given in (5.129b) and (5.129a), respectively. Obviously we have that

$$G_{i,\lambda} = G^*_{\lambda} \cap G_i, \tag{9.19}$$

for all $i = 0, 1, \dots, n$. Furthermore,

**Lemma 9.20.**

$$G^*_{\lambda} = G^*(\lambda), \tag{9.21a}$$
$$G^*_{i,\lambda} = G^*_i(\lambda), \tag{9.21b}$$

*for all* $i = 0, 1, \dots, n$.

*Proof.* The fact that $\chi_{i,\lambda} \in \mathrm{Irr}(G_{i,\lambda}|\lambda) = \mathrm{Irr}(G_i(\lambda)|\lambda)$ is the $\lambda$-Clifford correspondent of $\chi_i \in \mathrm{Irr}(G_i)$ implies that

$$G(\lambda, \chi_{1,\lambda}, \dots, \chi_{i,\lambda}) = G(\chi_1, \dots, \chi_i)(\lambda),$$

But $G^* = G(\chi_1, \dots, \chi_n)$, by (5.129a), while $G(\lambda) = G_{\lambda}$ by (9.14a). Thus (9.21a) follows.

Equation (9.21b) follows easily from (9.21a) and (9.19). $\square$

We can now prove

**Proposition 9.22.** *For every* $r = 0, 1, \dots, k'$ *and* $i = 1, \dots, l'$ *we have that*

$$P^*_{2r} \in \mathrm{Hall}_{\pi}(G^*_{2r,\lambda}), \tag{9.23a}$$
$$Q^*_{2i-1}(\lambda) \in \mathrm{Hall}_{\pi'}(G^*_{2i-1,\lambda}). \tag{9.23b}$$

*Therefore the triangular set* (9.17b) *can be chosen among the sets in its* $G_\lambda$*-conjugacy class so that it satisfies*

$$P^*_{2r} = P^*_{2r,\lambda}, \tag{9.24a}$$

$$Q^*_{2i-1}(\lambda) = Q^*_{2i-1,\lambda}, \tag{9.24b}$$

*whenever* $0 \le r \le k'$ *and* $1 \le i \le l'$.

*Proof.* According to Proposition 5.132, the group $P^*_{2k'}$ is a $\pi$-Hall subgroup of $G^*_{2k'}$. Furthermore, $\lambda$ is fixed by $P^*_{2k'} = \mathbf{A}(\chi_1, \dots, \chi_n) \le \mathbf{A}(\chi_1)$. Hence, $P^*_{2k'}$ is also a $\pi$-Hall subgroup of $G^*_{2k'}(\lambda)$. As $P^*_{2r} = P^*_{2k'} \cap G_{2r}$, while $G^*_{2r}(\lambda) = G^*_{2k'}(\lambda) \cap G_{2r} \trianglelefteq G^*_{2k'}(\lambda)$, we also get that $P^*_{2r}$ is a $\pi$-Hall subgroup of $G^*_{2r}(\lambda)$ for each $r = 0, 1, \dots, k'$ . So (9.23a) holds.

By Corollary 5.157 we have that $G^*_{2l'-1} = P^*_{2l'-2} \cdot Q^*_{2l'-1}$. This, along with the fact that $P^*_{2l'-2} \le P^*_{2k'}$ fixes $\lambda$, implies that $G^*_{2l'-1}(\lambda) = P^*_{2l'-2} \cdot Q^*_{2l'-1}(\lambda)$. Thus $Q^*_{2l'-1}(\lambda)$ is a $\pi'$-Hall subgroup of $G^*_{2l'-1}(\lambda) = G^*_{2l'-1,\lambda}$. Furthermore, for all $i = 1, \dots, l'$, we have that $Q^*_{2i-1}(\lambda) = Q^*_{2l'-1}(\lambda) \cap G_{2i-1}$, where $G_{2i-1} \trianglelefteq G_{2l'-1}$. Thus $Q^*_{2i-1}(\lambda)$ is a also $\pi'$-Hall subgroup of $G^*_{2i-1}(\lambda)$ for all $i = 1, \dots, l'$. This completes the proof of (9.23).

The groups $P^*_{2k',\lambda} = P_{2,\lambda} \cdots P_{2k',\lambda}$ and $Q^*_{2l'-1,\lambda} = Q_{1,\lambda} \cdots Q_{2l'-1,\lambda}$ satisfy the conditions in Propositions 5.132 and 5.155 in the $\lambda$-situation, that is, $P^*_{2k',\lambda} \in \mathrm{Hall}_\pi(G^*_{2k',\lambda})$ and $Q^*_{2l'-1,\lambda} \in \mathrm{Hall}_{\pi'}(G^*_{2l'-1,\lambda})$. Therefore, there exist $G(\lambda)$-conjugates, $(P^*_{2k',\lambda})^s, (Q^*_{2l'-1,\lambda})^s$, of $P^*_{2k',\lambda}$ and $Q^*_{2l'-1,\lambda}$ respectively, such that $P^*_{2k'} = (P^*_{2k',\lambda})^s$ and $Q^*_{2l'-1}(\lambda) = (Q^*_{2l'-1,\lambda})^s$. Hence, we also get that $P^*_{2r} = P^*_{2k'} \cap G^*_{2r} = (P^*_{2r,\lambda})^s$ and $Q^*_{2i-1}(\lambda) = Q^*_{2l'-1}(\lambda) \cap G_{2l'-1} = (Q^*_{2i-1,\lambda})^s$, whenever $0 \le r \le k'$ and $1 \le i \le l'$. The set (9.17b) was picked as any representative of a $G_\lambda = G(\lambda)$-conjugacy class of triangular sets. Thus we can pick (9.17b) to be the one that satisfies (9.24).

So Proposition 9.22 holds. □

The following is a straightforward but useful lemma:

**Lemma 9.25.** *Assume that* $Q_1 \le T \le G(\beta_1)$. *Then* $T = T(\lambda) \cdot Q_1$. *Furthermore, if* $S$ *satisfies* $Q_{1,2i} \le S \le N(P^*_{2i} \text{ in } G(\beta_{1,2i}))$, *for some* $i = 1, \dots, k'$, *then* $S = S(\lambda_{2i}) \cdot Q_{1,2i} = S(\lambda) \cdot Q_{1,2i}$.

*Proof.* As the group $T$ fixes $\beta_1$, it permutes among themselves the $Q_1$-conjugacy class of characters in $\mathrm{Irr}(E)$ lying under $\beta_1 = \chi_1$. Since $\lambda$ is one of these characters, we have $T \le T(\lambda) \cdot Q_1$. The other inclusion is trivial. So $T = T(\lambda) \cdot Q_1$.

The group $S$ normalizes $P^*_{2i}$ and thus normalizes $E_{2i} = C(P^*_{2i} \text{ in } E)$. Furthermore, it fixes $\beta_{1,2i}$. Hence $S$ permutes among themselves the $Q_{1,2k}$-conjugacy class of characters in $\mathrm{Irr}(E_{2i})$ lying under $\beta_{1,2i}$. Since $\lambda_{2i}$ lies in that class, we have $S \le S(\lambda_{2i}) \cdot Q_{1,2i}$. As the other inclusion is trivial, we get $S = S(\lambda_{2i}) \cdot Q_{1,2i}$. Since $S$ normalizes $P^*_{2i}$, and $\lambda_{2i}$ is the $P^*_{2i}$-Glauberman correspondent of $\lambda$, we obviously have that $S(\lambda_{2i}) = S(\lambda)$. Hence the lemma follows. □

After these preliminary comments we are ready to state and prove

**Theorem 9.26.** *The set* (9.17b) *chosen in Proposition 9.22 satisfies*

$$P_{2r,\lambda} = P_{2r} \tag{9.27a}$$

$$\alpha_{2r,\lambda} = \alpha_{2r}, \tag{9.27b}$$

*for all* $r = 0, 1, \dots, k'$. *And*

$$Q_{2i-1,\lambda} = Q_{2i-1}(\lambda) = Q_{2i-1}(\lambda_{2i-2}), \tag{9.28a}$$

$$\beta_{2i-1,\lambda} \in \mathrm{Irr}(Q_{2i-1,\lambda}) \text{ is the } \lambda_{2i-2}\text{-Clifford correspondent of } \beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1}), \tag{9.28b}$$

*for all* $i = 1, \dots, l'$. *(By convention* $\lambda_0 := \lambda$.*) Hence* $\beta_{2i-1,\lambda}$ *induces* $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$.

*Proof.* According to Lemma 9.25 for $T = Q^*_{2r-1}$ we get that $T = Q^*_{2r-1}(\lambda)Q_1$. Since $Q_1 = G_1 \trianglelefteq G$, it follows that $N(Q^*_{2r-1}(\lambda) \text{ in } P^*_{2r}) \leq N(Q^*_{2r-1} \text{ in } P^*_{2r})$. But $P^*_{2r}$ fixes $\lambda$. So the other direction of the above inclusion also holds. Thus

$$N(Q^*_{2r-1} \text{ in } P^*_{2r}) = N(Q^*_{2r-1}(\lambda) \text{ in } P^*_{2r}). \tag{9.29}$$

The triangular set (9.17b) satisfies the equivalent of Proposition 5.159 for the $\lambda$-situation. Hence

$$\begin{aligned} P_{2r,\lambda} &= N(Q^*_{2r-1,\lambda} \text{ in } P^*_{2r,\lambda}) && \text{by (5.160)} \\ &= N(Q^*_{2r-1}(\lambda) \text{ in } P^*_{2r}) && \text{by (9.24)} \\ &= N(Q^*_{2r-1} \text{ in } P^*_{2r}) && \text{by (9.29)} \\ &= P_{2r}, && \text{by (5.160)} \end{aligned}$$

for all $r = 1, \dots, k'$. Thus (9.27a) holds for all $r \geq 1$. It also holds for $r = 0$, since $P_{0,\lambda} = 1 = P_0$.

Similarly, for the $\pi'$-groups we have

$$\begin{aligned} Q_{2i-1,\lambda} &= N(P^*_{2i-2,\lambda} \text{ in } Q^*_{2i-1,\lambda}) && \text{by (5.161)} \\ &= N(P^*_{2i-2} \text{ in } Q^*_{2i-1}(\lambda)) && \text{by (9.24)} \\ &= N(P^*_{2i-2} \text{ in } Q^*_{2i-1})(\lambda) && \\ &= Q_{2i-1}(\lambda), && \text{by (5.161)} \end{aligned}$$

for all $i = 1, \dots, l'$. The group $Q_{2i-1}$ normalizes $P_2, \dots, P_{2i-2}$ and thus normalizes their product $P^*_{2i-2}$. Hence (9.13), with $Q_{2i-1}$ in the place of $N$, implies that $Q_{2i-1}(\lambda) = Q_{2i-1}(\lambda_{2i-2})$. So (9.28a) holds.

It remains to show (9.27b) and (9.28b) for the $\lambda$-characters. This will be done by induction on $i$ and $r$, with the help of various observations that we write here separately as steps.

**Step 1.** *For every* $i = 1, \dots, n$, *the* $cQ_{1,\lambda}$*-correspondent* $\chi_{i,1,\lambda} \in \mathrm{Irr}(G_{i,1,\lambda})$ *of* $\chi_{i,\lambda} \in \mathrm{Irr}(G_{i,\lambda})$ *induces the* $cQ_1$*-correspondent* $\chi_{i,1} \in \mathrm{Irr}(G_{i,1})$ *of* $\chi_i \in \mathrm{Irr}(G_i)$. *Even more,*

$$G_{i,1,\lambda} = G_{i,1}(\lambda) \ \textit{and} \tag{9.30a}$$

$$\chi_{i,1,\lambda} \in \mathrm{Irr}(G_{i,1,\lambda}) \textit{ is the } \lambda\textit{-Clifford correspondent of } \chi_{i,1} \in \mathrm{Irr}(G_{i,1}). \tag{9.30b}$$

*Proof.* We first remark that the $cQ_{1,\lambda}$-correspondent (which is analogous to the $cQ_1$-correspondent for the $\lambda$-case), is nothing else but a Clifford correspondent, as we can see in Table 5.1. That is, $G_{i,1,\lambda} = G_{i,\lambda}(\chi_{1,\lambda})$ and $\chi_{i,1,\lambda} \in \mathrm{Irr}(G_{i,1,\lambda})$ (see (5.65a), and (5.67)), is the $\chi_{1,\lambda}$-Clifford correspondent of $\chi_{i,\lambda} \in \mathrm{Irr}(G_{i,\lambda}|\chi_{i,\lambda})$, for all $i = 1, \dots, n$. Of course, $G_{1,1,\lambda} = G_{1,\lambda}$ and $\chi_{1,1,\lambda} = \chi_{1,\lambda}$.

Furthermore, for all $i = 1, \dots, n$, we have that

$$\begin{aligned} G_{i,1,\lambda} &= G_{i,\lambda}(\chi_{1,\lambda}) && \text{by (5.65a) for the } \lambda\text{-case} \\ &= G_i(\chi_{1,\lambda})(\lambda) && \text{as } G_{1,\lambda} = G_1(\lambda), \text{ by (9.14b)} \\ &= G_i(\chi_1, \lambda) && \text{by Clifford's theory, since } \chi_{1,\lambda} \text{ is the } \lambda\text{-Clifford correcpondent of } \chi_1 \\ &= G_{i,1}(\lambda). && \text{by (5.65a)} \end{aligned}$$

Therefore (9.30a) holds.

Let $i = 1, \dots, n$ be fixed. The character $\chi_{i,1,\lambda} \in \operatorname{Irr}(G_{i,1,\lambda}) = \operatorname{Irr}(G_i(\chi_1, \lambda))$ induces $\chi_{i,\lambda} \in \operatorname{Irr}(G_{i,\lambda}) = \operatorname{Irr}(G_i(\lambda))$, by (5.67). Also the character $\chi_{i,\lambda} \in \operatorname{Irr}(G_i(\lambda))$ induces $\chi_i \in \operatorname{Irr}(G_i)$ by (9.16a). Therefore, $\chi_{i,1,\lambda} \in \operatorname{Irr}(G_i(\chi_1, \lambda))$ induces $\chi_i \in \operatorname{Irr}(G_i)$. Hence $\chi_{i,1,\lambda} \in \operatorname{Irr}(G_i(\chi_1, \lambda))$ induces a character $\Psi \in \operatorname{Irr}(G_i(\chi_1))$. Note that the induced character $\Psi^{G_i}$ is $\chi_{i,1,\lambda}^{G_i} = \chi_i$. Even more, the character $\Psi$ lies above $\chi_1 = \beta_1$. To see this, first note that, according to Lemma 9.25 for $T = G_i(\chi_1)$, we have $G_i(\chi_1) = G_i(\chi_1)(\lambda) \cdot Q_1 = G_i(\chi_1, \lambda) \cdot G_1$, while $G_i(\chi_1, \lambda) \cap G_1 = G_1(\lambda)$. This, along with Mackey's Theorem (see Problem 5.6 in [12], or the special case in Problem 5.2 in [12]), implies $(\chi_{i,1,\lambda}^{G_i(\chi_1)})|_{G_1} = (\chi_{i,1,\lambda}|_{G_1(\lambda)})^{G_1}$. Hence

$$\langle \Psi|_{G_1}, \chi_1 \rangle = \langle (\chi_{i,1,\lambda}^{G_i(\chi_1)})|_{G_1}, \chi_1 \rangle = \langle (\chi_{i,1,\lambda}|_{G_1(\lambda)})^{G_1}, \chi_1 \rangle \neq 0,$$

where the last inequality holds as $\chi_1 = \chi_{1,\lambda}^{G_1}$ (by (9.16a)), and $\chi_{1,\lambda} \in \operatorname{Irr}(G_1(\lambda))$ lies under $\chi_{i,1,\lambda}$. Thus the character $\Psi \in \operatorname{Irr}(G_i(\chi_1))$, induces $\chi_i \in \operatorname{Irr}(G_i)$ and lies above $\chi_1$. Hence Clifford's theorem implies that $\Psi$ is the unique $\chi_1$-Clifford correspondent of $\chi_i$. This, along with (5.67), implies that $\Psi = \chi_{i,1}$, i.e., $\chi_{i,1,\lambda}^{G_{i,1}} = \Psi = \chi_{i,1}$.

So $\chi_{i,1,\lambda} \in \operatorname{Irr}(G_{i,1,\lambda}) = \operatorname{Irr}(G_{i,1}(\lambda))$ induces $\chi_{i,1} \in \operatorname{Irr}(G_{i,1})$, and lies above $\lambda$. Thus $\chi_{i,1,\lambda}$ is the $\lambda$-Clifford correspondent of $\chi_{i,1}$.

This completes the proof of the first step. $\square$

**Step 2.** *For every $i = 2, \dots, l'$ we have that*

$$Q_{2i-1} = Q_{2i-1}(\lambda) \cdot Q_{1,2i-2}. \tag{9.31}$$

*Furthermore, for every $r = 1, \dots, k'$ and every $i = 1, \dots, l' - 1$ we have that*

$$\begin{aligned} &N(P_{0,\lambda}, P_{2,\lambda}, \dots, P_{2i,\lambda}, Q_{1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})) = \\ &\qquad N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})), \end{aligned} \tag{9.32a}$$

*and*

$$\begin{aligned} &N(P_{0,\lambda}, P_{2,\lambda}, \dots, P_{2r-2,\lambda}, Q_{1,\lambda}, \dots, Q_{2r-1,\lambda} \text{ in } G_{2r,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2r-1,\lambda})) = \\ &\qquad N(P_0, P_2, \dots, P_{2r-2}, Q_1, \dots, Q_{2r-1} \text{ in } G_{2r,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2r-1,\lambda})). \end{aligned} \tag{9.32b}$$

*Thus, for all $i = 1, \dots, l' - 1$, the $cP_{2,\lambda}, \dots, cP_{2i,\lambda}, cQ_{3,\lambda}, \dots, cQ_{2i-1,\lambda}$-correspondent $\chi_{2i+1,2i,\lambda}$, of $\chi_{2i+1,1,\lambda}$ coincides with the $cP_2, \dots, cP_{2i}, cQ_3, \dots, cQ_{2i-1}$-correspondent of $\chi_{2i+1,1,\lambda}$. By convention, if $i = 1$ this is only the $cP_2 = cP_{2,\lambda}$-correspondence. Similarly, for all $r = 2, \dots, k'$, the $cP_{2,\lambda}, \dots, cP_{2r-2,\lambda}, cQ_{3,\lambda}, \dots, cQ_{2r-1,\lambda}$-correspondent $\chi_{2r,2r-1,\lambda}$, of $\chi_{2r,1,\lambda}$ coincides with the $cP_2, \dots, cP_{2r-2}, cQ_3, \dots, cQ_{2r-1}$-correspondent of $\chi_{2r,1,\lambda}$.*

*Proof.* For all $t = 2, \dots, l'$ we have that

$$Q_{1,2t-2} = Q_1 \cap Q_{2t-1} \leq Q_{2t-1} \leq N(P^*_{2t-2} \text{ in } G(\beta_{1,2t-2})),$$

(by (5.33) we get the equality, while Proposition 5.55 shows that $Q_{2t-1}$ fixes $\beta_{1,2t-2}$). Hence Lemma 9.25, for $Q_{2t-1}$ in the place of $S$, implies that

$$Q_{2t-1} = Q_{2t-1}(\lambda) \cdot Q_{1,2t-2},$$

for all $t = 2, \dots, l'$. This proves (9.31).

We have already seen that $P_{2r,\lambda} = P_{2r}$ while $Q_{2i-1,\lambda} = Q_{2i-1}(\lambda)$, whenever $1 \leq r \leq k'$ and $1 \leq i \leq l'$ (by (9.27a) and (9.28a)). Thus (9.31) implies that

$$Q_{2t-1} = Q_{2t-1,\lambda} \cdot Q_{1,2t-2},$$

for all $t = 2, \dots, l'$. Any subgroup of $G$ that normalizes $P_2, \dots, P_{2t-2}$ and $Q_{2t-1,\lambda}$ also normalizes $Q_{1,2t-2} = N(P^*_{2t-2}$ in $Q_1)$. Thus it normalizes $Q_{2t-1} = Q_{2t-1,\lambda} \cdot Q_{1,2t-2}$, whenever $t = 2, \dots, l'$. Since $Q_1 \trianglelefteq G$, is normalized by any subgroup of $G$, we therefore get

$$\begin{aligned} N(P_{0,\lambda}, P_{2,\lambda}, \dots, P_{2i,\lambda}, Q_{1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})) = \\ N(P_0, P_2, \dots, P_{2i}, Q_{1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})) \leq \\ N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})). \end{aligned}$$

The other inclusion is trivial as $G_{2i+1,\lambda} = G_{2i+1}(\lambda)$. So everything in $G_{2i+1,\lambda}$ that normalizes $Q_{2t-1}$ also normalizes $Q_{2t-1,\lambda} = Q_{2t-1}(\lambda)$, for all $t = 1, \dots, l'$. Also $Q_1 \trianglelefteq G$ is normalized by any subgroup of $G$. Hence we have equality, and (9.32a) is proved.

The proof for (9.32b) is similar. So we omit it.

Let $i = 1, \dots, l' - 1$ be fixed. For all $t = 1, \dots, i$, we have $P_{2t} = P_{2t,\lambda}$. So to prove the next two statements of Step 2, it suffices to show that the $cQ_{2t-1}$-correspondence coincides with the $cQ_{2t-1,\lambda}$-correspondence, for all $t = 2 \dots, i$. We remark that the $cQ_{2t-1,\lambda}$-correspondence was used, in Theorem 5.88 and Table 5.5 for the $\lambda$-situation, to get the irreducible character $\chi_{2i+1,2t-1,\lambda}$ of $N(Q_{2t-1,\lambda}$ in $G_{2i+1,2t-2,\lambda}(\chi_{2t-1,\lambda})) = G_{2i+1,2t-1,\lambda}$ from $\chi_{2i+1,2t-2,\lambda} \in \mathrm{Irr}(G_{2i+1,2t-2,\lambda})$, whenever $t = 2, \dots, i$. That is, we applied Lemma 5.56 to the groups

$$G_{2t-1,2t-2,\lambda} = Q_{2t-1,\lambda} \ltimes P_{2t-2,\lambda} \trianglelefteq G_{2t,2t-2,\lambda} \trianglelefteq \cdots \trianglelefteq G_{2i-1,2t-2,\lambda}$$

and their irreducible characters

$$\chi_{2t-1,2t-2,\lambda} = \beta_{2t-1,\lambda} \cdot \alpha^e_{2t-2,\lambda}, \chi_{2t,2t-2,\lambda}, \dots, \chi_{2i-1,2t-2,\lambda},$$

in the place of $N \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_r$ and $\{\chi = \alpha \cdot \beta^e, \chi_1, \dots, \chi_r\}$ respectively. On the other hand, for all $t = 2, \dots, i$, the character $\chi_{2i+1,2t-2,\lambda}$ has a $cQ_{2t-1}$-correspondent. Indeed, the group $Q_{2t-1}$ acts on $P_{2t-2} = P_{2t-1,\lambda}$, while the semidirect product $Q_{2t-1} \ltimes P_{2t-1,\lambda}$ is normalized by all the groups in the normal series

$$P_{2t-1,\lambda} \trianglelefteq G_{2t,2t-1,\lambda} \trianglelefteq \cdots \trianglelefteq G_{2i-1,2t-1,\lambda}.$$

Notice that $Q_{2t-1,\lambda}$ and $Q_{2t-1}$ have the same image in $\mathrm{Aut}(P_{2t-2,\lambda})$, as $Q_{2t-1} = Q_{2t-1,\lambda} \cdot Q_{1,2t-2}$ (by (9.31)), and $Q_{1,2t-2}$ centralizes $P_{2t-2,\lambda} = P_{2t-2}$, whenever $2 \leq t \leq i$. Furthermore, $G_{2i+1,2t-2,\lambda}$ normalizes $P_2, \dots, P_{2t-2}$ (see (5.91)), and thus normalizes $Q_{1,2t-2}$. Hence $N(Q_{2t-1,\lambda}$ in $G_{2i+1,2t-2,\lambda}) \leq N(Q_{2t-1}$ in $G_{2i+1,2t-2,\lambda})$. The other inclusion holds trivially, as $G_{2i+1,2t-2,\lambda} \leq G_{2i+1,\lambda} = G_{2i+1}(\lambda)$ fixes $\lambda$ and $Q_{2t-1,\lambda} = Q_{2t-1}(\lambda)$. Hence $N(Q_{2t-1,\lambda}$ in $G_{2i+1,2t-2,\lambda}) = N(Q_{2t-1}$ in $G_{2i+1,2t-2,\lambda})$, for all $t = 2, \dots, i$. Therefore Proposition 3.9 implies that the $cQ_{2t-1,\lambda}$-correspondent $\chi_{2i+1,2t-1,\lambda}$ of $\chi_{2i+1,2t-2,\lambda}$ coincides with its $cQ_{2t-1}$-correspondent, whenever $2 \leq t \leq i$.

We conclude that the $cP_{2,\lambda}, \dots, cP_{2i,\lambda}, cQ_{3,\lambda}, \dots, cQ_{2i-1,\lambda}$-correspondent $\chi_{2i+1,2i,\lambda}$, of $\chi_{2i+1,1,\lambda}$ coincides with the $cP_2, \dots, cP_{2i}, cQ_3, \dots, cQ_{2i-1}$-correspondent of $\chi_{2i+1,1,\lambda}$, for all $i = 1, \dots, l' - 1$.

Similarly we can work with the character $\chi_{2r,2r-1,\lambda}$, the group $G_{2r,2t-1,\lambda}$ and its normal subgroup

$G_{2t,2t-1,\lambda} = P_{2t,\lambda} \ltimes Q_{2t-1,\lambda}$, for all $t = 2, \dots, r$, for some fixed $r = 2, \dots, k'$. Thus we get that that the $cQ_{2t-1,\lambda}$-correspondent $\chi_{2r,2t-1,\lambda}$ of $\chi_{2r,2t-2,\lambda}$ coincides with the $cQ_{2t-1}$-correspondent. We conclude similarly that the $cP_{2,\lambda}, \dots, cP_{2r-2,\lambda}, cQ_{3,\lambda}, \dots, cQ_{2r-1,\lambda}$-correspondent $\chi_{2r,2r-1,\lambda}$, of $\chi_{2r,1,\lambda}$ coincides with the $cP_2, \dots, cP_{2r-2}, cQ_3, \dots, cQ_{2r-1}$-correspondent of $\chi_{2r,1,\lambda}$, for all $r = 2, \dots, k'$.

This completes the proof of Step 2. □

**Step 3.** *For all $i = 1, \dots, l' - 1$, we have that $G_{2i+1,2i,\lambda} = G_{2i+1,2i}(\lambda)$, while the character $\chi_{2i+1,2i,\lambda} \in \mathrm{Irr}(G_{2i+1,2i,\lambda})$ induces $\chi_{2i+1,2i} \in \mathrm{Irr}(G_{2i+1,2i})$. Similarly, for all $r = 1, \dots, k'$, we get that $G_{2r,2r-1,\lambda} = G_{2r,2r-1}(\lambda)$, while the character $\chi_{2r,2r-1,\lambda} \in \mathrm{Irr}(G_{2r,2r-1,\lambda})$ induces $\chi_{2r,2r-2} \in \mathrm{Irr}(G_{2r,2r-1})$.*

*Proof.* According to (5.91) for the $\lambda$-case we have that

$$G_{2i+1,2i,\lambda} = N(P_{0,\lambda}, P_{2,\lambda}, \dots, P_{2i,\lambda}, Q_{1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})).$$

This, along with (9.32a), (9.18) and (9.21b), implies that

$$\begin{aligned} &G_{2i+1,2i,\lambda} = N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G_{2i+1,\lambda}(\chi_{1,\lambda}, \dots, \chi_{2i,\lambda})) = \\ &N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G^*_{2i+1,\lambda}) = N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G^*_{2i+1}(\lambda)) = \\ &\qquad N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G^*_{2i+1})(\lambda). \end{aligned}$$

As $G_{2i+1,2i} = N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G^*_{2i+1})$ (by (5.91)), we have that $G_{2i+1,2i,\lambda} = G_{2i+1,2i}(\lambda)$ , for all $i = 1, \dots, l' - 1$. Furthermore, according to Step 2, the character $\chi_{2i+1,2i,\lambda}$ is the $cP_2, \dots, cP_{2i}, cQ_3, \dots, cQ_{2i-1}$-correspondent of $\chi_{2i+1,1,\lambda}$, whenever $i = 1, \dots, l' - 1$. But $\chi_{2i+1,1,\lambda} \in \mathrm{Irr}(G_{2i+1,1}(\lambda))$ induces $\chi_{2i+1,1}$ in $G_{2i+1,1}$ according to Step 1, for all such $i$. Furthermore, the $cA$-correspondence (for arbitrary $A$) respects induction (see Theorem 3.13). Hence $\chi_{2i+1,2i,\lambda}$ induces the $cP_2, \dots, cP_{2i}, cQ_3, \dots, cQ_{2i-1}$-correspondent character of $\chi_{2i+1,1}$ in the normalizer $N(P_0, P_2, \dots, P_{2i}, Q_1, \dots, Q_{2i-1} \text{ in } G_{2i+1}(\chi_1, \dots, \chi_{2i})) = G_{2i+1,2i}$. As this correspondent character of $\chi_{2i+1,1}$ is $\chi_{2i+1,2i}$ (by Theorem 5.88), we conclude that $\chi_{2i+1,2i,\lambda}$ induces $\chi_{2i+1,2i}$, for all $i = 1, \dots, l' - 1$. This completes the first part of Step 3.

For the second part, we first remark that the case $r = 1$ has been done in Step 1. Indeed, by (9.30a), we have that $G_{2,1,\lambda} = G_{2,1}(\lambda)$, while by (9.30b) the character $\chi_{2,1,\lambda}$ induces $\chi_{2,1}$. The rest of proof for $r = 2, \dots, k'$ is analogous to the proof of the first part, with the use of (9.32b) in the place of (9.32a). So we omit it. □

We can now continue with the proof of (9.27b) and (9.28b) for the $\lambda$-characters. If $r = 0$ then $\alpha_{0,\lambda} = 1 = \alpha_0$. Hence (9.27b) holds trivially for $r = 0$. Furthermore, $\beta_{1,\lambda} = \chi_{1,\lambda}$, by (5.17b). But $\chi_{1,\lambda}$ is the $\lambda$-Clifford correspondent of $\chi_1 \in \mathrm{Irr}(Q_1|\lambda)$. Thus (9.28b) holds for $i = 1$.

Using an inductive argument we will prove that, if (9.28b) holds when $i$ is some integer $t = 1, \dots, k' - 1$, then (9.27b) holds for $r = t$. Symmetrically if (9.27b) holds when $r$ is some integer $s = 1, \dots, l' - 1$, then (9.28b) holds for $i = s + 1$. This is enough to prove that (9.27b) and (9.28b) hold for all $r = 0, \dots, k'$ and all $i = 1, \dots, l'$, respectively.

Assume that (9.28b) holds for $i = t$. That is, $\beta_{2t-1,\lambda}$ is the $\lambda_{2t-2}$-Clifford correspondent of $\beta_{2t-1}$. Therefore

$$\beta_{2t-1,\lambda}^{Q_{2t-1}} = \beta_{2t-1}. \tag{9.33}$$

Furthermore, Theorem 5.88, and in particular (5.92), implies that the character $\alpha_{2t,\lambda}$ was picked as the unique character of $P_{2t,\lambda}$ that satisfies

$$\chi_{2t,2t-1,\lambda} = \alpha_{2t,\lambda} \cdot \beta^e_{2t-1,\lambda}, \tag{9.34a}$$

where $\beta^e_{2t-1,\lambda}$ is the canonical extension of $\beta_{2t-1,\lambda} \in \mathrm{Irr}(Q_{2t-1,\lambda})$ to $G_{2t,2t-1,\lambda}$. Similarly, $\alpha_{2t}$ is the unique character of $P_{2t}$ such that

$$\chi_{2t,2t-1} = \alpha_{2t} \cdot \beta^e_{2t-1}. \tag{9.34b}$$

According to Step 3 the character $\chi_{2t,2t-1,\lambda}$ induces $\chi_{2t,2t-1} \in \mathrm{Irr}(G_{2t,2t-1})$. Also $G_{2t,2t-1} = P_{2t}Q_{2t-1}$, see (5.92). This, along with (9.34), implies

$$\chi_{2t,2t-1} = (\chi_{2t,2t-1,\lambda})^{G_{2t,2t-1}} = (\alpha_{2t,\lambda} \cdot \beta^e_{2t-1,\lambda})^{G_{2t,2t-1}}. \tag{9.35}$$

Lemma 2.21 can be applied to the group $G_{2t,2t-1} = P_{2t} \ltimes Q_{2t-1}$ and the characters $\alpha_{2t,\lambda} \in \mathrm{Irr}(P_{2t,\lambda}) = \mathrm{Irr}(P_{2t})$ and $\beta^e_{2t-1,\lambda} \in \mathrm{Irr}(P_{2t} \ltimes Q_{2t-1,\lambda})$. Thus

$$(\alpha_{2t,\lambda} \cdot \beta^e_{2t-1,\lambda})^{G_{2t,2t-1}} = \alpha_{2t,\lambda} \cdot (\beta^e_{2t-1,\lambda})^{G_{2t,2t-1}}. \tag{9.36}$$

The groups $G_{2t,2t-1} = P_{2t} \ltimes Q_{2t-1} = (P_{2t}Q_{2t-1,\lambda}) \cdot Q_{2t-1}, Q_{2t-1}, P_{2t}Q_{2t-1,\lambda} = G_{2t,2t-1,\lambda}$ and $Q_{2t-1,\lambda}$, along with the character $\beta_{2t-1,\lambda} \in \mathrm{Irr}(Q_{2t-1,\lambda})$, satisfy the hypothesis of Proposition 2.16 in the place of the groups $G, N, K$ and $H$ respectively. So we conclude that

$$(\beta^e_{2t-1,\lambda})^{G_{2t,2t-1}} = (\beta^{Q_{2t-1}}_{2t-1,\lambda})^e,$$

where $(\beta^{Q_{2t-1}}_{2t-1,\lambda})^e$ is the canonical extension of $\beta^{Q_{2t-1}}_{2t-1,\lambda} \in \mathrm{Irr}(Q_{2t-1})$ to $G_{2t,2t-1}$. But, according to the inductive hypothesis, the character $\beta_{2t-1,\lambda}$ satisfies (9.33). Therefore

$$(\beta^e_{2t-1,\lambda})^{G_{2t,2t-1}} = (\beta^{Q_{2t-1}}_{2t-1,\lambda})^e = \beta^e_{2t-1},$$

where $\beta^e_{2t-1}$ is the canonical extension of $\beta_{2t-1}$ to $G_{2t,2t-1}$. This, along with (9.36) and (9.35), implies that

$$\chi_{2t,2t-1} = (\alpha_{2t,\lambda} \cdot \beta^e_{2t-1,\lambda})^{G_{2t,2t-1}} = \alpha_{2t,\lambda} \cdot \beta^e_{2t-1}.$$

Therefore (9.34b) holds with $\alpha_{2t,\lambda}$ in the place of $\alpha_{2t}$. As $\alpha_{2t}$ is the unique character of $P_{2t}$ satisfying (9.34b), we must have $\alpha_{2t} = \alpha_{2t,\lambda}$. Therefore, (9.27b) holds for $r = t$.

Now assume that (9.27b) holds when $r$ is some integer $s = 1, \ldots, l' - 1$. We will prove, in an argument similar to the one we just gave, that (9.28b) also holds for $i = s + 1$.

According to (9.27b) for $r = s$ we get

$$\alpha_{2s} = \alpha_{2s,\lambda}. \tag{9.37}$$

Furthermore, equations (5.93) imply that $\beta_{2s+1,\lambda}$ and $\beta_{2s+1}$ are the unique characters of $Q_{2s+1,\lambda}$ and $Q_{2s+1}$, respectively, that satisfy

$$\chi_{2s+1,2s,\lambda} = \alpha^e_{2s,\lambda} \cdot \beta_{2s+1,\lambda} \tag{9.38a}$$
$$\chi_{2s+1,2s} = \alpha^e_{2s} \cdot \beta_{2s+1}, \tag{9.38b}$$

where $\alpha^e_{2s,\lambda}$ and $\alpha^e_{2s}$ are the canonical extensions of $\alpha_{2s,\lambda} \in \mathrm{Irr}(P_{2s,\lambda})$ and $\alpha_{2s} \in \mathrm{Irr}(P_{2s})$ to $G_{2s+1,2s,\lambda} = P_{2s,\lambda} \rtimes Q_{2s+1,\lambda}$ and $G_{2s+1,2s} = P_{2s} \rtimes Q_{2s+1}$, respectively. But $P_{2s,\lambda} = P_{2s}$ and

$\alpha_{2s,\lambda} = \alpha_{2s}$ by (9.37). Furthermore, $G_{2s+1,2s,\lambda} \leq G_{2s+1,2s}$ (see Step 3). Therefore

$$\alpha^e_{2s,\lambda} = \alpha^e_{2s}|_{G_{2s+1,2s,\lambda}}. \tag{9.39}$$

The fact that $\chi_{2s+1,2s,\lambda}$ induces $\chi_{2s+1,2s}$ (see Step 3), along with (9.38) and (9.39), implies

$$\chi_{2s+1,2s} = (\alpha^e_{2s}|_{G_{2s+1,2s,\lambda}} \cdot \beta_{2s+1,\lambda})^{G_{2s+1,2s}}. \tag{9.40}$$

Using the isomorphism $Q_{2s+1,\lambda} \cong Q_{2s+1,\lambda} \ltimes P_{2s}/P_{2s}$, we denote the inflation of $\beta_{2s+1,\lambda}$ to $Q_{2s+1,\lambda} \ltimes P_{2s,\lambda} = Q_{2s+1,\lambda} \ltimes P_{2s,\lambda} = G_{2s+1,2s,\lambda}$ as $\beta^i_{2s+1,\lambda}$. So (9.40) becomes

$$\chi_{2s+1,2s} = (\alpha^e_{2s}|_{G_{2s+1,2s,\lambda}} \cdot \beta^i_{2s+1,\lambda})^{G_{2s+1,2s}}.$$

Hence we can apply Lemma 2.20 to get

$$(\alpha^e_{2s}|_{G_{2s+1,2s,\lambda}} \cdot \beta^i_{2s+1,\lambda})^{G_{2s+1,2s}} = \alpha^e_{2s} \cdot (\beta^i_{2s+1,\lambda})^{G_{2s+1,2s}}.$$

This, along with (9.40), implies that

$$\chi_{2s+1,2s} = \alpha^e_{2s} \cdot (\beta^i_{2s+1,\lambda})^{G_{2s+1,2s}} = \alpha^e_{2s} \cdot (\beta^i_{2s+1,\lambda})^{P_{2s} \rtimes Q_{2s+1}} = \alpha^e_{2s} \cdot (\beta^{Q_{2s+1}}_{2s+1,\lambda})^i,$$

where $(\beta^{Q_{2s+1}}_{2s+1,\lambda})^i$ denotes the inflation of $\beta^{Q_{2s+1}}_{2s+1,\lambda} \in \mathrm{Irr}(Q_{2s+1}) = \mathrm{Irr}(P_{2s}Q_{2s+1}/P_{2s})$ to $P_{2s} \rtimes Q_{2s+1}$. Note that $\alpha^e_{2s} \cdot (\beta^{Q_{2s+1}}_{2s+1,\lambda})^i$ is equal to $\alpha^e_{2s} \cdot (\beta^{Q_{2s+1}}_{2s+1,\lambda})$ by the definition of the ltter product. But $\beta_{2s+1}$ is the unique character of $Q_{2s+1}$ that satisfies (9.38b). Hence

$$(\beta^{Q_{2s+1}}_{2s+1,\lambda})^i = \beta^i_{2s+1}, \text{ and thus } \beta^{Q_{2s+1}}_{2s+1,\lambda} = \beta_{2s+1}. \tag{9.41}$$

Furthermore, the character $\chi_{2s+1,2s}$ lies above $\chi_{1,2s} \in \mathrm{Irr}(G_{1,2s}) = \mathrm{Irr}(Q_{1,2s})$. Also $\chi_{1,2s} = \beta_{1,2s}$ (as $\chi_1 = \beta_1$), lies above $\lambda_{2s}$ by (9.12). This, along with the fact that $\alpha^e_{2s,\lambda}$ is trivial on $Q_{1,2s} = G_{1,2s}$, implies that $\beta_{2s+1,\lambda} \in \mathrm{Irr}(Q_{2s+1,\lambda})$ lies above $\lambda_{2s}$ and induces $\beta_{2s+1} \in \mathrm{Irr}(Q_{2s+1})$. As $Q_{2s+1,\lambda} = Q_{2s+1}(\lambda) = Q_{2s+1}(\lambda_{2s})$, by (9.13) with $N = Q_{2s+1}$, we conclude that $\beta_{2s+1,\lambda}$ is the $\lambda_{2s}$-Clifford correspondent of $\beta_{2s+1}$. Hence (9.28b) holds for $i = s + 1$.

This completes the proof of Theorem 9.26. □

Assume that $i, j$ satisfy $1 \leq i \leq j \leq k'$. According to Lemma 2.5 in [22], Glauberman correspondence is compatible with Clifford theory. This, along with (9.28b) and (9.27a), implies that the $P_{2i} \cdots P_{2j} = P_{2i,\lambda} \cdots P_{2j,\lambda}$-Glauberman correspondent $\beta_{2i-1,2j}$ of $\beta_{2i-1}$ is induced by the $P_{2i,\lambda} \cdots P_{2j,\lambda}$-Glauberman correspondent $\beta_{2i-1,2j,\lambda}$ of $\beta_{2i-1,\lambda}$. Furthermore, $\beta_{2i-1,2j,\lambda}$ lies above the $P_{2i} \cdots P_{2j}$-Glauberman correspondent $\lambda_{2j}$ of $\lambda_{2i-2}$ (see (9.10)), as $\beta_{2i-1,\lambda}$ lies above $\lambda_{2i-2}$. Since $Q_{2i-1,\lambda} = Q_{2i-1}(\lambda)$, we also have that

$$Q_{2i-1,2j,\lambda} = C(P_{2i} \cdots P_{2j} \text{ in } Q_{2i-1,\lambda}) = C(P_{2i} \cdots P_{2j} \text{ in } Q_{2i-1})(\lambda) = Q_{2i-1,2j}(\lambda),$$

whenever $1 \leq i \leq j \leq k'$. But $Q_{2i-1,2j}(\lambda) = Q_{2i-1,2j}(\lambda_{2j})$, as $Q_{2i-1,2j}$ normalizes $E$ and $P^*_{2j}$. Therefore,

**Remark 9.42.** For every $i, j$ with $1 \leq i \leq j \leq k'$, we have

$$Q_{2i-1,2j,\lambda} = Q_{2i-1,2j}(\lambda) = Q_{2i-1,2j}(\lambda_{2j}),$$

while the character $\beta_{2i-1,2j,\lambda}$ is the $\lambda_{2j}$-Clifford correspondent of $\beta_{2i-1,2j}$.

Furthermore, $Q_{1,2i-2}$ centralizes $P_2, \dots, P_{2i-2}$ by (5.23a), and $Q_{2i-1} = Q_{2i-1}(\lambda) \cdot Q_{1,2i-2}$ by (9.31). We conclude that

$$C(Q_{2i-1}(\lambda) \text{ in } P_{2r}) = C(Q_{2i-1} \text{ in } P_{2r}), \tag{9.43}$$

whenever $1 \leq r < i \leq l'$. Hence,

$$\begin{aligned} P_{2r,2i-1,\lambda} &= C(Q_{2r+1,\lambda}, \dots, Q_{2i-1,\lambda} \text{ in } P_{2r,\lambda}) \\ &= C(Q_{2r+1}(\lambda), \dots, Q_{2i-1}(\lambda) \text{ in } P_{2r}) && \text{by (9.27a) and (9.28a)} \\ &= C(Q_{2r+1}, \dots, Q_{2i-1} \text{ in } P_{2r}) && \text{by (9.43)} \\ &= P_{2r,2i-1}, \end{aligned}$$

whenever $1 \leq r < i \leq l'$.

Even more, the $Q_{2r+1} \cdots Q_{2i-1}$-Glauberman correspondent $\alpha_{2r,2i-1} \in \mathrm{Irr}(P_{2r,2i-1})$ of the irreducible character $\alpha_{2r}$ of $P_{2r}$, (see (5.52)), coincides with the $Q_{2r+1}(\lambda) \cdots Q_{2i-1}(\lambda)$-Glauberman correspondent of $\alpha_{2r}$, by Corollary 3.10. In conclusion,

**Remark 9.44.** For every $r, i$ with $1 \leq r < i \leq l'$, we have

$$P_{2r,2i-1,\lambda} = P_{2r,2i-1} \text{ and } \alpha_{2r,2i-1,\lambda} = \alpha_{2r,2i-1}.$$

The relation between $\alpha^*_{2i}$ and $\alpha^*_{2i,\lambda}$ is an easy corollary of Theorem 9.26.

**Corollary 9.45.** *For all $r = 0, \dots, k'$ we have*

$$\alpha^*_{2r,\lambda} = \alpha^*_{2r} \in \mathrm{Irr}(P^*_{2r}) = \mathrm{Irr}(P^*_{2r,\lambda}).$$

*Proof.* By (9.24a) and (9.27b), we have $P^*_{2r} = P^*_{2r,\lambda}$ and $\alpha_{2r,\lambda} = \alpha_{2r}$, for all $r = 0, 1, \dots, k'$. Furthermore, the character $\alpha^*_{2r}$ is uniquely determined by $\alpha_{2r}$ and the one to one $Q_{2j+1}$-correspondence

$$\alpha^*_{2r,2j-1} \overleftrightarrow{Q_{2j+1}} \alpha^*_{2r,2j+1}.$$

The latter is a correspondence between all characters $\alpha^*_{2r,2j+1} \in \mathrm{Irr}(P_{2j+2} \cdots P_{2r})$ and all characters $\alpha^*_{2r,2j-1} \in \mathrm{Irr}(P_{2j} \cdot P_{2j+2} \cdots P_{2r})$ lying over some $Q_{2j+1}$-invariant character of $P_{2j}$, for all $j = 1, \dots, r-1$, as Lemma 5.142 and Theorem 5.143 imply. (Note that $\alpha^*_{2r} \in \mathrm{Irr}(P^*_{2r})$ is the $Q_3, Q_5, \dots, Q_{2r-1}$-correspondent of $\alpha_{2r} \in \mathrm{Irr}(P_{2i})$.)

Because $P_{2j+2} \cdots P_{2r}$ normalizes $Q_{2j+1}$ and fixes $\lambda$, (by Remark 9.8), it normalizes $Q_{2j+1}(\lambda) = Q_{2j+1,\lambda}$. The subgroup $Q_{1,2j} = C(P_2 \cdots P_{2j} \text{ in } Q_1)$ centralizes $P_{2j}$. But $Q_{2j+1} = Q_{2j+1}(\lambda) \cdot Q_{1,2j} = Q_{2j+1,\lambda} \cdot Q_{1,2j}$, according to (9.31), for all $j = 1, \dots, r-1$. Hence

$$\begin{aligned} P_{2j} \cap P_{2j+2} \cdots P_{2r} &= N(Q_{2j+1} \text{ in } P_{2j}) \\ &= C(Q_{2j+1} \text{ in } P_{2j}) = C(Q_{2j+1,\lambda} \text{ in } P_{2j}) = N(Q_{2j+1,\lambda} \text{ in } P_{2j}). \end{aligned}$$

Therefore

$$\begin{aligned} N(Q_{2j+1,\lambda} \text{ in } P_{2j} \cdot P_{2j+2} \cdots P_{2r}) &= N(Q_{2j+1,\lambda} \text{ in } P_{2j}) \cdot P_{2j+2} \cdot P_{2r} \\ &= N(Q_{2j+1} \text{ in } P_{2j}) \cdot P_{2j+2} \cdots P_{2r} = N(Q_{2j+1} \text{ in } P_{2j} \cdot P_{2j+2} \cdots P_{2r}). \end{aligned}$$

This, along with Proposition 3.9, ,

implies that the above $Q_{2j+1}$-correspondence coincides with the $Q_{2j+1,\lambda}$-correspondence, for all $j = 1, \ldots, r-1$. Hence the $Q_3, Q_5, \ldots, Q_{2r-1}$-correspondent, $\alpha_{2r}^* \in \mathrm{Irr}(P_{2r}^*)$, of $\alpha_{2r} = \alpha_{2r,\lambda} \in \mathrm{Irr}(P_{2r})$ coincides with the $Q_{3,\lambda}, Q_{5,\lambda}, \ldots, Q_{2r-1,\lambda}$-correspondent $\alpha_{2r,\lambda}^* \in \mathrm{Irr}(P_{2r,\lambda}^*)$ of $\alpha_{2r} = \alpha_{2r,\lambda} \in \mathrm{Irr}(P_{2r,\lambda})$, i.e., $\alpha_{2r}^* = \alpha_{2r,\lambda}^*$. So the corollary follows. $\square$

What about the groups $\mathbf{A}$ and $\mathbf{B}$? How is a Hall system for $G(\lambda)$ that satisfies the analogue of (9.2) for the $\lambda$-case related to $\mathbf{A}, \mathbf{B}$? The answer is given in

**Theorem 9.46.** *We can find $\mathbf{A}_\lambda \in \mathrm{Hall}_\pi(G_\lambda)$ and $\mathbf{B}_\lambda \in \mathrm{Hall}_{\pi'}(G_\lambda)$ satisfying the equivalent of (9.2) for the $\lambda$-groups, along with*

$$\mathbf{A}_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda}) = \mathbf{A}(\chi_1, \ldots, \chi_h), \tag{9.47a}$$

$$\mathbf{B}_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda}) = \mathbf{B}(\chi_1, \ldots, \chi_h, \lambda), \tag{9.47b}$$

*for all $h = 1, \ldots, n$.*

*Proof.* It suffices to show that $\mathbf{A}(\chi_1, \ldots, \chi_h)$ and $\mathbf{B}(\chi_1, \ldots, \chi_h, \lambda)$ satisfy (9.2b,c) for the $\lambda$-groups. We already know, by (9.2c), that $\mathbf{A}(\chi_1, \ldots, \chi_n) = P_{2k'}^*$, while $\mathbf{B}(\chi_1, \ldots, \chi_n)(\lambda) = Q_{2l'-1}^*(\lambda)$. But $P_{2k'}^* = P_{2k',\lambda}^*$ and $Q_{2l'-1}^*(\lambda) = Q_{2l'-1,\lambda}^*$, by (9.23). Thus

$$\mathbf{A}(\chi_1, \ldots, \chi_n) = P_{2k',\lambda}^*, \text{ and } \mathbf{B}(\chi_1, \ldots, \chi_n)(\lambda) = Q_{2l'-1,\lambda}^*.$$

Thus they satisfy (9.2c) for the $\lambda$-groups.

The fact that $\chi_{i,\lambda}$ is the $\lambda$-Clifford correspondent of $\chi_i$, whenever $i = 1, \ldots, h$, implies

$$G(\chi_1, \ldots, \chi_h, \lambda) = G_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda}) \leq G(\chi_1, \ldots, \chi_h),$$

for all $h = 1, \ldots, n$. As $\mathbf{A}(\chi_1)$ fixes $\lambda$, the group $\mathbf{A}(\chi_1, \ldots, \chi_h)$ is a subgroup of the first group in the above list. It is also a $\pi$-Hall subgroup of $G(\chi_1, \ldots, \chi_h)$, by (9.2b). Hence it is a $\pi$-Hall subgroup of $G_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda})$. Furthermore,

$$G(\chi_1, \ldots, \chi_h) = \mathbf{A}(\chi_1, \ldots, \chi_h) \cdot \mathbf{B}(\chi_1, \ldots, \chi_h),$$

by (9.2b). Hence $G_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda}) = G(\chi_1, \ldots, \chi_h, \lambda) = \mathbf{A}(\chi_1, \ldots, \chi_h) \cdot \mathbf{B}(\chi_1, \ldots, \chi_h, \lambda)$. So $\mathbf{A}(\chi_1, \ldots, \chi_h)$ and $\mathbf{B}(\chi_1, \ldots, \chi_h, \lambda)$ form a Hall system for $G_\lambda(\chi_{1,\lambda}, \ldots, \chi_{h,\lambda})$, for all $h = 1, \ldots, n$. This completes the proof of Theorem 9.46. $\square$

From now until the end of the section we restrict our attention to the smaller system (9.3). It is clear that the subset

$$\{Q_{2i-1,\lambda}, P_{2r,\lambda} | \beta_{2i-1,\lambda}, \alpha_{2r,\lambda}\}_{i=1,r=0}^{l,k},$$

of (9.17b), is a triangular set for the normal series $G_0 \trianglelefteq G_{1,\lambda} \trianglelefteq \cdots \trianglelefteq G_{m,\lambda} \trianglelefteq G_\lambda$, corresponding to the tower $\{\chi_{i,\lambda}\}_{i=0}^m$. Hence Theorem 9.26 implies that the above triangular set satisfies (9.27) and (9.28) for all $r = 0, \ldots, k$ and all $i = 1, \ldots, l$, respectively, since (9.17b) satisfies them. The groups in question now are $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$ along with their corresponding groups $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$ and $\widehat{P}_\lambda(\alpha_{2l-2,2l-2,\lambda})$, in the $\lambda$-situation. Their relation is described in the next two theorems.

**Theorem 9.48.** *Assume that $\{\mathbf{A}_\lambda, \mathbf{B}_\lambda\}$ is a Hall system for $G_\lambda$ that is derived from $\{\mathbf{A}, \mathbf{B}\}$ and satisfies the conditions in Theorem 9.46. Assume further that, for every $m = 1, \ldots, n$, the group*

$\widehat{Q}$ *is picked to satisfythe conditions in Theorem 8.13 for the smaller system* (9.3)*, while the group* $\widehat{Q}_\lambda$ *is picked to satisfy similar conditions for the* $\lambda$*-groups. Then*

$$\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}) = \widehat{Q}(\beta_{2k-1,2k}, \lambda). \tag{9.49}$$

*So* $\mathcal{Q}_\lambda = \widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}) \cdot Q^*_{2l-1,\lambda} \leq \mathcal{Q}(\lambda)$*, where* $\mathcal{Q} = \widehat{Q}(\beta_{2k-1,2k}) \cdot Q^*_{2l-1}$.

*Proof.* Assume that $\widehat{Q}_\lambda$ satisfies the conditions in Theorem 8.13 for the $\lambda$-situation. Of course it satisfies the equivalent of the conditions in Theorem 6.19 for the $\lambda$-groups. Furthermore,

$$\begin{aligned}
\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}) &= N(P^*_{2k,\lambda} \text{ in } \mathbf{B}_\lambda(\chi_{1,\lambda}, \dots, \chi_{2k,\lambda})) && \text{by Theorem 8.13} \\
&= N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \dots, \chi_{2k}, \lambda)) && \text{by (9.24a) and (9.47)} \\
&= N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \dots, \chi_{2k}))(\lambda) \\
&= \widehat{Q}(\beta_{2k-1,2k})(\lambda). && \text{by Theorem 8.13}
\end{aligned}$$

This proves the first part of the theorem. The last part follows from the first and (9.24b). ∎

Similarly to the above Theorem 9.48 we have

**Theorem 9.50.** *Assume that* $\{\mathbf{A}_\lambda, \mathbf{B}_\lambda\}$ *is a Hall system for* $G_\lambda$ *that is derived from* $\{\mathbf{A}, \mathbf{B}\}$ *and satisfies the conditions in Theorem 9.46. Assume further that, for every* $m = 1, \dots, n$*, the group* $\widehat{P}$ *is picked to satisfy the conditions in Theorem 8.15 for the smaller system* (9.3)*, while the group* $\widehat{P}_\lambda$ *is picked to satisfy the similar conditions for the* $\lambda$*-groups. Then Then*

$$\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda}) = \widehat{P}(\alpha_{2l-2,2l-1}). \tag{9.51}$$

*So* $\mathcal{P}_\lambda = \widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda}) \cdot P^*_{2k,\lambda} = \mathcal{P} = \widehat{P}(\alpha_{2l-1,2l-1}) \cdot P^*_{2k}$.

*Proof.* Assume that $\widehat{Q}_\lambda$ satisfies the conditions in Theorem 8.15 for the $\lambda$-situation. Of course it satisfies the equivalent of the conditions in Theorem 6.19 for the $\lambda$-groups. Furthermore,

$$\begin{aligned}
\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda}) &= N(Q^*_{2l-1,\lambda} \text{ in } \mathbf{A}_\lambda(\chi_{1,\lambda}, \dots, \chi_{2l-1,\lambda})) && \text{by Theorem 8.15} \\
&= N(Q^*_{2l-1}(\lambda) \text{ in } \mathbf{A}(\chi_1, \dots, \chi_{2l-1})) && \text{by (9.24b) and (9.47)}
\end{aligned}$$

Clearly $Q_1 \leq Q^*_{2l-1} \leq G(\beta_1)$. Thus Lemma 9.25 implies

$$Q^*_{2l-1} = Q^*_{2l-1}(\lambda) \cdot Q_1.$$

As $\mathbf{A}(\chi_1, \dots, \chi_{2l-1})$ is contained in $\mathbf{A}(\chi_1)$, it fixes $\lambda$ since $\mathbf{A}(\chi_1)$ fixes $\lambda$. It also normalizes $Q_1 \trianglelefteq G$. Therefore

$$N(Q^*_{2l-1}(\lambda) \text{ in } \mathbf{A}(\chi_1, \dots, \chi_{2l-1})) = N(Q^*_{2l-1} \text{ in } \mathbf{A}(\chi_1, \dots, \chi_{2l-1})).$$

According to Theorem 8.15, the latter group is $\widehat{P}(\alpha_{2l-2,2l-1})$. Hence

$$\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda}) = N(Q^*_{2l-1}(\lambda) \text{ in } \mathbf{A}(\chi_1, \dots, \chi_{2l-1})) = \widehat{P}(\alpha_{2l-2,2l-1}).$$

So the first part of Theorem 9.50 follows. This, along with (9.24a) implies the rest of the theorem. ∎

The fact that $\widehat{Q} \leq G' = G(\alpha^*_{2k})$ normalizes $P^*_{2k}$, along with (9.13), obviously implies

$$\widehat{Q}(\beta_{2k-1,2k}, \lambda) = \widehat{Q}(\beta_{2k-1,2k}, \lambda_{2k}). \tag{9.52}$$

We define

$$I := \text{the image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \operatorname{Aut}(P^*_{2k}). \tag{9.53}$$

Obviously, the group $I$ is well defined, as $\widehat{Q} \leq G(\alpha^*_{2k})$ normalizes $P^*_{2k}$. Then as an easy corollary of Theorem 9.48 we get

**Corollary 9.54.** *The groups $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$ and $\widehat{Q}(\beta_{2k-1,2k})$, chosen in Theorem 9.48, have the same image in $\operatorname{Aut}(P^*_{2k})$. In particular,*

$$I = I_\lambda, \tag{9.55}$$

*where $I_\lambda$ is the image of $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$ in $\operatorname{Aut}(P^*_{2k,\lambda})$.*

*Proof.* The group $\widehat{Q}(\beta_{2k-1,2k})$ is a subgroup of $G'(\beta_{2k-1,2k}) = G(\alpha^*_{2k}, \beta_{2k-1,2k})$. By Proposition 5.50, the character $\beta_{1,2k}$ is the unique character of $Q_{1,2k}$ lying under $\beta_{2k-1,2k}$. Thus $G'(\beta_{2k-1,2k})$ fixes $\beta_{1,2k}$. Hence

$$\widehat{Q}(\beta_{2k-1,2k}) \leq G'(\beta_{2k-1,2k}) \leq N(P^*_{2k} \text{ in } G(\beta_{1,2k})).$$

Furthermore, according to (6.34) and the Definition 6.28, the group $Q_{1,2k}$ is a subgroup of $\widehat{Q}$. It also fixes $\beta_{2k-1,2k} \in \operatorname{Irr}(Q_{2k-1,2k})$, as $Q_{1,2k} \leq Q_{2k-1,2k}$. Thus $Q_{1,2k}$ is a subgroup of $\widehat{Q}(\beta_{2k-1,2k})$. In conclusion,

$$Q_{1,2k} \leq \widehat{Q}(\beta_{2k-1,2k}) \leq N(P^*_{2k} \text{ in } G(\beta_{1,2k})).$$

Therefore Lemma 9.25 can be applied with $\widehat{Q}(\beta_{2k-1,2k})$ in the place of $S$. So

$$\widehat{Q}(\beta_{2k-1,2k}) = \widehat{Q}(\beta_{2k-1,2k}, \lambda) \cdot Q_{1,2k}.$$

This, along with Theorem 9.48 implies

$$\widehat{Q}(\beta_{2k-1,2k}) = \widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}) \cdot Q_{1,2k}.$$

The fact that $Q_{1,2k} = C(P^*_{2k} \text{ in } Q_1)$ centralizes $P^*_{2k}$ implies the first part of the corollary immediately. This, along with (9.24a) and the definitions of $I$ and $I_\lambda$, implies equation (9.55). ☐

Theorem 9.50 easily implies

**Corollary 9.56.** *The groups $\widehat{P}(\alpha_{2l-2,2l-1})$ and $\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda})$, chosen in Theorem 9.50, have the same images in $\operatorname{Aut}(Q^*_{2l-1})$ and the same images in $\operatorname{Aut}(Q^*_{2l-1,\lambda})$.*

We finish this section by checking a special case of extendibility in the $\lambda$-situation. The following is a well known result.

**Theorem 9.57.** *Assume that $G$ is a finite group and that $S \leq H$ are subgroups of $G$ with $S$ normal in $G$. Assume further that $\theta \in \operatorname{Irr}(H)$ lies above $\lambda \in \operatorname{Irr}(S)$. Let $\theta_\lambda \in \operatorname{Irr}(H(\lambda))$ denote the unique $\lambda$-Clifford correspondent of $\theta$.*

*If $\theta$ extends to its stabilizer $G(\theta)$ in $G$, then $\theta_\lambda$ also extends to $G(\theta, \lambda)$.*

*Proof.* A straight forward application of Clifford Theory implies that $G(\theta, \lambda) \leq G(\theta_\lambda)$. Furthermore, as $G(\theta)$ fixes $\theta$ it permutes among themselves the members of the $H$-conjugacy class of characters in $\mathrm{Irr}(S)$ lying under $\theta$. Since $\lambda \in \mathrm{Irr}(S)$ lies under $\theta$ we get

$$G(\theta) = H \cdot G(\theta, \lambda) \leq H \cdot G(\theta_\lambda). \tag{9.58a}$$

In addition,

$$G(\theta, \lambda) \cap H = H(\lambda). \tag{9.58b}$$

Let $\theta^i \in \mathrm{Irr}(G(\theta))$ be an extension of $\theta$ to $G(\theta)$. Then $\theta^i$ lies above $\lambda$. Let $\Psi \in \mathrm{Irr}(G(\theta, \lambda))$ denote the unique $\lambda$-Clifford correspondent of $\theta^i \in \mathrm{Irr}(G(\theta))$. So $\Psi$ lies above $\lambda$ and induces $\theta^i$. Therefore,

$$(\Psi^{G(\theta)})|_H = \theta^i|_H = \theta.$$

Mackey's Theorem, along with (9.58), implies that

$$(\Psi^{G(\theta)})|_H = (\Psi|_{H(\lambda)})^H.$$

Hence $(\Psi|_{H(\lambda)})^H = \theta$ is an irreducible character of $H$. So the restriction $\Psi|_{H(\lambda)}$ is an irreducible character of $H(\lambda)$ that induces $\theta$ and lies above $\lambda$ (as $\Psi$ lies above $\lambda$). We conclude that $\Psi|_{H(\lambda)}$ is the $\lambda$-Clifford correspondent of $\theta$. Hence $\Psi|_{H(\lambda)} = \theta_\lambda$. Thus $\Psi$ is an extension of $\theta_\lambda$ to $G(\theta, \lambda)$, and Theorem 9.57 follows. $\square$

As a consequence of Theorem 9.57 we get

**Theorem 9.59.** *Assume that $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$. Let $\widehat{Q}_\lambda$ be a $\pi'$-Hall subgroup of $G'_\lambda = G(\lambda, \alpha^*_{2k,\lambda})$ that satisfies the conditions in Theorem 9.48. Then the character $\beta_{2k-1,2k,\lambda}$ extends to $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$.*

*Proof.* According to (6.33) we get that

$$Q_{2k-1,2k} = \widehat{Q}_{2k-1,2k}(\beta_{2k-1,2k}) = \widehat{Q}(\beta_{2k-1,2k}) \cap G_{2k-1} \trianglelefteq \widehat{Q}(\beta_{2k-1,2k}).$$

In view of Remark 9.42, this implies

$$E_{2k} \trianglelefteq Q_{2k-1,2k,\lambda} = Q_{2k-1,2k}(\lambda_{2k}) \leq Q_{2k-1,2k} \trianglelefteq \widehat{Q}(\beta_{2k-1,2k}).$$

As $\beta_{2k-1,2k,\lambda} \in \mathrm{Irr}(Q_{2k-1,2k,\lambda})$ is the $\lambda_{2k}$-Clifford correspondent of $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$, Theorem 9.57 implies that $\beta_{2k-1,2k,\lambda}$ extends to $\widehat{Q}(\beta_{2k-1,2k}, \lambda_{2k})$, when $\beta_{2k-1,2k}$ extends to $\widehat{Q}(\beta_{2k-1,2k})$. But

$$\widehat{Q}(\beta_{2k-1,2k}, \lambda_{2k}) = \widehat{Q}(\beta_{2k-1,2k}, \lambda) = \widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}),$$

by (9.52) and Theorem 9.48.

This completes the proof of the theorem. $\square$

Furthermore, Remark 9.44 easily implies

**Theorem 9.60.** *Assume that $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$. Let $\widehat{P}_\lambda$ be the $\pi$-Hall subgroup of $G_\lambda(\beta^*_{2l-1})$ that satisfies the conditions in Theorem 9.50. Then the character $\alpha_{2l-2,2l-1,\lambda}$ extends to $\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda})$.*

## 9.2 Normal $\pi$-subgroups inside $P_2$

Assume now that we are in a situation where the fixed normal series (9.1a), i.e., $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n \trianglelefteq G$, satisfies (8.18). So teh following two onditions hold

$$G_2 = G_{2,\pi} \times G_{2,\pi'}, \tag{9.61}$$

$$G \text{ fixes } \chi_1. \tag{9.62}$$

We assume fixed the character tower (9.1b) and its corresponding triangular set (9.1c). We also fix the Hall system $\{\mathbf{A}, \mathbf{B}\}$ that satisfies (9.2).

The additional hypothesis on the group $G_2$ give more specific information on $\chi_2$ (see (8.19)). Thus we have

$$G_2 = P_2 \times Q_1 = P_2 \times G_1, \tag{9.63a}$$

$$\chi_2 = \alpha_2 \times \beta_1. \tag{9.63b}$$

Furthermore

$$G(\chi_2) = G(\alpha_2). \tag{9.64}$$

We also fix an integer $m = 2, \ldots, n$ and we consider the normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G. \tag{9.65a}$$

We also fix the subtower

$$\{1 = \chi_0, \chi_1, \ldots, \chi_m\} \tag{9.65b}$$

of (9.1b) and the subset

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l,\, k} \tag{9.65c}$$

of (9.1c) . The subtower (9.65b) is a character tower of (9.65a), and the subset (9.65c) is is a representative of the conjugacy class of triangular sets that corresponds uniquely to (9.65b). We also assume known the groups $\widehat{Q}$ and $\widehat{P}$ satsfying (6.20) and (8.1) respectively for the system (9.65). Also known are assume the groups $\mathcal{P}$ and $\mathcal{Q}$, as these were defined in (9.4a), i.e.,

$$\mathcal{P} = \widehat{P}(\alpha_{2l-2,2l-1}) \cdot P^*_{2k} \text{ and } \mathcal{Q} = \widehat{Q}(\beta_{2k-1,2k}) \cdot Q^*_{2l-1}. \tag{9.66}$$

So $\mathcal{P} \leq \mathbf{A}(\chi_1, \ldots, \chi_{2l-1})$ and $\mathcal{Q} \leq \mathbf{B}(\chi_1, \ldots, \chi_{2k})$, by (9.1).

As the series (9.1a) satisfies (9.61), we can apply all the results of Section 8.3. So according to Theorem 8.29, we can drop the group $Q_1 = G_1$ (i.e. shift the above series by one), without any loss. Thus the original system (9.1) reduces to

$$\begin{gathered} 1 \trianglelefteq P_2 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_n = G, \\ 1, \alpha_2, \chi_3, \ldots, \chi_n, \\ \{Q^s_1 = 1, Q_{2i-1}, P_{2r} | \beta^s_1 = 1, \beta_{2i-1}, \alpha_{2r}\}_{i=2,r=0}^{l',k'} \end{gathered} \tag{9.67a}$$

Similarly the subsystem (9.65) reduces to

$$\begin{gathered} 1 \trianglelefteq P_2 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G, \\ 1, \alpha_2, \chi_3, \ldots, \chi_m, \\ \{Q^s_1, Q_{2i-1}, P_{2r} | \beta^s_1, \beta_{2i-1}, \alpha_{2r}\}_{i=2,r=0}^{l,k} \end{gathered} \tag{9.67b}$$

They both satisfy the conditions in Theorem 8.29. Therefore the Hall system $\{\mathbf{A}, \mathbf{B}\}$ remains the same as well as the groups $P^*_{2k}, Q^*_{2l-1}, \widehat{Q}, \widehat{Q}(\beta_{2k-1,2k}), \widehat{P}$ and $\widehat{P}(\alpha_{2l-1,2l-1})$. Note also that the shifted systems satisfy

$$G^s_1 = Q^s_1 = 1 \text{ and } \chi^s_1 = \beta^s_1 = 1 \tag{9.68}$$

$$G^s_2 = P_2 = P^*_2 \text{ and } \chi^s_2 = \alpha_2 = \alpha^*_2. \tag{9.69}$$

As with the previous section, the goal is to understand the behavior of the above systems when we apply Clifford's theory to a normal subgroup of $G$. This time the normal subgroup is contained in $P_2$, and not $Q_1$ as was the case with Section 9.1. The results we obtain are similar to those of the previous section, and their proofs are identical, (the only modification being, whenever necessary, the interchanged role of $\pi$ and $\pi'$). As with the group $S$ and the character $\zeta$ earlier, we fix (until the end of this section) a subgroup $R$ of $G_2$ and a character $\eta$ of $R$ that satisfy

$$R \trianglelefteq G \text{ and } R \leq P_2, \tag{9.70a}$$

$$\eta \in \mathrm{Irr}(R) \text{ is } G\text{-invariant and lies under } \alpha_2. \tag{9.70b}$$

We also assume that $M$ is a normal subgroup of $G$ with

$$R \leq M \leq P_2. \tag{9.70c}$$

The role of $\mathbf{A}(\beta_1)$ is played here by the group $\mathbf{B}(\alpha_2)$. So, as in the case of $\lambda$ and $E$ in the previous section, we have

**Lemma 9.71.** *There is an irreducible character $\mu \in \mathrm{Irr}(M)$ such that $\mu$ is $\mathbf{B}(\alpha_2)$-invariant and lies under $\alpha_2$ and under $\chi_i$, for all $i = 3, \ldots, n$. Any such $\mu$ lies above $\eta \in \mathrm{Irr}(R)$.*

*Proof.* The proof is similar to that of Lemma 9.6. If $\mu_1$ is any irreducible character of $M$ lying under $\alpha_2$, then $\mathbf{B}(\alpha_2)$ permutes among themselves the $P_2$-conjugates of $\mu_1$, as it normalizes $P_2 \trianglelefteq G$ and fixes $\alpha_2$. So Glauberman's Lemma implies that $\mathbf{B}(\alpha_2)$ fixes at least one $P_2$-conjugate of $\mu_1$. The lemma follows if we observe that any character of $M$ that lies under $\alpha_2$ also lies under $\chi_i$, for all $i = 3, \ldots, n$, while any character of $M$ lying under $\alpha_2$ lies necessarily above $\eta \in \mathrm{Irr}(R)$, as $\eta$ is $G$-invariant. ☐

Note also that

**Remark 9.72.** Assume that $\mu_1 \in \mathrm{Lin}(M)$ is a linear character of $M$ lying under $\alpha_2$. Then there exists a $P_2$-conjugate $\mu \in \mathrm{Lin}(M)$ of $\mu_1$, such that $\mu$ is $\mathbf{B}(\alpha_2)$-invariant, and lies under $\alpha_2$ and above $\eta$.

**Remark 9.73.** The $\pi'$-group $Q^*_{2l'-1} = 1 \cdot Q_3 \cdot Q_5 \cdots Q_{2l'-1}$ fixes $\alpha_2$, as every one of its factors does, by (5.17e). Thus $Q^*_{2l'-1} \leq \mathbf{B}(\alpha_2)$. So $Q^*_{2l-1}$ fixes $\mu$.

**Remark 9.74.** For any $m \geq 2$, the $\pi'$-group $\mathcal{Q}$ is a subgroup of $\mathbf{B}(\chi_1, \chi_2, \ldots, \chi_{2k})$, by (9.4d). Thus $\mathcal{Q} \leq \mathbf{B}(\chi_1, \chi_2) = \mathbf{B}(\chi_2) = \mathbf{B}(\alpha_2)$. So $\mathcal{Q}$ fixes $\mu$.

From now on we fix a character $\mu \in \mathrm{Irr}(M)$ having all the properties in Lemma 9.71.

Since the $\pi'$-group $Q^*_{2i-1}$, as a subgroup of $Q^*_{2l'-1} \leq \mathbf{B}(\alpha_2)$, fixes the character $\mu$, we can define $M_{2i-1}$ and $\mu_{2i-1}$ by

$$M_{2i-1} = C(Q^*_{2i-1} \text{ in } M) \text{ and} \tag{9.75a}$$

$$\mu_{2i-1} \in \mathrm{Irr}(M_{2i-1}) \text{ is the } Q^*_{2i-1}\text{-Glauberman correspondent of } \mu \in \mathrm{Irr}(M), \tag{9.75b}$$

for all $i = 2, \dots, l'$. We also define $M_1$ and $\mu_1$ as

$$M_1 = M \text{ and } \mu_1 = \mu. \tag{9.75c}$$

Furthermore,

$$M_{2i-1} = P_{2,2i-1} \cap M = C(Q^*_{2i-1} \text{ in } P_2) \cap M, \tag{9.76a}$$

for all $i = 2, \dots, l'$, while

$$\mu_1 \text{ lies under } \alpha_2 \in \mathrm{Irr}(P_2), \tag{9.76b}$$

$$\mu_{2i-1} \text{ lies under } \alpha_{2,2i-1} \in \mathrm{Irr}(P_{2,2i-1}), \tag{9.76c}$$

whenever $2 \leq i \leq l'$, as $\alpha_{2,2i-1}$ is the $Q^*_{2i-1}$-Glauberman correspondent of $\alpha_2$, and $\alpha_2$ lies over $\mu$. In view of (9.75) we have

$$N(\mu) = N(\mu_{2i-1}), \tag{9.77}$$

for all groups $N$ with $N \leq N(Q^*_{2i-1}$ in $G)$, and all $i = 2, \dots, l'$.

As in the previous section, we define

$$G_\mu := G(\mu), \tag{9.78a}$$

$$G_{i,\mu} := G_i(\mu) = G_\mu \cap G_i, \tag{9.78b}$$

$$P_{2,\mu} := P_2(\mu), \tag{9.78c}$$

whenever $3 \leq i \leq n$. This way we can form the series

$$1 \trianglelefteq P_{2,\mu} \trianglelefteq G_{3,\mu} \trianglelefteq \cdots \trianglelefteq G_{n,\mu} = G_\mu, \tag{9.78d}$$

of normal subgroups of $G_\mu = G(\mu)$.

We also write

$$G_{1,\mu} := Q^s_1 = 1, \tag{9.79a}$$

and

$$G_{2,\mu} := P_2(\mu) = P_{2,\mu}. \tag{9.79b}$$

Furthermore, we can apply Clifford's Theorem to the groups $G_i$ and the characters $\chi_i$, for all $i = 3, \dots, n$. Thus there exist unique irreducible characters $\chi_{i,\mu} \in \mathrm{Irr}(G_{i,\mu})$ lying above $\mu$ and inducing $\chi_i$

$$\chi_{i,\mu} \in \mathrm{Irr}(G_{i,\mu}|\mu) \text{ and } (\chi_{i,\mu})^{G_i} = \chi_i, \tag{9.80a}$$

for all $i = 3, \dots, n$. We complete this list by setting

$$\chi_{1,\mu} = \beta^s_1 = 1 \tag{9.80b}$$

We also write

$$\chi_{2,\mu} = \alpha_{2,\mu} \in \mathrm{Irr}(P_2(\mu)) \tag{9.80c}$$

for the $\mu$-Clifford correspondent of $\alpha_2$, i.e., $\chi_{2,\mu}$ lies above $\mu$ and induces $\alpha_2 \in \mathrm{Irr}(P_2)$. Clearly $\chi_{2,\mu}$ lies above $\chi_{1,\mu} = 1$. Furthermore, as the $\chi_i$ lie above each other, the same holds for the characters $\chi_{i,\mu}$, by Clifford's theory, for all $i = 2, \dots, n$. Therefore, we have formed a character tower

$$\{1, \chi_{2,\mu} = \alpha_{2,\mu}, \chi_{3,\mu}, \dots, \chi_{n,\mu}\} \tag{9.81a}$$

for the series (9.78d). Hence Theorem 5.6, applied to the tower (9.81a), implies the existence of a unique $G_\mu$-conjugacy class of triangular sets of (9.78d) that corresponds to the tower (9.81a). Let

$$\{Q_{2i-1,\mu}, P_{2r,\mu} | \beta_{2i-1,\mu}, \alpha_{2r,\mu}\}_{i=1,r=0}^{l',\,k'} \tag{9.81b}$$

be a representative of this class.

All the groups, the characters and their properties that were described in Chapter 5 are valid for the $\mu$-situation. We follow the same notation as in the previous section, with an extra $\mu$ in the place of the $\lambda$ there. The goal is the same as in the previous section, that is, to compare the triangular set in (9.67) with that in (9.81b). As many of the results here have proofs analogous to those in Section 9.1, we give them briefly or skip them.

The first steps in that direction follow from (9.79) and (9.80). In particular, these relations clearly imply

$$Q_{1,\mu} = G_{1,\mu} = G_1^s = Q_1^s = 1, \tag{9.82a}$$
$$\beta_{1,\mu} = \beta_1^s = 1, \tag{9.82b}$$
$$P_{2,\mu} = P_2(\mu), \tag{9.82c}$$
$$\alpha_{2,\mu} \in \mathrm{Irr}(P_{2,\mu}) \text{ is the } \mu\text{-Clifford correspondent of } \alpha_2 \in \mathrm{Irr}(P_2). \tag{9.82d}$$

As in (5.129b) and (5.129a), we get the groups $G^*_{i,\mu}$ and $G^*_\mu$, defined as

$$\begin{aligned} G^*_{0,\mu} &:= G_{0,\mu} = 1, \\ G^*_{i,\mu} &:= G_{i,\mu}(\chi_{1,\mu}, \dots \chi_{i-1,\mu}) = G_{i,\mu}(\chi_{1,\mu}, \dots, \chi_{n,\mu}), \\ G^*_\mu &:= G_\mu(\chi_{1,\mu}, \dots, \chi_{n,\mu}), \end{aligned} \tag{9.83}$$

for all $i = 1, \dots, n$. Clearly we have

$$G_{i,\mu} = G^*_\mu \cap G_i, \tag{9.84a}$$
$$G^*_{1,\mu} = G_{1,\mu} = Q_{1,\mu} = 1, \tag{9.84b}$$
$$G^*_{2,\mu} = G_{2,\mu}(\chi_1) = P_{2,\mu}(1) = P_{2,\mu}, \tag{9.84c}$$

whenever $3 \leq i \leq n$.

As with the $\lambda$-groups and Lemma 9.20, the following holds:

**Lemma 9.85.** *For any $i = 3, \dots, n$ we have*

$$G^*_\mu = G^*(\mu) \tag{9.86a}$$
$$G^*_{i,\mu} = G^*_i(\mu). \tag{9.86b}$$

*Proof.* The proof is similar to that of Lemma 9.20. So we omit it. □

We can now prove

**Proposition 9.87.** *For every $r = 0, 1, \dots, k'$ and $i = 2, \dots, l'$ we have that*

$$P^*_{2r}(\mu) \in \mathrm{Hall}_\pi(G^*_{2r,\mu}), \tag{9.88a}$$
$$Q^*_{2i-1} \in \mathrm{Hall}_{\pi'}(G^*_{2i-1,\mu}). \tag{9.88b}$$

*Therefore the triangular set* (9.81b) *can be chosen among the sets in its $G_\mu$-conjugacy class so that it satisfies*

$$P^*_{2r}(\mu) = P^*_{2r,\mu}, \tag{9.89a}$$

$$Q^*_{2i-1} = Q^*_{2i-1,\mu}, \tag{9.89b}$$

*whenever $0 \leq r \leq k'$ and $2 \leq i \leq l'$.*

*Proof.* The cases $P^*_0$ and $P^*_2$ are easy (see (9.82c) ). The rest of the proof is similar to that of Proposition 9.22, with the roles of $P^*_{2i,\mu}$ and $Q^*_{2i-1,\mu}$ interchanged. □

The analogue of Lemma 9.25 is

**Lemma 9.90.** *Assume that $P_2 \leq T \leq G(\alpha_2)$. Then $T = T(\mu) \cdot P_2$. Furthermore, if $S$ satisfies $P_{2,2i-1} \leq S \leq N(Q^*_{2i-1} \text{ in } G(\alpha_{2,2i-1}))$, for some $i = 2, \ldots, l'$, Then $S = S(\mu_{2i-1}) \cdot P_{2,2i-1} = S(\mu) \cdot P_{2,2i-1}$.*

*Proof.* Similar to the proof of Lemma 9.90. We only remark that the role of $P_2$ here is the same as that of $Q_1$ there, as the group $P_2$ is a normal subgroup of $G$. □

To compare the groups $P_{2i,\mu}$ and $Q_{2i-1,\mu}$ with $P_{2i}$ and $Q_{2i-1}$ we have, as we would have guessed

**Theorem 9.91.** *The set* (9.81b) *chosen in Proposition 9.87 satisfies*

$$Q_{2i-1,\mu} = Q_{2i-1}, \tag{9.92a}$$

$$\beta_{2i-1,\mu} = \beta_{2i-1}, \tag{9.92b}$$

*for all $i = 2, \ldots, l'$, and*

$$P_{2r,\mu} = P_{2r}(\mu) = P_{2r}(\mu_{2r-1}), \tag{9.93a}$$

$$\alpha_{2r,\mu} \in \mathrm{Irr}(P_{2r,\mu}) \textit{ is the } \mu_{2r-1}\textit{-Clifford correspondent of } \alpha_{2r} \in \mathrm{Irr}(P_{2r}), \tag{9.93b}$$

*for all $r = 1, \ldots, k'$. Hence $\alpha_{2r,\mu}$ induces $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$.*

*Proof.* The proof is similar to that of Theorem 9.26. So we omit it. We only remark that we need to interchange the role of the $\pi$-and $\pi'$-groups. Note also that the case $i = 1$, that is omitted here, was already done, along with the case $r = 1$, in (9.82). □

The analogue of (9.31), that would also appear as a step if we had given the proof of the above theorem in detail, is

**Remark 9.94.** For all $r = 2, \ldots, k'$, we get

$$P_{2r} = P_{2r}(\mu) \cdot P_{2,2r-1} = P_{2r}(\mu_{2r-1}) \cdot P_{2,2r-1}.$$

*Proof.* For all $r = 2, \ldots, k'$ we have that

$$P_{2,2r-1} = P_2 \cap P_{2r} \leq P_{2r} \leq N(Q^*_{2r-1} \text{ in } G(\alpha_{2,2r-1})),$$

by (5.34) and Proposition 5.55. Hence we can apply Lemma 9.90, with $P_{2r}$ in the place of $S$. So the remark follows. □

Furthermore, the analogue of Remark 9.42 for the $\pi$-groups is

**Remark 9.95.** For all $r, i$ with $1 \leq r < i \leq l'$ we have

$$P_{2r,2i-1,\mu} = P_{2r,2i-1}(\mu) = P_{2r,2i-1}(\mu_{2i-1}),$$

while the character $\alpha_{2r,2i-1,\mu}$ is the $\mu_{2i-1}$-Clifford correspondent of $\alpha_{2r,2i-1}$.

*Proof.* The proof is the same as that of Remark 9.42, so we only sketch it.

$$\begin{aligned} P_{2r,2i-1,\mu} &= C(Q_{2r+1,\mu}, \ldots, Q_{2i-1,\mu} \text{ in } P_{2r,\mu}) = C(Q_{2r+1} \cdots Q_{2i-1} \text{ in } P_{2r}(\mu)) \\ &= C(Q_{2r+1} \cdots Q_{2i-1} \text{ in } P_{2r})(\mu) = P_{2r,2i-1}(\mu) = P_{2r,2i-1}(\mu_{2i-1}), \end{aligned} \tag{9.96}$$

where the last equation follows from (9.77), along with the fact that $P_{2r,2i-1}$ normalizes $Q^*_{2i-1}$.

Furthermore, the fact that Clifford theory is compatible with Glauberman correspondence (Lemma 2.5 in [22]), along with (9.93b), implies that the $Q_{2r+1} \cdots Q_{2i-1} = Q_{2r+1,\mu} \cdots Q_{2i-1,\mu}$-Glauberman correspondent $\alpha_{2r,2i-1}$ of $\alpha_{2r}$ is induced by the $Q_{2r+1,\mu} \cdots Q_{2i-1,\mu}$-Glauberman correspondent $\alpha_{2r,2i-1,\mu}$ of $\alpha_{2r,\mu}$, and, in addition, $\alpha_{2r,2i-1,\mu}$ lies above the $Q_{2r+1} \cdots Q_{2i-1}$-Glauberman correspondent $\mu_{2i-1}$ of $\mu_{2r-1}$. Hence Remark 9.95 follows. $\square$

Now we can prove two corollaries that follow from Theorem 9.91,

**Corollary 9.97.** *For all $i, j$ with $2 \leq i \leq j \leq k'$ we have that*

$$Q_{2i-1,2j,\mu} = Q_{2i-1,2j} \text{ and } \beta_{2i-1,2j,\mu} = \beta_{2i-1,2j}.$$

*In addition, $Q_{1,2j,\mu} = Q^s_{1,2j} = 1$ and $\beta_{1,2j,\mu} = \beta^s_{1,2j} = 1$, whenever $1 \leq j \leq k'$.*

*Proof.* For all $t, i$ with $2 \leq i \leq t \leq k'$, the group $Q_{2i-1}$ centralizes $P_{2,2t-1} = C(Q_3, \ldots, Q_{2t-1} \text{ in } P_2)$. As $P_{2t} = P_{2t}(\mu) \cdot P_{2,2t-1}$, we conclude that

$$C(P_{2t} \text{ in } Q_{2i-1}) = C(P_{2t}(\mu) \text{ in } Q_{2i-1}), \tag{9.98}$$

whenever $2 \leq i \leq t \leq k'$. So we get

$$\begin{aligned} Q_{2i-1,2j,\mu} &= C(P_{2i,\mu} \cdots P_{2j,\mu} \text{ in } Q_{2i-1,\mu}) && \text{by (5.33), for the } \mu\text{-case} \\ &= C(P_{2i}(\mu) \cdots P_{2j}(\mu) \text{ in } Q_{2i-1}) && \text{by (9.92a) and (9.93a)} \\ &= C(P_{2i} \cdots P_{2j} \text{ in } Q_{2i-1}) && \text{by (9.98)} \\ &= Q_{2i-1,2j} && \text{by (5.33)} \end{aligned}$$

whenever $2 \leq i \leq j \leq k'$.

The character $\beta_{2i-1,2j,\mu}$ is the $P_{2i,\mu} \cdots P_{2j,\mu}$-Glauberman correspondent of $\beta_{2i-1,\mu}$, by Definition 5.49 for the $\mu$-case. Hence it is also the $P_{2i}(\mu) \cdots P_{2j}(\mu)$-Glauberman correspondent of $\beta_{2i-1}$, according to Theorem 9.91. This, along with (9.98), implies that $\beta_{2i-1,2j,\mu}$ is the $P_{2i} \cdots P_{2j}$-Glauberman correspondent of $\beta_{2i-1}$. Thus $\beta_{2i-1,2j,\mu}$ equals $\beta_{2i-1,2j}$, as the latter was also defined as the $P_{2i} \cdots P_{2j}$-Glauberman correspondent of $\beta_{2i-1}$. This completes the proof of the first part of the corollary. The rest holds trivially. So Corollary 9.97 holds. $\square$

**Corollary 9.99.** *For all $r = 1, \ldots, k'$, the character $\alpha^*_{2r,\mu} \in \text{Irr}(P^*_{2r,\mu})$ is the $\mu$-Clifford correspondent of $\alpha^*_{2r} \in \text{Irr}(P^*_{2r})$.*

*Proof.* The character $\alpha^*_{2r,\mu}$ is defined (see Definition 5.147), as the $Q_{3,\mu},\dots,Q_{2r-1,\mu}$-correspondent of $\alpha_{2r,\mu}$, whenever $1 \leq r \leq k'$. Hence, (9.92a) implies that $\alpha^*_{2r,\mu}$ is the $Q_3,\dots,Q_{2i-1}$-correspondent of $\alpha_{2r,\mu}$. But $\alpha_{2r,\mu}$ is the $\mu$-Clifford correspondent of $\alpha_{2r}$, and the groups $Q_3,\dots,Q_{2r-1}$ fix $\mu$. According to Proposition 3.12 the $A$-correspondence is compatible with the Clifford correspondence. Thus, taking $A$ as $Q_3, Q_5,\dots,Q_{2r-1}$ in turn, we conclude that the $Q_3,\dots,Q_{2r-1}$-correspondent $\alpha^*_{2r,\mu}$ of $\alpha_{2r,\mu} \in \mathrm{Irr}(P_{2r}(\mu))$ is the $\mu$-Clifford correspondent of the $Q_3,\dots,Q_{2r-1}$-correspondent $\alpha^*_{2r}$ of $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$. Hence Corollary 9.99 follows. $\square$

As far as the Hall system $\{\mathbf{A}, \mathbf{B}\}$ is concerned, we have, similarly to Theorem 9.46, the following

**Theorem 9.100.** *We can find new $\mathbf{A}_\mu \in \mathrm{Hall}_\pi(G_\mu)$ and $\mathbf{B}_\mu \in \mathrm{Hall}_{\pi'}(G_\mu)$ satisfying the equivalent of* (9.2) *for the $\mu$-groups, along with*

$$\mathbf{A}_\mu(\chi_{1,\mu},\dots,\chi_{h,\mu}) = \mathbf{A}(\chi_1,\dots,\chi_h,\mu), \tag{9.101a}$$
$$\mathbf{B}_\mu(\chi_{1,\mu},\dots,\chi_{h,\mu}) = \mathbf{B}(\chi_1,\dots,\chi_h), \tag{9.101b}$$

*for all $h = 2,\dots,n$. Hence*

$$\mathbf{A}_\mu(\chi_{2,\mu},\dots,\chi_{h,\mu}) = \mathbf{A}(\chi_2,\dots,\chi_h,\mu),$$
$$\mathbf{B}_\mu(\chi_{2,\mu},\dots,\chi_{h,\mu}) = \mathbf{B}(\chi_2,\dots,\chi_h),$$

*for all such $h$.*

*Proof.* The proof is similar to that of Theorem 9.46, with the roles of $\mathbf{A}$ and $\mathbf{B}$ interchanged. Just observe, for the last part, that $\chi_{1,\mu} = 1$, while $\chi_1$ is $G$-invariant (and thus $\mathbf{A}$-and $\mathbf{B}$-invariant). $\square$

We restrict our attention to the smaller system (9.65). The subset

$$\{Q_{2i-1,\mu}, P_{2r,\mu} | \beta_{2i-1,\mu}, \alpha_{2r,\mu}\}_{i=1,r=0}^{l,\,k}$$

of (9.81b), is clearly a triangular set of the normal series $1 = G_0 \trianglelefteq G_{1,\mu} \trianglelefteq G_{2,\mu} \trianglelefteq \cdots \trianglelefteq G_{m,\mu} \trianglelefteq G_\mu$, and the tower $\{\chi_{i,\mu}\}_{i=0}^m$. Of course, (9.79) and (9.82) imply that $G_{1,\mu} = 1 = Q_{1,\mu}$ and $\chi_{1,\mu} = 1 = \beta_{1,\mu}$, while $G_{2,\mu} = P_{2,\mu}$ and $\chi_{2,\mu} = \alpha_{2,\mu}$. In view of Theorem 9.91, the above system can be chosen to satisfy (9.92) and (9.93), for all $r = 0,\dots,k$ and all $i = 1,\dots,l$. As in the previous section, we can chose the groups $\widehat{Q}$ and $\widehat{P}$ along with their corresponding in the $\mu$-case groups $\widehat{Q}_\mu$ and $\widehat{P}_\mu$ to satisfy theorems analogous to Theorems 9.48 and 9.50, that is,

**Theorem 9.102.** *Assume that $\{\mathbf{A}_\mu, \mathbf{B}_\mu\}$ is a Hall system for $G_\mu$ that is derived from $\{\mathbf{A}, \mathbf{B}\}$ and satisfies the conditions in Theorem 9.100. Assume further that for every $m = 1,\dots,n$, the group $\widehat{Q}$ is picked to satisfy the conditions in Theorem 8.13 for the smaller system* (9.65)*, while the group $\widehat{Q}_\lambda$ is picked to satisfy the equivalence of teh conditions in Theorem 8.13 for the $\mu$-groups. Then*

$$\widehat{Q}_\mu(\beta_{2k-1,2k,\mu}) = \widehat{Q}(\beta_{2k-1,2k}).$$

*So $\mathcal{Q}_\mu = \widehat{Q}_\mu(\beta_{2k-1,2k,\mu}) \cdot Q^*_{2l-1,\mu} = \mathcal{Q}$.*

*Proof.* Same as the proof of Theorem 9.50, with the roles of $\widehat{P}$ and $\widehat{Q}$ interchanged. Note that, when passing to the shifted system, the groups $\widehat{P}$ and $\widehat{Q}$ remain the same by Theorem 8.29. $\square$

We also have

**Theorem 9.103.** *Let $\{\mathbf{A}_\mu, \mathbf{B}_\mu\}$ be as above, and let $m = 1, \ldots, n$ be fixed. If $\widehat{P}$ is picked to satisfy the conditions in Theorem 8.15, for the smaller system* (9.65)*, while $\widehat{P}_\mu$ is picked to satisfy the equivalent of the conditions in Theorem 8.15 for the $\mu$-groups, then*

$$\widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu}) = \widehat{P}(\alpha_{2l-2,2l-1}, \mu).$$

*So $\mathcal{P}_\mu = \widehat{P}_\mu(\alpha_{2l-1,2l-1,\mu}) \cdot P^*_{2k,\mu} \leq \mathcal{P}(\mu)$.*

*Proof.* See the proof of Theorem 9.48. $\square$

As a corollary of Theorem 9.102, we get

**Corollary 9.104.** *The groups $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu})$ and $\widehat{Q}(\beta_{2k-1,2k})$ have the same image in $\mathrm{Aut}(P^*_{2k})$ . They also have the same image in $\mathrm{Aut}(P^*_{2k,\mu})$. Thus*

$$I = \text{ the image of } \widehat{Q}_\mu(\beta_{2k-1,2k,\mu}) \text{ in } \mathrm{Aut}(P^*_{2k}). \tag{9.105}$$

*Proof.* This is trivially true, as according to Theorem 9.102 the two groups $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu})$ and $\widehat{Q}(\beta_{2k-1,2k})$ coincide. Hence (9.105) follows. $\square$

We define

$$J := \text{ the image of } \widehat{P}(\alpha_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(Q^*_{2l-1}). \tag{9.106}$$

Note that this is the analogue to the definition of $I$ in (9.53). Of course $J$ is well defined as $\widehat{P}$ is a subgroup of $G(\beta^*_{2l-1})$, and thus normalizes $Q^*_{2l-1}$. Furthermore, Theorem 9.103 easily implies

**Corollary 9.107.** *The groups $\widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$ have the same image inside $\mathrm{Aut}(Q^*_{2l-1}) = \mathrm{Aut}(Q^*_{2l-1,\mu})$. Hence*

$$J = J_\mu,$$

*where $J_\mu$ is the image of $\widehat{P}_\mu(\alpha_{2l-1,2l-1,\mu})$ in $\mathrm{Aut}(Q^*_{2l-1,\mu})$.*

*Proof.* Same as the proof of Corollary 9.54. So, in view of (9.89b), it is clear that $\mathrm{Aut}(Q^*_{2l-1}) = \mathrm{Aut}(Q^*_{2l-1,\mu})$, as $Q^*_{2l-1} = Q^*_{2l-1,\mu}$. While

$$P_{2,2l-1} \leq \widehat{P}(\alpha_{2l-2,2l-1}) \leq N(Q^*_{2l-1} \text{ in } G(\alpha_{2,2l-1})).$$

Thus

$$\widehat{P}(\alpha_{2l-2,2l-1}) = \widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu}) \cdot P_{2,2l-1},$$

where $P_{2,2l-1} = C(Q_3 \cdot Q_5 \cdots Q_{2l-1} \text{ in } P_2)$. As $Q_1$ centralizes $P_2$, we conclude that $P_{2,2l-1} = C(Q^*_{2l-1} \text{ in } P_2)$. Therefore, Corollary 9.107 follows. $\square$

We conclude this section with the analogue to Theorems 9.59 and 9.60.

**Theorem 9.108.** *Assume that the character $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$. Let $\widehat{Q}_\mu$ be the group picked in Theorem 9.102. Then the character $\beta_{2k-1,2k,\mu} \in \mathrm{Irr}(Q_{2k-1,\mu})$ extends to the group $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu})$.*

*Proof.* The proof here is as trivial was that of Theorem 9.60. So, $\beta_{2k-1,2k,\mu} = \beta_{2k-1,2k}$, by Corollary 9.97, and $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu}) = \widehat{Q}(\beta_{2k-1,2k})$, by Theorem 9.102. Thus Theorem 9.108 holds. $\square$

Theorem 9.57 and Remark 9.95 imply

**Theorem 9.109.** *Assume that* $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ *extends to* $\widehat{P}(\alpha_{2l-1,2l-1})$. *Let* $\widehat{P}_\mu$ *be a* $\pi$*-Hall subgroup of* $G_{\mu,\beta^*_{2l-1}}$, *chosen so that the conditions in Theorem 9.103 hold. Then the character* $\alpha_{2l-2l-1,\mu}$ *extends to* $\widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu})$.

*Proof.* Same as that of Theorem 9.59, so we omit it. ∎

## 9.3 Kernels

### 9.3.1 Inside $Q_1$

In our common situation, we have a finite group $G$, and the fixed system described in (9.1) and (9.2), that is

$$\text{the normal series: } 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{9.110a}$$

$$\text{the character tower:} \{1 = \chi_0, \chi_1, \dots, \chi_n\} \tag{9.110b}$$

$$\text{the triangular set: } \{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1, r=0}^{l', k'} \tag{9.110c}$$

and the Hall system

$$\{\mathbf{A}, \mathbf{B}\}, \tag{9.110d}$$

that satisfies (9.2). As usual, we assume known all the groups and their characters that accompany the above setting. In particular, we assume known the groups $P^*_{2k'}, Q^*_{2l'-1}, P_{2i,2j+1}, Q_{2i-1,2j}$, as well as $\widehat{Q}, \widehat{P}$, and the characters $\alpha^*_{2k'}, \beta^*_{2l'-1}, \alpha_{2i,2j+1}$ and $\beta_{2i-1,2j}$.

We also assume that $S$ and $\zeta \in \mathrm{Irr}(S)$ satisfy (9.5), i.e., $S$ is a normal subgroup of $G$ contained in $G_1$, and $\zeta$ is a $G$-invariant character of $S$ lying under $\beta_1 = \chi_1$. If $K$ denotes the kernel of $\zeta$, then $K$ is a normal subgroup of $G$, as $\zeta$ is $G$-invariant, and $K$ is contained in $G_1 \leq G_i$, for all $i = 1, \dots, n$. Thus we can define the factor groups

$$G_K = G/K \text{ and } G_{i,K} = G_i/K, \tag{9.111a}$$

for all $i = 1, \dots, n$. Then $G_{i,K}$ is the image of $G_i$ in the factor group $G_K$. This way we have created a normal series

$$G_{0,K} = 1 \trianglelefteq G_{1,K} \trianglelefteq G_{2,K} \trianglelefteq \cdots \trianglelefteq G_{n,K} = G_K \tag{9.111b}$$

of $G_K$, that clearly satisfies Hypothesis 5.1. Along with that series we can associate a character tower that arises from (9.110b). Indeed, $K = \mathrm{Ker}(\zeta)$ is contained in the kernel of $\chi_i$, for all $i = 1, \dots, n$, as $\chi_i$ lies above the $G$-invariant character $\zeta$. Thus there exists a unique character $\chi_{i,K}$ of the factor group $G_{i,K} = G_i/K$, that inflates to $\chi_i \in \mathrm{Irr}(G_i)$, whenever $1 \leq i \leq n$. Hence the set

$$\{1 = \chi_{0,K}, \chi_{1,K}, \dots, \chi_{n,K}\}, \tag{9.111c}$$

forms a character tower for the series (9.111b).

As in the earlier sections, we fix an integer $m = 1, \dots, n$, and consider the smaller system

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G, \tag{9.112a}$$

$$\{1 = \chi_0, \chi_1, \dots, \chi_m\}, \tag{9.112b}$$

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1, r=0}^{l, k}. \tag{9.112c}$$

Clearly the series (9.111b) and the tower (9.111c) provide the smaller reduced system

$$G_{0,K} = 1 \trianglelefteq G_{1,K} \trianglelefteq G_{2,K} \trianglelefteq \cdots \trianglelefteq G_{m,K} \trianglelefteq G_K \tag{9.113a}$$

$$\{1 = \chi_{0,K}, \chi_{1,K}, \dots, \chi_{m,K}\}. \tag{9.113b}$$

The aim of this section is to give a "nice" triangular set for (9.111b), that corresponds to the tower (9.111c), so that we can control the groups $P^*_{2k}, Q^*_{2l-1}, I$ and $J$ of the system (9.112). This is done using the natural group epimorphism

$$\rho : G \to G/K. \tag{9.114a}$$

Assume that $H$ is a subgroup of $G$ and $\theta$ an irreducible character of $H$ having $K \cap H$ in its kernel. Let $\mathrm{Irr}(H|H \cap K)$ be the set of all such characters of $H$. Then, as it is well known, any $\theta \in \mathrm{Irr}(H|H \cap K)$ determines, in a natural way, a unique irreducible character $\rho(\theta) \in \mathrm{Irr}((HK)/K)$ such that

$$[\rho(\theta)](\sigma K) = \theta(\sigma), \tag{9.114b}$$

for all $\sigma \in H$. We remark that an arbitrary element $\tau \in G$ fixes $\theta$ if and only if it normalizes $H$ and its image $\rho(\tau)$ in $G/K$ fixes $\rho(\theta)$. (Note that we could have $\tau \in G$ such that $\rho(\tau)$ fixes $\rho(\theta)$, and thus normalizes $HK$, but $\tau$ moves $H$ around inside $HK$.) Therefore

**Remark 9.115.** The homomorphism

$$\rho : G(\theta) \to (G/K)(\rho(\theta)) = G_K(\rho(\theta))$$

is onto if and only if for every $x \in G$ with $\rho(x) \in G_K(\rho(\theta))$, there exists an element $\tau$ such that $\tau \in xK \cap N(H \text{ in } G)$.

Some sufficient conditions that make the above homomorphism onto are given in the next lemmas.

**Lemma 9.116.** *If $\theta \in \mathrm{Irr}(H|H \cap K)$ and $H$ is a $\hat{\pi}$-Hall subgroup of $HK$, for some set of primes $\hat{\pi}$, then the map*

$$\rho : G(\theta) \to G_K(\rho(\theta))$$

*is an epimorphism.*

*Proof.* Let $x \in G$ be such that $\rho(x)$ fixes $\rho(\theta) \in \mathrm{Irr}(\rho(H))$. Then $\rho(x)$ normalizes $\rho(H)$. So $x$ normalizes the inverse image $HK$ of $\rho(H)$ in $G$. Therefore

$$H^x \leq H^x K = (HK)^x = HK.$$

As $H$ is a $\hat{\pi}$-Hall subgroup of $HK$, we conclude that $H^x = H^k$, for some $k \in K$. Hence the element $\tau = xk^{-1}$ normalizes $H$ and lies in $xK$. In view of Remark 9.115 the proof is complete. □

As a generalization of Lemma 9.116 we have

**Lemma 9.117.** *Assume that $H_i$ is a subgroup of $G$ and $\theta_i \in \mathrm{Irr}(H_i|H_i \cap K)$, for all $i = 1, \dots, s$, and some integer $s \geq 1$. Assume further that the product $H_1 \cdot H_2 \cdots H_s$ is a $\hat{\pi}$-Hall subgroup of $H_1 \cdot H_2 \cdots H_s \cdot K$, for some set of primes $\hat{\pi}$, and that $H_1 \cdots H_s \cap H_i K = H_i$ for all $i = 1, \dots, s$. Let $x$ be an element in $G$ that normalizes $H_i K$, for all $i = 1, \dots, s$. Then there exists some $\tau \in xK$ such that $\tau$ normalizes $H_i$, for all $i = 1, \dots, s$. Hence the map*

$$\rho : G(\theta_1, \dots, \theta_s) \to G_K(\rho(\theta_1), \dots, \rho(\theta_s))$$

*is an epimorphism.*

*Proof.* Since $x$ normalizes each $H_i K$, it also normalizes the product group $H_1 \cdot H_2 \cdots H_s \cdot K$. As $H_1 \cdots H_s$ is a $\hat{\pi}$-Hall subgroup of $H_1 \cdots H_s \cdot K$, we get that $(H_1 \cdots H_s)^x = (H_1 \cdots H_s)^k$, for some

$k \in K$. Thus the element $\tau = xk^{-1}$ normalizes the product $H_1 \cdots H_s$. But $\tau$ also normalizes $H_iK$, as $x$ does, for all $i = 1, \dots, s$. Hence $\tau$ normalizes the intersection $H_i = H_1 \cdots H_s \cap H_iK$, for each $i = 1, \dots, s$. This completes the proof of the first part of the lemma.

To show that the desired map is an epimorphism it suffices to see, (according to Remark 9.115), that for any $x \in G$ with $\rho(x) \in G_K(\rho(\theta_1), \dots, \rho(\theta_s))$, there exists some $\tau \in xK$ that normalizes the groups $H_i$, for all $i = 1, \dots, s$. But any such element $x$ normalizes $H_iK$, as $\rho(x)$ fixes $\rho(\theta_i) \in \mathrm{Irr}(H_iK/K)$, for all $i = 1, \dots, s$. Therefore, the first part of the lemma applies, and guarantees the existence of such a $\tau$. □

As we did with the system (9.111), we will follow the same, standard by now, notation as in Chapters 5 and 6, with the addition of an extra $K$ in the subscripts. We first observe

**Lemma 9.118.** *For all $j = 1, \dots, n$, all $i = 1, \dots, l'$, and all $r = 1, \dots, k'$, we have that*

$$G^*_{j,K} := G_{j,K}(\chi_{1,K}, \dots, \chi_{j-1,K}) = G^*_j/K,$$
$$(Q^*_{2i-1}K)/K \in \mathrm{Hall}_{\pi'}(G^*_{2i-1,K}),$$
$$P^*_{2r} \cong (P^*_{2r}K)/K \in \mathrm{Hall}_{\pi}(G^*_{2r,K}).$$

*Proof.* Since each $\chi_i$, for $i = 1, \dots, n$, is a character of a normal subgroup $G_i$ of $G$ containing $K$, its stabilizer $G(\chi_i)$ is the unique image of $G_K(\chi_{i,K}) = G(\chi_i)/K$ in $G$. The first part follows from this and the definition of $G^*_j$ in (5.129b). The other two parts are implied by the first, and the fact that $Q^*_{2i-1}$ and $P^*_{2r}$ are $\pi'$-and $\pi$-Hall subgroups of $G^*_{2i-1}$ and $G^*_{2r}$, respectively. □

We also note

**Lemma 9.119.** *The intersection $K \cap Q_{2i-1}$ is a subgroup of the kernel* $\mathrm{Ker}(\beta_{2i-1})$ *of $\beta_{2i-1} \in$* $\mathrm{Irr}(Q_{2i-1})$, *whenever $1 \le i \le l'$.*

*Proof.* Indeed, assume $i = 1, \dots, l'$ is fixed. Then $K \cap Q_{2i-1} \le Q_1 \cap Q_{2i-1} = Q_{1,2i}$, (see (5.33) for the last equality). So $K \cap Q_{2i-1} \le \mathrm{Ker}(\beta_1|_{Q_{1,2i}})$, as $K \le \mathrm{Ker}(\beta_1) = \mathrm{Ker}(\chi_1)$. Since $\beta_{1,2i}$ is the $P^*_{2i}$-Glauberman correspondent of $\beta_1$, it is a constituent of the restriction $\beta_1|_{Q_{1,2i}}$ of $\beta_1$ to $Q_{1,2i}$. Thus $K \cap Q_{2i-1}$ is also a subgroup of the kernel, $\mathrm{Ker}(\beta_{1,2i})$, of this constituent $\beta_{1,2i}$. The character $\beta_{1,2i}$ is the unique character of $Q_{1,2i}$ lying under $\beta_{2i-1}$, according to Proposition 5.55. Thus $\mathrm{Ker}(\beta_{1,2i}) \le \mathrm{Ker}(\beta_{2i-1})$. We conclude that

$$K \cap Q_{2i-1} \le \mathrm{Ker}(\beta_{1,2i}) \le \mathrm{Ker}(\beta_{2i-1}), \tag{9.120}$$

whenever $1 \le i \le l'$. Thus Lemma 9.119 follows. □

In view of above lemma, we see that the character $\beta_{2i-1}$ determines a unique character

$$\beta_{2i-1,K} := \rho(\beta_{2i-1}) \in \mathrm{Irr}(\rho(Q_{2i-1})), \tag{9.121a}$$

by (9.114b).

Since $P_{2r}$ is a $\pi$-group, for all $r = 0, 1, \dots, k'$, we have that $P_{2r} \cong (P_{2r}K)/K$, as $K$ is a $\pi'$-group. Thus the character $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$ determines, under the above isomorphism, a unique character

$$\alpha_{2r,K} := \rho(\alpha_{2k}) \in \mathrm{Irr}((P_{2r}K)/K). \tag{9.121b}$$

Now we can define the desired $K$-triangular set for (9.111b).

**Theorem 9.122.** *For every $r = 0, 1, \ldots, k'$ and $i = 1, \ldots, l'$ we define*

$$\begin{aligned} P_{2r,K} &= \rho(P_{2r}) = (P_{2r}K)/K \cong P_{2r}, \\ Q_{2i-1,K} &= \rho(Q_{2i-1}) = (Q_{2i-1}K)/K. \end{aligned} \tag{9.123}$$

*Then the set*

$$\{Q_{2i-1,K}, P_{2r,K} | \beta_{2i-1,K}, \alpha_{2r,K}\}_{i=1,r=0}^{l',k'} \tag{9.124}$$

*is representative of the unique $G_K$-conjugacy class of triangular sets that corresponds to* (9.111c).

*Proof.* It suffices to verify all the relations in (5.17) and in Theorem 5.88, for the $K$ case. We will do that using the map $\rho$ and the fact that the same relations hold for the set (9.110c).

The first two relations (5.17a, b) of (5.17), hold trivially. To see that (5.17c) and (5.17e) hold, it is enough to show that the maps

$$\begin{aligned} \rho : G_{2r}(\alpha_2, \ldots, \alpha_{2r-2}, \beta_1, \ldots, \beta_{2r-1}) &\to G_{2r,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2r-2}), \rho(\beta_1), \ldots, \rho(\beta_{2r-1})) \\ \rho : G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3}) &\to G_{2i-1,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2i-2}), \rho(\beta_1), \ldots, \rho(\beta_{2i-3})) \end{aligned} \tag{9.125}$$

are onto, whenever $1 \leq r \leq k'$ and $2 \leq i \leq l'$. Indeed, that would be enough to guarantee that if we apply $\rho$ to (5.17c) and (5.17e) we get

$$\begin{aligned} \rho(P_{2r}) = P_{2r,K} &\in \operatorname{Hall}_{\pi}(G_{2r,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2r-2}), \rho(\beta_1), \ldots, \rho(\beta_{2r-1}))), \\ \rho(Q_{2i-1}) = Q_{2i-1,K} &\in \operatorname{Hall}_{\pi'}(G_{2i-1,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2i-2}), \rho(\beta_1), \ldots, \rho(\beta_{2i-3}))). \end{aligned}$$

We first fix some $i = 2, \ldots, l'$ and consider the map

$$\rho : G_{2i-1}(\alpha_2, \ldots, \alpha_{2i-2}, \beta_1, \ldots, \beta_{2i-3}) \to G_{2i-1,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2i-2}), \rho(\beta_1), \ldots, \rho(\beta_{2i-3})). \tag{9.126}$$

The groups $P_{2j}$ and their characters $\alpha_{2j}$, with $1 \leq j \leq i-1$, satisfy the hypotheses of Lemma 9.117. That is, the product $P_2 \cdots P_{2i-2} = P^*_{2i-2}$ forms a group, that is actually a $\pi$-Hall subgroup of $P^*_{2i-2}K$. Furthermore, $P^*_{2i-2} \cap P_{2j}K = P_{2j}$, for all $j = 1, \ldots, i-1$. We conclude that the map

$$\rho : G(\alpha_2, \ldots, \alpha_{2i-2}) \to G_K(\rho(\alpha_2), \ldots, \rho(\alpha_{2i-2})), \tag{9.127}$$

is an epimorphism.

Let $x \in G$ with $\rho(x) \in G_{2i-1,K}(\rho(\alpha_2), \ldots, \rho(\alpha_{2i-2}), \rho(\beta_1), \ldots, \rho(\beta_{2i-3}))$. Then the epimorphism of (9.127) allows to assume that $x$ is an element of $G(\alpha_2, \ldots \alpha_{2i-2})$. Thus to prove that the map in (9.126) is onto, it suffices to show that $x$ normalizes $Q_{2j-1}$, for all $j = 1, \ldots, i-1$. Clearly $x$ normalizes $Q_1 \trianglelefteq G$. For the rest we will induct on $j$.

Assume that $x$ normalizes $Q_1, \ldots, Q_{2r-1}$, for some $r$ with $1 \leq r < i-1$. Then $x$ fixes $\beta_1, \ldots, \beta_{2r-1}$. Hence $x$ normalizes the group $G_{2r+1}(\alpha_2, \ldots, \alpha_{2r}, \beta_1, \ldots, \beta_{2r-1})$. But $Q_{2r+1}$ is a $\pi'$-Hall subgroup of this latter group, by (5.17e), and $x$ normalizes the $\pi'$-group $Q_{2r+1}K$, as $\rho(x)$ fixes $\rho(\beta_{2r+1}) \in \operatorname{Irr}((Q_{2r+1}K)/K)$. Therefore $x$ normalizes the intersection

$$Q_{2r+1}K \cap G_{2r+1}(\alpha_2, \ldots, \alpha_{2r}, \beta_1, \ldots, \beta_{2r-1}) = Q_{2r+1}.$$

This completes the proof of the inductive argument, and thus shows that the map in (9.126) is onto.

A similar proof shows that the other map in (9.125) is also onto.

To prove (5.17f), we have to show that $\beta_{2i-1,K}$ lies above the $P_{2i-2,K}$-Glauberman correspondent

$\beta_{2i-3,2i-2,K} \in \mathrm{Irr}(Q_{2i-3,2i-2,K})$ of $\beta_{2i-3,K}$, for all $i = 2, \ldots, l'$. First notice that

$$Q_{2i-3,2i-2,K} = C(P_{2i-2,K} \text{ in } Q_{2i-3,K}) = C((P_{2i-2}K)/K \text{ in } (Q_{2i-3}K)/K).$$

The fact that $(|Q_{2i-3}|, |P_{2i-2}|) = 1$ and Glauberman's lemma imply that

$$C((P_{2i-2}K)/K \text{ in } (Q_{2i-3}K)/K) = (C(P_{2i-2} \text{ in } Q_{2i-3})K)/K = (Q_{2i-3,2i-2}K)/K = \rho(Q_{2i-3,2i-2}).$$

Hence $Q_{2i-3,2i-2,K} = \rho(Q_{2i-3,2i-2})$, whenever $2 \leq i \leq l'$. Furthermore, $\beta_{2i-3,2i-2}$ is the $P_{2i-2}$-Glauberman correspondent of $\beta_{2i-3}$. Thus $\rho(\beta_{2i-3,2i-2})$ is the $\rho(P_{2i-2}) = P_{2i-2,K}$-Glauberman correspondent of $\rho(\beta_{2i-3})$, by Proposition 3.5. In conclusion,

$$\rho(\beta_{2i-3,2i-2}) = \beta_{2i-3,2i-2,K}, \tag{9.128}$$

for all $i = 2, \ldots, l'$. This, along with the fact that $\beta_{2i-1}$ lies above $\beta_{2i-3,2i-2}$, implies that $\beta_{2i-1,K} = \rho(\beta_{2i-1})$ lies above $\beta_{2i-3,2i-2,K} = \rho(\beta_{2i-3,2i-2})$. Thus (5.17f) holds.

Similarly we can show

$$\rho(\alpha_{2r-2,2r-1}) = \alpha_{2r-2,2r-1,K}, \tag{9.129}$$

for all $r = 1, \ldots, k'$. From this (5.17d) follows.

As far as Theorem 5.88 is concerned, the relations in (5.92), and (5.93), are easily translated to the $K$-case using $\rho$ (for groups and characters). For example, we have

$$G_{2i,2i-1,K} := \rho(G_{2i,2i-1}) = \rho(P_{2i} \rtimes Q_{2i-1}) = P_{2i,K} \rtimes Q_{2i-1,K},$$

whenever $1 \leq i \leq k'$. So the first part of (5.92) holds for the $K$-case. The proof for the rest is analogous, and we leave it to the reader.

We need to work more to show that (5.91) holds for the $K$-case, i.e., to show that

$$\begin{aligned} G_{i,2j-1,K} &:= G_{i,K}(\alpha_{2,K}, \ldots, \alpha_{2j-2,K}, \beta_{1,K}, \ldots, \beta_{2j-1,K}) \\ &= N(P_{2,K}, \ldots, P_{2j-2,K}, Q_{1,K}, \ldots, Q_{2j-1,K} \text{ in } G_{i,K}(\chi_{1,K}, \ldots, \chi_{2j-1,K})), \end{aligned} \tag{9.130a}$$

and

$$\begin{aligned} G_{i,2r,K} &:= G_{i,K}(\alpha_{2,K}, \ldots, \alpha_{2r,K}, \beta_{1,K}, \ldots, \beta_{2r-1,K}) \\ &= N(P_{2,K}, \ldots, P_{2r,K}, Q_{1,K}, \ldots, Q_{2r-1,K} \text{ in } G_{i,K}(\chi_{1,K}, \ldots, \chi_{2r,K})), \end{aligned} \tag{9.130b}$$

whenever $j = 1, \ldots, l'$, $r = 1, \ldots, k'$, and $i = 1, \ldots, n$. (Note that we have separated the odd from the even case in (5.91).)

We have already seen that the maps in (9.125) are onto. So if we prove that the maps

$$\begin{aligned} \rho : &N(P_2, \ldots, P_{2j-2}, Q_1, \ldots, Q_{2j-1} \text{ in } G_i(\chi_1, \ldots, \chi_{2j-1})) \\ &\to N(P_{2,K}, \ldots, P_{2j-2,K}, Q_{1,K}, \ldots, Q_{2j-1,K} \text{ in } G_{i,K}(\chi_{1,K}, \ldots, \chi_{2j-1,K})), \end{aligned} \tag{9.131a}$$

and

$$\begin{aligned} \rho : &N(P_2, \ldots, P_{2r}, Q_1, \ldots, Q_{2r-1} \text{ in } G_i(\chi_1, \ldots, \chi_{2r})) \\ &\to N(P_{2,K}, \ldots, P_{2r,K}, Q_{1,K}, \ldots, Q_{2r-1,K} \text{ in } G_{i,K}(\chi_{1,K}, \ldots, \chi_{2r,K})), \end{aligned} \tag{9.131b}$$

are onto, then the equations in (9.130) hold, as we can apply $\rho$ to the equation (5.91).

We will prove that the map in (9.131a) is onto and leave the proof of (9.131b) to the reader. The idea for the proof is the same as that used to prove the maps in (9.125) were onto.

So assume that $x \in G$ is such that $\rho(x)$ lies $G_i(\chi_{1,K}, \dots, \chi_{2j-1,K})$ and normalizes the groups $P_{2,K}, \dots, P_{2j-2,K}$ and $Q_{1,K}, \dots, Q_{2j-1,K}$, for some fixed $i = 1, \dots, n$ and $j = 1, \dots, l'$. Then it is easy to see that $x$ lies in $G_i$ and fixes $\chi_1, \dots, \chi_{2j-1}$. Furthermore, $x$ normalizes $P_2K, \dots, P_{2j-2}K$, while the groups $P_2, \dots, P_{2j-2}$ and the characters $\alpha_2, \dots, \alpha_{2i-2}$, satisfy the hypotheses of Lemma 9.117. Hence we can assume that $x$ normalizes the groups $P_2, \dots, P_{2j-2}$.

It remains to show that $x$ normalizes the groups $Q_1, \dots, Q_{2j-1}$. We use induction on $j$. Clearly $x$ normalizes $Q_1 \trianglelefteq G$. Now assume that $x$ normalizes $Q_1, \dots, Q_{2r-1}$ for some $r$ with $1 \leq r < j$. Then $x$ normalizes $N(P_2, \dots, P_{2r}, Q_1, \dots, Q_{2r-1}$ in $G_{2r+1}(\chi_1, \dots, \chi_{2r-1}))$, as $G_{2r+1} \trianglelefteq G$. But this normalizer equals $G_{2r+1,2r} = P_{2r} \rtimes Q_{2r+1}$, having $Q_{2r+1}$ as a $\pi'$-Hall subgroup. As $x$ also normalizes the $\pi'$-group $Q_{2r+1}K$, (since $\rho(x)$ normalizes $Q_{2r+1,K}$), we conclude that $x$ normalizes the intersection $(P_{2r} \rtimes Q_{2r+1}) \cap Q_{2r+1}K = Q_{2r+1}$. This completes the proof of the inductive step, thus proving that the map (9.131a) is an epimorphism.

So, with analogous proofs left to the reader, Theorem 9.122 follows. ☐

Clearly we have

**Remark 9.132.** For any fixed $m = 1, \dots, n$, the smaller set

$$\{Q_{2i-1,K}, P_{2r,K} | \beta_{2i-1,K}, \alpha_{2r,K}\}_{i=1,r=0}^{l,k}$$

is a triangular set for (9.113a) that corresponds to the character tower (9.113b). So now we have a complete smaller $K$-system

$$G_{0,K} = 1 \trianglelefteq G_{1,K} \trianglelefteq G_{2,K} \trianglelefteq \cdots \trianglelefteq G_{m,K} \trianglelefteq G_K, \tag{9.133a}$$

$$\{1 = \chi_{0,K}, \chi_{1,K}, \dots, \chi_{m,K}\}, \tag{9.133b}$$

$$\{Q_{2i-1,K}, P_{2r,K} | \beta_{2i-1,K}, \alpha_{2r,K}\}_{i=1,r=0}^{l,k}. \tag{9.133c}$$

For any $k = 1, \dots, k'$, we write $\alpha^*_{2k,K}$ for the $Q_{3,K}, \dots, Q_{2k-1,K}$-correspondent of $\alpha_{2k,K}$ (see Definition 5.147). Then the above theorem implies

**Corollary 9.134.**

$$P^*_{2k,K} = \rho(P^*_{2k}) \cong P^*_{2k},$$
$$\alpha^*_{2k,K} = \rho(\alpha^*_{2k}) \in \mathrm{Irr}((P^*_{2k}K)/K),$$

*for all $k = 1, \dots, k'$.*

*Proof.* In view of Lemma 9.118 and (9.123) we have that

$$P^*_{2k,K} = P_{2,K} \cdots P_{2k,K} = \rho(P_2) \cdots \rho(P_{2k}) = \rho(P^*_{2k}) \cong P^*_{2k},$$

for all $k = 1, \dots, k'$. The character $\alpha^*_{2k}$ was defined as the $Q_3, \dots, Q_{2k-1}$-correspondent of $\alpha_{2k}$, for all such $k$. Hence Proposition 3.5 implies that $\rho(\alpha^*_{2k})$ is the $\rho(Q_3), \dots, \rho(Q_{2k-1})$-correspondent of $\rho(\alpha_{2k})$, whenever $k = 1, \dots, k'$. But $\rho(\alpha_{2k}) = \alpha_{2k,K}$, by (9.121b), and $\rho(Q_{2i-1}) = Q_{2i-1,K}$, by (9.123), for all $i = 1, \dots, k' \leq l'$. Thus, the $\rho(Q_3), \dots, \rho(Q_{2k-1})$-correspondent of $\rho(\alpha_{2k})$ is nothing else but the $Q_{3,K}, \dots, Q_{2k-1,K}$-correspondent $\alpha^*_{2k,K}$ of $\alpha_{2k,K}$. We conclude that $\alpha^*_{2k,K} = \rho(\alpha^*_{2k})$, for all $k = 1, \dots, k'$, and the corollary follows. ☐

Furthermore,

**Corollary 9.135.** *For all $l = 1, \dots, l'$, the character $\beta^*_{2l-1,K} \in \mathrm{Irr}(Q^*_{2l-1,K})$ is the unique character of $Q^*_{2l-1,K} = (Q^*_{2l-1}K)/K$ that inflates to $\beta^*_{2l-1} \in \mathrm{Irr}(Q^*_{2l-1})$. Hence $\beta^*_{2l-1,K} = \rho(\beta^*_{2l-1})$.*

*Proof.* In view of (9.123), the product group $Q^*_{2l-1,K} = Q_{1,K} \cdots Q_{2l-1,K}$ is the image under $\rho$ of $Q^*_{2l-1} = Q_1 \cdots Q_{2l-1}$, that is,

$$Q^*_{2l-1,K} = \rho(Q^*_{2l-1}) = (Q^*_{2l-1}K)/K,$$

for all $l = 1, \dots, l'$. Furthermore, $K$ is a subgroup of $Q_1 = Q^*_1$, and is contained in $\mathrm{Ker}(\beta_1)$. As $\beta_1 = \beta^*_1$ is the unique character of $Q^*_1$ lying under $\beta^*_{2l-1}$, we conclude that $K$ is a subgroup of $\mathrm{Ker}(\beta^*_{2l-1})$, for all $l = 1, \dots, l'$. Hence there is a unique character, $\rho(\beta^*_{2l-1})$ of $(Q^*_{2l-1}K)/K$ that inflates to $\beta^*_{2l-1}$. It suffices to show that $\rho(\beta^*_{2l-1}) = \beta^*_{2l-1,K}$.

Indeed, since $\beta^*_{2l-1}$ is the $P_2, \dots, P_{2l-2}$-correspondent of $\beta_{2l-1}$, we get that $\rho(\beta^*_{2l-1})$ is the $\rho(P_2), \dots, \rho(P_{2l-2})$-correspondent of $\rho(\beta_{2l-1})$, for all $l = 1, \dots, l'$, (see Proposition 3.5). But $\rho(P_{2i}) = P_{2i,K}$, for all $i = 1, \dots, k'$, by (9.123), and $\rho(\beta_{2l-1}) = \beta_{2l-1,K}$, by (9.121). Hence, $\rho(\beta^*_{2l-1})$ is the $P_{2,K}, \dots, P_{2l-2,K}$-correspondent of $\beta_{2l-1,K}$, for all $l = 1, \dots, l'$. This completes the proof of the corollary. $\square$

Even more, the Hall system $\{\mathbf{A}, \mathbf{B}\}$ is nicely transfered via $\rho$ to a Hall system of (9.111), as the next theorem shows.

**Theorem 9.136.** *Let*

$$\mathbf{A}_K := \rho(\mathbf{A}) = (\mathbf{A}K)/K \cong \mathbf{A} \text{ and } \mathbf{B}_K = \rho(\mathbf{B}) = (\mathbf{B}K)/K = \mathbf{B}/K.$$

*Then $\{\mathbf{A}_K, \mathbf{B}_K\}$ is a Hall system for $G_K$ that satisfies the equivalent of (9.2) for the $K$-case.*

*Proof.* The maps

$$\begin{aligned}
\rho &: G(\chi_1, \dots, \chi_h) \to G_K(\chi_{1,K}, \dots, \chi_{h,K}),\\
\rho &: \mathbf{A}(\chi_1, \dots, \chi_h) \to \mathbf{A}_K(\chi_{1,K}, \dots, \chi_{h,K}),\\
\rho &: \mathbf{B}(\chi_1, \dots, \chi_h) \to \mathbf{B}_K(\chi_{1,K}, \dots, \chi_{h,K}),
\end{aligned}$$

are clearly onto, as $\chi_i \in \mathrm{Irr}(G_i)$ with $G_i \trianglelefteq G$, for all $i = 1, \dots, h$, and all $h = 1, \dots, n$. This, along with (9.2a, b), implies

$$\mathbf{A}_K \in \mathrm{Hall}_\pi(G_K), \quad \mathbf{B}_K \in \mathrm{Hall}_{\pi'}(G_K),$$
$$\mathbf{A}_K(\chi_{1,K}, \chi_{2,K}, \dots, \chi_{h,K}) \text{ and } \mathbf{B}(\chi_{1,K}, \chi_{2,K}, \dots, \chi_{h,K}) \text{ form a Hall system for } G_K(\chi_{1,K}, \chi_{2,K}, \dots, \chi_{h,K}),$$

for all $h = 1, \dots, n$.

In addition, (9.2c) and Corollary 9.134 imply

$$\mathbf{A}_K(\chi_{1,K}, \dots, \chi_{n,K}) = \rho(\mathbf{A}(\chi_1, \dots, \chi_n)) = \rho(P^*_{2k'}) = P^*_{2k',K}.$$

Similarly we get that $\mathbf{B}_K(\chi_{1,K}, \dots, \chi_{n,K}) = \rho(Q^*_{2l'-1}) = Q^*_{2l'-1,K}$. Hence the groups $\mathbf{A}_K, \mathbf{B}_K$ form a Hall system for $G_K$, and satisfy (9.2) for the $K$-case.

Since $(|\mathbf{A}|, |K|) = 1$, we clearly have $(\mathbf{A}K)/K \cong \mathbf{A}$. This completes the proof of the theorem. $\square$

As a corollary of the above theorem we have

**Corollary 9.137.** *For any $k = 1, \ldots, k'$ and any $l = 1, \ldots, l'$ we have*

$$\rho(N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \ldots, \chi_{2k}))) = N(P^*_{2k,K} \text{ in } \mathbf{B}_K(\chi_{1,K}, \ldots, \chi_{2k,K})),$$
$$\rho(N(Q^*_{2l-1} \text{ in } \mathbf{A}(\chi_1, \ldots, \chi_{2l-1}))) = N(Q^*_{2l-1,K} \text{ in } \mathbf{A}_K(\chi_{1,K}, \ldots, \chi_{2l-1,K})).$$

*Proof.* As $(|P^*_{2k}|, |K|) = 1$, we have

$$(N(P^*_{2k} \text{ in } \mathbf{B}(\chi_1, \ldots, \chi_{2k}))K)/K = N((P^*_{2k}K)/K \text{ in } (\mathbf{B}(\chi_1, \ldots, \chi_{2k})K)/K).$$

Also, $N((P^*_{2k}K)/K$ in $(\mathbf{B}(\chi_1, \ldots, \chi_{2k})K)/K)$ equals $N(P^*_{2k,K}$ in $\mathbf{B}_K(\chi_{1,K}, \ldots, \chi_{2k,K}))$, as $P^*_{2k,K} = (P^*_{2k}K)/K$, by Corollary 9.134, and $(\mathbf{B}(\chi_1, \ldots, \chi_{2k})K)/K = \mathbf{B}_K(\chi_{1,K}, \ldots, \chi_{2k,K})$, by Theorem 9.136. Furthermore, as $(N(P^*_{2k}$ in $\mathbf{B}(\chi_1, \ldots, \chi_{2k}))K)/K = \rho(N(P^*_{2k}$ in $\mathbf{B}(\chi_1, \ldots, \chi_{2k})))$, the first part of the corollary follows.

The proof for the second equation is similar. ∎

Working on the smaller system (9.112), we can now prove

**Theorem 9.138.** *Let $\{\mathbf{A}_K, \mathbf{B}_K\}$ be a Hall system of $G_K$ that arises from $\{\mathbf{A}, \mathbf{B}\}$ via Theorem 9.136. For any fixed $m = 1, \ldots, n$, we choose the groups $\widehat{Q}$ and $\widehat{Q}_K$ to satisfy the conditions in Theorem 8.13 for the systems* (9.112) *and* (9.133)*, respectively. Then*

$$\widehat{Q}_K(\beta_{2k-1,2k,K}) = \rho(\widehat{Q}(\beta_{2k-1,2k})) = (\widehat{Q}(\beta_{2k-1,2k})K)/K. \tag{9.139}$$

*Proof.* We choose $\widehat{Q}$ and $\widehat{Q}_K$ to satisfy the conditions Theorem 8.13 for the systems (9.112) and (9.133), respectively. Therefore $N(P^*_{2k}$ in $\mathbf{B}(\chi_1, \ldots, \chi_{2k})) = \widehat{Q}(\beta_{2k-1,2k})$. In addition, we have $N(P^*_{2k,K}$ in $\mathbf{B}_K(\chi_{1,K}, \ldots, \chi_{2k,K})) = \widehat{Q}_K(\beta_{2k-1,2k,K})$. This, along with Corollary 9.137, implies Theorem 9.138 ∎

Similarly we can show

**Theorem 9.140.** *Assume that $\{\mathbf{A}_K, \mathbf{B}_K\}$ are as above. Assume further that, for any fixed $m = 1, \ldots, n$, we choose the groups $\widehat{P}$ and $\widehat{P}_K$ to satisfy the conditions in Theorem 8.15 for the systems* (9.112) *and* (9.133)*, respectively. Then*

$$\widehat{P}_K(\alpha_{2l-2,2l-1,K}) = \rho(\widehat{P}(\alpha_{2l-2,2l-1})) = (\widehat{P}(\alpha_{2l-2,2l-1})K)/K$$

*Hence*

$$\widehat{P}_K(\alpha_{2l-2,2l-1,K}) \cong \widehat{P}(\alpha_{2l-2,2l-1}).$$

*Proof.* Choose $\widehat{P}$ and $\widehat{P}_K$ to satisfy the conditions in Theorem 8.15, for the systems (9.112) and (9.133), respectively. Then the first part of Theorem 9.140 follows from Corollary 9.137. Note that $(\widehat{P}(\alpha_{2l-2,2l-1})K)/K \cong \widehat{P}(\alpha_{2l-2,2l-1})$, as $(|\widehat{P}(\alpha_{2l-2,2l-1})|, |K|) = 1$. Hence the theorem follows. ∎

As $\widehat{Q}_K(\beta_{2k-1,2k,K}) = \rho(\widehat{Q}(\beta_{2k-1,2k}))$, while $P^*_{2k} \cong P^*_{2k,K} = \rho(P^*_{2k})$, the action of $\widehat{Q}(\beta_{2k-1,2k})$ on $P^*_{2k}$ is carried onto the action of $\widehat{Q}_K(\beta_{2k-1,2k,K})$ on $P^*_{2k,K} \cong P^*_{2k}$, in the sense that

$$\rho(\sigma^\tau) = \rho(\sigma)^{\rho(\tau)} \in P^*_{2k,K} \tag{9.141}$$

for any $\sigma \in P^*_{2k}$ and any $\tau \in \widehat{Q}(\beta_{2k-1,2k})$. Let $I_K$ be the image of $\widehat{Q}_K(\beta_{2k-1,2k,K})$ in the automorphism group $\mathrm{Aut}(P^*_{2k,K})$. As the isomorphism $\rho$ of $P^*_{2k}$ onto $P^*_{2k,K}$ induces an isomorphism of $\mathrm{Aut}(P^*_{2k})$ onto $\mathrm{Aut}(P^*_{2k,K})$, we conclude that this isomorphism carries the image of $\widehat{Q}(\beta_{2k-1,2k})$ in the former automorphism group onto the image of $\widehat{Q}_K(\beta_{2k-1,2k,K})$ in the latter such group. So we have an isomorphism

$$\begin{aligned}\rho_K : I = \text{ the image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(P^*_{2k})&\\ \to I_K = \text{ the image of } \widehat{Q}_K(\beta_{2k-1,2k,K}) \text{ in } \mathrm{Aut}(P^*_{2k,K}).&\end{aligned} \tag{9.142}$$

Furthermore, identifying $P^*_{2k}$ with $P^*_{2k,K}$ and $\mathrm{Aut}(P^*_{2k})$ with $\mathrm{Aut}(P^*_{2k,K})$ we conclude

**Corollary 9.143.** *For any fixed $m = 1, \ldots, n$, the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{Q}_K(\beta_{2k-1,2k,K})$ have the same image in $\mathrm{Aut}(P^*_{2k,K}) = \mathrm{Aut}(P^*_{2k})$.*

Similarly, Theorem 9.140 implies

**Corollary 9.144.** *For any fixed $m = 1, \ldots, n$, the groups $\widehat{P}(\alpha_{2l-2,2l-1})$ and $\widehat{P}_K(\alpha_{2l-2,2l-1,K})$ have the same image in $\mathrm{Aut}(Q^*_{2l-1,K})$.*

*Proof.* As we have seen in Corollary 9.135 and Theorem 9.140

$$\begin{aligned} Q^*_{2l-1,K} &= \rho(Q^*_{2l-1}) = (Q^*_{2l-1}K)/K, \\ \widehat{P}_K(\alpha_{2l-2,2l-1,K}) &= \rho(\widehat{P}(\alpha_{2l-2,2l-1})) \cong \widehat{P}(\alpha_{2l-2,2l-1}). \end{aligned}$$

So the action of $\widehat{P}_K(\alpha_{2l-2,2l-1,K})$ on $Q^*_{2l-1,K}$ is given (similarly to (9.141)) as

$$\rho(x)^{\rho(y)} = \rho(x^y) \in Q^*_{2l-1,K} \tag{9.145}$$

for any $x \in Q^*_{2l-1}$ and $y \in \widehat{P}(\alpha_{2l-2,2l-1})$. Furthermore, $\widehat{P}(\alpha_{2l-2,2l-1})$ acts also on $Q^*_{2l-1,K}$ via

$$\rho(x)^y = \rho(x^y),$$

for any $x$ and $y$ as above. As the map

$$\rho : \widehat{P}(\alpha_{2l-2,2l-1}) \to \widehat{P}_K(\alpha_{2l-2,2l-1,K}),$$

sending $y \in \widehat{P}(\alpha_{2l-2,2l-1})$ to $\rho(y) \in \widehat{P}_K(\alpha_{2l-2,2l-1,K})$, is an isomorphism, Corollary 9.144 follows. □

We conclude this section with

**Theorem 9.146.** *Assume that the character $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$. Then the character $\beta_{2k-1,2k,K} \in \mathrm{Irr}(Q_{2k-1,2k,K})$ extends to the group $\widehat{Q}_K(\beta_{2k-1,2k,K})$.*

*Proof.* Obvious, since $beta_{2k-1,2k,K} = \rho(\beta_{2k-1,2k})$ by (9.128). □

Similarly we have.

**Theorem 9.147.** *Assume that the character $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$. Then the character $\alpha_{2l-2,2l-1,K} \in \mathrm{Irr}(P_{2l-2,2l-1,K})$ extends to the group $\widehat{P}_K(\alpha_{2l-2,2l-1,K})$.*

### 9.3.2 Inside $P_2$

Assume now that the system (9.110) for the normal series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$ is fixed, but in addition $G_2$ satisfies (9.61), i.e., $G_2$ is the direct product of a $\pi$-and a $\pi'$-group, while $\chi_1 = \beta_1$ is $G$-invariant. Then the triangular set (9.110c) satisfies (9.63), i.e.,

$$G_2 = G_2(\beta_1) = P_2 \times Q_1 = P_2 \times G_1, \tag{9.148a}$$

$$\chi_2 = \alpha_2 \times \beta_1. \tag{9.148b}$$

Assume further that $R$ is a normal subgroup of $G$ and $\eta$ is a character in $\mathrm{Irr}(R)$ satisfying (9.70), i.e., $R$ is a subgroup of $P_2$ while the irreducible character $\eta$ of $R$ is $G$-invariant and lies under $\alpha_2$. We write $K$ for the kernel of $\eta$. (Note this is not the same $K$ as in Section 9.3.1.) Then $K$ is a normal subgroup of $G$, as $\eta$ is $G$-invariant. As in the previous section, we are interested at the factor group $G_K = G/K$. (Note that this time $K$ is a $\pi$-group.) So we define the factor groups $G_{i,K} = G_i/K$, for all $i = 2, \dots, n$. We also write $G_{1,K} = (G_1K)/K$. Then (9.148a) implies

$$G_{1,K} = (G_1 \times K)/K \cong G_1. \tag{9.149}$$

The series

$$1 = G_{0,K} \trianglelefteq G_{1,K} \trianglelefteq G_{2,K} \trianglelefteq \cdots \trianglelefteq G_{n,K} = G_K \tag{9.150a}$$

is a normal series of $G$ that satisfies Hypothesis 5.1. Furthermore, for every $i = 2, \dots, n$ the character $\chi_i$ lies above the $G$-invariant character $\zeta$. Hence $K$ is a subgroup of $\mathrm{Ker}(\chi_i)$, for all such $i$. So we can define again the character $\chi_{i,K} \in \mathrm{Irr}(G_{i,K})$, to be the unique character of $G_i/K$ that inflates to $\chi_i$, whenever $2 \leq i \leq n$. As $G_{1,K} \cong G_1$, we denote by $\chi_{1,K}$ the unique irreducible character of $G_{1,K}$ that corresponds to $\chi_1$, via that isomorphism. So we get a character tower for (9.150),

$$\{1 = \chi_{0,K}, \chi_{1,K}, \chi_{2,K}, \dots, \chi_{n,K}\}, \tag{9.150b}$$

that arises from the original tower (9.110b).

In conclusion, we have created a similar system to that of Section 9.3.1, with the only important difference being that $K$ is a $\pi$-group instead of a $\pi'$-group. The natural map $\rho$, on groups and characters, that was defined at (9.114), is carried unchanged in this situation. Of course Remark 9.115 and Lemmas 9.116 and 9.117 are still valid. With the help of the same map $\rho$ we will define a triangular set for (9.150a) that corresponds to (9.150b). As we would expect, this set is going to be the mirror of the set (9.124), with the roles of the $\pi$-and the $\pi'$-groups interchanged.

We start with

**Lemma 9.151.** *For all $j = 2, \dots, n$, all $i = 1, \dots, l'$ and all $r = 1, \dots k'$, we have*

$$G^*_{j,K} := G_{j,K}(\chi_{1,K}, \dots, \chi_{j-1,K}) = G^*_j/K,$$
$$Q^*_{2i-1} \cong (Q^*_{2i-1}K)/K \in \mathrm{Hall}_{\pi'}(G^*_{2i-1,K}),$$
$$P^*_{2r}/K = (P^*_{2r}K)/K \in \mathrm{Hall}_{\pi}(G^*_{2r,K}).$$

*Proof.* See the proof of Lemma 9.118. □

Also in the same way we worked to prove Lemma 9.119 and (9.121), but this time using the characters $\alpha_{2r}, \alpha_2$ and $\alpha_{2,2r-1}$ in the place of $\beta_{2i-1}, \beta_1$ and $\beta_{1,2i}$ respectively, we can see that

**Remark 9.152.** The intersection $K \cap P_{2r}$ is a subgroup of the kernel $\mathrm{Ker}(\alpha_{2r})$ of $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$, whenever $1 \leq r \leq k'$. Thus, for all such $r$, there exists a unique character

$$\alpha_{2r,K} := \rho(\alpha_{2r}) \in \mathrm{Irr}((P_{2r}K)/K), \tag{9.153}$$

that inflates to $\alpha_{2r} \in \mathrm{Irr}(P_{2r})$.

Furthermore, $Q_{2i-1}$ is a $\pi'$-group, and thus has order coprime to $|K|$, for all $i = 1, \ldots, l'$. Hence $Q_{2i-1} \cong (Q_{2i-1}K)/K$, for all such $i$. So the character $\beta_{2i-1} \in \mathrm{Irr}(Q_{2i-1})$ determines, under the above isomorphism, a unique character

$$\beta_{2i-1,K} := \rho(\beta_{2i-1}) \in \mathrm{Irr}((Q_{2i-1}K)/K), \tag{9.154}$$

for all $i = 1, \ldots, l'$.

We can now prove the main theorem of this section, the analogue of Theorem 9.122.

**Theorem 9.155.** *For every $r = 0, 1, \ldots, k'$ and $i = 1, \ldots, l'$ we define*

$$P_{2r,K} = \rho(P_{2r}) = (P_{2r}K)/K, \tag{9.156}$$
$$Q_{2i-1,K} = \rho(Q_{2i-1}) = (Q_{2i-1}K)/K \cong Q_{2i-1}. \tag{9.157}$$

*Then the set*

$$\{Q_{2i-1,K}, P_{2r,K} | \beta_{2i-1,K}, \alpha_{2r,K}\}_{i=1,r=0}^{l',k'} \tag{9.158}$$

*is a representative of the unique $G_K$-conjugacy class of triangular sets that corresponds to* (9.150b).

*Proof.* The proof is the same as that of Theorem 9.122, if we interchange the roles of $P_{2r}$ and $\alpha_{2r}$ with those of $Q_{2i-1}$ and $\beta_{2i-1}$, respectively. ☐

We also get

**Corollary 9.159.** *For all $k = 1, \ldots, k'$, we have $P^*_{2k,K} = \rho(P^*_{2k})$. Furthermore, the character $\alpha^*_{2k,K}$ is the unique character $\rho(\alpha^*_{2k})$ of $P^*_{2k,K}$ that inflates to $\alpha^*_{2k} \in \mathrm{Irr}(P^*_{2k})$.*

*Proof.* Same as that of Corollary 9.135, with the roles of $\pi$ and $\pi'$ -interchanged. ☐

and

**Corollary 9.160.**

$$Q^*_{2l-1,K} = \rho(Q^*_{2l-1}) \cong Q^*_{2l-1},$$
$$\beta^*_{2l-1,K} = \rho(\beta^*_{2l-1}),$$

*for all $l = 1, \ldots, l'$.*

*Proof.* See Corollary 9.134. ☐

The same argument as that of Theorem 9.136 implies

**Theorem 9.161.** *Let*

$$\mathbf{A}_K := \rho(\mathbf{A}) = (\mathbf{A}K)/K \text{ and } \mathbf{B}_K = \rho(\mathbf{B}) = (\mathbf{B}K)/K \cong \mathbf{B}.$$

*Then $\{\mathbf{A}_K, \mathbf{B}_K\}$ forms a Hall system for $G_K$ that satisfies the equivalent of* (9.2) *for the $K$-case.*

Until the end of the section, we fix an integer $m = 2, \ldots, n$ and consider the smaller system (9.112). Of course, as before, we get a smaller $K$-system (9.133), where now the triangular set is picked to be a subset of (9.158). So as in Theorems 9.138 and 9.140, we have

**Theorem 9.162.** *Let $\{\mathbf{A}_K, \mathbf{B}_K\}$ be tha Hall system of $G_K$ that arises from $\{\mathbf{A}, \mathbf{B}\}$ via Theorem 9.161. For any fixed $m = 1, \ldots, n$, we choose the groups $\widehat{Q}$ and $\widehat{Q}_K$ to satisfy the conditions in Theorem 8.13 for the systems* (9.112) *and* (9.133)*, respectively. Then*

$$\widehat{Q}_K(\beta_{2k-1,2k,K}) = \rho(\widehat{Q}(\beta_{2k-1,2k})) = (\widehat{Q}(\beta_{2k-1,2k})K)/K. \tag{9.163}$$

*Hence*

$$\widehat{Q}_K(\beta_{2k-2,2k,K}) \cong \widehat{Q}(\beta_{2k-1,2k}).$$

*Proof.* See Theorem 9.140 □

and

**Theorem 9.164.** *Assume that $\{\mathbf{A}_K, \mathbf{B}_K\}$ are as above. Assume further that, for any fixed $m = 2, \ldots, n$, we choose the groups $\widehat{P}$ and $\widehat{P}_K$ to satisfy the conditions in Theorem 8.15 for the systems* (9.112) *and* (9.133)*, respectively. Then*

$$\widehat{P}_K(\alpha_{2l-2,2l-1,K}) = \rho(\widehat{P}(\alpha_{2l-2,2l-1})) = (\widehat{P}(\alpha_{2l-2,2l-1})K)/K.$$

*Proof.* Same as that of Theorem 9.138. □

So we get the next two corollaries

**Corollary 9.165.** *The groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{Q}_K(\beta_{2k-1,2k,K})$, have the same image in the group of automorphisms $\mathrm{Aut}(P^*_{2k,K})$.*

*Proof.* See the proof of Corollary 9.144. □

and

**Corollary 9.166.** *For any fixed $m = 2, \ldots, n$, the groups $\widehat{P}(\alpha_{2l-2,2l-1})$ and $\widehat{P}_K(\alpha_{2l-2,2l-1,K})$ have the same image in $\mathrm{Aut}(Q^*_{2l-1,K}) = \mathrm{Aut}(Q^*_{2l-1})$.*

*Proof.* See Corollary 9.143. □

We conclude the section and the chapter with

**Theorem 9.167.** *Assume that $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$. Then the character $\beta_{2k-1,2k,K} \in \mathrm{Irr}(Q_{2k-1,2k,K})$ extends to $\widehat{Q}_K(\beta_{2k-1,2k,K})$.*

*Proof.* Obvious. □

and

**Theorem 9.168.** *Assume that the character $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$. Then the character $\alpha_{2l-2,2l-1,K} \in \mathrm{Irr}(P_{2l-2,2l-1,K})$ extends to the group $\widehat{P}_K(\alpha_{2l-2,2l-1,K})$.*

# Chapter 10

# Linear Limits

## 10.1 Basic properties

We say that $(G, A, \phi, N, \psi)$ is a *linear quintuple* if $A \leq N$ are normal subgroups of a finite group $G$, $\phi \in \mathrm{Lin}(A)$ is a $G$-invariant linear character of $A$ and $\psi \in \mathrm{Irr}(N|\phi)$. Note that as $\phi$ is $G$-invariant, $\mathrm{Ker}(\phi)$ is a normal subgroup of $G$. Furthermore, $\mathrm{Ker}(\phi) = \mathrm{Ker}(\psi|_A) \leq \mathrm{Ker}(\psi)$. Hence $\mathrm{Ker}(\phi)$ is contained in the largest normal subgroup $M$ of $G$ contained in $\mathrm{Ker}(\psi)$. (Note that $M = \bigcap_{x \in G}(\mathrm{Ker}(\psi))^x$.) This, along with the fact that $\phi$ is linear, implies that $A$ is abelian modulo $M$, i.e., $(AM)/M \cong A/(A \cap M)$ is abelian.

Let $A' \trianglelefteq G$ with $A \leq A' \leq N$. Let $\phi' \in \mathrm{Irr}(A')$ be a linear character of $A'$ extending $\phi$ and lying under $\psi$. Then $(A'M)/M \cong A'/(A' \cap M)$ is also abelian. Indeed, $[A', A']$ is contained in $\mathrm{Ker}((\phi')^n)$, for every $n \in N$, since $\phi'$ is linear. Thus $[A', A'] \leq \bigcap_{n \in N} \mathrm{Ker}((\phi')^n)$. As the restriction of $\psi$ to $A'$ is a sum of $N$-conjugates of $\phi'$, we conclude that $[A', A']$ is contained in $\mathrm{Ker}(\psi|_{A'}) = \bigcap_{n \in N} \mathrm{Ker}((\phi')^n)$. So $[A', A'] \leq \mathrm{Ker}(\psi)$. Furthermore, $[A', A']$ is a normal subgroup of $G$, as $A' \trianglelefteq G$. Hence $[A', A']$ is a subgroup of $M$, which implies that $A'/(A' \cap M)$ is abelian.

Furthermore, we can use Clifford theory to form a new linear quintuple $(G', A', \phi', N', \psi')$, where $G' = G(\phi')$ and $N' = N(\phi')$ are the stabilizers of $\phi'$ in $G$ and $N$ respectively, and $\psi'$ is the $\phi'$-Clifford correspondent of $\psi \in \mathrm{Irr}(N|\phi')$. We say that $(G', A', \phi', N', \psi')$ is a *linear reduction* of $(G, A, \phi, N, \psi)$. We call this reduction *proper* if the reduced linear quintuple is different from the original one, i.e., if and only if $A < A'$. We can repeat this process and consider a linear reduction of the linear reduction $(G', A', \phi', N', \psi')$. Any linear quintuple that we reach after a series of such linear reductions, is called a *multiple linear reduction* of $(G, A, \phi, N, \psi)$. A "minimal" multiple linear reduction is a linear quintuple that has no proper linear reductions We call such a minimal linear quintuple a *linear limit* of $(G, A, \phi, N, \psi)$. We denote by $LL(G, A, \phi, N, \psi)$ the set of all the linear limits of $(G, A, \phi, N, \psi)$, and by

$$(l(G), l(A), l(\phi), l(N), l(\psi))$$

any element of that set. Assume that $H$ is a subgroup of $G$ with $N \leq H \leq G$. Then the quintuple $(H, A, \phi, N, \psi)$ is clearly a linear one. The definition of linear limits clearly implies

**Remark 10.1.** If $(l(G), l(A), l(\phi), l(N), l(\psi))$ is a linear limit of $(G, A, \phi, N, \psi)$, then $(l(G) \cap H, l(A), l(\phi), l(N), l(\psi))$ is a linear limit of $(H, A, \phi, N, \psi)$.

The following is also straight forward:

**Remark 10.2.** If we reach the linear quintuple $(G', A', \phi', N', \psi')$ after a series of linear reductions starting with the quintuple $(G, A, \phi, N, \psi)$, then $LL(G', A', \phi', N', \psi') \subseteq LL(G, A, \phi, N, \psi)$.

If $(l(G), l(A), l(\phi), l(N), l(\psi))$ is a linear limit of $(G, A, \phi, N, \psi)$, then we can form the quintuple $(l(G)/K, l(A)/K, l(\phi)/K, l(N)/K, l(\psi)/K)$ where $K = \mathrm{Ker}(l(\phi))$ is the kernel of $l(\phi)$ (and thus a normal subgroup of $l(G)$) while $l(\phi)/K$ and $l(\psi)/K$ are the unique characters of the factor groups $l(A)/K$ and $l(N)/K$ that inflate to $l(\phi)$ and $l(\psi)$, respectively. It is clear that the quintuple $(l(G)/K, l(A)/K, l(\phi)/K, l(N)/K, l(\psi)/K)$ is linear, and that $l(\phi)/K$ is faithful. We call this triple a *faithful linear limit* of $(G, A, \phi, N, \psi)$ We denote by $FLL(G, A, \phi, N, \psi)$ the set of all faithful linear limits of $(G, A, \phi, N, \psi)$, and by

$$(fl(G), fl(A), fl(\phi), fl(N), fl(\psi)) \tag{10.3}$$

any element of that set.

Now assume that $N \trianglelefteq H \trianglelefteq G$ and $\chi \in \mathrm{Irr}(H|\psi)$. Then any linear reduction $(G', A', \phi', N', \psi')$ of $(G, A, \phi, N, \psi)$ provides an irreducible character $\chi' \in \mathrm{Irr}(H')$, where $H' = G' \cap H = H(\phi')$ and $\chi'$ is the $\phi'$-Clifford correspondent of $\chi$, i.e., $\chi'$ lies above $\phi'$ and induces $\chi$. We can repeat this process and consider the Clifford correspondent for $\chi'$ in the next linear reduction of $(G', A', \phi', N', \psi')$ that we perform. When we reach a linear limit $(l(G), l(A), l(\phi), l(N), l(\psi))$, of $(G, A, \phi, N, \psi)$ we have also reached a character $\theta \in \mathrm{Irr}(l(G) \cap H)$ that induces $\chi$. Any such character $\theta$, that arises by repeated Clifford correspondences on linear reductions, we call a *linear limit* of $\chi$. We write it as $\theta = l_{\phi,\psi}(\chi)$, or more simply as $l(\chi)$ if the starting linear quintuple is clear. We also write as $l_{\phi,\psi}(H) = l(H)$ the domain of $l(\chi)$, i.e., $l(H) = l(G) \cap H$. Clearly $l(\chi)$ lies above $l(\phi)$ and $l(\psi)$, and induces $\chi$. The collection of all linear limits of $\chi$ we write as $LL(\chi)$. Note that $LL(\chi)$ is a subset of $CCC_N(\chi)$ as this was defined in [14]. Furthermore, let $(fl(G), fl(A), fl(\phi), fl(N), fl(\psi))$ be a faithful linear limit, i.e.,

$$(fl(G), fl(A), fl(\phi), fl(N), fl(\psi)) = (l(G)/K, l(A)/K, l(\phi)/K, l(N)/K, l(\psi)/K) \tag{10.4}$$

where $(l(G), l(A), l(\phi), l(N), l(\psi))$ is a linear limit of $(G, A, \phi, N, \psi)$ and $K = \mathrm{Ker}(l(\phi))$. Then $K$ is a subgroup of $\mathrm{Ker}(l(\chi))$ the kernel of the linear limit $l(\chi) \in \mathrm{Irr}(l(H))$ of $\chi$, as $\chi$ lies above the $l(G)$-invariant character $l(\phi) \in \mathrm{Irr}(l(A))$. Thus $l(\chi)$ is inflated from a unique character $l(\chi)/K$ of the factor group $l(H)/K$ that we call *faithful linear limit* of $\chi$ and write as $fl(\chi)$. We also write $fl(H)$ for the domain of $fl(\chi)$, i.e., $fl(H) = l(H)/K$. The set of all faithful linear limits of $\chi$ we denote by $FLL(\chi)$.

We conclude these preliminary definitions of linear limits with the following straight forward observations.

**Remark 10.5.** Let $(l(G), l(A), l(\phi), l(N), l(\psi))$ be a linear limit of $(G, A, \phi, N, \psi)$. Let $K$ be the kernel $\mathrm{Ker}(l(\phi))$, and let $(fl(G), fl(A), fl(\phi), fl(N), fl(\psi))$ be the faithful linear limit defined in (10.4). Then $l(G)$ is a subgroup of $G$ while $fl(G)$ is the section $l(G)/K$ of $G$. Furthermore, any subgroup $H$ of $G$ with $N \trianglelefteq H$, has a limit and a faithful limit group $l(H) = l(G) \cap H$ and $fl(H) = l(H)/K$, respectively, that satisfy $l(N) \trianglelefteq l(H) < l(G)$ and $fl(N) \trianglelefteq fl(H) < fl(G)$. In addition, $(l(H), l(A), l(\phi), l(N), l(\psi))$ and $(fl(H), fl(A), fl(\phi), fl(N), fl(\psi))$ are a linear and a faithful linear limit respectively, of $(H, A, \phi, N, \psi)$. If $H$ is normal in $G$, then $l(H)$ and $fl(H)$ are normal in $l(G)$ and $fl(G)$, respectively.

**Definition 10.6.** By convention, whenever $N \leq H \leq G$ and $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ is a faithful linear limit of $(G, A, \phi, N, \psi)$, we write $(\mathbb{G} \cap H, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ for the faithful linear limit of $(H, A, \phi, N, \psi)$ (described in Remark 10.5) that $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ induces.

We can say a little more for a special type of subgroup $B$ of $G$. Assume that $B \leq C_G(N)$, i.e., $B$ centralizes $N$. Then $B$ centralizes any $A'$ with $A \leq A' \leq N$. Hence it fixes any character $\phi' \in \mathrm{Irr}(A')$, and, in particular, those that extend $\phi$. Hence $B$ is also a subgroup of $G(\phi')$. Furthermore, it centralizes $N(\phi') \leq N$. Repeating the same argument at every linear reduction we perform (note that all such are inside $N$), we see that $B$ is a subgroup of $l(G)$, and centralizes $l(N)$. Furthermore, as $K \leq l(N)$, we get that $fl(B) := (BK)/K$ is a subgroup of $fl(G)$ that centralizes $fl(N) = l(N)/K$. Thus we have shown

**Remark 10.7.** If $B$ is a subgroup of $G$ that centralizes $N$, i.e., $B \leq C_G(N)$, then $B \leq l(G)$ centralizes $l(N)$ while $fl(B) = (BK)/K \leq fl(G)$ centralizes $fl(N)$. If in addition $(|B|, |K|) = 1$ then $B \cong fl(B) \leq fl(G)$.

The next two lemmas are straight forward applications of the above definitions.

**Lemma 10.8.** *Any faithful linear limit of $(G, A, \phi, N, \psi)$ is a minimal linear quintuple, that is, no proper linear reductions can be made to a faithful linear limit of $(G, A, \phi, N, \psi)$.*

*Proof.* Let

$$(fl(G), fl(A), fl(\phi), fl(N), fl(\psi)) = (\mathcal{G}/K, \mathcal{A}/K, \Phi/K, \mathcal{N}/K, \Psi/K),$$

be a faithful linear limit of $(G, A, \phi, N, \psi)$, where $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$ is a linear limit of the latter, and $K = \mathrm{Ker}(\Phi)$. It is not hard to see that any linear reduction of $(\mathcal{G}/K, \mathcal{A}/K, \Phi/K, \mathcal{N}/K, \Psi/K)$ provides a linear reduction of $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$. Indeed, if $\hat{\gamma}$ is a linear extension of $\Phi/K$ to a normal subgroup $\hat{\Gamma}$ of $\mathcal{G}/K$, and lies under $\Psi/K$, then $\hat{\Gamma} = \Gamma/K$, where $\Gamma$ is a normal subgroup of $\mathcal{G}$. Furthermore, $\hat{\gamma}$ inflates to a unique character $\gamma \in \mathrm{Irr}(\Gamma)$, i.e., $\hat{\gamma} = \gamma/K$. Also, $\gamma$ is linear, as $\hat{\gamma}$ is, and lies under $\Psi \in \mathrm{Irr}(\mathcal{N})$, as $\hat{\gamma}$ lies under $\Psi/K \in \mathrm{Irr}(\mathcal{N}/K)$. Hence we can form a linear reduction of $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$, using the extension $\gamma$ of $\Phi$ to $\Gamma$. As no proper linear reductions can be made to the quintuple $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$, the lemma follows. $\square$

**Corollary 10.9.** *Let $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ be a faithful linear limit of $(G, A, \phi, N, \psi)$. Then $\mathbb{A}$ is a cyclic central subgroup of $\mathbb{G}$ (it could be trivial), and is maximal among the abelian $\mathbb{G}$-invariant subgroups of $\mathbb{N}$. Hence $\mathbb{A} = Z(\mathbb{N})$.*

*Proof.* Clearly $\mathbb{A}$ is a normal subgroup of $\mathbb{G}$, while $\mathbf{\Phi}$ is a linear faithful $\mathbb{G}$-invariant character of $\mathbb{A}$. Then $\mathbb{A}$ is cyclic, as it affords a faithful linear character. The additional fact that $\mathbf{\Phi}$ is $\mathbb{G}$-invariant implies that $\mathbb{A}$ is a subgroup of the center $Z(\mathbb{G})$ of $\mathbb{G}$.

Now assume that $B$ is an abelian $\mathbb{G}$-invariant subgroup of $\mathbb{N}$ that contains $\mathbb{A}$. Let $\beta \in \mathrm{Irr}(B)$ be any character of $B$ that lies above $\mathbf{\Phi}$ and under $\mathbf{\Psi}$. (Clearly such a character exists if the group $B$ exists.) Then $\beta$ is an extension of $\mathbf{\Phi}$ to $B$. Furthermore, if $\mathbf{\Psi}_\beta$ is the $\beta$-Clifford correspondent of $\mathbf{\Psi}$, then the quintuple $(\mathbb{G}(\beta), B, \beta, \mathbb{N}(\beta), \mathbf{\Psi}_\beta)$ is a linear reduction of $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$. According to Lemma 10.8 the latter quintuple can't have any proper reductions. Therefore, $B = \mathbb{A}$, and $\mathbb{A}$ is maximal among the abelian $\mathbb{G}$-invariant subgroups of $\mathbb{N}$.

As the center $Z(\mathbb{N})$ is an abelian characteristic subgroup of $\mathbb{N}$, it is clearly $\mathbb{G}$-invariant. Furthermore, $\mathbb{A}$ is a subgroup of $Z(\mathbb{N})$, as $\mathbb{A}$ is a subgroup of $Z(\mathbb{G})$. So $\mathbb{A} = Z(\mathbb{N})$, and Corollary 10.9 follows. $\square$

**Corollary 10.10.** *Let $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ be a faithful linear limit of $(G, A, \phi, N, \psi)$. Assume further that $N$ is a $p$-group, for some odd prime $p$. Then either $\mathbb{N} = Z(\mathbb{N})$ is cyclic, or $\mathbb{N}$ is the central product $\mathbb{N} = E \odot \mathbb{A}$, of a non-trivial extra special $p$-group $E$ of exponent $p$, and $\mathbb{A} = Z(\mathbb{N})$. In both* *cases the irreducible character* $\mathbf{\Psi} \in \mathrm{Irr}(\mathbb{N})$ *is zero on* $\mathbb{N} - \mathbb{A}$ *and a multiple of* $\mathbf{\Phi}$ *on* $\mathbb{A}$*. Hence* $\mathbf{\Psi}$ *is* $\mathbb{G}$*-invariant.*

*Proof.* According to Corollary 10.9, the group $\mathbb{A} = Z(\mathbb{N})$ is cyclic, central in $\mathbb{G}$ and maximal among the abelian $\mathbb{G}$-invariant subgroups of $\mathbb{N}$. Furthermore, the fact that $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ is a linear quintuple, implies that $\mathbf{\Phi}$ is $\mathbb{G}$-invariant.

If $\mathbb{N} = \mathbb{A} = Z(\mathbb{N})$, then $\mathbb{N}$ is cyclic and $\mathbf{\Phi} = \mathbf{\Psi}$. In this case the corollary holds trivially.

If $\mathbb{N} > \mathbb{A}$, then $\mathbb{A} \neq 1$, or else $\mathbb{A} = \mathbb{N} = 1$, since $\mathbb{N}$ is a $p$-group. Furthermore, $\mathbf{\Phi}$ is a faithful linear character of the cyclic group $\mathbb{A}$. The fact that the cyclic group $\mathbb{A} = Z(\mathbb{N})$ is maximal among the abelian subgroups of $\mathbb{N}$, normal in $\mathbb{G}$ implies that every characteristic abelian subgroup of the $p$-group $\mathbb{N}$ is cyclic and central. Hence P.Hall's theorem (see Theorem 4.22 pp.75 in [19]) implies that $\mathbb{N}$ is the central product

$$\mathbb{N} = E \odot Z(\mathbb{N}) = E \odot \mathbb{A},$$

where $E$ is a non-trivial (or else $\mathbb{N}$ is abelian), extra special $p$-group of exponent $p$. Hence $\mathbf{\Psi}$ must be the unique character of $\mathbb{N}$ that lies above the faithful character $\mathbf{\Phi}$ of the center, (see Theorem 7.5 in [11]) Even more, this unique character is zero outside $\mathbb{A}$, while its restriction to $\mathbb{A}$ is a multiple of $\mathbf{\Phi}$. In addition, the fact that $\mathbf{\Phi}$ is $\mathbb{G}$-invariant, while $\mathbb{N}$ is a normal subgroup of $\mathbb{G}$, makes $\mathbf{\Psi}$ also $\mathbb{G}$-invariant. So Corollary 10.10 follows. □

**Lemma 10.11.** *Let* $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ *be a faithful linear limit of the linear quintuple* $(G, A, \phi, N, \psi)$*. If* $\Theta \in \mathrm{Irr}(\mathbb{G}|\mathbf{\Phi})$*, then there exists a unique* $\chi \in \mathrm{Irr}(G|\phi)$*, so that* $\Theta$ *is a faithful linear limit of* $\chi$*, that is,* $\Theta = fl(\chi)$*. If, in addition,* $\Theta$ *lies above* $\mathbf{\Psi}$*, then* $\chi$ *lies above* $\psi$*.*

*Proof.* Let

$$(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi}) = (\mathcal{G}/K, \mathcal{A}/K, \Phi/K, \mathcal{N}/K, \Psi/K),$$

where $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$ is a linear limit of $(G, A, \phi, N, \psi)$, and $K = \mathrm{Ker}(\Phi)$. Then $\Theta$ inflates to a unique character $\theta \in \mathrm{Irr}(\mathcal{G})$. Clearly $\theta$ lies above $\Phi$. If $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi) = (G, A, \phi, N, \psi)$, i.e., the starting quintuple was already minimal, then the lemma obviously holds with $\chi = \theta$.

If $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$ is a linear reduction of $(G, A, \phi, N, \psi)$, that is we reach the limit quintuple after only one proper reduction, then $\mathcal{G} = G(\Phi)$. Hence Clifford's theorem implies that $\theta \in \mathrm{Irr}(\mathcal{G}|\Phi)$ induces irreducibly to $G$. Therefore the character $\theta^G = \chi$ is the only character in $\mathrm{Irr}(G|\phi)$ having $\theta$ as its $\Phi$-Clifford correspondent. Hence $\chi = \theta^G$ is an irreducible character of $G$ that lies above $\phi$, since $\Phi$ is an extension of $\phi$. It is also obvious that $l(\chi) = \theta$, while $fl(\chi) = \Theta$.

If we need a series of linear reductions to reach the limit quintuple $(\mathcal{G}, \mathcal{A}, \Phi, \mathcal{N}, \Psi)$, then we repeat Clifford's theorem as many times as the number of proper linear reductions we perform. In conclusion, the character $\theta^G = \chi$ is an irreducible character of $G$ that lies above $\phi$, and satisfies the conditions in Lemma 10.11.

If, in addition, $\Theta$ lies above $\mathbf{\Psi}$, then the inflation $\theta$ of $\Theta$ to $\mathcal{G}$, lies above $\Psi$. Since $l(\psi) = \Psi$, the character $\Psi$ induces $\psi$ in $N$, i.e., $\Psi^N = \psi$. We conclude that $\theta^G = \chi$ lies above $\psi$ whenever $\theta$ lies above $\Psi$. This completes the proof of Lemma 10.11. □

The following is straight forward.

**Lemma 10.12.** *Assume that* $(G, A, \phi, N, \psi)$ *and* $(H, B, \beta, M, \mu)$ *are two linear quintuples. Assume further that there exists an epimorphism of linear quintuples*

$$\rho : (G, A, \phi, N, \psi) \to (H, B, \beta, M, \mu).$$

*By this we mean that $\rho$ is an epimorphism of the group $G$ onto $H$ sending $A$ onto $B$ and $N$ onto $M$. Furthermore, $\phi = \beta \circ \rho_A$ and $\psi = \mu \circ \rho_N$. The restriction of $\rho$ to any linear reduction (or multiple linear reduction) $(G', A', \phi', N', \psi')$ of $(G, A, \phi, N, \psi)$ is an epimorphism onto a linear reduction (or multiple linear reduction respectively) $(H', B', \beta', M', \mu')$ of $(H, B, \beta, M, \mu)$. In this way $\rho$ induces a one to one correspondence between linear reductions ( or multiple linear reductions) of $(G, A, \phi, N, \psi)$ and linear reductions (respectively multiple linear reductions) of $(H, B, \beta, M, \mu)$. Hence $\rho$ induces a one to one correspondence between linear limits of $(G, A, \phi, N, \psi)$ and linear limits of $(H, B, \beta, M, \mu)$.*

*Proof.* If $(G', A', \phi', N', \psi')$ is a linear reduction of $(G, A, \phi, N, \psi)$, it is easy to check that its image $\rho((G', A', \phi', N', \psi'))$ under $\rho$ is a linear reduction of $(H, B, \beta, M, \mu)$.

Let $(H', B', \beta', M', \mu')$ be a linear reduction of $(H, B, \beta, M, \mu)$. If $G', A'$ and $N'$ are the inverse images, under $\rho$, of $H', B'$ and $M'$, respectively, then the quintuple $(G', A', \beta' \circ \rho'_A, N', \mu' \circ \rho'_N)$ is a linear reduction of $(G, A, \phi, N, \psi)$, and its image under $\rho$ equals $(H', B', \beta', M', \mu')$. We conclude that there exists a one to one correspondence between linear reductions of $(G, A, \phi, N, \psi)$ and linear reductions of $(H, B, \beta, M, \mu)$.

Because a multiple linear reduction is reached after a series of linear reductions, repeated applications of the one to one correspondence on linear reductions implies the existence of a one to one correspondence between multiple linear reductions of $(G, A, \phi, N, \psi)$ and $(H, B, \beta, M, \mu)$. Furthermore, since any linear limit of $(G, A, \phi, N, \psi)$ is a minimal multiple linear reduction of the latter quintuple, we also get a one to one correspondence between the linear limits of $(G, A, \phi, N, \psi)$ and those of $(H, B, \beta, M, \mu)$. Hence the lemma holds. $\square$

**Corollary 10.13.** *Assume that the linear quintuples $(G, A, \phi, N, \psi)$ and $(H, B, \beta, M, \mu)$ satisfy the hypothesis in Lemma 10.12. Then any faithful linear limit of $(H, B, \beta, M, \mu)$ is isomorphic to a faithful linear limit of $(G, A, \phi, N, \psi)$.*

*Proof.* Let $(H', B', \beta', M', \mu')$ be a linear limit of $(H, B, \beta, M, \mu)$. Then according to Lemma 10.12 it corresponds to a linear limit $(G', A', \phi', N', \psi')$ of $(G, A, \phi, N, \psi)$ . Because $\rho$ maps the latter linear limit onto the former, we get that $\rho$ maps $A'$ onto $B'$, while $\phi' = \beta' \circ \rho'_A$. We conclude that the kernel $\mathrm{Ker}(\phi')$ of $\phi'$ is mapped, under $\rho$, onto the kernel $\mathrm{Ker}(\beta')$ of $\beta'$, i.e., $\rho(\mathrm{Ker}(\phi')) = \mathrm{Ker}(\beta')$. Furthermore, if $S$ equals the kernel of $\rho_{G'}$ then $S$ is a normal subgroup of $G'$ that is contained in $\mathrm{Ker}(\phi')$, (since for all $s \in S$ we get $\phi'(s) = \beta'(\rho(s)) = \beta'(1) = 1$). Hence the following holds

$$
\begin{aligned}
G' / \mathrm{Ker}(\phi') &\cong H' / \mathrm{Ker}(\beta'), \\
A' / \mathrm{Ker}(\phi') &\cong B' / \mathrm{Ker}(\beta'), \\
N' / \mathrm{Ker}(\phi') &\cong M' / \mathrm{Ker}(\beta').
\end{aligned}
$$

In addition, the unique characters $\phi' / \mathrm{Ker}(\phi')$ and $\psi' / \mathrm{Ker}(\phi')$ of the factor groups $A' / \mathrm{Ker}(\phi')$ and $N' / \mathrm{Ker}(\phi')$ that inflate to $\phi'$ and $\psi'$, respectively, correspond under the above isomorphisms, to the unique characters $\beta' / \mathrm{Ker}(\beta')$ and $\mu' / \mathrm{Ker}(\beta')$ of the factor groups $B' / \mathrm{Ker}(\beta')$ and $M' / \mathrm{Ker}(\beta')$ that inflate to $\beta'$ and $\mu'$, respectively. This completes the proof of the corollary. $\square$

**Proposition 10.14.** *Let $(G, A, \phi, N, \psi)$ be a linear quintuple, and $T = \mathrm{Ker}(\phi)$. Then the factor quintuple $(G/T, A/T, \phi/T, N/T, \psi/T)$ is well defined. Furthermore, any faithful linear limit of the factor quintuple is isomorphic to a faithful linear limit of the original one.*

*Proof.* Observe that the natural epimorphism from $G$ to $G/T$ provides an epimorphism

$$\rho : (G, A, \phi, N, \psi) \to (G/T, A/T, \phi/T, N/T, \psi/T)$$

of linear quintuples, so that $(G, A, \phi, N, \psi)$ and $(G/T, A/T, \phi/T, N/T, \psi/T)$ satisfy the hypothesis in Lemma 10.12. The rest of proof is a simple application of Lemma 10.12 and Corollary 10.13. □

From now on, whenever needed, we will identify any two faithful linear limits of $(G, A, \phi, N, \psi)$ and $(G/T, A/T, \phi/T, N/T, \psi/T)$ which are isomorphic under the preceding proposition.

**Corollary 10.15.** *Assume that the linear quintuple $(G', A', \phi', N', \psi')$ is a multiple linear reduction of $(G, A, \phi, N, \psi)$, and let $T = \mathrm{Ker}(\phi')$. Then the factor quintuple $(G'/T, A'/T, \phi'/T, N'/T, \psi'/T)$ is well defined. Furthermore, any faithful linear limit of the factor quintuple is, under some identification, also a faithful linear limit of $(G, A, \phi, N, \psi)$, i.e.,*

$$FLL(G'/T, A'/T, \phi'/T, N'/T, \psi'/T) \leq FLL(G, A, \phi, N, \psi).$$

*Proof.* Follows immediately from Remark 10.2 and Proposition 10.14. □

For the next proposition we will need a nice observation of I.M.Isaacs, that is actually the exercise (6.11) in [12].

**Lemma 10.16.** *Let $B$ be a normal subgroup of a finite group $G$, $\gamma \in \mathrm{Lin}(B)$ a linear character of $B$ and $\chi \in \mathrm{Irr}(G|\gamma)$ an irreducible character of $G$ lying above $\gamma$. If $\chi_\gamma \in \mathrm{Irr}(G(\gamma))$ is the $\gamma$-Clifford correspondent of $\chi$ in the stabilizer $G(\gamma)$ of $\gamma$ in $G$, then $\chi$ is monomial if and only if $\chi_\gamma$ is monomial.*

*Proof.* It is clear that if $\chi_\gamma$ is monomial then $\chi$ is monomial, as $\chi_\gamma$ induces $\chi$ in $G$.

So we assume that $\chi$ is a monomial character, and we will show that $\chi_\gamma$ is also monomial. Let $K = \mathrm{Ker}(\chi)$. Of course $K \trianglelefteq G$. It is clear that $\chi_\gamma$ is monomial if and only if the irreducible character $\chi_\gamma/K$ of the factor group $G(\gamma)/K$ that inflates to $\chi_\gamma$, is monomial. Hence it suffices to prove the lemma in the case of a faithful irreducible character $\chi$, as we can pass to the factor groups $G/K$ and $(BK)/K$. So in the rest of the proof we assume that $K = 1$.

Clifford's Theorem implies that the restriction $\chi|_B$ of $\chi$ to $B$ is a sum of $G$-conjugates of $\gamma$. Thus $1 = \mathrm{Ker}(\chi|_B) = \bigcap_{s \in G/G(\gamma)}(\mathrm{Ker}(\gamma^s))$. But the derived group $[B, B]$ of $B$ is contained in the kernel of $\gamma^s$ for every $s \in G$, as $\gamma$ is linear. Thus $[B, B] \leq \mathrm{Ker}(\chi|_B) = 1$. So $B$ is abelian.

We can now follow the hint of problem 6.11 in [12]. As $\chi$ is monomial, there exists $H \leq G$ and $\lambda \in \mathrm{Lin}(H)$ with $\chi = \lambda^G$. Thus the irreducible character $\lambda^{HB}$ of $HB$ lies above a $G$-conjugate $\gamma^s$ of $\gamma$, where $s \in G$. As the $G$-conjugate $\lambda^{s^{-1}} \in \mathrm{Lin}(H^{s^{-1}})$ of $\lambda$ also induces $\chi$, we can replace $H$ by $H^{s^{-1}}$ and $\lambda$ by $\lambda^{s^{-1}}$. This way $\lambda^{HB}$ is replaced by $(\lambda^{s^{-1}})^{H^{s^{-1}}B} = (\lambda^{HB})^{s^{-1}}$, which lies above $\gamma$.

According to Mackey's Theorem

$$\lambda^{HB}|_B = (\lambda|_{H \cap B})^B. \tag{10.17}$$

As $B$ is abelian, the right hand side of (10.17) equals the sum of $|B : H \cap B|$ distinct character extensions of $\lambda|_{H \cap B}$ to $B$, each one appearing with multiplicity one. Thus every irreducible constituent of $\lambda^{HB}|_B$ appears with multiplicity one. This, along with Clifford's theorem, (as $\lambda^{HB}$ lies above $\gamma$), implies that

$$\lambda^{HB}|_B = e \cdot \sum_{s \in S} \gamma^s = \sum_{s \in S} \gamma^s,$$

where $S$ is a family of representatives for the cosets $H(\gamma)Bs$ of $H(\gamma)B = (HB)(\gamma)$ in $HB$, and $e$ is a positive integer. Furthermore, Clifford's theorem implies the existence of an irreducible character $\theta \in \mathrm{Irr}(HB(\gamma))$ lying above $\gamma$ and inducing $\lambda^{HB}$. The fact that $e = 1$ implies that $\theta|_B = \gamma$, i.e.,

$\theta \in \mathrm{Irr}(HB(\gamma))$ is an extension of $\gamma \in \mathrm{Irr}(B)$ to $HB(\gamma)$. Thus $\theta \in \mathrm{Lin}(HB(\gamma)|\gamma)$ induces $\lambda^{HB}$. Hence $\theta^G = \chi$, as $\lambda$ induces $\chi$. Therefore, $\theta^{G(\gamma)}$ is an irreducible character of $G(\gamma)$ lying above $\gamma$ and inducing $\chi$. As the $\gamma$-Clifford correspondent $\chi_\gamma$ of $\chi$ is unique, we conclude that $\theta^{G(\gamma)} = \chi_\gamma$. Hence $\chi_\gamma$ is induced from the linear character $\theta$, and thus is monomial.

This completes the proof of the lemma in the case of an abelian $B$. So the lemma follows. □

**Proposition 10.18.** *Let $(G, A, \phi, N, \psi)$ be a linear quintuple. Let $\chi \in \mathrm{Irr}(G|\phi)$ an irreducible character of $G$ lying above $\phi$, $l(\chi) \in \mathrm{Irr}(l(G)|l(\phi))$ be a linear limit of $\chi$, and $fl(\chi) \in \mathrm{Irr}(fl(G)|fl(\phi))$ be the corresponding faithful linear limit of $\chi$. Then the following are equivalent*

*1) $\chi$ is monomial*

*2) $l(\chi)$ is monomial*

*3) $fl(\chi)$ is monomial*

*Proof.* Let $(G', A', \phi', N', \psi')$ be a linear reduction of $(G, A, \phi, N, \psi)$. According to Lemma 10.16, the character $\chi \in \mathrm{Irr}(G|\phi)$ is monomial if and only its $\phi'$-Clifford correspondent $\chi'$ is monomial. This is true for every linear reduction, so at the end we get that $\chi$ is monomial if and only if any linear limit $l(\chi)$ of $\chi$ is monomial.

Let $fl(\chi) \in \mathrm{Irr}(fl(G)) = \mathrm{Irr}(l(G)/K)$ be the faithful linear limit of $\chi$ corresponding to $l(\chi)$. It is obvious that $l(\chi)$ is monomial if and only if $fl(\chi)$ is monomial. This, along with the already proved first equivalence, implies that $fl(\chi)$ is monomial if and only if $\chi$ is monomial. As this is true for any faithful linear limit $fl(\chi)$ of $\chi$, the proof of Proposition 10.18 is complete. □

## 10.2 Linear limits of characters of $p$-groups

Assume that $(G, A, \phi, N, \psi)$ is a linear quintuple. For the rest of this section we suppose that $N$ is a $p$-group, for some odd prime $p$. The main result of this section is

**Theorem 10.19.** *Suppose that $(G, A, \phi, N, \psi)$ is a linear quintuple with $N$ a $p$-group, for some odd prime $p$. Assume further that $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ and $(\mathbb{G}', \mathbb{A}', \mathbf{\Phi}', \mathbb{N}', \mathbf{\Psi}')$ are two faithful linear limits of $(G, A, \phi, N, \psi)$. Then both $\mathbb{N}/\mathbb{A}$ and $\mathbb{N}'/\mathbb{A}'$ are naturaly symplectic $\mathbb{Z}_p(G(\psi)/N)$-modules. Furthermore, $\mathbb{N}/\mathbb{A}$ is isomorphic to $\mathbb{N}'/\mathbb{A}'$ as a symplectic $\mathbb{Z}_p(G(\psi)/N)$-module.*

To prove it we will use strongly Theorem 8.4 in [3]. We remark here that the definitions of a “stabilizer limit” and an “elementary stabilizer limit” that were given in Sections 2 and 3, respectively, in [3], are related to but not the same as our definition of linear limits.

We start with an elementary construction of symplectic modules, and the associate notation. Assume that a $p$-group $R$ is a normal subgroup of some finite group $X$, where $p$ is an odd prime. Assume further that the center $Z(R)$ of $R$ is central in $X$, while $R$ is a central product

$$R = E \odot Z(R),$$

where either $E$ is 1, or else $E$ is an extra special group of exponent $p$ and

$$E \cap Z(R) = Z(E).$$

Then $R/Z(R)$ is an elementary abelian $p$-group, which may be trivial. So $R/Z(R)$, when written additively, can be considered as a vector space over the field $\mathbb{Z}_p$ of $p$ elements. This way $R/Z(R)$

becomes a $\mathbb{Z}_p(X)$-module. Moreover, $[R,R]=[E,E]=Z(E)$ is either trivial or a cyclic group of order $p$. If $Z(E)=1$, i.e., $R/Z(R)=1$, then $R/Z(R)$ becomes trivially a symplectic $\mathbb{Z}_p(X)$-module. If $|Z(E)|=p$, then we can still make $R/Z(R)$ a symplectic $\mathbb{Z}_p(X)$-module. Indeed, if $\lambda \in \mathrm{Irr}(Z(E))$ is any faithful linear character of $Z(E)$, then we can define a bilinear form $<,>$ from $(R/Z(R)) \times (R/Z(R))$ to the multiplicative group $\mathbb{C}_p$ of complex $p$-roots of unity as

$$< \bar{x}, \bar{y} >= \lambda([x,y]) \in \mathbb{C}_p, \tag{10.20}$$

for all $x,y \in R$, where $\bar{x}$ denotes the image of $x \in R$ in the factor group $R/Z(R)$, and $[x,y]$ is the commutator of $x$ and $y$ in $R$. Note that, as the multiplicative group $\mathbb{C}_p$ of $p$-roots of unity is isomorphic to the additive group $\mathbb{Z}_p^+$ of $\mathbb{Z}_p$, we can identify these two isomorphic groups, and consider the bilinear form $<,>$ as a symplectic form in $\mathbb{Z}_p$. As $Z(R)$ is central in $X$, this form is $X$-invariant. So $R/Z(R)$ is a symplectic $\mathbb{Z}_p(X)$-module.

We assume that $R$ and $X$ are as above, with $Z(R)$ central in $X$. Let $U$ be a subgroup of $R$ containing $Z(R)$, and normal in $X$. Then, (see the notation in [1]), we call the symplectic $\mathbb{Z}_p(X)$-submodule $U/Z(R)$ of $R/Z(R)$ **isotropic** if $U \leq R$ is an abelian subgroup of $R$. We call $U/Z(R)$ **anisotropic** if 0 is its only isotropic $\mathbb{Z}_p(X)$-submodule, i.e., every abelian subgroup of $U$ which is normal in $X$ is contained in $Z(R)$. Observe that 0 is an anisotropic symplectic $\mathbb{Z}_p(X)$-module.

Now we go back to the linear quintuple $(G,A,\phi,N,\psi)$. Let $(\mathcal{G},\mathcal{A},\Phi,\mathcal{N},\Psi)$ be a linear limit of $(G,A,\phi,N,\psi)$, and

$$(\mathbb{G},\mathbb{A},\mathbf{\Phi},\mathbb{N},\mathbf{\Psi}) = (\mathcal{G}/K,\mathcal{A}/K,\Phi/K,\mathcal{N}/K,\Psi/K), \tag{10.21}$$

be the corresponding faithful linear limit of $(G,A,\phi,N,\Psi)$, where $K=\mathrm{Ker}(\Phi)$. Assume further that $(G_i,A_i,\phi_i,N_i,\psi_i)$, is a chain of linear quintuples, for all $i=0,\dots,n$, such that

$$(G_0,A_0,\phi_0,N_0,\psi_0) = (G,A,\phi,N,\psi), \tag{10.22a}$$

$$(G_n,A_n,\phi_n,N_n,\psi_n) = (\mathcal{G},\mathcal{A},\Phi,\mathcal{N},\Psi), \text{ and} \tag{10.22b}$$

$$(G_i,A_i,\phi_i,N_i,\psi_i) \text{ is a proper linear reduction of } (G_{i-1},A_{i-1},\phi_{i-1},N_{i-1},\psi_{i-1}), \tag{10.22c}$$

whenever $i=1,\dots,n$. These objects stay fixed until the end of the section. We also keep fixed an arbitrary $\mathbb{C}G(\psi)$-elementary stabilizer limit $\Lambda \in ESL(\psi|\mathbb{C}G(\psi))$ of $\psi$, in the sense of [3], and in particular (3.7) of that paper. (Note that the ordered triple $(G(\psi),N,\psi)$ is a member of the family defined in (2.1) of [3]. Thus we can define a $\mathbb{C}G(\psi)$-elementary stabilizer limit of $\psi$.)

We start with some results following (10.22).

**Lemma 10.23.** *Let $M$ be a subgroup of $G$ with $A_i \leq M$, for some $i=0,1,\dots,n$. Assume further that an irreducible character $\chi \in \mathrm{Irr}(M)$, when restricted to $A_i$, is a multiple of $\phi_i$. Then $G(\chi)=G(\chi,\phi_1,\dots,\phi_i)=G_i(\chi)$. In particular, $G(\phi_i)=G_i$, for all $i=0,1,\dots,n$.*

*Proof.* Clearly (10.22c) implies that $G_i=G(\phi_0,\phi_1,\dots,\phi_i)$ for every $i=0,1,\dots,n$. For all $i=1,\dots,n$, the linear character $\phi_i \in \mathrm{Irr}(A_i)$ is an extension of $\phi_{i-1} \in \mathrm{Irr}(A_{i-1})$. Even more, for all such $i$ the group $A_i$ is a normal subgroup of $G(\phi_1,\dots,\phi_{i-1})=G_{i-1}$.

Assume that $i=0,1,\dots,n$ is fixed. Let $M \geq A_i$ and $\chi \in \mathrm{Irr}(M)$ with $\chi|_{A_i}=m\phi_i$ for some integer $m \geq 1$. For any $j=0,1,\dots,i-1$, the linear character $\phi_i$ is an extension of $\phi_j \in \mathrm{Irr}(A_j)$ to $A_i$. Hence $\chi|_{A_j}=m\phi_j$ for all $j=0,1,\dots,i$. Therefore $G(\chi)$ fixes any such $\phi_j$ if and only if it normalizes $A_j$. Clearly $G(\chi)$ fixes the $G$-invariant character $\phi_0$ of $A_0=A$. It also normalizes $A_1$, as the latter is normal in $G$. Thus $G(\chi)$ fixes $\phi_1$. Suppose now that $G(\chi)$ normalizes $A_1,\dots,A_{j-1}$, where $j=2,\dots,i$. Then it fixes $\phi_1,\dots,\phi_{j-1}$, i.e., $G(\chi) \leq G(\chi)(\phi_0,\phi_1,\dots,\phi_{j-1})$. But $A_j$ is a normal subgroup of $G(\phi_0, \phi_1, \dots, \phi_{j-1}) = G_{j-1}$. Hence $G(\chi)$ normalizes $A_j$ as well, and therefore also fixes $\phi_j$ . As this holds for all $j = 1, \dots, i$, the first statement of the lemma follows.

The second part of the lemma follows from the first, if we take $\chi = \phi_i$ and $M = A_i$. $\square$

**Proposition 10.24.** *For every $i = 0, 1, \dots, n$, we have*

$$G(\psi_i) = G_i(\psi_i) = G_i(\psi) \text{ and } G(\psi_i)N = G(\psi). \tag{10.25}$$

*Hence $G(\Psi) = \mathcal{G}(\Psi) = \mathcal{G}(\psi)$ and $G(\Psi)N = G(\psi)$.*

*Proof.* As $\mathcal{G} = G_n$ and $\Psi = \psi_n$, it suffices to prove (10.25). For this proof we will use induction on $i$. As $G_0 = G$ and $\psi_0 = \psi$, the equations in (10.25) hold trivially for $i = 0$. Suppose (10.25) is true for all $i = 0, \dots, t-1$, where $t = 1, \dots, n$. We will show it holds for $i = t$.

By (10.22c), both groups $A_t$ and $N_{t-1}$ are normal subgroups of $G_{t-1}$. Furthermore, $\phi_t \in \mathrm{Irr}(A_t)$ is a linear extension of $\phi_{t-1} \in \mathrm{Irr}(A_{t-1})$ and lies under $\psi_{t-1}$. In addition, $\psi_t \in \mathrm{Irr}(N_t)$ is the $\phi_t$-Clifford correspondent of $\psi_{t-1} \in \mathrm{Irr}(N_{t-1})$. Hence

$$G_{t-1}(\psi_t) = G_{t-1}(\psi_{t-1}, \phi_t). \tag{10.26}$$

As $G_t$ is the subgroup $G_{t-1}(\phi_t)$ of $G_{t-1}$, both sides of this equation are equal to

$$G_t(\psi_{t-1}) = G_t(\psi_t). \tag{10.27}$$

Furthermore, any element of $G_{t-1}$ that fixes $\psi_{t-1}$ permutes among themselves the $N_{t-1}$-conjugates of $\phi_t$, as $A_t$ is normal in $G_{t-1}$. We conclude that $G_{t-1}(\psi_{t-1}) = G_{t-1}(\psi_{t-1}, \phi_t)N_{t-1}$. This, along with (10.26), implies

$$G_{t-1}(\psi_{t-1}) = G_{t-1}(\psi_{t-1}, \phi_t)N_{t-1} = G_{t-1}(\psi_t)N_{t-1}. \tag{10.28}$$

Hence

$$\begin{aligned} G_t(\psi_t) &= G_t(\psi_{t-1}) && \text{by (10.27)} \\ &= G_{t-1}(\phi_t, \psi_{t-1}) && \text{as } G_t = G_{t-1}(\phi_t) \\ &= G_{t-1}(\phi_t, \psi) && \text{by induction for } i = t-1 \\ &= G_t(\psi). \end{aligned}$$

Lemma 10.23 implies that $G(\psi_t) = G_t(\psi_t)$, as $\psi_t|_{A_t}$ is a multiple of $\phi_t$. We conclude that

$$G_t(\psi_t) = G_t(\psi) = G(\psi_t). \tag{10.29}$$

Hence the first part of (10.25) follows for the inductive step. For the second part we get

$$\begin{aligned}
G(\psi) &= G(\psi_{t-1})N && \text{by induction for } i = t-1\\
&= G_{t-1}(\psi_{t-1})N\\
&= G_{t-1}(\psi_{t-1}, \phi_t)N_{t-1}N && \text{by (10.28)}\\
&= G_{t-1}(\psi_{t-1}, \phi_t)N\\
&= G_t(\psi_{t-1})N && \text{as } G_t = G_{t-1}(\phi_t)\\
&= G_t(\psi_t)N && \text{by (10.27)}\\
&= G(\psi_t)N && \text{by (10.29)}
\end{aligned}$$

This completes the inductive proof of (10.25) for $i = t$. Hence Proposition 10.24 follows. □

**Corollary 10.30.** *The inclusion $\mathcal{G}(\Psi) \to G(\psi)$ induces an isomorphism of $\mathcal{G}(\Psi)/\mathcal{N}$ onto $G(\psi)/N$, where $\bar{s}$ maps to $\bar{s}N$, for any $\bar{s} \in \mathcal{G}(\Psi)/\mathcal{N}$.*

*Proof.* Obvious, as $\mathcal{G}(\Psi)N = G(\Psi)N = G(\psi)$, while $\mathcal{G}(\Psi) \cap N = \mathcal{G} \cap N = \mathcal{N}$. □

**Corollary 10.31.** *The character $\Psi \in \mathrm{Irr}(\mathcal{N})$ is $\mathcal{G}$-invariant. Furthermore, $\Psi$ is zero on $\mathcal{N} - \mathcal{A}$, and it is a multiple of $\Phi$ on $\mathcal{A}$. Hence*

$$\mathcal{G} = \mathcal{G}(\Psi) = \mathcal{G}(\psi) = G(\Psi).$$

*Proof.* According to (10.21), we have $\mathbb{G} = \mathcal{G}/K, \mathbb{N} = \mathcal{N}/K$ and $\mathbb{A} = \mathcal{A}/K$, where $K = \mathrm{Ker}(\Phi)$. Furthermore, $\boldsymbol{\Psi}$ is the unique character of the factor group $\mathcal{N}/K$ that inflates to $\Psi \in \mathrm{Irr}(\mathcal{A})$. But as $N$ is a $p$-group, Corollary 10.10 implies that the character $\boldsymbol{\Psi} \in \mathrm{Irr}(\mathbb{N})$ is $\mathbb{G}$-invariant. Hence $\Psi$ is $\mathcal{G}$-invariant. The same corollary implies that $\boldsymbol{\Psi} \in \mathrm{Irr}(\mathbb{N})$ vanishes outside $\mathbb{A}$ and is a multiple of $\boldsymbol{\Phi}$ on $\mathbb{A}$. Thus a similar property holds for its unique inflation $\Psi \in \mathrm{Irr}(\mathcal{N})$. The rest of the corollary follows easily from Proposition 10.24. □

The next result follows immediately from the above two corollaries.

**Corollary 10.32.** *The isosmorphism of $G(\psi)/N$ onto $\mathcal{G}(\Psi)/\mathcal{N}$ in Corollary 10.30, composed with the natural isomorphism of $\mathcal{G}/\mathcal{N} = \mathcal{G}(\Psi)/\mathcal{N}$ onto $\mathbb{G}/\mathbb{N}$, provides an isomorphism $j$ from $G(\psi)/N$ onto the factor group $\mathbb{G}/\mathbb{N}$. So any coset $t \in G(\psi)/N$ gets mapped under $j$, to the image of the coset $(t \cap \mathcal{G}(\Psi))/\mathcal{N}$ under the natural isomorphism of $\mathcal{G}/\mathcal{N}$ onto $\mathbb{G}/\mathbb{N}$.*

**Proposition 10.33.** *The factor group $\mathbb{N}/\mathbb{A}$ is an anisotropic symplectic $\mathbb{Z}_p(\mathbb{G}/\mathbb{N})$-module, which may be 0, with respect to the bilinear $\mathbb{G}$-invariant form defined, as in (10.20), by*

$$< x\mathbb{A}, y\mathbb{A} >= \boldsymbol{\Phi}([x, y]) \in \mathbb{C}_p \cong \mathbb{Z}_p^+, \tag{10.34}$$

*for any $x, y \in \mathbb{N}$. Here the $\mathbb{G}/\mathbb{N}$-action is induced by conjugation in $\mathbb{G}$.*

*Proof.* As $N$ is a $p$-group, Lemmas 10.9 and 10.10 imply that either $\mathbb{N} = \mathbb{A}$ or $\mathbb{N}$ is the central product $\mathbb{N} = E \odot Z(\mathbb{N}) = E \odot \mathbb{A}$, of a non-trivial extra special $p$-group $E$ of exponent $p$, and $\mathbb{A} = Z(\mathbb{N})$ which is central in $\mathbb{G}$. Furthermore, $\boldsymbol{\Phi} \in \mathrm{Irr}(\mathbb{A})$ is a faithful linear character of $\mathbb{A}$. In both cases, the factor group $\mathbb{N}/\mathbb{A}$, becomes a symplectic $\mathbb{Z}_p(\mathbb{G})$-module, where $\mathbb{G}$ acts on $\mathbb{N}/\mathbb{A}$ by conjugation, and the symplectic form is defined via commutation in $\mathbb{N}$, (see (10.20) and the paragraph that follows it). In addition, Lemma 10.9 implies that $\mathbb{A}$ is maximal among the abelian subgroups of $\mathbb{N}$ which are normal in $\mathbb{G}$. Hence $\mathbb{N}/\mathbb{A}$ is an anisotropic symplectic $\mathbb{Z}_p(\mathbb{G})$-module, which may be 0.

Clearly $\mathbb{N}$ centralizes the factor group $\mathbb{N}/\mathbb{A}$, as $[\mathbb{N}, \mathbb{N}] \leq \mathbb{A}$. Hence the action of $\mathbb{G}$ on $\mathbb{N}/\mathbb{A}$ induces one of $\mathbb{G}/\mathbb{N}$ on that symplectic group. So Proposition 10.33 follows. □

Proposition 10.33, along with the isomorphism $j$ defined in Corollary 10.32, implies

**Corollary 10.35.** *The factor group $\mathbb{N}/\mathbb{A}$ is an anisotropic symplectic $\mathbb{Z}_p(G(\psi)/N)$-module, with respect to the bilinear form defined in* (10.34). *Here the $G(\psi)/N$-action is defined as through the isomorphism $j$, defined in Corollary 10.32 as*

$$(x\mathbb{A})^{\bar{s}} = x^{j(\bar{s})}\mathbb{A} \in \mathbb{N}/\mathbb{A}$$

*for all $x \in \mathbb{N}$ and $\bar{s} \in G(\psi)/N$.*

**Corollary 10.36.** *The factor group $\mathcal{N}/\mathcal{A}$ is an anisotropic symplectic $\mathbb{Z}_p(\mathcal{G}/\mathcal{N})$-module, that may be* 0, *with respect to the bilinear form defined by*

$$< s\mathcal{A}, t\mathcal{A} >= \Phi([s,t]) \in \mathbb{C}_p \cong \mathbb{Z}_p^+, \tag{10.37}$$

*for any $s, t \in \mathcal{N}$. Here the $\mathcal{G}/\mathcal{N}$-action is induced by conjugation in $\mathcal{G}$. Hence with respect to the same form, $\mathcal{N}/\mathcal{A}$ is an anisotropic symplectic $\mathbb{Z}_p(G(\psi)/N)$-module, where the action of $G(\psi)/N \cong \mathcal{G}(\Psi)/\mathcal{N} = \mathcal{G}/\mathcal{N}$ is defined by*

$$(s\mathcal{A})^{rN} = (s^r)\mathcal{A} \in \mathcal{N}/\mathcal{A},$$

*for any $s, t \in \mathcal{N}$ and $r \in \mathcal{G}$.*

*Proof.* Let $K = \text{Ker}(\Phi)$. Then, as we have already seen,

$$\mathcal{N}/\mathcal{A} \cong (\mathcal{N}/K)/(\mathcal{A}/K) = \mathbb{N}/\mathbb{A}, \tag{10.38}$$

where the isomorphism is $\mathcal{G}$-invariant. Furthermore, $\mathbf{\Phi}$ is the unique character of the factor group $\mathcal{A}/K = \mathbb{A}$ that inflates to $\Phi \in \text{Irr}(\mathcal{A})$. Hence, under the isomorphism in (10.38), Proposition 10.33 implies that $\mathcal{N}/\mathcal{A}$ is an anisotropic symplectic $\mathbb{Z}_p(\mathcal{G}/\mathcal{N})$-module, with respect to the bilinear form that (10.34) determines. (Note that this bilinear form translates to (10.37).) Furthermore, $\mathcal{G}$ acts on $\mathcal{N}/\mathcal{A}$ by conjugation, while $\mathcal{N}$ centralizes it. But $\mathcal{G} = \mathcal{G}(\Psi) = G(\Psi)$, by Corollary 10.31, while Corollary 10.30 implies that $G(\psi)/N$ is naturally isomorphic to $\mathcal{G}(\Psi)/\mathcal{N} = \mathcal{G}/\mathcal{N}$. Hence $\mathcal{N}/\mathcal{A}$ becomes an anisotropic $G(\psi)/N$-module. This completes the proof of the corollary. □

**Proposition 10.39.** *If $N$ is a $p$-group, then $\Psi$ is a $\mathbb{C}G(\psi)$-stabilizer limit of $\psi$ as this is defined in [3], that is, $\Psi \in SL(\psi|\mathbb{C}G(\psi))$.*

*Proof.* According to (10.22c), for every $i = 1, \ldots, n$, we have a normal subgroup $A_i$ of $G_{i-1}$ contained in $N_{i-1}$, and a linear character $\phi_i \in \text{Irr}(A_i)$ lying under $\psi_{i-1} \in \text{Irr}(N_{i-1})$. Furthermore, $\psi_i \in \text{Irr}(N_i)$ is the $\phi_i$-Clifford correspondent of $\psi_{i-1}$. As the ordered triple $(G_{i-1}(\psi_{i-1}), N_{i-1}, \psi_{i-1})$ is a member of the family defined in (2.1) of [3], while $A_i$ is a normal subgroup of $G_{i-1}(\psi_{i-1})$, we conclude that $\psi_i$ is an element of the set $\text{DCC}(\psi_{i-1}|\mathbb{C}G_{i-1}(\psi_{i-1}))$ defined in (2.2) of [3]. According to Proposition 10.24 we have $G_{i-1}(\psi_{i-1}) = G(\psi_{i-1})$. Hence we get a sequence of characters $\psi = \psi_0, \psi_1, \ldots, \psi_n = \psi$, such that

$$\psi_i \in \text{DCC}(\psi_{i-1}|\mathbb{C}G(\psi_{i-1})),$$

for all $i = 1, \dots, n$. Hence $\Psi$ lies in the set $\mathrm{CC}(\psi|\mathbb{C}G(\psi))$, (see (2.3) in [3]).

According to the definition of stabilizer limits, in (2.16) of [3], we can complete the proof of the proposition by showing that $\Psi \in \mathrm{Irr}(\mathcal{N})$ is the only member of $\mathrm{DCC}(\Psi|\mathbb{C}G(\Psi))$. By (2.14) and (2.15) in [3], it suffices to show that whenever $M$ is a normal subgroup of $G(\Psi)$ contained in $\mathcal{N}$, the restriction $\Psi|_M$ is a multiple of of a single irreducible character. Suppose such an $M \leq \mathcal{N}$ is fixed. Let $\theta \in \mathrm{Irr}(M)$ be an irreducible character of $M$ that lies under $\Psi$. It is enough to show that $\theta$ is $G(\Psi)$-invariant. We know from Corollary 10.31 that $G(\Psi) = \mathcal{G}(\Psi) = \mathcal{G}$. So it suffices to show that $\mathcal{G}(\theta) = \mathcal{G}$.

$\Psi$ lies above the $\mathcal{G}$-invariant linear character $\Phi \in \mathrm{Irr}(\mathcal{A})$. Hence we can replace $M$ with $M \cdot \mathcal{A}$ and $\theta \in \mathrm{Irr}(M)$ with $\theta \cdot \Phi \in \mathrm{Irr}(M\mathcal{A})$ (where $(\theta \cdot \Phi)(ma) = \theta(m)\Phi(a)$, for all $m \in M$ and $a \in \mathcal{A}$ ). This way $\mathcal{G}(\theta) = \mathcal{G}(\theta \cdot \Phi)$ remains the same. So we may assume that $\mathcal{A} \leq M \leq \mathcal{N}$, and that $\theta$ lies above $\Phi$. Then $M/\mathcal{A}$ is a $\mathbb{Z}_p(\mathcal{G}/\mathcal{N})$-submodule of $\mathcal{N}/\mathcal{A}$. But the latter is an anisotropic $\mathbb{Z}_p(\mathcal{G}/\mathcal{N})$ module by Corollary 10.36. Hence its symplectic form $<,>$ (see (10.37)), restricts to a non-singular bilinear alternating form on $(M/\mathcal{A}) \times (M/\mathcal{A})$. It follows that $\theta$ is zero on $M - \mathcal{A}$ and a multiple of $\Phi$ on $\mathcal{A}$. Therefore $\mathcal{G}(\theta) = \mathcal{G}$, and the proposition follows. ☐

According to (2.12) in [3], we may define another triple, denoted by $(G(\psi)\{\Psi\}^*, N\{\Psi\}^*, \Psi^*)$, using the $\mathbb{C}G(\psi)$-stabilizer limit $\Psi$ of $\psi$. The star groups are defined in (2.12) of [3], as the factor groups we get when we divide the triple $(G(\psi)\{\Psi\}, N\{\Psi\}, \Psi)$ by $\mathrm{Ker}(\Psi)$. So $\Psi^*$ in [3] denotes the unique character $\Psi/\mathrm{Ker}(\Psi)$ from which $\Psi$ is inflated. Note also that $X\{\theta\}$ denotes in [3] the stabilizer $X(\theta)$ of $\theta$ in $X$, for any group $X$ and any irreducible character $\theta$ of any subgroup of $X$. In our case, where $N$ is a $p$-group, the kernel $\mathrm{Ker}(\Psi)$ of $\Psi$ coincides with $K = \mathrm{Ker}(\Phi)$, by Corollary 10.31. Furthermore, the same corollary implies that $G(\psi)\{\Psi\} = G(\psi)(\Psi) = \mathcal{G}(\Psi) = \mathcal{G}$. Of course $N\{\Psi\} = N(\Psi) = \mathcal{G}(\Psi) \cap N = \mathcal{G} \cap N = \mathcal{N}$. Hence the star triple $(G(\psi)\{\Psi\}^*, N\{\Psi\}^*, \Psi^*)$ in [3], is what we write as $(\mathbb{G}, \mathbb{N}, \boldsymbol{\Psi})$ (see (10.21)).

Even more, according to (2.13a) in [3] the stabilizer limit $\Psi$ of $\psi$ defines a natural isomorphism denoted by $\cdot/\Psi$ from $G(\psi)/N$ to $G(\psi, \Psi)^*/N(\Psi)^* = \mathbb{G}/\mathbb{N}$. Observe that this is exactly the isomorphism $j$ defined in Corollary 10.32. Having explained this, we can now prove

**Theorem 10.40.** *Let $\Lambda \in$ $ESL(\psi|\mathbb{C}G(\psi))$ be a $\mathbb{C}G(\psi)$-elementary stabilizer limit of $\psi$, with $K_0 = \mathrm{Ker}(\Lambda)$ and $N(\Lambda)^* = N(\Lambda)/K_0$. Then $N(\Lambda)^*/Z(N(\Lambda)^*)$ is isomorphic to $\mathbb{N}/\mathbb{A}$ as symplectic $\mathbb{Z}_p(G(\psi)/N)$-modules.*

*Proof.* We are going to apply Theorem 8.4 in [3], for the triple $(G(\psi), N, \psi)$ here in the place of $(G, N, \psi)$ there, the $\mathbb{C}G(\psi)$-elementary stabilizer limit $\Lambda$ of $\psi$ here, in the place of the $\mathbb{C}G$-elementary stabilizer limit $\phi$ of $\psi$ there, and the $\mathbb{C}G(\psi)$-stabilizer limit $\Psi$ of $\psi$ here, in the place of $\theta$ there. (Note that the hypotheses (7.1) and (7.2a) in [3] are satisfied.) Observe also that $\Lambda$ is an irreducible character of $N(\Lambda)$, by (2.4c) in [3], as $\Lambda \in$ $\mathrm{CC}(\psi|\mathbb{C}G(\psi))$. Hence Theorem 8.4 gives us a monomorphism $\mu$ of the group $G(\psi)\{\Lambda\}^* = G(\psi, \Lambda)/K_0$ into the group $G(\psi)\{\Psi\}^* = \mathbb{G}$ that satisfies the equivalent of (6.1) in [3]. Furthermore, the relations (8.5)in [3] tell us that

$$\mathbb{G} = \mathbb{A}\ \mu(G(\psi, \Lambda)^*) = \mathbb{A}\ \mu(G(\psi, \Lambda)/K_0) \text{ and } \mathbb{N} = \mathbb{A}\ \mu(N(\Lambda)^*) = \mathbb{A}\ \mu(N(\Lambda)/K_0). \tag{10.41}$$

(Note that in our case $Z(N\{\Psi\}^*) = Z(\mathbb{N}) = \mathbb{A}$.) Furthermore, $\mu$ satisfies (6.1), and, in particular, (6.1a), of [3]. Hence the triple $(\mu(G(\psi, \Lambda)^*), \mu(N(\Lambda)^*), \mu(\Lambda^*))$ is a restrictor of $(\mathbb{G}, \mathbb{N}, \boldsymbol{\Psi})$, in the sense of (5.1) in [3]. (Where the irreducible character $\Lambda^*$ is the unique character of the factor group $N(\Lambda)^* = N(\Lambda)/K_0$ from which $\Lambda \in \mathrm{Irr}(N(\Lambda))$ is inflated, and $\mu(\Lambda^*) \in \mathrm{Irr}(\mu(N(\Lambda)^*))$ is the unique character of $\mu(N(\Lambda)^*)$ whose composition with $\mu$ is $\Lambda^*$.) Therefore (5.1) of [3] implies

$$\mu(N(\Lambda)^*) = \mathbb{N} \cap \mu(G(\psi, \Lambda)^*) \text{ and } \mu(\Lambda^*) = \mathbf{\Psi}|_{\mu(N(\Lambda)^*)}. \tag{10.42}$$

Hence $\mu$ restricts to an isomorphism

$$N(\Lambda)/K_0 = N(\Lambda)^* \cong \mu(N(\Lambda)^*) = \mathbb{N} \cap \mu(G(\psi, \Lambda)^*) \tag{10.43}$$

that sends the irreducible character $\Lambda^* \in \mathrm{Irr}(N(\Lambda)^*)$ to the restriction of $\mathbf{\Psi}$ to $\mu(N(\Lambda)^*)$. Even more, in view of (10.41) we have $\mathbb{A} \cap \mu(N(\Lambda)^*) = Z(\mu(N(\Lambda)^*))$, and thus

$$\mathbb{N}/\mathbb{A} \cong \mu(N(\Lambda)^*)/(\mathbb{A} \cap \mu(N(\Lambda)^*)) = \mu(N(\Lambda)^*)/Z(\mu(N(\lambda)^*)).$$

According to (10.43), the group $N(\Lambda)^* = N(\Lambda)/K_0$ is isomorphic to $\mu(N(\Lambda)^*)$. Hence the inverse image under $\mu$ of $Z(\mu(N(\Lambda)^*)) = \mathbb{A} \cap \mu(N(\Lambda)^*)$ in $N(\Lambda)^*$ is the center $Z(N(\Lambda)^*)$. Furthermore,

$$N(\Lambda)^*/Z(N(\Lambda)^*) \cong \mu(N(\Lambda)^*)/Z(\mu(N(\Lambda)^*)) \cong \mathbb{N}/\mathbb{A}. \tag{10.44}$$

Let $i$ be the above isomorphism that sends the factor group $N(\Lambda)^*/Z(N(\Lambda)^*)$ onto $\mathbb{N}/\mathbb{A}$. (Of course $i$ is induced by the restriction of $\mu$ to $N(\Lambda)^*$.) As we have seen (at (10.42)), the character $\Lambda^*$ maps, under $\mu$, to the restriction of $\mathbf{\Psi}$ to $\mu(N(\Lambda)^*)$. Hence $\Lambda^*$ has a structure similar to that of $\mathbf{\Psi}$, ie., $\Lambda^* \in \mathrm{Irr}(N(\Lambda)^*)$ lies above the unique linear character $\lambda^*$ of $Z(N(\Lambda)^*)$ that is carried, under $\mu$, to the restriction of $\mathbf{\Phi}$ to $\mathbb{A} \cap \mu(N(\Lambda)^*) = Z(\mu(N(\Lambda)^*))$. There is a natural alternating bilinear form on $N(\Lambda)^*/Z(N(\Lambda)^*) \times N(\Lambda)^*/Z(N(\Lambda)^*)$ defined by

$$< x\, Z(N(\Lambda)^*),\ y\, Z(N(\Lambda)^*) >= \lambda^*([x, y]) = \mathbf{\Phi}([\mu(x), \mu(y)]) \in \mathbb{Z}_p, \tag{10.45}$$

for all $x, y \in N(\Lambda)^*$. The isomorphism $i$ carries this bilinear form onto the form $<,>$ of $\mathbb{N}/\mathbb{A} \times \mathbb{N}/\mathbb{A}$, defined in (10.34). Hence $N(\Lambda)^*/Z(N(\Lambda)^*)$ is a symplectic group isomorphic to the symplectic group $\mathbb{N}/\mathbb{A}$.

In view of (10.41) and (10.42), we get a natural isomorphism between the groups $\mathbb{G}/\mathbb{N}$ and $\mu(G(\psi, \Lambda)^*)/\mu(N(\Lambda)^*)$. This, composed with $\mu$, provides an isomorphism $\mu^*$ of $G(\psi, \Lambda)^*/N(\Lambda)^*$ onto $\mathbb{G}/\mathbb{N}$. The group $G(\psi, \Lambda)^*/N(\Lambda)^*$ acts on $N(\Lambda)^*/Z(N(\Lambda)^*)$ via conjugation in $G(\psi, \Lambda)^*$, and leaves the form (10.45) invariant. As $\mu$ preserves conjugation, and induces the isomorphism $i$, it follows that $\mu^*$ and $i$ send the action of $G(\psi, \Lambda)^*/N(\Lambda)^*$ on $N(\Lambda)^*/Z(N(\Lambda)^*)$ to the action of $\mathbb{G}/\mathbb{N}$ on $\mathbb{N}/\mathbb{A}$ in the sense that

$$i(\bar{x}^{\bar{s}}) = i(\bar{x})^{\mu^*(\bar{s})} \in \mathbb{N}/\mathbb{A}, \tag{10.46}$$

for all $\bar{x} \in N(\Lambda)^*/Z(N(\Lambda)^*)$ and $\bar{s} \in G(\psi, \Lambda)^*/N(\Lambda)^*$.

The group $G(\psi)/N$ is naturally isomorphic to the factor group $G(\psi, \Lambda)^*/N(\Lambda)^*$, via the isomorphism $\cdot/\Lambda$ in (2.13a) of [3]. Any coset $\gamma \in G(\psi)/N$ gets mapped under $\cdot/\Lambda$, to the image $\gamma/\Lambda$ of the coset $\gamma \cap G(\psi, \Lambda) in G(\psi, \Lambda)/N(\Lambda)$ under the natural epimorphism of $G(\psi, \Lambda)$ onto $G(\psi, \Lambda)^*$. We use this isomorphism to make the symplectic $\mathbb{Z}_p(G(\psi, \Lambda)^*/N(\Lambda)^*)$-module $N(\Lambda)^*/Z(N(\Lambda)^*)$ into a symplectic $\mathbb{Z}_p(G(\psi)/N)$-module. As we have already seen in Corollary 10.35, we may turn the $\mathbb{Z}_p(\mathbb{G}/\mathbb{N})$-module $\mathbb{N}/\mathbb{A}$ into a $\mathbb{Z}_p(G(\psi)/N)$-module, using the isomorphism $j$ of Corollary 10.32. But $j$ is the natural isomorphism $\cdot/\Psi$, as this is defined in (2.13a) of[3]. According to (6.1b) in [3], the isomorphism $\cdot/\Lambda$ is the composition of $\cdot/\Psi = j$ with $\mu^*$. We conclude that $i$ is an isomorphism of $N(\Lambda)^*/Z(N(\Lambda)^*)$ onto $\mathbb{N}/\mathbb{A}$ as symplectic $\mathbb{Z}_p(G(\psi)/N)$-modules. So Theorem 10.40 follows. □

Theorem 10.19 is now an easy corollary of Theorem 10.40, as

$$\mathbb{N}/\mathbb{A} \cong N(\Lambda)^*/Z(N(\Lambda)^*) \cong \mathbb{N}'/\mathbb{A}',$$

as symplectic $G(\psi)/N$-modules.

We conclude this section with a characterization of any faithful linear limit $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ of $(G, A, \phi, N, \psi)$ when $N$ is nearly extra special.

**Proposition 10.47.** *Assume that $(G, A, \phi, N, \psi)$ is a linear quintuple with $N$ a $p$-group, such that $A = Z(N)$ is cyclic and central in $G$. Assume further that $A$ is maximal among the abelian characteristic subgroups of $N$, while $\phi$ is a faithful linear character of $A$. Then $V = N/A$ is a symplectic $\mathbb{Z}_p(G/N)$-space with the symplectic form $< wA, yA >= \phi([w, y])$, for any $w, y \in N$. If $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ is a faithful linear limit of $(G, A, \phi, N, \psi)$, then $\mathbb{N}/\mathbb{A}$ is isomorphic as a symplectic $\mathbb{Z}_p(G/N)$-module to $W^{\perp}/W$, where $W$ is a maximal $G/N$-invariant totally isotropic subspace of $V$, and $W^{\perp}$ is the perpendicular subspace to $W$ with respect to the above bilinear form.*

*Proof.* As $A = Z(N)$ is maximal characteristic abelian subgroup of $N$, we conclude that $N$ is the central product of $A$ with an extra special $p$-group of exponent $p$. Hence the factor group $V = N/A$ is a $\mathbb{Z}_p(G)$-module and thus a $\mathbb{Z}_p(G/N)$ module (see the discussion after Theorem 10.19). Note also that $\psi$ is the unique character of $N$ that lies above $\phi$, and thus is $G$-invariant as $\phi$ is.

Let $W$ be maximal among the $G/N$-invariant totally isotropic subspaces of $V$. If $X$ is the inverse image of $W$ in $N$, then $X$ is an abelian normal subgroup of $N$ that contains $A = Z(N)$. (Note that $X$ could be $A$.) Then $\phi \in \mathrm{Lin}(A)$ extends to a linear character $\lambda$ of $X$. In addition, the stabilizer $X'$ of $\lambda$ in $N$ is the inverse image in $N$ of $W^{\perp}$, while $X'/X$ is naturally isomorphic to the factor symplectic space $W^{\perp}/W$. Furthermore, if $\psi_\lambda \in \mathrm{Irr}(X')$ is the $\lambda$-Clifford correspondent of $\psi$, then the quintuple $(G(\lambda), X, \lambda, X', \psi_\lambda)$ is a linear reduction of $(G, A, \phi, N, \psi)$. Now, $G = G(\lambda) \cdot N$ as $G$ fixes the unique character $\psi$ of $N$ that lies above $\lambda$. As $W$ is a maximal $G/N$-invariant totally isotropic subspace of $V$, we conclude that $(G(\lambda), X, \lambda, X', \psi_\lambda)$ is a linear limit of $(G, A, \phi, N, \psi)$. (Or else, $\lambda$ would be extended to an abelian normal subgroup $B$ of $G(\lambda)$ contained in $N(\lambda)$. Thus the image of $B$ in $V$ would be a $G(\lambda)$-invariant, and thus $G$-invariant, totally isotropic subspace of $V$, contradicting the maximality of $W$.) Hence, if $K = \mathrm{Ker}(\lambda)$, then $(G(\lambda)/K, X/K, \lambda/K, X'/K, \psi_\lambda/K)$ is a faithful linear limit of $(G, A, \phi, N, \psi)$. But $(X'/K)/(X/K)$ is isomorphic to $X'/X$ (see (10.38)), and this isomorphism is $G(\lambda)$-, and thus $G$-,invariant. We conclude that, for the faithful linear limit $(G(\lambda)/K, X/K, \lambda/K, X'/K, \psi_\lambda/K)$ of $(G, A, \phi, N, \psi)$, the proposition holds, that is, $(X'/K)/(X/K)$ is isomorphic to $W^{\perp}/W$ for some maximal $G/N$-invariant totally isotropic subspace of $N/A$.

According to Theorem 10.19, if $(\mathbb{G}, \mathbb{A}, \mathbf{\Phi}, \mathbb{N}, \mathbf{\Psi})$ is another faithful linear limit of $(G, A, \phi, N, \psi)$, then $\mathbb{N}/\mathbb{A}$ is isomorphic to $(X'/K)/(X/K)$, and this isomorphism is invariant under $G(\psi)/N = G/N$. This completes the proof of Proposition 10.47. □

## 10.3 Linear limits, character towers and triangular sets

Assume that we have the same situation as in Chapter 9. That is, we have a fixed normal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{10.48a}$$

of $G$ that satisfies Hypothesis 5.1. We also fix a character tower

$$\{1 = \chi_0, \chi_1, \ldots, \chi_n\} \tag{10.48b}$$

for that series, along with a representative of its corresponding conjugacy class of triangular sets

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l',\,k'}. \tag{10.48c}$$

Along with the above system we fix a Hall system $\{\mathbf{A}, \mathbf{B}\}$ of $G$ that satisfies (9.2), that is,

$$\mathbf{A} \in \mathrm{Hall}_{\pi}(G), \mathbf{B} \in \mathrm{Hall}_{\pi'}(G), \tag{10.48d}$$

$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_h) \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_h) \text{ form a Hall system for } G(\chi_1, \chi_2, \ldots, \chi_h), \tag{10.48e}$$

$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_n) = P^*_{2k'} \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_n) = Q^*_{2l'-1}, \tag{10.48f}$$

for all $h = 1, \ldots, n$. The way the above character tower, its triangular set, and the Hall system change, if we take a linear limit with respect to a subgroup $G_i$ of $G$, is in general arbitrary. In some special cases we can control these changes, as we will see in the next two subsections. The basic results were already proved in Chapter 9. Here we will apply them multiple times and translate them into the language of "linear limits".

For the rest of the chapter, we fix an integer $m = 1, \ldots, n$. Whenever necessary we consider the smaller system

$$\begin{gathered} 1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G, \\ \{1 = \chi_0, \chi_1, \ldots, \chi_m\}, \\ \{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l,\,k}, \end{gathered} \tag{10.49}$$

where the integers $k, l$ are related to $m$ via (5.7). Of course, as always, along with the above system the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$ are uniquely defined, up to conjugation, via Theorem 8.13 and Theorem 8.15, respectively.

We first work, as in Chapter 9, inside a $\pi'$-group.

### 10.3.1 "$\mathbf{A}(\beta_1)$"-invariant linear reductions

Assume that the normal series (10.48a), its character tower (10.48b), the triangular set (10.48c) and the Hall system (10.48d) are fixed. In addition, we assume that $S$ is a subgroup of $G$ satisfying

$$S \trianglelefteq G \text{ with } S \leq Q_1, \text{ and} \tag{10.50a}$$

$$\zeta \in \mathrm{Lin}(S) \text{ is } G\text{-invariant and lies under } \beta_1. \tag{10.50b}$$

Note that (10.50a,b) are the conditions in (9.5). Furthermore, the quintuple $(G, S, \zeta, Q_1, \beta_1)$ is a linear one.

Let $E$ be a normal subgroup of $G$ with $S \leq E \leq Q_1$, and $\lambda_1 \in \mathrm{Lin}(E)$ be a linear character of $E$ lying above $\zeta$ and under $\beta_1$. So $\lambda_1$ is an extension of $\zeta$ to $E$. Then we can use all the results of the first section of Chapter 9. We also use the same notation as that introduced in Section 9.1. In particular, Remark 9.7 implies that some $G_1$-conjugate $\lambda \in \mathrm{Irr}(E)$ of $\lambda_1$ is $\mathbf{A}(\chi_1) = \mathbf{A}(\beta_1)$-invariant and extends $\zeta$. So the quintuple $(G_\lambda, E, \lambda, Q_{1,\lambda}, \beta_{1,\lambda})$ is a linear reduction of $(G, S, \zeta, Q_1, \beta_1)$. We call it an "$\mathbf{A}(\beta_1)''$*-invariant linear reduction,* as $\lambda$ was picked, among its $G_1$-conjugates, to be $\mathbf{A}(\beta_1)$-invariant. We saw in (9.15) that the series $1 = G_{0,\lambda} \trianglelefteq G_{1,\lambda} \trianglelefteq \cdots \trianglelefteq G_{n,\lambda} = G_\lambda$, formed by the stabilizers of $\lambda$ in the various subgroups $G_i$ of $G$, is a normal series of $G_\lambda$. Along with that series of groups, we get the tower of characters $\{\chi_{i,\lambda} \in G_{i,\lambda}\}_{i=0}^n$, where $\chi_{i,\lambda}$ is the $\lambda$-Clifford correspondent of $\chi_i$ (see (9.17a)). As in Section 9.1, we add a subscript $\lambda$ to any object such as $P_{2r}, Q_{2i-1}, \alpha_{2r}, \beta_{2i-1}$ etc, to indicate the corresponding object for the $\lambda$-situation. We pick the groups $\{P_{2r,\lambda}, Q_{2i-1,\lambda}\}$, for all $r = 1, \ldots, k$ and all $i = 1, \ldots, l$, to satisfy the conditions in Proposition 9.22. In particular, we get $P^*_{2r} = P^*_{2r,\lambda}$ by (9.24a), while $Q^*_{2i-1} = Q^*_{2i-1,\lambda}$ by (9.24b), whenever $r = 1, \ldots, k'$ and $i = 1, \ldots, l'$, respectively. Then the triangular set

$$\{Q_{2i-1,\lambda}, P_{2r,\lambda} | \beta_{2i-1,\lambda}, \alpha_{2r,\lambda}\}_{i=1,r=0}^{l',\,k'}$$

satisfies the conditions in Theorem 9.26. In addition, the $\lambda$-Hall system $\{\mathbf{A}_\lambda, \mathbf{B}_\lambda\}$ for $G_\lambda$ can be chosen to satisfy the conditions in Theorem 9.46. In particular (9.47) implies

$$\mathbf{A}_\lambda(\chi_{1,\lambda}) = \mathbf{A}(\chi_1). \tag{10.51}$$

Also for the fixed smaller system 10.49, all the conclusions of Theorems 9.48 and 9.50 hold. Hence the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-1,2l-1})$ and their $\lambda$-correspondents can be chosen to satisfy the conditions in Theorems 8.13 and 8.15, respectively, along with (9.49) and (9.51). Thus

$$\widehat{Q}(\beta_{2k-1,2k})(\lambda) = \widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda}), \tag{10.52a}$$

$$\widehat{P}(\alpha_{2l-2,2l-1}) = \widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda}), \tag{10.52b}$$

$$\mathcal{Q}(\lambda) \geq \mathcal{Q}_\lambda \tag{10.52c}$$

$$\mathcal{P} = \mathcal{P}_\lambda, \tag{10.52d}$$

where the groups $\mathcal{Q}$ and $\mathcal{P}$ are defined as in (9.4). Also (9.4c), along with (10.51), implies

$$\widehat{P}(\alpha_{2l-2,2l-1}) \leq \mathcal{P} \leq \mathbf{A}(\chi_1) = \mathbf{A}_\lambda(\chi_{1,\lambda}). \tag{10.52e}$$

Furthermore, Corollary 9.54 implies that the image $I$ of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P^*_{2k})$ equals the image $I_\lambda$ of $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$ in $\mathrm{Aut}(P^*_{2k,\lambda}) = \mathrm{Aut}(P^*_{2k})$. Similarly, the images of $\widehat{P}(\alpha_{2l-2,2l-1})$ and $\widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda})$ in $\mathrm{Aut}(Q^*_{2l-1,\lambda})$ coincide, by Corollary 9.56.

The following observation turns out to be very important for the ultimate proof of Main Theorem 1. We define the group $U$ as

$$U := Q^*_{2l-1} \rtimes J, \tag{10.53}$$

where $J$ is the image of $\widehat{P}(\alpha_{2l-2,2l-1})$ in $\mathrm{Aut}(Q^*_{2l-1})$, as this was defined in (9.106). (Clearly the group $U$ depends on the smaller system (10.49) and thus on $m$). We observe that the quintuple $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$ is a linear one. The "$\mathbf{A}(\beta_1)$"-invariant linear reduction $(G_\lambda, E, \lambda, Q_{1,\lambda}, \beta_{1,\lambda})$ of $(G, S, \zeta, Q_1, \beta_1)$ determines naturally the linear reduction $(U(\lambda), E, \lambda, Q^*_{2l-1,\lambda}, \beta^*_{2l-1,\lambda})$ of the quintuple $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$. Note that $U(\lambda) = Q^*_{2l-1,\lambda} \rtimes J$, as $Q^*_{2l-1,\lambda} = Q^*_{2l-1}(\lambda)$ and $\lambda$ is

$\mathbf{A}(\beta_1) \geq \widehat{P}(\alpha_{2l-2,2l-1})$-invariant. We call such a reduction a *$G$-associate linear reduction* of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$, as the $E$-group we are choosing for this reduction is not only normal in $U$, as it is the common case in linear reductions, but it is normal in $G$. We remark here that the group $U$ is isomorphic to a section of $G$.

Assume now that there exists another pair $(E', \lambda'_1)$, where $E'$ is a subgroup of $Q_{1,\lambda}$, normal in $G_\lambda$, such that $S \leq E \leq E' \leq Q_{1,\lambda}$, and $\lambda'_1$ is a linear character of $E'$ that extends $\lambda$ and lies under $\beta_{1,\lambda}$. Then, by Remark 9.7, we can replace $\lambda'_1$ with one of its $Q_{1,\lambda}$-conjugates $\lambda'$, that is $\mathbf{A}_\lambda(\chi_{1,\lambda})$-invariant and also lies under $\beta_{1,\lambda}$. By (10.51) the above character is $\mathbf{A}(\beta_1) = \mathbf{A}(\chi_1) = \mathbf{A}_\lambda(\chi_{1,\lambda})$-invariant. So we can repeat the same process and consider an "$\mathbf{A}(\beta_1)$"-invariant linear reduction $(G_{\lambda'}, E', \lambda', Q_{1,\lambda,\lambda'}, \beta_{1,\lambda,\lambda'})$ of the "$\mathbf{A}(\beta_1)$"-linear reduction $(G_\lambda, E, \lambda, Q_{1,\lambda}, \beta_{1,\lambda})$. That is, we apply again the methods of Section 9.1, but this time for the normal series $1 = G_{0,\lambda} \trianglelefteq G_{1,\lambda} \trianglelefteq \cdots \trianglelefteq G_{n,\lambda} = G_\lambda$, the normal subgroup $E$ of $G_\lambda$ in the place of $S$, and the $G_\lambda$-invariant character $\lambda$ in the place of $\zeta$. Clearly $E'$ satisfies (9.5c). So Proposition 9.22, Theorems 9.26, 9.48 and 9.50 along with their Corollaries 9.54 and 9.56 can be applied. We conclude that

$$P^*_{2k,\lambda,\lambda'} = P^*_{2k,\lambda} = P^*_{2k}, \tag{10.54a}$$

$$Q^*_{2l-1,\lambda,\lambda'} = Q^*_{2l-1,\lambda}(\lambda') = Q^*_{2l-1}(\lambda, \lambda'), \tag{10.54b}$$

We also have a Hall system $\{\mathbf{A}_{\lambda,\lambda'}, \mathbf{B}_{\lambda,\lambda'}\}$ for $G_{\lambda,\lambda'}$ that satisfies the conditions Theorem 9.46 and is derived from $\{\mathbf{A}_\lambda, \mathbf{B}_\lambda\}$. For any fixed $m = 1, \ldots, n$, the groups $\widehat{Q}_{\lambda,\lambda'}$ and $\widehat{P}_{\lambda,\lambda'}$ can be chosen with respect to the above Hall system. Hence they satisfy

$$\widehat{Q}_{\lambda,\lambda'}(\beta_{2k-1,2k,\lambda,\lambda',\lambda}) = \widehat{Q}(\beta_{2k-1,2k})(\lambda, \lambda'), \tag{10.54c}$$

$$\widehat{P}_{\lambda,\lambda'}(\alpha_{2l-2,2l-1,\lambda,\lambda',\lambda}) = \widehat{P}_\lambda(\alpha_{2l-2,2l-1,\lambda,\lambda}) = \widehat{P}(\alpha_{2l-2,2l-1}), \tag{10.54d}$$

$$\mathcal{Q}_{\lambda,\lambda'} \leq \mathcal{Q}(\lambda, \lambda'), \tag{10.54e}$$

$$\mathcal{P}_{\lambda,\lambda'} = \mathcal{P}. \tag{10.54f}$$

Hence (10.54d) and repeated applications of (9.4c) and (9.47a) imply

$$\widehat{P}(\alpha_{2l-2,2l-1}) \text{ is a subgroup of } \mathbf{A}(\beta_1) = \mathbf{A}(\chi_1) = \mathbf{A}_\lambda(\chi_{1,\lambda}) = \mathbf{A}_{\lambda,\lambda'}(\chi_{1,\lambda'}). \tag{10.54g}$$

Furthermore, (10.54a) implies that $\mathrm{Aut}(P^*_{2k,\lambda,\lambda'}) = \mathrm{Aut}(P^*_{2k})$. Thus Corollary 9.54 implies

$$I = I_\lambda = \text{ the image of } \widehat{Q}_{\lambda,\lambda'}(\beta_{2k-1,2k,\lambda,\lambda'}) \text{ in } \mathrm{Aut}(P^*_{2k,\lambda,\lambda'}), \tag{10.54h}$$

where $I$ is the image of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P^*_{2k})$ and $I_\lambda$ that of $\widehat{Q}_\lambda(\beta_{2k-1,2k,\lambda})$ in $\mathrm{Aut}(P^*_{2k,\lambda})$. Also

$$\text{The subgroups } \widehat{P}_{\lambda,\lambda'}(\alpha_{2l-2,2l-1,\lambda,\lambda'}) \text{ and } \widehat{P}(\alpha_{2l-2,2l-1}) \text{ have the same images in } \mathrm{Aut}(Q^*_{2l-1,\lambda,\lambda'}), \mathrm{Aut}(Q^*_{2l-1,\lambda}) \text{ and } \mathrm{Aut}(Q^*_{2l-1}). \tag{10.54i}$$

As far as the quintuple $(U(\lambda), E, \lambda, Q^*_{2l-1,\lambda}, \beta^*_{2l-1,\lambda})$ is concerned, we clearly have that one of its $G_\lambda$-associate linear reductions is the quintuple $(U(\lambda, \lambda'), E', \lambda', Q^*_{2l-1,\lambda,\lambda'}, \beta^*_{2l-1,\lambda,\lambda'})$. Furthermore, the group $\widehat{P}(\alpha_{2l-2,2l-1})$ fixes $\lambda$, as it is a subgroup of $\mathbf{A}(\chi_1)$, by (10.54h). It also fixes $\lambda'$, as $\widehat{P}(\alpha_{2l-2,2l-1})$ is a subgroup of $\mathbf{A}_\lambda(\chi_{1,\lambda})$, by (10.54h). Hence the image $J$ of $\widehat{P}(\alpha_{2l-2,2l-1})$ in $\mathrm{Aut}(Q^*_{2l-1})$, fixes $\lambda, \lambda'$. Thus $J \leq U(\lambda, \lambda')$. Even more, $Q^*_{2l-1,\lambda,\lambda'} = Q^*_{2l-1}(\lambda, \lambda')$, by (10.54b). We conclude that

$$U(\lambda, \lambda') = Q^*_{2l-1,\lambda,\lambda'} \rtimes J.$$

We can continue this process until we reach a linear limit

$$(l(G), l(S), l(\zeta), l(Q_1), l(\beta_1)) \in LL(G, S, \zeta, Q_1, \beta_1) \tag{10.55}$$

of $(G, S, \zeta, Q_1, \beta_1)$. As this was done in a very specific way, at every linear reduction we were using a character that is invariant, under $\mathbf{A}(\beta_1)$, we call any such limit an "$\mathbf{A}(\beta_1)$"*-invariant linear limit* of $(G, S, \zeta, Q_1, \beta_1)$. The fact that we only consider "$\mathbf{A}(\beta_1)$"-invariant linear characters in every linear reduction does not restrict our options in the possible linear reduction we can perform, as, according to Remark 9.7, we can always replace any given linear character with one of its conjugates that is "$\mathbf{A}(\beta_1)$"-invariant.

Of course, along with the limit group $l(G)$, we reach a limit normal series of $l(G)$

$$1 = l(G_0) \trianglelefteq l(G_1) \trianglelefteq l(G_2) \trianglelefteq \cdots \trianglelefteq l(G_n) = l(G), \tag{10.56}$$

where $l(G_i) = G_i \cap l(G)$.

Along with the series (10.56) we get a limit character tower

$$\{l(\chi_i) \in \mathrm{Irr}(l(G_i))\}_{i=0}^n, \tag{10.57a}$$

where $l(\chi_0) = 1$ and

$$l(\chi_i) \in LL(\chi_i) \text{ is a linear limit of } \chi_i, \tag{10.57b}$$

for all $i = 1, \ldots, n$. We also write

$$\{l(Q_{2i-1}), l(P_{2r}) | l(\beta_{2i-1}), l(\alpha_{2r})\}_{i=1,r=0}^{l',k'} \tag{10.58}$$

for a representative of the unique $l(G)$-conjugacy class of triangular sets of (10.56) that corresponds to the character tower (10.57a), and is derived from the original triangular set (10.48c) following the rules in Theorem 9.26. In addition we write $\{l(\mathbf{A}), l(\mathbf{B})\}$ for a Hall system of $l(G)$ that satisfies (8.4) for the limit case.

Of course, the above system restricts to the smaller

$$\begin{gathered} 1 = l(G_0) \trianglelefteq l(G_1) \trianglelefteq l(G_2) \trianglelefteq \cdots \trianglelefteq l(G_m) \trianglelefteq l(G), \\ \{l(\chi_i) \in \mathrm{Irr}(l(G_i))\}_{i=0}^m, \\ \{l(Q_{2i-1}), l(P_{2r}), l(P_0) | l(\beta_{2i-1}), l(\alpha_{2r})\}_{i=1,r=0}^{l,k}. \end{gathered} \tag{10.59}$$

We also write $l(P^*_{2k})$ for the product group $l(P_0) \cdot l(P_2) \cdots l(P_{2k})$, and $l(Q^*_{2l-1})$ for the product $l(Q_1) \cdots l(Q_{2l-1})$. Similarly, working for a fixed $m$ and looking at the smaller system (10.59), we denote by $l(\widehat{Q})$ the analogue of $\widehat{Q}$ in this limit case, and by $l(\widehat{P})$ the analogue of $\widehat{P}$, i.e., $l(\widehat{Q})$ and $l(\widehat{P})$, satisfy Theorems 6.19 and 8.1, respectively, for the limit case.

Using this notation we can easily see that results similar to (10.54) hold. In particular,

**Theorem 10.60.** *Assume that the normal series, the character tower and the triangular set in* (10.48) *satisfy the conditions* (10.50). *Assume further that* (10.55) *is an* "$\mathbf{A}(\beta_1)$"*-invariant linear limit of* $(G, S, \zeta, Q_1, \beta_1)$, *and* (10.57a) *a character tower that arises as a linear limit of* (10.48b) *(see* (10.57b)*). Then the triangular set* (10.58), *that corresponds to the tower* (10.57a), *can be* *chosen to satisfy*

$$l(P^*_{2k}) = P^*_{2k}. \tag{10.61a}$$

*Furthermore, a Hall system* $\{l(\mathbf{A}), l(\mathbf{B})\}$ *of* $l(G)$ *can be derived from* $\{\mathbf{A}, \mathbf{B}\}$*, so that at every linear reduction the conditions in Theorem 9.46 hold. Then, for any* $m = 1, \dots, n$*, the groups* $l(\widehat{Q})(l(\beta_{2k-1,2k}))$ *and* $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$*, for the smaller system* (10.59)*, can be chosen in association to* $l(\mathbf{B})$ *and* $l(\mathbf{A})$ *respectively. Therefore we have*

$$l(\widehat{P})(l(\alpha_{2l-2,2l-1})) = \widehat{P}(\alpha_{2l-2,2l-1}), \tag{10.61b}$$
$$l(\mathcal{P}) = \mathcal{P}, \tag{10.61c}$$
$$l(I) = I, \tag{10.61d}$$

*where* $l(I)$ *is the image of* $l(\widehat{Q})(l(\beta_{2k-1,2k}))$ *in* $\mathrm{Aut}(l(P^*_{2k}))$.

*Proof.* Follows immediately by repeated applications of Proposition 9.22, Theorems 9.26, 9.46, 9.48, 9.50 and Corollaries 9.54 and 9.56, at every "$\mathbf{A}(\beta_1)$"-invariant linear reduction that we perform. ☐

**Remark 10.62.** The fact $l(P^*_{2k}) = P^*_{2k}$ implies that the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $l(\widehat{Q})(l(\beta_{2k-1,2k}))$ have the same image, that is $I$, in the automorphism group $\mathrm{Aut}(P^*_{2k}) = \mathrm{Aut}(l(P^*_{2k}))$.

Similarly,

**Remark 10.63.** The equation $l(\widehat{P})(l(\alpha_{2l-2,2l-1})) = \widehat{P}(\alpha_{2l-2,2l-1})$ implies that both these groups have the same images in the automorphism groups $\mathrm{Aut}(Q^*_{2l-1})$ and $\mathrm{Aut}(l(Q^*_{2l-1}))$.

Along with the limit (10.55), we reach the quintuple

$$(l(U), l(S), l(\zeta), l(Q^*_{2l-1}), l(\beta^*_{2l-1})), \tag{10.64}$$

This is a multiple linear reduction, but not necessarily a linear limit, of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$, as we could possibly reduce it further using a normal subgroup of $l(T)$, that is not normal in $l(G)$, and a linear extension of $l(\zeta)$ to that normal subgroup. We call $(l(U), l(S), l(\zeta), l(Q^*_{2l-1}), l(\beta^*_{2l-1}))$ a *G-associate limit* of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$. Note that $l(U)$ is isomorphic to a section of $l(G)$ . We clearly have

**Remark 10.65.** The $G$-associate linear limit $(l(U), l(S), l(\zeta), l(Q^*_{2l-1}), l(\beta^*_{2l-1}))$ of the quintuple $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$ satisfies

$$l(Q^*_{2l-1}) = l(G) \cap Q^*_{2l-1}, \tag{10.66a}$$
$$l(U) = l(Q^*_{2l-1}) \rtimes J. \tag{10.66b}$$

Repeated applications of Theorems 9.59 and 9.60, at every "$\mathbf{A}(\beta_1)$"-linear reduction that we perform, imply

**Theorem 10.67.** *If* $\beta_{2k-1,2k}$ *extends to* $\widehat{Q}(\beta_{2k-1,2k})$*, then the character* $l(\beta_{2k-1,2k})$ *also extends to* $l(\widehat{Q})(l(\beta_{2k-1,2k}))$*. Similarly, if* $\alpha_{2l-2,2l-1}$ *extends to* $\widehat{P}(\alpha_{2l-2,2l-1})$*, then* $l(\alpha_{2l-2,2l-1})$ *also extends to* $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$.

Now let $K$ be the kernel of the limit character $l(\zeta)$. As we have seen in the previous section (see (10.3)), we can form a faithful linear limit

$$(fl(G), fl(S), fl(\zeta), fl(Q_1), fl(\beta_1)) = (l(G)/K, l(S)/K, l(\zeta)/K, l(Q_1)/K, l(\beta_1)/K)$$

of the linear quintuple $(G, S, \zeta, Q_1, \beta_1)$. We call such a limit an "$\mathbf{A}(\beta_1)$"*-invariant faithful linear limit* , as it is obtained from an "$\mathbf{A}(\beta_1)$"-invariant linear limit. Along with that we have a normal series of $fl(G)$

$$0 = fl(G_0) \trianglelefteq fl(G_1) \trianglelefteq fl(G_2) \trianglelefteq \cdots \trianglelefteq fl(G_n) = fl(G), \tag{10.68a}$$

where $fl(G_i) = l(G_i)/K$, for all $i = 1, \dots, n$. Along with the series (10.68a) we get a character tower

$$\{fl(\chi_i) \in \mathrm{Irr}(fl(G_i))\}_{i=0}^{n} \tag{10.68b}$$

where $fl(\chi_i)$ is the unique character of $fl(G_i) = l(G_i)/K$ that inflates to $l(\chi_i) \in \mathrm{Irr}(l(G_i))$, for each $i = 1, \dots, n$. Let $fl(\chi_0) = 1$, then

$$fl(\chi_i) \in FLL(\chi_i) \text{ is a faithful linear limit of } \chi_i \tag{10.68c}$$

for all $i = 0, 1, \dots, n$. Let

$$\{fl(Q_{2i-1}), fl(P_{2r}) | fl(\beta_{2i-1}), fl(\alpha_{2r})\}_{i=1,r=0}^{l',k'} \tag{10.68d}$$

be the representative of the unique $fl(G)$-conjugate class of triangular sets that corresponds to (10.68b), that is derived from the set (10.58).

The fact that the quintuple $(l(G), l(S), l(\zeta), l(Q_1), l(\beta_1))$ is a linear one implies that the group $l(S)$ and its irreducible character $l(\zeta)$ satisfy (9.5). Thus we can apply the results of Section 9.3. In particular, Theorem 9.122 implies that the set (10.68d) satisfies

$$fl(Q_{2i-1}) = (l(Q_{2i-1})K)/K, \tag{10.69a}$$
$$fl(P_{2r}) = (l(P_{2r})K)/K \cong l(P_{2r}). \tag{10.69b}$$

whenever $1 \leq i \leq l'$ and $1 \leq r \leq k'$. Hence

$$fl(Q^*_{2l-1}) = (l(Q^*_{2l-1})K)/K, \tag{10.69c}$$
$$fl(P^*_{2k}) = (l(P^*_{2k})K)/K \cong l(P^*_{2k}). \tag{10.69d}$$

Even more, we can pick a Hall system $\{fl(\mathbf{A}), fl(\mathbf{B})\}$ of $fl(G)$ to satisfy the conditions in Theorem 9.136, i.e.,

$$fl(\mathbf{A}) = (l(\mathbf{A})K)/K \cong l(\mathbf{A}) \text{ and } fl(\mathbf{B}) = (l(\mathbf{B})K)/K. \tag{10.70}$$

For every fixed $m = 1, \dots, n$, the smaller limit system (10.59) provides the faithful limit system

$$0 = fl(G_0) \trianglelefteq fl(G_1) \trianglelefteq fl(G_2) \trianglelefteq \cdots \trianglelefteq fl(G_m) \trianglelefteq fl(G), \tag{10.71a}$$
$$\{fl(\chi_i) \in \mathrm{Irr}(fl(G_i))\}_{i=0}^{m} \tag{10.71b}$$
$$\{fl(Q_{2i-1}), fl(P_{2r}) | fl(\beta_{2i-1}), fl(\alpha_{2r})\}_{i=1,r=0}^{l,k} \tag{10.71c}$$

where its triangular set (10.71c) satisfies (10.69). Even more, having fixed the Hall system $\{fl(\mathbf{A}), fl(\mathbf{B})\}$, Theorem 9.138 implies that the group $fl(\widehat{Q})$ can be chosen, (in relation to $fl(\mathbf{B})$), to satisfy the conditions of Theorem 6.19 for the faithful linear situation (10.71), along with

$$fl(\widehat{Q})(fl(\beta_{2k-1,2k})) = (l(\widehat{Q})(l(\beta_{2k-1,2k}))K)/K. \tag{10.72}$$

Even more, if

$$fl(I) := \text{ the image of } fl(\widehat{Q})(fl(\beta_{2k-1,2k})) \text{ in the automorphism group } \mathrm{Aut}(fl(P^*_{2k})), \qquad (10.73)$$

then identifying $l(P^*_{2k})$ with $fl(P^*_{2k})$, Corollary 9.143 implies that

$$fl(I) \cong \text{ Image of } l(\widehat{Q})(l(\beta_{2k-1,2k})) \text{ in } \mathrm{Aut}(l(P^*_{2k})).$$

But this last group equals the image $I$ of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P^*_{2k})$, by Remark 10.62. Hence

$$fl(I) \cong l(I) = I. \qquad (10.74)$$

In addition, Theorem 9.140 implies that we may choose the group $fl(\widehat{P})$, (in relation to $fl(\mathbf{A})$), for the system (10.71) to satisfy (8.1), along with

$$fl(\widehat{P})(fl(\alpha_{2l-2,2l-1})) \cong l(\widehat{P})(l(\alpha_{2l-2,2l-1})). \qquad (10.75)$$

Thus (see Corollary 9.144), the above two isomorphic groups have the same image in the automorphism group $\mathrm{Aut}(fl(Q^*_{2l-1}))$. This, along with Remark 10.63, implies that the groups $fl(\widehat{P})(fl(\alpha_{2l-2,2l-1}))$ and $\widehat{P}(\alpha_{2l-2,2l-1})$ have the same image in $\mathrm{Aut}(fl(Q^*_{2l-1}))$.

In conclusion we get

**Theorem 10.76.** *Assume that the normal series, the character tower, the triangular set and the Hall system in* (10.48) *are fixed. Along with them we fix $S$ and $\zeta \in \mathrm{Irr}(S)$ to satisfy* (10.50)*. Let $(fl(G), fl(S), fl(\zeta), fl(Q_1), fl(\beta_1))$ be an "$\mathbf{A}(\beta_1)$"-invariant faithful linear limit of $(G, S, \zeta, Q_1, \beta_1)$ and* (10.68b) *be the character tower for the normal series* (10.68a)*, that arises as the faithful linear limit of the tower* (10.48b)*. Then we can pick the triangular set* (10.68d) *to satisfy* (10.69)*. In particular $P^*_{2r}$ is naturally isomorphic to $fl(P^*_{2r})$, for all $r = 1, \ldots, k'$. We also derive a Hall system $\{fl(\mathbf{A}), fl(\mathbf{B})\}$ of $fl(G)$ from the original $\{\mathbf{A}, \mathbf{B}\}$, via* (10.70) *and Theorem 10.60. Then for any $m = 1, \ldots, n$, the groups $fl(\widehat{Q})$ and $fl(\widehat{P})$, for the smaller faithful system* (10.71)*, can be chosen, (in association to $fl(\mathbf{B})$ and $fl(\mathbf{A})$), to satisfy*

1. *the associated isomorphism of $\mathrm{Aut}(P^*_{2k})$ onto $\mathrm{Aut}(fl(P^*_{2k}))$ sends the image of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P^*_{2k})$ onto that of $fl(\widehat{Q})(fl(\beta_{2k-1,2k}))$ in $\mathrm{Aut}(fl(P^*_{2k}))$, i.e.,*

$$fl(I) \cong I.$$

2. *$\widehat{P}(\alpha_{2l-2,2l-1}) \cong fl(\widehat{P})(fl(\alpha_{2l-2,2l-1}))$, and they both have the same image in $\mathrm{Aut}(fl(Q^*_{2l-1}))$.*

*Proof.* Follows from Theorem 10.60, and equations (10.71d), (10.74) and (10.75). ☐

Furthermore, Theorems 9.146 and 9.147, along with Theorem 10.67, easily imply

**Theorem 10.77.** *If the character $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$, then the character $fl(\beta_{2k-1,2k}) \in \mathrm{Irr}(fl(Q_{2k-1,2k}))$ extends to $fl(\widehat{Q})(fl(\beta_{2k-1,2k}))$. Similarly, if the character $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$, then the irreducible character $fl(\alpha_{2l-2,2l-1})$ of $fl(P_{2l-2,2l-1})$ extends to $fl(\widehat{P})fl(\alpha_{2l-2,2l-1})$.*

Finally, the group $K = \mathrm{Ker}(l(\zeta))$ is a normal subgroup of $l(G)$, as $l(\zeta)$ is $l(G)$-invariant. Furthermore, $K \leq l(Q_1) \leq l(Q^*_{2l-1})$, thus $K \trianglelefteq l(Q^*_{2l-1})$. Since the group $l(U)$ is isomorphic to a section of $l(G)$, and $K$ is a subgroup of $l(Q^*_{2l-1}) \leq l(U)$, we conclude that $K$ is also a normal subgroup of $l(U)$. Hence we can form the faithful linear quintuple

$$(l(U)/K, l(S)/K, l(\zeta)/K, l(Q^*_{2l-1})/K, l(\beta^*_{2l-1})/K) = (fl(U), fl(S), fl(\zeta), fl(Q^*_{2l-1}), fl(\beta^*_{2l-1}). \tag{10.78a}$$

We call the above quintuple a *G-associate faithful linear limit* of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$. The fact that $K$ is a $\pi'$-normal subgroup of $l(Q^*_{2l-1})$, while $J$ is the image of the $\pi$-group $\widehat{P}(\alpha_{2l-2,2l-1}) = l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$ in $\mathrm{Aut}(Q^*_{2l-1})$, along with (10.66b), implies

$$fl(U) = l(U)/K = (l(Q^*_{2l-1})/K) \rtimes J = fl(Q^*_{2l-1}) \rtimes J. \tag{10.78b}$$

Corollary 10.15 clearly implies

**Proposition 10.79.** *Let $(fl(Q^*_{2l-1}) \rtimes J, fl(S), fl(\zeta), fl(Q^*_{2l-1}), fl(\beta^*_{2l-1})$ be a G-associate faithful linear limit of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$. Then any faithful linear limit of the former quintuple is also a faithful linear limit of $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$.*

### 10.3.2 "$\mathbf{B}(\alpha_2)$"-invariant linear reductions

Assume that the normal series (10.48a), its character tower (10.48b) and the triangular set (10.48c) are fixed. In addition, we assume that $G_2$ is a direct product

$$G_2 = G_{2,\pi} \times G_{2,\pi'} \tag{10.80a}$$

while

$$\chi_1 \text{ is } G\text{-invariant}, \tag{10.80b}$$

that is, (9.61) holds. Hence (9.63) holds for the triangular set (10.48c). In particular we have

$$G_2 = P_2 \times G_1 = P_2 \times Q_1, \tag{10.80c}$$

$$\chi_2 = \alpha_2 \times \beta_1, \tag{10.80d}$$

$$G(\chi_2) = G(\alpha_2). \tag{10.80e}$$

Furthermore, we assume that the normal subgroup $R$ of $G$ and its irreducible character $\eta \in \mathrm{Irr}(R)$, satisfy (9.70), that is,

$$R \trianglelefteq G \text{ with } R \leq P_2, \tag{10.80f}$$

$$\eta \in \mathrm{Lin}(R) \text{ is } G\text{-invariant and lies under } \alpha_2. \tag{10.80g}$$

The quintuple $(G, R, \eta, P_2, \alpha_2)$ is clearly a linear one. As with the "$\mathbf{A}(\beta_1)$"-invariant linear reductions, we will get a linear limit of the above quintuple with respect to the group $\mathbf{B}(\alpha_2) = \mathbf{B}(\chi_2)$.

To get a linear reduction of $(G, R, \eta, P_2, \alpha_2)$ we start with a normal subgroup $M$ of $G$ contained in $P_2$ and a linear character $\mu_1$ of $M$ that extends $\eta$ and lies under $\alpha_2$. Note that all the hypothesis of Section 9.2 are satisfied, and therefore all the results of that section hold. Thus, according to Remark 9.72, there exists a $P_2$-conjugate $\mu \in \mathrm{Lin}(M)$ of $\mu_1$, such that $\mu$ is $\mathbf{B}(\alpha_2)$-invariant, extends $\eta$, and lies under $\alpha_2$. We proceed using the same notation as that of Section 9.2. As in (9.78d) we form the series

$$1 = G_{0,\mu} \trianglelefteq G_{1,\mu} \trianglelefteq \cdots \trianglelefteq G_{n,\mu} = G_\mu, \tag{10.81a}$$

consisting of the stabilizer of $\mu$ in the groups $G_i$ and $G$ for $i = 2, 3, \ldots, n$. In addition (see (9.79)), we write

$$G_{1,\mu} = 1, \tag{10.81b}$$

$$G_{2,\mu} = P_2(\mu). \tag{10.81c}$$

Along with that we get, as in (9.80) and (9.81a), the $\mu$-character tower $\{\chi_{i,\mu}\}_{i=0}^{n}$, where

$$\chi_{0,\mu} = 1, \tag{10.81d}$$

$$\chi_{1,\mu} = 1, \tag{10.81e}$$

$$\chi_{2,\mu} = \alpha_{2,\mu}. \tag{10.81f}$$

Furthermore, $\alpha_{2,\mu}$ and $\chi_{i,\mu}$ are the $\mu$-Clifford correspondents of $\alpha_2$ and $\chi_i$, respectively, for all $i = 3, \ldots, n$. Proposition 9.87 and Theorem 9.91 show that we can choose a triangular set $\{Q_{1,\mu} = 1, Q_{2i-1,\mu}, P_{2r,\mu} | \beta_{1,\mu} = 1, \beta_{2i-1,\mu}, \alpha_{2r,\mu}\}_{i=2,r=0}^{l',k'}$, that corresponds to the above $\mu$-character tower, so that $P^*_{2r,\mu} = P^*_{2r}(\mu)$, while $Q^*_{2i-1,\mu} = Q^*_{2i-1}$, whenever $1 \leq r \leq k'$ and $1 \leq i \leq l'$. In addition, Theorem 9.100 implies that the $\mu$-Hall system $\{\mathbf{A}_\mu, \mathbf{B}_\mu\}$ for $G_\mu$ can be chosen to satisfy (9.101). Then $\mathbf{B}_\mu(\chi_{1,\mu}, \chi_{2,\mu}) = \mathbf{B}(\chi_1, \chi_2)$. As $\chi_2 = \alpha_2 \times \chi_1$, where $\chi_1$ is $G$-invariant and $\chi_{2,\mu} = \alpha_{2,\mu}$, we conclude that

$$\mathbf{B}_\mu(\alpha_{2,\mu}) = \mathbf{B}_\mu(\chi_{2,\mu}) = \mathbf{B}(\chi_1, \chi_2) = \mathbf{B}(\alpha_2). \tag{10.82}$$

We assume fixed the smaller system (10.49). In addition, we assume that $m$ is any integer so that

$$m \geq 2.$$

Then Theorems 9.102 and 9.103 hold for this smaller system. Hence the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$, along with their $\mu$-correspondents $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu})$ and $\widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu})$, can be chosen to satisfy

$$\widehat{Q}(\beta_{2k-1,2k}) = \widehat{Q}_\mu(\beta_{2k-1,2k,\mu}), \tag{10.83a}$$

$$\widehat{P}(\alpha_{2l-2,2l-1}, \mu) = \widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu}), \tag{10.83b}$$

$$\mathcal{Q} = \mathcal{Q}_\mu, \tag{10.83c}$$

$$\mathcal{P}(\mu) \geq \mathcal{P}_\mu. \tag{10.83d}$$

Equation (10.82), along with Remark 9.74, implies

$$\widehat{Q}(\beta_{2k-1,2k}) \leq \mathcal{Q} \leq \mathbf{B}(\chi_2) = \mathbf{B}_\mu(\alpha_{2,\mu}). \tag{10.83e}$$

Furthermore, Corollary 9.104 implies that $\widehat{Q}_\mu(\beta_{2k-1,2k,\mu})$ and $\widehat{Q}(\beta_{2k-1,2k})$ have the same images in both $\mathrm{Aut}(P^*_{2k})$ and $\mathrm{Aut}(P^*_{2k,\mu})$. Similarly Corollary 9.107 implies that the image $J$ of $\widehat{P}(\alpha_{2l-2,2l-1})$ in $\mathrm{Aut}(Q^*_{2l-1})$ equals the image $J_\mu$ of $\widehat{P}_\mu(\alpha_{2l-2,2l-1,\mu})$ in $\mathrm{Aut}(Q^*_{2l-1,\mu}) = \mathrm{Aut}(Q^*_{2l-1})$. The quintuple $(G_\mu, M, \mu, P_{2,\mu}, \alpha_{2,\mu})$ is clearly a linear reduction of $(G, R, \eta, P_2, \alpha_2)$. We call it a "$\mathbf{B}(\alpha_2)$"*-invariant linear reduction,* as $\mu$ is $\mathbf{B}(\alpha_2)$-invariant.

Similarly to the group $U$, we write $T$ for group

$$T = P^*_{2k} \rtimes I \tag{10.84}$$

where (as always)

$$I = \text{ Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(P^*_{2k}).$$

It is clear that the quintuple $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$ is a linear one. Furthermore, the "$\mathbf{B}(\alpha_2)$"-invariant linear reduction $(G_\mu, M, \mu, P_{2,\mu}, \alpha_{2,\mu})$ of $(G, R, \eta, P_2, \alpha_2)$ determines naturally a linear reduction $(T(\mu), M, \mu, P^*_{2k,\mu}, \alpha^*_{2k,\mu})$ of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$. Note that, as $\widehat{Q}(\beta_{2k-1,2k})$ fixes $\mu$ by (10.83e), its image $I$ in $\mathrm{Aut}(P^*_{2k})$ also fixes $\mu$. As $P^*_{2k,\mu} = P^*_{2k}(\mu)$, we conclude that the stabilizer $T(\mu)$ of $\mu$ in $T$ satisfies $T(\mu) = P^*_{2k,\mu} \rtimes I$. We call such a reduction a *$G$-associate linear reduction* of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$, as the group $M$ we are choosing for this reduction is normal in $G$ We also remark that $T$ and $T_\mu$ are isomorphic to a section of $G$ and $G_\mu$, respectively.

Now we can repeat the procedure. So assume that there exists another pair $(M', \mu'_1)$, such that $M'$ is a normal subgroup of $G_\mu$ satisfying $R \leq M \leq M' \leq P_{2,\mu}$, and $\mu'_1 \in \mathrm{Lin}(M')$ is an extension of $\mu$, and thus an extension of $\eta$, that lies under $\alpha_{2,\mu}$. Again, using Remark 9.72, we can replace $\mu'_1$ with a $P_{2,\mu}$-conjugate $\mu'$ of $\mu'_1$ that is $\mathbf{B}_\mu(\alpha_{2,\mu})$-invariant, extends $\mu$, and lies under $\alpha_{2,\mu}$. (So $\mu'$ is $\mathbf{B}(\alpha_2) = \mathbf{B}_\mu(\alpha_{2,\mu})$-invariant, by (10.82)). We apply the results of Section 9.2 to the series $G_{0,\mu} = 1 \trianglelefteq G_{1,\mu} = 1 \trianglelefteq G_{2,\mu} = P_{2,\mu} \trianglelefteq G_{3,\mu} \trianglelefteq \cdots \trianglelefteq G_{n,\mu} = G_\mu$, its character tower $\{\chi_{i,\mu}\}_{i=0}^n$, and the triangular set $\{Q_{2i-1,\mu}, P_{2r,\mu} | \beta_{2i-1,\mu}, \alpha_{2r,\mu}\}_{i=1,r=0}^{l',k'}$, already picked at the previous reduction. We also use the normal subgroup $M$ of $G_\mu$ in the place of $R$, the $G_\mu$-invariant character $\mu$ in the place of $\eta$, and the normal subgroup $M'$ of $G_\mu$ in the place of $M$. Notice that (9.70) holds, with $M'$ here in the place of $M$ there, and $M$ here in the place of $R$ there. Furthermore, the group $G_{2,\mu}$ splits trivialy as the product $G_{2,\mu} \times 1$ of a $\pi$-and a $\pi'$-group. Thus the conditions (9.61) and (9.70) are satisfied. Hence all the results of Section 9.2 hold. In particular, we have a normal series

$$1 = G_{0,\mu,\mu'} \trianglelefteq G_{1,\mu,\mu'} \trianglelefteq G_{2,\mu,\mu'} \trianglelefteq \cdots \trianglelefteq G_{n,\mu,\mu'} = G_{\mu,\mu'},$$

of the stabilizer $G_{\mu,\mu'} = G(\mu, \mu')$ of $\mu'$ in $G_\mu = G(\mu)$. In addition,

$$G_{1,\mu,\mu'} = G_{1,\mu} = 1, \tag{10.85a}$$
$$G_{2,\mu,\mu'} = P_2(\mu, \mu'). \tag{10.85b}$$

and

$$G_{i,\mu,\mu'} = G_{i,\mu} \cap G_{\mu,\mu'} = G_i(\mu, \mu'),$$

for all $i = 2, 3, \ldots, n$. We also get a character tower $\{\chi_{i,\mu,\mu'}\}_{i=0}^n$ for that series, where $\chi_{i,\mu,\mu'}$ is the $\mu'$-Clifford correspondent of $\chi_{i,\mu}$, for each $i = 2, \ldots, n$. Furthermore, as in (10.81), we have

$$\chi_{1,\mu,\mu'} = 1, \tag{10.85c}$$
$$\chi_{2,\mu,\mu'} = \alpha_{2,\mu,\mu'}, \tag{10.85d}$$

where $\alpha_{2,\mu,\mu'}$ is the $\mu'$-Clifford correspondent of $\alpha_{2,\mu}$.

Hence Proposition 9.87, Theorems 9.91, 9.102 and 9.103, along with their Corollaries 9.104 and 9.107, imply that we can pick a triangular set $\{Q_{2i-1,\mu,\mu'}, P_{2r,\mu,\mu'} | \beta_{2i-1,\mu,\mu'}, \alpha_{2r,\mu,\mu'}\}_{i=1,r=0}^{l',k'}$ that corresponds to the character tower $\{\chi_{\mu,\mu'}\}_{i=0}^n$, so that

$$P^*_{2k,\mu,\mu'} = P^*_{2k,\mu}(\mu') = P^*_{2k}(\mu, \mu') = P^*_{2k} \cap G_{\mu,\mu'}, \tag{10.86a}$$
$$Q^*_{1,\mu,\mu'} = Q^*_{1,\mu} = Q_{1,\mu} = 1, \tag{10.86b}$$
$$Q^*_{2l-1,\mu,\mu'} = Q^*_{2l-1,\mu} = Q^*_{2l-1}. \tag{10.86c}$$

We also pick a Hall system $\{\mathbf{A}_{\mu,\mu'}, \mathbf{B}_{\mu,\mu'}\}$ of $G_{\mu,\mu'}$, that satisfies the conditions in Theorem 9.100 and is derived from $\{\mathbf{A}_\mu, \mathbf{B}_\mu\}$. So it is derived from the original $\{\mathbf{A}, \mathbf{B}\}$. Therefore, for any fixed $m = 1, \ldots, n$, the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{P}(\alpha_{2l-2,2l-1})$ for the smaller system (10.49), can be chosen via $\mathbf{B}_{\mu,\mu'}$ and $\mathbf{A}_{\mu,\mu'}$, respectively, (see Theorems 8.13 and 8.15). Hence, as in Theorem 9.102 and 9.103, they satisfy

$$\widehat{Q}_{\mu,\mu'}(\beta_{2k-1,2k,\mu,\mu'}) = \widehat{Q}_{\mu}(\beta_{2k-1,2k,\mu}) = \widehat{Q}(\beta_{2k-1,2k}), \tag{10.86d}$$
$$\widehat{P}_{\mu,\mu'}(\alpha_{2l-2,2l-1,\mu,\mu'}) = \widehat{P}(\alpha_{2l-2,2l-1})(\mu,\mu'), \tag{10.86e}$$
$$\mathcal{Q}_{\mu,\mu'} = \mathcal{Q}_{\mu} = \mathcal{Q}, \tag{10.86f}$$
$$\mathcal{P}_{\mu,\mu'} \leq \mathcal{P}(\mu,\mu'). \tag{10.86g}$$

Furthermore,

$$\widehat{Q}(\beta_{2k-1,2k}) \text{ is a subgroup of } \mathbf{B}(\alpha_2) = \mathbf{B}_{\mu}(\alpha_{2,\mu}) = \mathbf{B}_{\mu,\mu'}(\alpha_{2,\mu,\mu'}), \tag{10.86h}$$
$$G_{2,\mu,\mu'} = P_{2,\mu,\mu'}. \tag{10.86i}$$

Equation (10.86d) implies that the groups $\widehat{Q}(\beta_{2k-1,2k})$ and $\widehat{Q}_{\mu,\mu'}(\beta_{2k-1,2k,\mu,\mu'})$ have the same image in $\mathrm{Aut}(P^*_{2k,\mu,\mu'})$. Also, (10.86c), along with Corollary 9.107, implies that $J_\mu = J_{\mu,\mu'}$, where $J_{\mu,\mu'}$ denotes the image of $\widehat{P}_{\mu,\mu'}(\alpha_{2l-2,2l-1,\mu,\mu'})$ in $\mathrm{Aut}(Q^*_{2l-1,\mu,\mu'})$. So

$$J = J_\mu = J_{\mu,\mu'}. \tag{10.87}$$

As far as the linear reductions are concerned, we have that $(G_{\mu,\mu'}M', \mu', P_{2,\mu,\mu'}, \alpha_{2,\mu,\mu'})$ is a "$\mathbf{B}(\alpha_2)$"-invariant linear reduction of $(G_\mu, M, \mu, P_{2,\mu}, \alpha_{2,\mu})$. Furthermore, the reduced quintuple $(T(\mu,\mu'), M', \mu', P^*_{2k,\mu,\mu'}, \alpha^*_{2k,\mu,\mu'})$ is a $G_\mu$-associate linear reduction of $(T(\mu), M, \mu, P^*_{2k,\mu}, \alpha^*_{2k,\mu})$. Note that

$$T(\mu,\mu') = (P^*_{2k} \rtimes I)(\mu,\mu') = P^*_{2k,\mu,\mu'} \rtimes I, \tag{10.88}$$

as both $\mu$ and $\mu'$, are $\mathbf{B}(\alpha_2) \geq \widehat{Q}(\beta_{2k-1,2k})$-invariant, (by (10.86h)), and thus $I$-invariant.

We continue this process until we reach a linear limit

$$(l(G), l(R), l(\eta), l(P_2), l(\alpha_2)) \in LL(G, R, \eta, P_2, \alpha_2), \tag{10.89}$$

that we call a "$\mathbf{B}(\alpha_2)$"*-invariant linear limit* of the linear quintuple $(G, R, \eta, P_2, \alpha_2)$. We also reach a limit normal series for the group $l(G)$

$$1 = l(G_0) \trianglelefteq l(G_1) \trianglelefteq l(G_2) \trianglelefteq \cdots \trianglelefteq l(G_n) = l(G), \tag{10.90a}$$

where $l(G_i) = G_i \cap l(G)$, for all $i = 2, \dots, n$, and

$$l(G_1) = 1, \tag{10.90b}$$
$$l(G_2) = l(P_2), \tag{10.90c}$$

as the same holds at every linear reduction. Observe also that the above normal series has the same notation as the one in (10.56), but of course is produced in a different way.

Along with the series (10.90a) we get a character tower

$$\{l(\chi_i) \in \mathrm{Irr}(l(G_i))\}_{i=0}^n \tag{10.90d}$$

where

$$\begin{aligned}
&l(\chi_1) = 1 = l(\chi_0),\\
&l(\chi_2) = l(\alpha_2),\\
&l(\chi_i) \in LL(\chi_i) \text{ is a linear limit of } \chi_i
\end{aligned} \tag{10.90e}$$

for all $i = 3, \dots, n$. Let

$$\{l(Q_{2i-1}), l(P_{2r}) | l(\beta_{2i-1}), l(\alpha_{2r})\}_{i=1,r=0}^{l',k'} \tag{10.90f}$$

be the representative of the unique $l(G)$-conjugate class that corresponds to (10.90d), and is derived from the original triangular set (10.48c) following the rules in Theorem 9.91. . We also denote by

$$\{l(\mathbf{A}), l(\mathbf{B})\} \tag{10.90g}$$

a Hall system for $l(G)$ that satisfies (8.1), for the above limit case.

Of course the above system restricts to the smaller

$$\begin{gathered}
1 = l(G_0) \trianglelefteq l(G_1) \trianglelefteq l(G_2) \trianglelefteq \cdots \trianglelefteq l(G_m) \trianglelefteq l(G),\\
\{l(\chi_i) \in \mathrm{Irr}(l(G_i))\}_{i=0}^{m}\\
\{l(Q_{2i-1}), l(P_{2r}) | l(\beta_{2i-1}), l(\alpha_{2r})\}_{i=1,r=0}^{l,k}
\end{gathered} \tag{10.91a}$$

Note that we have the same notation as that in (10.59). Similar to the notation there, we write $l(P^*_{2k})$ and $l(Q^*_{2l-1})$ for the product groups $l(P_0) \cdot l(P_2) \cdots l(P_{2k})$ and $l(Q_1) \cdots l(Q_{2l-1})$, respectively. Also for any fixed $m$, we denote by $l(\widehat{Q})$ the analogue of $\widehat{Q}$ in this limit case, and by $l(\widehat{P})$ the analogue of $\widehat{P}$, for the smaller system (10.91) i.e., $l(\widehat{Q})$ and $l(\widehat{P})$, satisfy the conditions in Theorems 6.19 and 8.1, respectively, for the limit case.

Then

**Theorem 10.92.** *Assume that the normal series, the character tower, the triangular set and the Hall system in* (10.48) *satisfy the conditions* (10.80). *Assume further that* (10.89) *is a "$\mathbf{B}(\alpha_2)$"-invariant linear limit of* $(G, R, \eta, P_2, \alpha_2)$ *and* (10.90a) *a character tower that arises as a linear limit of* (10.48b) *(see* (10.90e)*). Then the triangular set* (10.90f)*, that corresponds to the tower* (10.90a)*, can be chosen to satisfy*

$$l(Q^*_{2i-1}) = Q^*_{2i-1}, \tag{10.93a}$$

*for all $i = 1, \dots, l'$. Furthermore, a Hall system $\{l(\mathbf{A}), l(\mathbf{B})\}$ for $l(G)$, can be derived from $\{\mathbf{A}, \mathbf{B}\}$ so that at every linear reduction theconditions in Theorem 9.100 hold. Then, for every $m = 1, \dots, n$, the groups $l(\widehat{Q})(l(\beta_{2k-1,2k}))$ and $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$ for the smaller system* (10.91)*, can be chosen, using the groups $l(\mathbf{B})$ and $l(\mathbf{A})$, respectively, to satisfy*

$$\widehat{Q}(\beta_{2k-1,2k}) = l(\widehat{Q})(l(\beta_{2k-1,2k})), \tag{10.93b}$$

$$\mathcal{Q} = l(\mathcal{Q}), \tag{10.93c}$$

$$J = l(J), \tag{10.93d}$$

*where $l(J)$ is the image of $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$ in* $\mathrm{Aut}(l(Q^*_{2l-1}))$.

*Proof.* We reach the linear limit (10.89) doing, at every step, "$\mathbf{B}(\alpha_2)$"-invariant linear reductions. Therefore at every step we are picking a triangular set that satisfies the conditions in Proposition 9.87 and Theorem 9.91. We also pick, at every linear redution, a Hall system that satisfies theconditions in Theorem 9.100. Furthermore, for any fixed $m = 1, \ldots, n$, the groups $\widehat{Q}$ and $\widehat{P}$ satisfy the conditions in Theorems 9.102 and 9.103. Hence at every step equations (10.86) hold. In particular, repeated applications of (10.86c), (10.86d) and (10.86f) imply (10.93a,b) and (10.93c), respectively. Similarly, repeated applications of (10.87) imply (10.93d). Hence Theorem 10.92 follows. $\square$

As an easy consequence of (10.93a) we have

**Remark 10.94.** The groups $\widehat{Q}(\beta_{2k-1,2k})$ and $l(\widehat{Q})(l(\beta_{2k-1,2k}))$ have the same images in both automorphism groups $\mathrm{Aut}(P^*_{2k})$ and $\mathrm{Aut}(l(P^*_{2k}))$.

Also repeated applications of Theorems 9.108 and 9.109 at every "$\mathbf{B}(\alpha_2)$"-invariant linear reduction imply

**Theorem 10.95.** *If $\beta_{2k-1,2k}$ extends to $\widehat{Q}(\beta_{2k-1,2k})$, then the character $l(\beta_{2k-1,2k})$ also extends to the limit group $l(\widehat{Q})(l(\beta_{2k-1,2k}))$. Similarly, if $\alpha_{2l-2,2l-1}$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$ then $l(\alpha_{2l-2,2l-1})$ also extends to $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$.*

Notice that, along with the limit in (10.89), we reach the quintuple

$$(l(T), l(R), l(\eta), l(P^*_{2k}), l(\alpha^*_{2k})), \tag{10.96}$$

that we call a *$G$-associate limit* of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$. Note that, as with (10.64), the $G$-associate limit is a multiple linear reduction, but not a linear limit, of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$. Because (10.88) holds for every "$\mathbf{B}(\alpha_2)$"-invariant linear reduction, we have

**Proposition 10.97.** *The $G$-associate linear limit $(l(T), l(R), l(\eta), l(P^*_{2k}), l(\alpha^*_{2k}))$ of the quintuple $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$, satisfies*

$$l(P^*_{2k}) = l(G) \cap P^*_{2k}, \tag{10.98a}$$
$$l(T) = l(P^*_{2k}) \rtimes I. \tag{10.98b}$$

We want to pass to a faithful linear limit of $(G, R, \eta, P_2, \alpha_2)$, as we did with the "$\mathbf{A}(\beta_1)$"-invariant case in (10.68). So we first note that $l(G_2) = l(P_2)$. So $l(G_2)$ is the product of a $\pi$-and a trivial $\pi'$-group. Furthermore, $l(R)$ is a normal subgroup of $l(G)$, while $l(\eta) \in \mathrm{Lin}(l(R))$ is an $l(G)$-invariant linear character that lies under $l(\alpha_2) \in \mathrm{Irr}(l(P_2))$, as $(l(G), l(R), l(\eta), l(P_2), l(\alpha_2))$ is a linear quintuple. Hence all the conditions of Section 9.3.2 are satisfied. Thus if $K = \mathrm{Ker}(l(\eta))$ is the kernel of $l(\eta)$, then we can form the faithful linear limit

$$(fl(G), fl(R), fl(\eta), fl(P_2), fl(\alpha_2)) = (l(G)/K, l(R)/K, l(\eta)/K, l(P_2)/K, l(\alpha_2)/K)$$

of the linear quintuple $(G, R, \eta, P_2, \alpha_2)$. We call this a "$\mathbf{B}(\alpha_2)$"*-invariant faithful linear limit*, as it is obtained from a "$\mathbf{B}(\alpha_2)$"-invariant linear limit. Along with it we have a normal series of $fl(G)$, as in (9.150),

$$1 = fl(G_0) \trianglelefteq fl(G_1) \trianglelefteq fl(G_2) \trianglelefteq \cdots \trianglelefteq fl(G_n) = fl(G), \tag{10.99a}$$

where

$$fl(G_1) = l(G_1)K/K = 1, \tag{10.99b}$$
$$fl(G_2) = l(G_2)K/K = l(P_2)/K \tag{10.99c}$$
$$fl(G_i) = l(G_i)/K, \tag{10.99d}$$

for all $i = 3, \dots, n$. Along with the series (10.99a) we get a character tower, see (9.150b),

$$\{fl(\chi_i) \in \mathrm{Irr}(fl(G_i))\}_{i=0}^{n} \tag{10.99e}$$

where

$$fl(\chi_1) = 1, \tag{10.99f}$$
$$fl(\chi_2) = fl(\alpha_2), \tag{10.99g}$$
$$fl(\chi_i) \in FLL(\chi_i) \text{ is a faithful linear limit of } \chi_i \tag{10.99h}$$

for all $i = 3, \dots, n$. That is, $fl(\chi_i)$ is the unique character of $fl(G_i) = G_i/K$ that inflates to $\chi_i \in \mathrm{Irr}(G_i)$. Let

$$\{fl(Q_{2i-1}), fl(P_{2r}) | fl(\beta_{2i-1}), fl(\alpha_{2r})\}_{i=1,r=0}^{l',k'} \tag{10.99i}$$

be a representative of the unique $fl(G)$-conjugate class that corresponds to (10.99e). Then Theorem 9.155 implies that we can pick the set (10.99i) so that

$$\begin{aligned} fl(Q_{2i-1}) &= (l(Q_{2i-1})K)/K \cong l(Q_{2i-1}), \\ fl(P_{2r}) &= (l(P_{2r})K)/K. \end{aligned} \tag{10.100a}$$

whenever $1 \leq i \leq l'$ and $1 \leq r \leq k'$. Hence

$$fl(Q^*_{2l-1}) = (l(Q^*_{2l-1})K)/K \cong l(Q^*_{2l-1}), \tag{10.100b}$$
$$fl(P^*_{2k}) = (l(P^*_{2k})K)/K = l(P^*_{2k})/K. \tag{10.100c}$$

Furthermore, we can pick a Hall system $\{fl(\mathbf{A}), fl(\mathbf{B})\}$ for $fl(G)$ to satisfy Theorem 9.161, that is

$$fl(\mathbf{A}) = (l(\mathbf{A})K)/K \text{ and } fl(\mathbf{B}) = (l(\mathbf{B})K)/K \cong l(\mathbf{B}). \tag{10.101}$$

For every fixed $m = 1, \dots, n$ we have the smaller faithful limit system

$$0 = fl(G_0) \trianglelefteq fl(G_1) \trianglelefteq fl(G_2) \trianglelefteq \cdots \trianglelefteq fl(G_m) \trianglelefteq fl(G), \tag{10.102a}$$
$$\{fl(\chi_i) \in \mathrm{Irr}(fl(G_i))\}_{i=0}^{m} \tag{10.102b}$$
$$\{fl(Q_{2i-1}), fl(P_{2r}) | fl(\beta_{2i-1}), fl(\alpha_{2r})\}_{i=1,r=0}^{l,k} \tag{10.102c}$$

withn the triangular set picked so that (10.100b) holds. Furthermore, if $fl(\widehat{Q})$ denotes the corresponding to $\widehat{Q}$ for the system (10.102), then Theorem 9.162 implies that, having fixed the Hall system 10.101, we can choose $fl(\widehat{Q})$ so that

$$fl(\widehat{Q}) = (l(\widehat{Q})K)/K \cong l(\widehat{Q}).$$

Even more, Corollary 9.165 implies that

$$\begin{aligned} &fl(\widehat{Q})(fl(\beta_{2k-1,2k})) \text{ and } l(\widehat{Q})(l(\beta_{2k-1,2k})) \\ &\qquad\qquad \text{have the same image in the automorphism group } \mathrm{Aut}(fl(P^*_{2k})). \end{aligned} \tag{10.103}$$

In addition, Theorem 9.164 implies that we may choose the group $fl(\widehat{P})$ for the smaller system

(10.102) so that

$$fl(\widehat{P})(fl(\alpha_{2l-2,2l-1})) = (l(\widehat{P})(l(\alpha_{2l-2,2l-1}))K)/K. \tag{10.104a}$$

Thus, as in Corollary 9.166, identifying $l(Q^*_{2l-1})$ with the isomorphic group $fl(Q^*_{2l-1})$, we conclude that the image $fl(J)$ of $fl(\widehat{P})(fl(\alpha_{2l-2,2l-1}))$ in $\mathrm{Aut}(fl(Q^*_{2l-1}))$ is isomorphic to the image $l(J)$ of $l(\widehat{P})(l(\alpha_{2l-2,2l-1}))$ in $\mathrm{Aut}(l(Q^*_{2l-1}))$. But the latter equals $J$, by (10.93d). Hence

$$fl(J) \cong l(J) = J. \tag{10.104b}$$

In conclusion we get

**Theorem 10.105.** *Assume that the normal series, the character tower, the triangular set and the Hall system in* (10.48) *are fixed and satisfy (10.80a). Along with them we fix $R$ and $\eta \in \mathrm{Irr}(R)$ to satisfy (10.80b,c). Let $(fl(G), fl(R), fl(\eta), fl(P_2), fl(\alpha_2))$ be a "$\mathbf{B}(\alpha_2)$"-invariant faithful linear limit of $(G, R, \eta, P_2, \alpha_2)$, and* (10.99e) *be a character tower for the normal series* (10.99a)*, arising as a faithful linear limit of the tower* (10.48b)*. Then we can pick the triangular set* (10.99i) *to satisfy* (10.100)*. In particular, $Q^*_{2i-1}$ is naturally isomorphic to $fl(Q^*_{2i-1})$, for all $i = 1, \dots, l'$. We also reach a Hall system $\{fl(\mathbf{A}), fl(\mathbf{B})\}$ for $fl(G)$, from $(\mathbf{A}, \mathbf{B})$ via* (10.101) *and Theorem 10.92. Then for any $m = 1, \dots, n$, the groups $fl(\widehat{Q})$ and $fl(\widehat{P})$ for the smaller system* (10.102)*, can be chosen, with respect to $fl(\mathbf{B})$ and $fl(\mathbf{A})$ respectively, to satisfy*

1. *$\widehat{Q}(\beta_{2k-1,2k})$ is isomorphic to $fl(\widehat{Q})(fl(\beta_{2k-1,2k}))$, and they both have the same image in $\mathrm{Aut}(fl(P^*_{2k}))$.*
2. *the associated isomorphism of $\mathrm{Aut}(Q^*_{2l-1})$ onto the group of automorphisms $\mathrm{Aut}(fl(Q^*_{2l-1}))$ sends the image of $\widehat{P}(\alpha_{2l-2,2l-1})$ inside $\mathrm{Aut}(Q^*_{2l-1})$ onto the image of $fl(\widehat{P})(fl(\alpha_{2l-2,2l-1}))$ inside $\mathrm{Aut}(fl(Q^*_{2l-1}))$, i.e.,*

$$fl(J) \cong J.$$

*Proof.* We have already seen that we can pick the set (10.99i) so that (10.100) holds. The rest follows from (10.103) and (10.104). ☐

Furthermore, Theorems 9.146 and 9.168, along with Theorem 10.95, easily imply

**Theorem 10.106.** *If the character $\beta_{2k-1,2k} \in \mathrm{Irr}(Q_{2k-1,2k})$ extends to $\widehat{Q}(\beta_{2k-1,2k})$, then the character $fl(\beta_{2k-1,2k}) \in \mathrm{Irr}(fl(Q_{2k-1,2k}))$ extends to $fl(\widehat{Q})(fl(\beta_{2k-1,2k}))$. Similarly, if the character $\alpha_{2l-2,2l-1} \in \mathrm{Irr}(P_{2l-2,2l-1})$ extends to $\widehat{P}(\alpha_{2l-2,2l-1})$, then the irreducible character $fl(\alpha_{2l-2,2l-1})$ of $fl(P_{2l-2,2l-1})$ extends to $fl(\widehat{P})fl(\alpha_{2l-2,2l-1})$.*

The character $l(\eta)$ is $l(T)$-invariant and thus $I$-invariant. Furthermore, $K = \mathrm{Ker}(l(\eta))$ is a subgroup of $l(P_2) \leq P_2$. Hence equation (10.98b) implies

$$fl(T) = l(T)/K = (l(P^*_{2k})/K) \rtimes I = fl(P^*_{2k}) \rtimes I.$$

So we can form the quintuple

$$(fl(T) = fl(P^*_{2k}) \rtimes I, fl(R), fl(\eta), fl(P^*_{2k}), fl(\alpha^*_{2k})), \tag{10.107}$$

that we call a $G$-associate faithful linear limit of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$. Corollary 10.15 clearly implies

**Proposition 10.108.** *Any faithful linear limit of $(fl(T), fl(R), fl(\eta), fl(P^*_{2k}), fl(\alpha^*_{2k}))$ is also a faithful linear limit of $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$.*

# Chapter 11

# Main Theorem

## 11.1 An outline of the proof

We start with a monomial group $G$ of order $p^a q^b$, for distinct odd primes $p, q$ and integers $a, b \geq 0$. Of course $G$ is solvable. Hence there exists some chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{11.1a}$$

of normal subgroups $G_i$ of $G$ that satisfy Hypothesis 5.1. So $G_1$ is a $q$-group, while $G_i/G_{i-1}$ is a $p$-group if $i$ is even, and a $q$-group if $i$ is odd, for each $i = 2, \ldots, n$. Let $\chi_0, \chi_1, \ldots, \chi_n$ satisfy

$$\chi_i \in \mathrm{Irr}(G_i) \text{ lies under } \chi_j \in \mathrm{Irr}(G_j) \tag{11.1b}$$

for any $i, j = 0, 1, 2, \ldots, n$ with $i \leq j$, i.e., the $\chi_i$ form a character tower for the series (11.1a). Assume further that the integers $k'$ and $l'$ are related to $n$ via (5.7), while the set

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l',k'} \tag{11.1c}$$

is a representative of the unique conjugacy class of triangular sets that correspond to (11.1b).

We fix a Sylow system $\{\mathbf{A}, \mathbf{B}\}$ of $G$ satisfying (8.4) with $\pi = \{p\}$, that is,

$$\mathbf{A} \in \mathrm{Syl}_p(G) \text{ and } \mathbf{B} \in \mathrm{Syl}_q(G), \tag{11.1d}$$

$$\mathbf{A}(\chi_1, \chi_2, \ldots, \chi_i) \in \mathrm{Syl}_p(G(\chi_1, \chi_2, \ldots, \chi_i)) \text{ and } \mathbf{B}(\chi_1, \chi_2, \ldots, \chi_i) \in \mathrm{Syl}_q(G(\chi_1, \chi_2, \ldots, \chi_i)), \tag{11.1e}$$

$$\mathbf{A}(\chi_1, \ldots, \chi_n) = P^*_{2k'} = P_2 \cdot P_4 \cdots P_{2k'} \text{ and } \mathbf{B}(\chi_1, \ldots, \chi_n) = Q^*_{2l'-1} = Q_1 \cdot Q_3 \cdots Q_{2l'-1}, \tag{11.1f}$$

for each $i = 1, 2, \ldots, n$. Therefore

$$G(\chi_1, \chi_2, \ldots, \chi_i) = \mathbf{A}(\chi_1, \chi_2, \ldots, \chi_i)\mathbf{B}(\chi_1, \chi_2, \ldots, \chi_i) \tag{11.1g}$$

for $i = 1, 2, \ldots, n$.

We are going to perform a series of linear reductions of a very special form. We set $S = 1$, and let $\zeta$ be the trivial linear character of $S$. We also set $R = 1$, and let $\eta$ be the trivial character of $R$. Then $(G, S, \zeta, G_1, \chi_1)$ is a quintuple satisfying (10.50). So we may pass to an arbitrary "$\mathbf{A}(\beta_1)$"-invariant faithful linear limit $(\mathbb{G}^{(1)}, \mathbb{S}^{(1)}, \boldsymbol{\zeta}^{(1)}, \mathbb{G}_1^{(1)}, \Theta_1^{(1)})$ of the quintuple $(G, S, \zeta, G_1, \chi_1)$.

Of course along with that we obtain (see (10.68)) a normal series

$$1 \trianglelefteq \mathbb{Q}_1^{(1)} = \mathbb{G}_1^{(1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(1)} = \mathbb{G}^{(1)}, \tag{11.2}$$

of $\mathbb{G}^{(1)}$, from the series (11.1a). In addition we reach a tower of characters $\Theta_i^{(1)} \in \mathrm{Irr}(\mathbb{G}_i^{(1)})$, for $i = 0, 1, \ldots, n$, such that $\Theta_0^{(1)} = 1$ and $\Theta_i^{(1)}$ lies under $\Theta_j^{(1)}$ and above $\boldsymbol{\zeta}^{(1)}$, whenever $1 \leq i \leq j \leq n$. We also get a triangular set

$$\{\mathbb{Q}_{2i-1}^{(1)}, \mathbb{P}_{2r}^{(1)} | \boldsymbol{\beta}_{2i-1}^{(1)}, \boldsymbol{\alpha}_{2r}^{(1)}\}_{i=1,r=0}^{k',l'} \tag{11.3}$$

for (11.2) that corresponds uniquely to the tower $\{\Theta_i^{(1)}\}_{i=0}^n$ and satisfies Theorem 10.76. Furthermore, the original Sylow system $\{\mathbf{A}, \mathbf{B}\}$ for $G$, (see (11.1d)), provides a Sylow system $\{\mathbf{A}^{(1)}, \mathbf{B}^{(1)}\}$ for $\mathbb{G}^{(1)}$, that also satisfies Theorem 10.76.
Obviously the trivial group $R$ is still a subgroup of $\mathbb{P}_2^{(1)}$, and its trivial character $\eta$ lies under $\boldsymbol{\alpha}_2^{(1)}$. We denote by $\mathbb{R}^{(1)} = 1 = R$ the trivial group, seen inside $\mathbb{P}_2^{(1)}$, and by $\boldsymbol{\eta}^{(1)} \in \mathrm{Irr}(\mathbb{R}^{(1)})$ its trivial character. Hence

$$1 \times \mathbb{S}^{(1)} = \mathbb{R}^{(1)} \times \mathbb{S}^{(1)} \text{ is a central subgroup of } \mathbb{G}^{(1)} \tag{11.4a}$$

$$1 \times \boldsymbol{\zeta}^{(1)} = \boldsymbol{\eta}^{(1)} \times \boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{R}^{(1)} \times \mathbb{S}^{(1)}). \tag{11.4b}$$

If $n \geq 2$ then in addition we have

$$\mathbb{R}^{(1)} \times \mathbb{S}^{(1)} \trianglelefteq \mathbb{P}_2^{(1)} \cdot \mathbb{Q}_1^{(1)} = \mathbb{G}_2^{(1)} \tag{11.4c}$$

$$\boldsymbol{\eta}^{(1)} \text{ and } \boldsymbol{\zeta}^{(1)} \text{ lie under } \boldsymbol{\alpha}_2^{(1)} \text{ and } \boldsymbol{\beta}_1^{(1)}, \text{ respectively .} \tag{11.4d}$$

Notice that, as $(\mathbb{G}^{(1)}, \mathbb{S}^{(1)}, \boldsymbol{\zeta}^{(1)}, \mathbb{G}_1^{(1)}, \Theta_1^{(1)})$ is a faithful linear limit of $(G, S, \zeta, G_1, \chi_1)$, Corollaries 10.9 and 10.10 imply

**Remark 11.5.** $\mathbb{S}^{(1)} = Z(\mathbb{G}_1^{(1)})$ is a cyclic central subgroup of $\mathbb{G}^{(1)}$, maximal among the abelian $\mathbb{G}^{(1)}$-invariant subgroups of $\mathbb{G}_1^{(1)}$. Furthermore, the character $\boldsymbol{\beta}_1^{(1)} = \Theta_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant.

Note also that Corollary 10.15 easily implies

**Remark 11.6.** Any faithful linear limit of $(\mathbb{G}^{(1)}, \mathbb{S}^{(1)}, \boldsymbol{\zeta}^{(1)}, \mathbb{G}_i^{(1)}, \Theta_i^{(1)})$ is also a faithful linear limit of $(G, 1, 1, G_i, \chi_i)$, for all $i = 1, 2, \ldots, n$.

As far as the monomial characters of $G$ are concerned we have

**Proposition 11.7.** *Any character $\Theta^{(1)} \in \mathrm{Irr}(\mathbb{G}^{(1)})$ that lies above $1 \times \boldsymbol{\zeta}^{(1)} = \boldsymbol{\eta}^{(1)} \times \boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{R}^{(1)} \times \mathbb{S}^{(1)})$ is monomial.*

*Proof.* Let $\Theta^{(1)}$ be an irreducible character of $\mathbb{G}^{(1)}$ that lies above $\boldsymbol{\eta}^{(1)} \times \boldsymbol{\zeta}^{(1)}$, and thus above $\boldsymbol{\zeta}^{(1)}$. According to Lemma 10.11, there exists an irreducible character $\chi \in \mathrm{Irr}(G)$, that lies above $\zeta = 1$ and satisfies $\Theta^{(1)} = fl(\chi)$, that is, $\Theta^{(1)}$ is the faithful linear limit of $\chi$. But $\chi$ is monomial, as $G$ is a monomial group. Therefore, Proposition 10.18 implies that $\Theta^{(1)}$ is also monomial. $\square$

The first critical result, which we will prove in Section 11.3, is

**Theorem 11.8.** *After the above reduction, the group $\mathbb{G}_2^{(1)}$ is nilpotent, if it exists, i.e., if $n \geq 2$.*

This theorem implies that $\mathbb{G}_2^{(1)}$ is the direct product

$$\mathbb{G}_2^{(1)} = \mathbb{P}_2^{(1)} \times \mathbb{Q}_1^{(1)} = \mathbb{P}_2^{(1)} \times \mathbb{G}_1^{(1)} \tag{11.9a}$$

of its $p$-Sylow subgroup $\mathbb{P}_2^{(1)}$ and its $q$-Sylow subgroup $\mathbb{Q}_1^{(1)} = \mathbb{G}_1^{(1)}$. It also implies that $\Theta_2^{(1)}$ is the direct product

$$\Theta_2^{(1)} = \boldsymbol{\alpha}_2^{(1)} \times \boldsymbol{\beta}_1^{(1)} \tag{11.9b}$$

of $\boldsymbol{\alpha}_2^{(1)} \in \mathrm{Irr}(\mathbb{P}_2^{(1)})$ and $\boldsymbol{\beta}_1^{(1)} = \Theta_1^{(1)} \in \mathrm{Irr}(\mathbb{Q}_1^{(1)}) = \mathrm{Irr}(\mathbb{G}_1^{(1)})$. Therefore, in the case of $n \geq 2$, the relations in (11.4c,d) imply

$$\begin{gathered} \mathbb{R}^{(1)} \times \mathbb{S}^{(1)} \trianglelefteq \mathbb{P}_2^{(1)} \times \mathbb{Q}_1^{(1)} \\ \boldsymbol{\eta}^{(1)} \times \boldsymbol{\zeta}^{(1)} \text{ lies under } \boldsymbol{\alpha}_2^{(1)} \times \boldsymbol{\beta}_1^{(1)} = \Theta_2^{(1)} \, . \end{gathered} \tag{11.10}$$

Furthermore, the character $\boldsymbol{\beta}_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant. Hence the normal series (11.2), along with its character tower $\{\Theta_i^{(1)}\}_{i=0}^n$, satisfies (8.18) of Section 8.3. So we shift the series by one, and consider the series

$$1 \trianglelefteq \mathbb{P}_2^{(1)} \trianglelefteq \mathbb{G}_3^{(1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(1)} = \mathbb{G}^{(1)}, \tag{11.11a}$$

of normal subgroups of $\mathbb{G}^{(1)}$, that satisfies Hypothesis 5.1. Then, as we have seen in Section 8.3, and in particular Theorem 8.29, the characters

$$1, \boldsymbol{\alpha}_2^{(1)}, \Theta_3^{(1)}, \ldots, \Theta_n^{(1)}, \tag{11.11b}$$

form a tower for the above series, with corresponding triangular set

$$\{\mathbb{Q}_{2i-1}^{(1)}, \mathbb{P}_{2r}^{(1)} | \boldsymbol{\beta}_{2i-1}^{(1)}, \boldsymbol{\alpha}_{2r}^{(1)}\}_{i=2,r=1}^{l',k'}. \tag{11.11c}$$

(Note that we have dropped the first $q$ and $p$ groups, $\mathbb{Q}_1^{(1),s}$ and $\mathbb{P}_0^{(1),s}$ (see (8.20c) and (8.30c)) respectively, along with their characters, as these are assume trivial for the shifted system.) Furthermore, as Theorem 8.29 implies, the above shifted system and the one for (11.2) have in common the Sylow system $\{\mathbf{A}^{(1)}, \mathbf{B}^{(1)}\}$, i.e., this Sylow system satisfies (8.4) for (11.11).

The quintuple $(\mathbb{G}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}_2^{(1)}, \boldsymbol{\alpha}_2^{(1)})$ is clearly a linear one. Therefore we may pass to a $\mathbf{B}^{(1)}(\boldsymbol{\alpha}_2^{(1)})$-invariant faithful linear limit $(\mathbb{G}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)})$ of the former quintuple. So, (see (10.99)), the chain (11.11a) reduces to a chain

$$1 \trianglelefteq \mathbb{P}_2^{(2)} \trianglelefteq \mathbb{G}_3^{(2)} \trianglelefteq \mathbb{G}_4^{(2)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(2)} = \mathbb{G}^{(2)}, \tag{11.12a}$$

of normal subgroups $\mathbb{G}_i^{(2)}$ of $\mathbb{G}^{(2)}$. The character tower (11.11b) reduces (see (10.99e)) to the tower

$$\{1, \boldsymbol{\alpha}_2^{(2)}, \Theta_i^{(2)}\}_{i=3}^n \tag{11.12b}$$

where $\boldsymbol{\alpha}_2^{(2)} \in \mathrm{Irr}(\mathbb{P}_2^{(2)})$ and $\Theta_i^{(2)} \in \mathrm{Irr}(\mathbb{G}_i^{(2)})$ , for all $i = 3, \ldots, n$. According to the Theorem 10.105, the Sylow system $\{\mathbf{A}^{(1)}, \mathbf{B}^{(1)}\}$ for $\mathbb{G}^{(1)}$ reduces to a Sylow system $\{\mathbf{A}^{(2)}, \mathbf{B}^{(2)}\}$ for $\mathbb{G}^{(2)}$. Furthermore, the same theorem provides a unique, up to conjugation, triangular set

$$\{\mathbb{Q}_{2i-1}^{(2)}, \mathbb{P}_{2r}^{(2)} | \boldsymbol{\beta}_{2i-1}^{(2)}, \boldsymbol{\alpha}_{2r}^{(2)}\}_{i=2,r=1}^{l',k'}, \tag{11.12c}$$

that corresponds to (11.12b). (Note that we have dropped two trivial groups and their characters.)

Clearly, the characters $\boldsymbol{\alpha}_2^{(2)}$ and $\Theta_i^{(2)}$ lie above the limit character $\boldsymbol{\eta}^{(2)}$, for all $i = 3, \dots, n$. As $(\mathbb{G}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)})$ is a faithful linear limit of $(\mathbb{G}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}_2^{(1)}, \boldsymbol{\alpha}_2^{(1)})$, Corollaries 10.9 and 10.10 imply

**Remark 11.13.** $\mathbb{R}^{(2)} = Z(\mathbb{P}_2^{(2)})$ is a cyclic central subgroup of $\mathbb{G}^{(2)}$, maximal among the abelian $\mathbb{G}^{(2)}$-invariant subgroups of $\mathbb{P}_2^{(2)}$, while the character $\boldsymbol{\alpha}_2^{(2)}$ is $\mathbb{G}^{(2)}$-invariant.

In addition, the way we perform the linear reductions, along with Remark 11.6 and Corollary 10.15, implies

**Remark 11.14.** Any faithful linear limit of $(\mathbb{G}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{G}_i^{(2)}, \Theta_i^{(2)})$ is also a faithful linear limit of $(G, 1, 1, G_i, \chi_i)$, for all $i = 3, 4, \dots, n$.

Furthermore, the fact that $\mathbb{S}^{(1)}$ is a normal subgroup of $\mathbb{G}^{(1)}$ that centralizes $\mathbb{P}_2^{(1)}$ implies, by Remarks 10.5 and 10.7, that there exists a subgroup $\mathbb{S}^{(2)}$ of $\mathbb{G}^{(2)}$ such that

$$\mathbb{S}^{(1)} \cong \mathbb{S}^{(2)} \trianglelefteq \mathbb{G}^{(2)} \tag{11.15}$$

and $\mathbb{S}^{(2)}$ centralizes $\mathbb{P}_2^{(2)}$. Thus, it also centralizes $\mathbb{R}^{(2)} \leq \mathbb{P}_2^{(2)}$. In addition, $\mathbb{S}^{(2)}$ is a central subgroup of $\mathbb{G}^{(2)}$, as $\mathbb{S}^{(1)}$ is a central subgroup of $\mathbb{G}^{(1)}$, and $\mathbb{G}^{(2)}$ is a section of $\mathbb{G}^{(1)}$. Under the isomorphism in (11.15), the irreducible character $\boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{S}^{(1)})$ maps to an irreducible character $\boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{S}^{(2)})$. If $n \geq 3$ we can say more about $\mathbb{S}^{(2)}$ and its character $\boldsymbol{\zeta}^{(2)}$. Indeed, the fact we used a $\mathbf{B}^{(1)}(\boldsymbol{\alpha}_2^{(1)})$-invariant faithful limit to get (11.12a), implies (see Theorem 10.105) that $\mathbb{Q}_{2l-1}^{(1),*} \cong \mathbb{Q}_{2l-1}^{(2),*}$ for all $l = 2, \dots, l'$. In particular we have

$$\mathbb{Q}_3^{(1)} \cong \mathbb{Q}_3^{(2)}.$$

The group $\mathbb{S}^{(1)}$ is a subgroup of $\mathbb{Q}_1^{(1)}$. But the latter is a subgroup of $\mathbb{Q}_3^{(1)}$, as $\mathbb{G}_2^{(1)} = \mathbb{P}_2^{(1)} \times \mathbb{Q}_1^{(1)} \leq \mathbb{G}_3^{(1)}$. Hence $\mathbb{S}^{(1)}$ is a subgroup of $\mathbb{Q}_3^{(1)}$. We conclude that its isomorphic image $\mathbb{S}^{(2)}$ is a subgroup of $\mathbb{Q}_3^{(2)}$. Furthermore, its irreducible character $\boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{S}^{(1)})$ lies under $\boldsymbol{\beta}_1^{(1)}$. But $\boldsymbol{\beta}_1^{(1)}$ lies under $\boldsymbol{\beta}_3^{(1)}$, as $\mathbb{Q}_1^{(1)} \leq \mathbb{Q}_3^{(1)}$. Hence $\boldsymbol{\zeta}^{(1)}$ lies under $\boldsymbol{\beta}_3^{(1)}$. As $\boldsymbol{\zeta}^{(1)}$ maps to $\boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{S}^{(2)})$, while $\boldsymbol{\beta}_3^{(1)}$ maps to $\boldsymbol{\beta}_3^{(2)} \in \mathrm{Irr}(\mathbb{Q}_3^{(2)})$, we conclude that $\boldsymbol{\zeta}^{(2)}$ lies under $\boldsymbol{\beta}_3^{(2)}$. Hence

$$\mathbb{R}^{(2)} \times \mathbb{S}^{(2)} \text{ is a central subgroup of } \mathbb{G}^{(2)} \tag{11.16a}$$

$$\boldsymbol{\eta}^{(2)} \times \boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{R}^{(2)} \times \mathbb{S}^{(2)}). \tag{11.16b}$$

If in addition $n \geq 3$ then

$$\mathbb{R}^{(2)} \times \mathbb{S}^{(2)} \trianglelefteq \mathbb{P}_2^{(2)} \cdot \mathbb{Q}_3^{(2)} = \mathbb{G}_3^{(2)}, \tag{11.16c}$$

$$\boldsymbol{\eta}^{(2)} \text{ and } \boldsymbol{\zeta}^{(2)} \text{ lie under } \boldsymbol{\alpha}_2^{(2)} \text{ and } \boldsymbol{\beta}_3^{(2)}, \text{ respectively.} \tag{11.16d}$$

Now we can extend Proposition 11.7 to

**Proposition 11.17.** *Every irreducible character $\Theta^{(2)} \in \mathrm{Irr}(\mathbb{G}^{(2)})$ that lies above $\boldsymbol{\eta}^{(2)} \times \boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{R}^{(2)} \times \mathbb{S}^{(2)})$ is monomial.*

*Proof.* Obviously any $\Theta^{(2)} \in \mathrm{Irr}(\mathbb{G}^{(2)} | \boldsymbol{\eta}^{(2)} \times \boldsymbol{\zeta}^{(2)})$ lies above $\boldsymbol{\eta}^{(2)} \in \mathrm{Irr}(\mathbb{R}^{(2)})$. As the quintuple $(\mathbb{G}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)})$ is a faithful linear limit of $(\mathbb{G}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}_2^{(1)}, \boldsymbol{\alpha}_2^{(1)})$, Lemma 10.11 implies the existence of an irreducible character $\Theta^{(1)} \in \mathrm{Irr}(\mathbb{G}^{(1)})$, lying above $\boldsymbol{\eta}^{(1)}$, so that $\Theta^{(2)}$ is a faithful linear limit of $\Theta^{(1)}$, under the $\mathbf{B}^{(1)}(\boldsymbol{\alpha}_2^{(1)})$-invariant reductions we performed. As we have already seen, under those reductions the $q$-subgroup $\mathbb{S}^{(1)}$ of $\mathbb{G}^{(1)}$ maps isomorphically to the subgroup $\mathbb{S}^{(2)}$ of $\mathbb{G}^{(2)}$. Hence the only way the faithful limit character $\Theta^{(2)} \in \mathrm{Irr}(\mathbb{G}^{(2)})$ can lie above $\boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{S}^{(2)})$, is if $\Theta^{(1)}$ lies above $\boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{S}^{(1)})$. In conclusion $\Theta^{(1)} \in \mathrm{Irr}(\mathbb{G}^{(1)})$ lies above $\boldsymbol{\eta}^{(1)} \times \boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{R}^{(1)} \times \mathbb{S}^{(1)})$. Now we can apply Proposition 11.7 to conclude that $\Theta^{(1)}$ is monomial. But $\Theta^{(2)}$ is a faithful linear limit of $\Theta^{(1)}$. Hence Proposition 11.7 implies that $\Theta^{(2)}$ is also monomial. This completes the proof of the proposition. $\square$

The next important theorem, that is proved in Section 11.4 is

**Theorem 11.18.** *After the above reductions, the group $\mathbb{G}_3^{(2)}$ is nilpotent if it exists, i.e., if $n \geq 3$.*

Hence $\mathbb{G}_3^{(2)}$ is the direct product

$$\mathbb{G}_3^{(2)} = \mathbb{P}_2^{(2)} \times \mathbb{Q}_3^{(2)}, \tag{11.19a}$$

of its $p$-Sylow subgroup $\mathbb{P}_2^{(2)}$ and its $q$-Sylow subgroup $\mathbb{Q}_3^{(2)}$. Furthermore, its irreducible character $\Theta_3^{(2)}$ is the direct product

$$\Theta_3^{(2)} = \boldsymbol{\alpha}_2^{(2)} \times \boldsymbol{\beta}_3^{(2)}, \tag{11.19b}$$

of $\boldsymbol{\alpha}_2^{(2)} \in \mathrm{Irr}(\mathbb{P}_2^{(2)})$ and $\boldsymbol{\beta}_3^{(2)} \in \mathrm{Irr}(\mathbb{Q}_3^{(2)})$. Hence (11.16) in the case of $n \geq 3$ becomes

$$\mathbb{R}^{(2)} \times \mathbb{S}^{(2)} \trianglelefteq \mathbb{P}_2^{(2)} \times \mathbb{Q}_3^{(2)} = \mathbb{G}_3^{(2)}, \tag{11.20}$$

$$\boldsymbol{\eta}^{(2)} \times \boldsymbol{\zeta}^{(2)} \text{ lies under } \boldsymbol{\alpha}_2^{(2)} \times \boldsymbol{\beta}_3^{(2)}. \tag{11.21}$$

The fact that $\mathbb{G}_3^{(2)}$ is a nilpotent group permits us to shift the series (11.12a) by one, and apply all the results of Section 8.3. (Note that the roles of $p$ and $q$ are interchanged.) Thus we get the series

$$1 \trianglelefteq \mathbb{Q}_3^{(2)} \trianglelefteq \mathbb{G}_4^{(2)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(2)} = \mathbb{G}^{(2)}, \tag{11.22a}$$

of normal subgroups of $\mathbb{G}^{(2)}$. Then, according to Section 8.3 and, in particular, Theorem 8.29, the characters

$$1, \boldsymbol{\beta}_3^{(2)}, \Theta_4^{(2)}, \dots, \Theta_n^{(2)}, \tag{11.22b}$$

form a tower for the above series, with corresponding triangular set

$$\{\mathbb{P}_{2r}^{(2)}, \mathbb{Q}_{2i-1}^{(2)} | \boldsymbol{\alpha}_{2r}^{(2)}, \boldsymbol{\beta}_{2i-1}^{(2)}\}_{r=2,i=2}^{k',l'}. \tag{11.22c}$$

(As expected, (see (8.20c) and (8.30c)), there are 3 trivial groups (the $\mathbb{P}_0^{(2),s}, \mathbb{P}_2^{(2),s}$ and $\mathbb{Q}_1^{(2),s}$) in (11.22c) that have been dropped.) Furthermore, the Hall system $\{\mathbf{A}^{(2)}, \mathbf{B}^{(2)}\}$ for $\mathbb{G}^{(2)}$, that was obtained via the second faithful linear limit, satisfies the equivalent of (11.1d-f) for (11.22) (see Theorem 8.29). Of course the groups $\mathbb{S}^{(2)}$ and $\mathbb{R}^{(2)}$ remain central subgroups of $\mathbb{G}^{(2)}$ that satisfy (11.16).

At this point we can repeat the process from the beginning, with $\mathbb{R}^{(2)}$ and $\mathbb{S}^{(2)}$ in the place of $R$ and $S$ respectively. (So the next step would be to take an $\mathbf{A}^{(2)}(\boldsymbol{\beta}_3^{(2)})$-invariant faithful linear limit $(\mathbb{G}^{(3)}, \mathbb{S}^{(3)}, \boldsymbol{\zeta}^{(3)}, \mathbb{Q}_3^{(3)}, \boldsymbol{\beta}_3^{(3)})$ of the quintuple $(\mathbb{G}^{(2)}, \mathbb{S}^{(2)}, \boldsymbol{\zeta}^{(2)}, \mathbb{Q}_3^{(2)}, \boldsymbol{\beta}_3^{(2)})$.)

Suppose, for the sake of our inductive hypothesis, that we have repeated this process $t-1$ times, for some integer $t$ with $2 < t \leq n$, i.e., we have taken $t-1$ invariant faithful limits and, after each such, we have shifted our series by one. So we arrive at a group $\mathbb{G}^{(t-1)} = \mathbb{G}_n^{(t-1)}$, that is a section of the original group $G$. Note that according to the inductive hypothesis, the last group shown to be nilpotent is the group $\mathbb{G}_t^{(t-1)}$. Therefore, depending on the parity of $t$, the group $\mathbb{G}_t^{(t-1)}$ equals

$$\mathbb{G}_t^{(t-1)} = \mathbb{P}_t^{(t-1)} \times \mathbb{Q}_{t-1}^{(t-1)}, \qquad \text{when } t \text{ is even} \tag{11.23a}$$

$$\mathbb{G}_t^{(t-1)} = \mathbb{P}_{t-1}^{(t-1)} \times \mathbb{Q}_t^{(t-1)}, \qquad \text{when } t \text{ is odd.} \tag{11.23b}$$

According to the inductive hypothesis, we can also generalize (11.1) and Remark 11.13. Thus we can assume that

$$\mathbb{R}^{(t-1)} \times \mathbb{S}^{(t-1)} \trianglelefteq \mathbb{P}_t^{(t-1)} \times \mathbb{Q}_{t-1}^{(t-1)} = \mathbb{G}_t^{(t-1)} \tag{11.24a}$$

$$\boldsymbol{\eta}^{(t-1)} \times \boldsymbol{\zeta}^{(t-1)} \text{ lies under } \boldsymbol{\alpha}_t^{(t-1)} \times \boldsymbol{\beta}_{t-1}^{(t-1)} = \Theta_t^{(t-1)}, \tag{11.24b}$$

$$\mathbb{S}^{(t-1)} = Z(\mathbb{Q}_{t-1}^{(t-1)}) \text{ and } \boldsymbol{\beta}_{t-1}^{(t-1)} \text{ is } \mathbb{G}^{(t-1)}\text{-invariant}, \tag{11.24c}$$

in the case of an even $t$. If $t$ is odd then

$$\mathbb{R}^{(t-1)} \times \mathbb{S}^{(t-1)} \trianglelefteq \mathbb{P}_{t-1}^{(t-1)} \times \mathbb{Q}_t^{(t-1)} = \mathbb{G}_t^{(t-1)} \tag{11.24d}$$

$$\boldsymbol{\eta}^{(t-1)} \times \boldsymbol{\zeta}^{(t-1)} \text{ lies under } \boldsymbol{\alpha}_{t-1}^{(t-1)} \times \boldsymbol{\beta}_t^{(t-1)} = \Theta_t^{(t-1)}, \tag{11.24e}$$

$$\mathbb{R}^{(t-1)} = Z(\mathbb{P}_{t-1}^{(t-1)}) \text{ and } \boldsymbol{\alpha}_{t-1}^{(t-1)} \text{ is } \mathbb{G}^{(t-1)}\text{-invariant.} \tag{11.24f}$$

Furthermore, the last group dropped is the $q$-group $\mathbb{Q}_{t-1}^{(t-1)}$ in the case of an even $t$, or the $p$-group $\mathbb{P}_{t-1}^{(t-1)}$ in the case of an odd $t$.

**Case 1: $t$ is even**

Assume first that $t$ is even. So we reach the series (after the $q$-group $\mathbb{Q}_{t-1}^{(t-1)}$ is dropped)

$$1 \trianglelefteq \mathbb{P}_t^{(t-1)} \trianglelefteq \mathbb{G}_{t+1}^{(t-1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t-1)} = \mathbb{G}^{(t-1)}, \tag{11.25a}$$

of normal subgroups of $\mathbb{G}^{(t-1)}$. Then $\mathbb{G}_i^{(t-1)}/\mathbb{G}_{i-1}^{(t-1)}$ is a $p$-group if $i$ is even, and a $q$-group if $i$ is odd, for each $i = t+2, \dots, n$, while for $i = t+1$ we get $\mathbb{G}_{t+1}^{(t-1)}/\mathbb{P}_t^{(t-1)}$ is a $q$-group with $\mathbb{P}_t^{(t-1)}$ a $p$-group. Along with the above series, we reach the characters

$$1, \boldsymbol{\alpha}_t^{(t-1)} \in \operatorname{Irr}(\mathbb{P}_t^{(t-1)}), \Theta_i^{(t-1)} \in \operatorname{Irr}(\mathbb{G}_i^{(t-1)}) \tag{11.25b}$$

for each $i = t+1, \dots, n$, that form a tower for (11.25a). Furthermore, we have the triangular set

$$\{\mathbb{Q}_{2i-1}^{(t-1)}, \mathbb{P}_{2r}^{(t-1)} | \boldsymbol{\beta}_{2i-1}^{(t-1)}, \boldsymbol{\alpha}_{2r}^{(t-1)}\}_{i=(t/2)+1, r=t/2}^{l',k'} \tag{11.25c}$$

that corresponds uniquely, up to conjugation, to (11.25b). (Note the first $t$ groups in (11.25c), have been dropped as they are trivial.) Also the Sylow system $\{\mathbf{A}, \mathbf{B}\}$ has been transfered to a Sylow system $\{\mathbf{A}^{(t-1)}, \mathbf{B}^{(t-1)}\}$ of $\mathbb{G}^{(t-1)}$ that satisfies the properties in (8.4), for (11.25). In addition, we reach two central subgroups of $\mathbb{G}^{(t-1)}$, the $p$-group $\mathbb{R}^{(t-1)}$ and the $q$-group $\mathbb{S}^{(t-1)}$, along with their characters $\boldsymbol{\eta}^{(t-1)} \in \operatorname{Irr}(\mathbb{R}^{(t-1)})$ and $\boldsymbol{\zeta}^{(t-1)} \in \operatorname{Irr}(\mathbb{S}^{(t-1)})$. We assume that $n \geq t$, so that our inductive step will be the $t$-th step. So we get the linear quintuple (see (11.24))

$$(\mathbb{G}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_t^{(t-1)}, \boldsymbol{\alpha}_t^{(t-1)}). \tag{11.26}$$

Furthermore, our inductive hypothesis implies that every irreducible character $\Theta^{(t-1)} \in \mathrm{Irr}(\mathbb{G}^{(t-1)})$ that lies above $\boldsymbol{\eta}^{(t-1)} \times \boldsymbol{\zeta}^{(t-1)} \in \mathrm{Irr}(\mathbb{R}^{(t-1)} \times \mathbb{S}^{(t-1)})$ is monomial. Also any faithful linear limit of the quintuple $(\mathbb{G}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{G}_i^{(t-1)}, \Theta_i^{(t-1)})$ or the $(\mathbb{G}^{(t-1)}, \mathbb{S}^{(t-1)}, \boldsymbol{\zeta}^{(t-1)}, \mathbb{G}_i^{(t-1)}, \Theta_i^{(t-1)})$ is also a faithful linear limit of $(G, 1, 1, G_i, \chi_i)$ for all $i = t-1, t, \dots, n$.

For the inductive step, we take a $\mathbf{B}^{(t-1)}(\boldsymbol{\alpha}_t^{(t-1)})$-invariant faithful linear limit

$$(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_t^{(t)}, \boldsymbol{\alpha}_t^{(t)})$$

of $(\mathbb{G}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_t^{(t-1)}, \boldsymbol{\alpha}_t^{(t-1)})$. The series (11.25a) reduces to

$$1 \trianglelefteq \mathbb{P}_t^{(t)} \trianglelefteq \mathbb{G}_{t+1}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t)} = \mathbb{G}^{(t)}. \tag{11.27a}$$

Furthermore, the character tower (11.25b) reduces to the tower

$$1, \boldsymbol{\alpha}_t^{(t)}, \Theta_i^{(t)} \in \mathrm{Irr}(\mathbb{G}_i^{(t)}) \tag{11.27b}$$

Along with the above character tower we get a triangular set

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)} | \boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t/2)+1, r=t/2}^{l',k'} \tag{11.27c}$$

that satisfies Theorem 10.105. Hence the Sylow system $\{\mathbf{A}^{(t-1)}, \mathbf{B}^{(t-1)}\}$ for $\mathbb{G}^{(t-1)}$, reduces to a Sylow system $\{\mathbf{A}^{(t)}, \mathbf{B}^{(t)}\}$ for $\mathbb{G}^{(t)}$, that satisfies (8.4) for the above reduced $t$-system. As with Remarks 11.5 and 11.13, the fact that we have taken faithful linear limits, along with Corollaries 10.9 and 10.10, implies

**Remark 11.28.** The group $\mathbb{R}^{(t)} = Z(\mathbb{P}_t^{(t)})$ is a cyclic central subgroup of $\mathbb{G}^{(t)}$, maximal among the abelian $\mathbb{G}^{(t)}$-invariant subgroups of $\mathbb{G}_t^{(t)}$, while the character $\boldsymbol{\alpha}_t^{(t)}$ is $\mathbb{G}^{(t)}$-invariant.

In addition, the way the reductions are done, along with Corollary 10.15, implies

**Remark 11.29.** Any faithful linear limit of the quintuple $(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{G}_i^{(t)}, \Theta_i^{(t)})$ or the quintuple $(\mathbb{G}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{G}_i^{(t)}, \Theta_i^{(t)})$ is also a faithful linear limit of $(G, 1, 1, G_i, \chi_i)$ for all $i = t, t+1, \dots, n$.

Even more, Theorem 10.105 implies that

$$\mathbb{Q}_{2l-1}^{(t-1),*} \cong \mathbb{Q}_{2l-1}^{(t),*}, \tag{11.30a}$$

for all $l = t+1, \dots, l'$, where $\mathbb{Q}_{2l-1}^{(t-1),*}$ and $\mathbb{Q}_{2l-1}^{(t),*}$ denote the product groups $\mathbb{Q}_{2l-1}^{(t-1),*} = \mathbb{Q}_{t+1}^{(t-1)} \cdot \mathbb{Q}_{t+3}^{(t-1)} \cdots \mathbb{Q}_{2l-1}^{(t-1)}$ and $\mathbb{Q}_{2l-1}^{(t),*} = \mathbb{Q}_{t+1}^{(t)} \cdot \mathbb{Q}_{t+3}^{(t)} \cdots \mathbb{Q}_{2l-1}^{(t)}$, respectively. In particular,

$$\mathbb{Q}_{t+1}^{(t)} \cong \mathbb{Q}_{t+1}^{(t-1)}. \tag{11.30b}$$

(Observe that $\mathbb{Q}_{t+1}^{(t-1),*} = \mathbb{Q}_{t+1}^{(t-1)}$ and $\mathbb{Q}_{t+1}^{(t),*} = \mathbb{Q}_{t+1}^{(t)}$.) The group $\mathbb{S}^{(t-1)}$ is a central subgroup of $\mathbb{G}^{(t-1)}$ that centralizes $\mathbb{P}_t^{(t-1)}$ (see (11.24a)). Hence Remarks 10.5 and 10.7 imply that $\mathbb{S}^{(t-1)}$ maps isomorphically to a normal subgroup

$$\mathbb{S}^{(t)} \cong \mathbb{S}^{(t-1)} \tag{11.30c}$$

of $\mathbb{G}^{(t)}$, that centralizes $\mathbb{P}_t^{(t)}$. In addition, $\mathbb{S}^{(t)}$ is a central subgroup of $\mathbb{G}^{(t)}$, as $\mathbb{S}^{(t-1)}$ is a central subgroup of $\mathbb{G}^{(t-1)}$, and $\mathbb{G}^{(t)}$ is a section of $\mathbb{G}^{(t-1)}$. Furthermore, under the group isomorphism in

(11.30c), the irreducible character $\boldsymbol{\zeta}^{(t-1)} \in \operatorname{Irr}(\mathbb{S}^{(t-1)})$ maps to

$$\boldsymbol{\zeta}^{(t)} \in \operatorname{Irr}(\mathbb{S}^{(t)}). \tag{11.30d}$$

As $n \geq t+1$, the group $\mathbb{Q}_{t+1}^{(t-1)}$ exists. Furthermore, as $\mathbb{G}_t^{(t-1)} = \mathbb{P}_t^{(t-1)} \times \mathbb{Q}_{t-1}^{(t-1)}$ is nilpotent we get that $\mathbb{Q}_{t-1}^{(t-1)}$ is a subgroup of $\mathbb{Q}_{t+1}^{(t-1)}$, while its irreducible character $\boldsymbol{\beta}_{t-1}^{(t-1)} \in \operatorname{Irr}(\mathbb{Q}_{t-1}^{(t-1)})$ lies under $\boldsymbol{\beta}_{t+1}^{(t-1)} \in \operatorname{Irr}(\mathbb{Q}_{t+1}^{(t-1)})$. Hence $\mathbb{S}^{(t-1)} \leq \mathbb{Q}_{t-1}^{(t-1)}$ is a subgroup of $\mathbb{Q}_{t+1}^{(t-1)}$. This along with (11.30) implies that $\mathbb{S}^{(t)}$ is a subgroup of $\mathbb{Q}_{t+1}^{(t)}$. Even more, its irreducible character $\boldsymbol{\zeta}^{(t)} \in \operatorname{Irr}(\mathbb{S}^{(t)})$ lies under $\boldsymbol{\beta}_{t+1}^{(t)}$, as $\boldsymbol{\zeta}^{(t-1)}$ lies under $\boldsymbol{\beta}_{t-1}^{(t-1)}$ (see (11.24b)) and that under $\boldsymbol{\beta}_{t+1}^{(t-1)}$. Hence

**Proposition 11.31.** *We have two central subgroups of $\mathbb{G}^{(t)}$, the $p$-group $\mathbb{R}^{(t)}$ and the $q$-group $\mathbb{S}^{(t)}$. Along with them we get their irreducible characters $\boldsymbol{\eta}^{(t)} \in \operatorname{Irr}(\mathbb{R}^{(t)})$ and $\boldsymbol{\zeta}^{(t)} \in \operatorname{Irr}(\mathbb{S}^{(t)})$. These groups and characters satisfy*

$$\mathbb{R}^{(t)} \times \mathbb{S}^{(t)} \trianglelefteq \mathbb{P}_t^{(t)} \cdot \mathbb{Q}_{t+1}^{(t)} = \mathbb{G}_{t+1}^{(t)},$$
$$\boldsymbol{\eta}^{(t)} \text{ and } \boldsymbol{\zeta}^{(t)} \text{ lie under } \boldsymbol{\alpha}_t^{(t)} \text{ and } \boldsymbol{\beta}_{t+1}^{(t)}, \text{ respectively.}$$

We can also show

**Proposition 11.32.** *Every irreducible character $\Theta^{(t)} \in \operatorname{Irr}(\mathbb{G}^{(t)})$ that lies above $\boldsymbol{\eta}^{(t)} \times \boldsymbol{\zeta}^{(t)} \in \operatorname{Irr}(\mathbb{R}^{(t)} \times \mathbb{S}^{(t)})$, is monomial.*

*Proof.* The quintuple $(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_t^{(t)}, \boldsymbol{\alpha}_t^{(t)})$ is a faithful linear limit of the linear quintuple $(\mathbb{G}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_t^{(t-1)}, \boldsymbol{\alpha}_t^{(t-1)})$. If $\Theta^{(t)} \in \operatorname{Irr}(\mathbb{G}^{(t)})$ lies above $\boldsymbol{\eta}^{(t)} \times \boldsymbol{\zeta}^{(t)}$, then it lies above $\boldsymbol{\eta}^{(t)}$, hence Lemma 10.11 implies the existence of an irreducible character $\Theta^{(t-1)} \in \operatorname{Irr}(\mathbb{G}^{(t-1)})$ above $\boldsymbol{\eta}^{(t-1)}$, so that $\Theta^{(t)}$ is a faithful linear limit of $\Theta^{(t-1)}$, under the $\mathbf{B}^{(t-1)}(\boldsymbol{\alpha}_t^{(t-1)})$-invariant reductions we performed. As we have already seen, under those reductions, the $q$-subgroup $\mathbb{S}^{(t-1)}$ of $\mathbb{G}^{(t-1)}$ maps isomorphically to the subgroup $\mathbb{S}^{(t)}$ of $\mathbb{G}^{(t)}$, see (11.30c). Hence the only way the faithful limit character $\Theta^{(t)} \in \operatorname{Irr}(\mathbb{G}^{(t)})$ can lie above $\boldsymbol{\zeta}^{(t)} \in \operatorname{Irr}(\mathbb{S}^{(t)})$, is if $\Theta^{(t-1)}$ lies above $\boldsymbol{\zeta}^{(t-1)} \in \operatorname{Irr}(\mathbb{S}^{(t-1)})$. In conclusion $\Theta^{(t-1)} \in \operatorname{Irr}(\mathbb{G}^{(t-1)})$ lies above $\boldsymbol{\eta}^{(t-1)} \times \boldsymbol{\zeta}^{(t-1)} \in \operatorname{Irr}(\mathbb{R}^{(t-1)} \times \mathbb{S}^{(t-1)})$. According to our inductive hypothesis, $\Theta^{(t-1)}$ is monomial. As $\Theta^{(t)}$ is a faithful linear limit of $\Theta^{(t-1)}$, Proposition 11.7 implies that $\Theta^{(t)}$ is also monomial. This completes the proof of the proposition. □

The main theorem for the inductive step, in the case of an even $t$, will be

**Theorem 11.33.** *After the above reductions, the group $\mathbb{G}_{t+1}^{(t)}$ is nilpotent if it exists, i.e., if $n \geq t+1$.*

Observe that, as before, the fact $\mathbb{G}_{t+1}^{(t)}$ is nilpotent allows us to shift the series (11.27a) by one. So, provided that $t+2 \leq n$, we get the series

$$1 \trianglelefteq \mathbb{Q}_{t+1}^{(t)} \trianglelefteq \mathbb{G}_{t+2}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t)} = \mathbb{G}^{(t)}. \tag{11.34a}$$

The character tower and its corresponding triangular set are carried over to the shifted series, as in the Sylow system. Hence the characters

$$1, \boldsymbol{\beta}_{t+1}^{(t)}, \Theta_{t+2}^{(t)}, \ldots, \Theta_n^{(t)} \tag{11.34b}$$

form a tower for the series (11.34a). Its corresponding triangular set is

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)} | \boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t/2)+1, r=(t/2)+1}^{l', k'} \tag{11.34c}$$

**Case 2: $t$ is odd**

We work similarly if $t$ is odd. Note that, after we drop the $p$-group $\mathbb{P}_{t-1}^{(t-1)}$ (see (11.23)), we reach the system

$$1 \trianglelefteq \mathbb{Q}_t^{(t-1)} \trianglelefteq \mathbb{G}_{t+1}^{(t-1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t-1)} = \mathbb{G}^{(t-1)}, \tag{11.35a}$$

$$1, \boldsymbol{\beta}_t^{(t-1)} \in \mathrm{Irr}(\mathbb{Q}_t^{(t-1)}), \Theta_i^{(t-1)} \in \mathrm{Irr}(\mathbb{G}_i^{(t-1)}) \tag{11.35b}$$

$$\{\mathbb{Q}_{2i-1}^{(t-1)}, \mathbb{P}_{2r}^{(t-1)} | \boldsymbol{\beta}_{2i-1}^{(t-1)}, \boldsymbol{\alpha}_{2r}^{(t-1)}\}_{i=(t+1)/2, r=(t+1)/2}^{l',k'} \tag{11.35c}$$

that is equivalent to (11.25) for the even case.

Observe that we can adjoint the trivial group at the bottom of (11.35a) and consider the series

$$1 \trianglelefteq 1 \trianglelefteq \mathbb{Q}_t^{(t-1)} \trianglelefteq \mathbb{G}_{t+1}^{(t-1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t-1)} = \mathbb{G}^{(t-1)}, \tag{11.36}$$

This way $t$ has become even. We can now interchange the roles of $p$ and $q$, and apply the already proved results of Case 1. For clarity we remark that, to prove the inductive step in this case of an odd $t$, we take an $\mathbf{A}^{(t-1)}(\boldsymbol{\beta}_t^{(t-1)})$-invariant faithful linear limit

$$(\mathbb{G}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{Q}_t^{(t)}, \boldsymbol{\beta}_t^{(t)})$$

of $(\mathbb{G}^{(t-1)}, \mathbb{S}^{(t-1)}, \boldsymbol{\zeta}^{(t-1)}, \mathbb{Q}_t^{(t-1)}, \boldsymbol{\beta}_t^{(t-1)})$. So the system (11.35) reduces to

$$1 \trianglelefteq \mathbb{Q}_t^{(t)} \trianglelefteq \mathbb{G}_{t+1}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_n^{(t)} = \mathbb{G}^{(t)}. \tag{11.37a}$$

$$1, \boldsymbol{\beta}_t^{(t)}, \Theta_i^{(t)} \in \mathrm{Irr}(\mathbb{G}_i^{(t)}) \tag{11.37b}$$

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)} | \boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t+1)/2, r=(t+1)/2}^{l',k'} \tag{11.37c}$$

All the conclusions of the even case are transfered to the odd case. In particular we get,

**Remark 11.38.** The group $\mathbb{S}^{(t)} = Z(\mathbb{Q}_t^{(t)})$ is a cyclic central subgroup of $\mathbb{G}^{(t)}$, maximal among the abelian $\mathbb{G}^{(t)}$-invariant subgroups of $\mathbb{G}_t^{(t)}$, while the character $\boldsymbol{\beta}_t^{(t)}$ is $\mathbb{G}^{(t)}$-invariant.

**Remark 11.39.** Any faithful linear limit of the quintuple $(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{G}_i^{(t)}, \Theta_i^{(t)})$ or the quintuple $(\mathbb{G}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{G}_i^{(t)}, \Theta_i^{(t)})$ is also a faithful linear limit of $(G, 1, 1, G_i, \chi_i)$ for all $i = t, t+1, \ldots, n$.

**Proposition 11.40.** *We have two central subgroups of $\mathbb{G}^{(t)}$, the $p$-group $\mathbb{R}^{(t)}$ and the $q$-group $\mathbb{S}^{(t)}$. Along with them we get their irreducible characters $\boldsymbol{\eta}^{(t)} \in \mathrm{Irr}(\mathbb{R}^{(t)})$ and $\boldsymbol{\zeta}^{(t)} \in \mathrm{Irr}(\mathbb{S}^{(t)})$. These groups and characters satisfy*

$$\mathbb{R}^{(t)} \times \mathbb{S}^{(t)} \trianglelefteq \mathbb{P}_{t+1}^{(t)} \cdot \mathbb{Q}_t^{(t)} = \mathbb{G}_{t+1}^{(t)}$$

$$\boldsymbol{\eta}^{(t)} \textit{ and } \boldsymbol{\zeta}^{(t)} \textit{ lie under } \boldsymbol{\alpha}_{t+1}^{(t)} \textit{ and } \boldsymbol{\beta}_t^{(t)}, \textit{ respectively.}$$

**Proposition 11.41.** *Every irreducible character $\Theta^{(t)} \in \mathrm{Irr}(\mathbb{G}^{(t)})$ that lies above $\boldsymbol{\eta}^{(t)} \times \boldsymbol{\zeta}^{(t)} \in \mathrm{Irr}(\mathbb{R}^{(t)} \times \mathbb{S}^{(t)})$, is monomial.*

The main theorem for the inductive step in the case of an odd $t$, will be

**Theorem 11.42.** *After the above reductions, the group $\mathbb{G}_{t+1}^{(t)}$ is nilpotent if it exists, i.e., if $n \geq t+1$.*

Observe that, as before, the fact $\mathbb{G}_{t+1}^{(t)}$ is nilpotent allows us to shift the series (11.37a) by one. So, provided that $t+2 \leq n$, we get the series

$$1 \trianglelefteq \mathbb{P}_{t+1}^{(t)} \trianglelefteq \mathbb{G}_{t+2}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_{n}^{(t)} = \mathbb{G}^{(t)}. \tag{11.43a}$$

The character tower and its corresponding triangular set are carried over to the shifted series, as in the Sylow system. Hence the characters

$$1, \boldsymbol{\alpha}_{t+1}^{(t)}, \Theta_{t+2}^{(t)}, \ldots, \Theta_{n}^{(t)} \tag{11.43b}$$

form a tower for the series (11.43a). Its corresponding triangular set is

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)} | \boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t+3)/2, r=(t+1)/2}^{l',k'} \tag{11.43c}$$

Now we can repeat the process at the $t+1$-st step.

## 11.2 Conclusions for the smaller systems

Before giving the proofs of the three theorems stated in the previous section, we will analyze the behavior, under the described reductions, of the smaller system

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_m \trianglelefteq G, \tag{11.44a}$$

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{m} \tag{11.44b}$$

$$\{Q_{2i-1}, P_{2r}, | \beta_{2i-1}, \alpha_{2r}\}_{i=1, r=0}^{l,k} \tag{11.44c}$$

for any fixed, but arbitrary, $m = 1, \ldots, n$. The integers $k$ and $l$ are, as usual, related to $m$ via (5.7). Associated to this system are the groups $T = P_{2k}^{*} \rtimes I$ and $U = Q_{2l-1}^{*} \rtimes J$, where $I$ is the image of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P_{2k}^{*})$ and $J$ that of $\widehat{P}(\alpha_{2l-2,2l-1})$ in $\mathrm{Aut}(Q_{2l-1}^{*})$, (for their definitions see (10.84) and (10.53)).

After the first $\mathbf{A}(\beta_1)$-invariant reductions the above smaller system reduces, along with the original (11.1), to the system

$$1 \trianglelefteq \mathbb{G}_{1}^{(1)} \trianglelefteq \mathbb{G}_{2}^{(1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_{m}^{(1)} \trianglelefteq \mathbb{G}^{(1)}, \tag{11.45a}$$

$$\{\Theta_{i}^{(1)} \in \mathrm{Irr}(\mathbb{G}_{i}^{(1)})\}_{i=0}^{m} \tag{11.45b}$$

$$\{\mathbb{Q}_{2i-1}^{(1)}, \mathbb{P}_{2r}^{(1)}, | \boldsymbol{\beta}_{2i-1}^{(1)}, \boldsymbol{\alpha}_{2r}^{(1)}\}_{i=1, r=0}^{l,k} \tag{11.45c}$$

(Note that this last triangular set is a subset of (11.3).) Of course the groups $T$ and $U$ reduce to $\mathbb{T}^{(1)}$ and $\mathbb{U}^{(1)}$, respectively. So $\mathbb{T}^{(1)} = \mathbb{P}_{2k}^{(1),*} \rtimes \mathbb{I}^{(1)}$, where $\mathbb{I}^{(1)}$ is the image of $\widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}_{2k-1,2k}^{(1)})$ in $\mathrm{Aut}(\mathbb{P}_{2k}^{(1),*})$. Similarly, $\mathbb{U}^{(1)} = \mathbb{Q}_{2l-1}^{(1),*} \rtimes \mathbb{J}^{(1)}$, where $\mathbb{J}^{(1)}$ is the image of $\widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(1)})$ in $\mathrm{Aut}(\mathbb{Q}_{2l-1}^{(1),*})$.

Then, according to Theorem 10.76,

**Theorem 11.46.** *The groups $\widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}_{2k-1,2k}^{(1)})$ and $\widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}_{2l-1,2l-1}^{(1)})$ can be chosen to satisfy*

*$P_{2k}^{*}$ is naturally isomorphic to $\mathbb{P}_{2k}^{(1),*}$.*

*This isosmorphism sends the image of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P_{2k}^{*})$ onto that of $\widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}_{2k-1,2k}^{(1)})$ in*

$\mathrm{Aut}(\mathbb{P}_{2k}^{(1),*})$. *So* $I \cong \mathbb{I}^{(1)}$. *In addition,*

$$T \cong \mathbb{T}^{(1)},$$

$$\textit{Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \textit{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(1),*}) \; = \mathbb{J}^{(1)} = \; \textit{Image of } \widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(1)}) \textit{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(1),*}) \; .$$

Note that as $P_{2k}^*$ is naturally isomorphic to $\mathbb{P}_{2k}^{(1),*}$, we get that

$$\text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(1),*}) \; = \mathbb{I}^{(1)} = \; \text{Image of } \widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}_{2k-1,2k}^{(1)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(1),*}) \qquad (11.47)$$

As we have already seen, the group $R = 1$ seen inside $\mathbb{P}_2^{(1)} \leq \mathbb{P}_{2k}^{(1),*}$ is denoted by $\mathbb{R}^{(1)}$, and its irreducible character $\eta = 1$, by $\boldsymbol{\eta}^{(1)}$. So we can form the linear quintuple $(\mathbb{T}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}_{2k}^{(1),*}, \boldsymbol{\alpha}_{2k}^{(1),*})$. Furthermore, $\mathbb{T}^{(1)} \cong T$, while $\mathbb{P}_{2k}^{(1),*}$ is naturally isomorphic to $P_{2k}^*$, (so that the character $\boldsymbol{\alpha}_{2k}^{(1),*}$ maps to $\alpha_{2k}^*$). This, along with the equations in (10.78), implies

**Corollary 11.48.** *After the first* $\mathbf{A}(\beta_1)$*-invariant reductions we have*

$$(\mathbb{T}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}_{2k}^{(1),*}, \boldsymbol{\alpha}_{2k}^{(1),*}) \cong (T, R, \eta, P_{2k}^*, \alpha_{2k}^*). \qquad (11.49)$$

*Furthermore, the quintuple* $(\mathbb{U}^{(1)}, \mathbb{S}^{(1)}, \boldsymbol{\zeta}^{(1)}, \mathbb{Q}_{2l-1}^{(1),*}, \boldsymbol{\beta}_{2l-1}^{(1),*})$ *is a* $G$*-associate faithful linear limit of* $(U, S, \zeta, Q_{2l-1}^*, \beta_{2l-1}^*)$. *Hence any faithful linear limit of the former quintuple is also a faithful linear limit of the latter one.*

From now on, we will identify the quintuples in (11.49).

Note that when we shift the series (11.2) by one in (11.11), the same happens to the series (11.44a). Thus we get the system

$$1 \trianglelefteq \mathbb{P}_2^{(1)} \trianglelefteq \mathbb{G}_3^{(1)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_m^{(1)} \trianglelefteq \mathbb{G}^{(1)}, \qquad (11.50a)$$

$$1, \boldsymbol{\alpha}_2^{(1)}, \Theta_3^{(1)}, \ldots, \Theta_m^{(1)}, \qquad (11.50b)$$

$$\{\mathbb{Q}_{2i-1}^{(1)}, \mathbb{P}_{2r}^{(1)}, |\boldsymbol{\beta}_{2i-1}^{(1)}, \boldsymbol{\alpha}_{2r}^{(1)}\}_{i=2,r=1}^{l,k} \qquad (11.50c)$$

that is a smaller system for (11.11). Observe also that, according to Theorem 8.29, none of the groups $\mathbb{I}^{(1)}, \mathbb{J}^{(1)}, \mathbb{T}^{(1)}$ and $\mathbb{U}^{(1)}$ changes passing to the shifted case.

When the next series of $\mathbf{B}^{(1)}(\boldsymbol{\alpha}_2^{(1)})$-invariant reductions is performed the system (11.50) reduces to

$$1 \trianglelefteq \mathbb{P}_2^{(2)} \trianglelefteq \mathbb{G}_3^{(2)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_m^{(2)} \trianglelefteq \mathbb{G}^{(2)}, \qquad (11.51a)$$

$$1, \boldsymbol{\alpha}_2^{(2)}, \Theta_3^{(2)}, \ldots, \Theta_m^{(2)}, \qquad (11.51b)$$

$$\{\mathbb{Q}_{2i-1}^{(2)}, \mathbb{P}_{2r}^{(2)}, |\boldsymbol{\beta}_{2i-1}^{(2)}, \boldsymbol{\alpha}_{2r}^{(2)}\}_{i=2,r=1}^{l,k} \qquad (11.51c)$$

The groups $\mathbb{I}^{(1)}, \mathbb{J}^{(1)}$ and $\mathbb{T}^{(1)}, \mathbb{U}^{(1)}$ reduce to the groups $\mathbb{I}^{(2)}, \mathbb{J}^{(2)}$ and $\mathbb{T}^{(2)}, \mathbb{U}^{(2)}$, respectively, so that Theorem 10.105 holds. (The notation is as expected, that is $\mathbb{T}^{(2)} = \mathbb{P}_{2k}^{(2),*} \rtimes \mathbb{I}^{(2)}$, where $\mathbb{I}^{(2)}$ is the image of $\widehat{\mathbb{Q}}^{(2)}(\boldsymbol{\beta}_{2k-1,2k}^{(2)})$ in $\mathrm{Aut}(\mathbb{P}_{2k}^{(2),*})$, and $\mathbb{U}^{(2)} = \mathbb{Q}_{2l-1}^{(2),*} \rtimes \mathbb{J}^{(2)}$, where $\mathbb{J}^{(2)}$ is the image of $\widehat{\mathbb{P}}^{(2)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(2)})$ in $\mathrm{Aut}(\mathbb{Q}_{2l-1}^{(2),*})$.) In view of Theorem 10.105, we get

**Theorem 11.52.** *The reduced system* (11.51) *satisfies*

$$\mathbb{Q}^{(1),*}_{2l-1} \text{ is naturally isomorphic to } \mathbb{Q}^{(2),*}_{2l-1},$$

*This isosmorphism sends the image of* $\widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}^{(1)}_{2l-2,2l-1})$ *in* $\mathrm{Aut}(\mathbb{Q}^{(1),*}_{2l-1})$ *onto that of* $\widehat{\mathbb{P}}^{(2)}(\boldsymbol{\alpha}^{(2)}_{2l-2,2l-1})$ *in* $\mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1})$*. So* $\mathbb{J}^{(1)} \cong \mathbb{J}^{(2)}$*. In addition,*

$$\mathbb{U}^{(1)} \cong \mathbb{U}^{(2)},$$
$$\textit{Image of } \widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}^{(1)}_{2k-1,2k}) \textit{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \ = \mathbb{I}^{(2)} = \ \textit{Image of } \widehat{\mathbb{Q}}^{(2)}(\boldsymbol{\beta}^{(2)}_{2k-1,2k}) \textit{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \ .$$

Note that similar to (11.47), the fact that $\mathbb{Q}^{(1),*}_{2l-1}$ is naturally isomorphic to $\mathbb{Q}^{(2),*}_{2l-1}$, along with Theorem 11.52, implies

$$\text{Image of } \widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}^{(1)}_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1}) \ = \mathbb{J}^{(2)} = \ \text{Image of } \widehat{\mathbb{P}}^{(2)}(\boldsymbol{\alpha}^{(1)}_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1}). \tag{11.53}$$

Furthermore, the group $\mathbb{S}^{(1)}$ is isomorphic to $\mathbb{S}^{(2)}$, (see (11.15)), while its irreducible character $\boldsymbol{\zeta}^{(1)} \in \mathrm{Irr}(\mathbb{S}^{(1)})$ maps under the above isomorphism to $\boldsymbol{\zeta}^{(2)} \in \mathrm{Irr}(\mathbb{S}^{(2)})$. This, along with the isomorphism between $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$, and the remarks following Theorem 10.105 (see (10.107)), implies

**Corollary 11.54.** *After the second* $\mathbf{B}^{(1)}(\boldsymbol{\alpha}^{(1)}_2)$*-invariant reductions we have*

$$(\mathbb{U}^{(1)}, \mathbb{S}^{(1)}, \boldsymbol{\zeta}^{(1)}, \mathbb{Q}^{(1),*}_{2l-1}, \boldsymbol{\beta}^{(1),*}_{2l-1}) \cong (\mathbb{U}^{(2)}, \mathbb{S}^{(2)}, \boldsymbol{\zeta}^{(2)}, \mathbb{Q}^{(2),*}_{2l-1}, \boldsymbol{\beta}^{(2),*}_{2l-1}). \tag{11.55}$$

*Furthermore, the quintuple* $(\mathbb{T}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}^{(2),*}_{2k}, \boldsymbol{\alpha}^{(2),*}_{2k})$ *is a* $\mathbb{G}^{(1)}$*-associate faithful linear limit of* $(\mathbb{T}^{(1)}, \mathbb{R}^{(1)}, \boldsymbol{\eta}^{(1)}, \mathbb{P}^{(1),*}_{2k}, \boldsymbol{\alpha}^{(1),*}_{2k})$*. Hence any faithful linear limit of the former quintuple is also a faithful linear limit of the latter one.*

From now on we will identify the quintuples in (11.55).

Corollary 11.48 and 11.54 combined (with the appropriate identifications) imply

**Theorem 11.56.** *Any faithful linear limit of the quintuple* $(\mathbb{T}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}^{(2),*}_{2k}, \boldsymbol{\alpha}^{(2),*}_{2k})$*, or the* $(\mathbb{U}^{(2)}, \mathbb{S}^{(2)}, \boldsymbol{\zeta}^{(2)}, \mathbb{Q}^{(2),*}_{2l-1}, \boldsymbol{\beta}^{(2),*}_{2l-1})$ *is also a faithful linear limit of the quintuple* $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$*, or* $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$*, respectively.*

Furthermore, for the image $\mathbb{J}^{(2)}$ of $\widehat{\mathbb{P}}^{(2)}(\boldsymbol{\alpha}^{(2)}_{2l-2,2l-1})$ in $\mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1})$ we have

$$\begin{aligned} \mathbb{J}^{(2)} &= \ \text{Image of } \widehat{\mathbb{P}}^{(1)}(\boldsymbol{\alpha}^{(1)}_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1}) && \text{by (11.53)} \\ &= \ \text{Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1}) && \text{by Theorem 11.46, since } \mathbb{Q}^{(1),*}_{2l-1} \cong \mathbb{Q}^{(2),*}_{2l-1} \end{aligned}$$

Even more, equation (11.47), along with the fact that $\mathbb{P}^{(2),*}_{2k}$ is a factor of $\mathbb{P}^{(1),*}_{2k}$, implies

$$\text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \ = \ \text{Image of } \widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}^{(1)}_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}).$$

Therefore, for the image $\mathbb{I}^{(2)}$ of $\widehat{\mathbb{Q}}^{(2)}(\boldsymbol{\beta}^{(2)}_{2k-1,2k})$ in $\mathrm{Aut}(\mathbb{P}^{(2),*}_{2k})$ we have

$$\begin{aligned}\mathbb{I}^{(2)} = &\text{ Image of } \widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}^{(1)}_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \qquad\qquad \text{by Theorem 11.52}\\ = &\text{ Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \ .\end{aligned}$$

In conclusion, we get

**Theorem 11.57.**

$$\mathbb{J}^{(2)} = \textit{ Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \textit{ in } \mathrm{Aut}(\mathbb{Q}^{(2),*}_{2l-1}) \ ,$$

$$\mathbb{I}^{(2)} = \textit{ Image of } \widehat{Q}(\beta_{2k-1,1k}) \textit{ in } \mathrm{Aut}(\mathbb{P}^{(2),*}_{2k}) \ .$$

When we shift the series (11.12a) by one to get (11.22a), the system (11.51) is also shifted by one. Thus we reach

$$1 \trianglelefteq \mathbb{Q}^{(2)}_3 \trianglelefteq \mathbb{G}^{(2)}_4 \trianglelefteq \cdots \trianglelefteq \mathbb{G}^{(2)}_m \trianglelefteq \mathbb{G}^{(2)}, \tag{11.58a}$$

$$1, \boldsymbol{\beta}^{(2)}_3, \Theta^{(2)}_4, \ldots, \Theta^{(2)}_m, \tag{11.58b}$$

$$\{\mathbb{Q}^{(2)}_{2i-1}, \mathbb{P}^{(2)}_{2r}, |\boldsymbol{\beta}^{(2)}_{2i-1}, \boldsymbol{\alpha}^{(2)}_{2r}\}^{l,k}_{i=2,r=2} \tag{11.58c}$$

According to Theorem 8.29, the groups $\mathbb{I}^{(2)}, \mathbb{J}^{(2)}, \mathbb{T}^{(2)}$ and $\mathbb{U}^{(2)}$ remain unchanged passing to the shifted system (11.58).

According to our inductive hypothesis, after $t-1$ steps (where $2 < t \leq m$), we reach the system

$$1 \trianglelefteq \mathbb{P}^{(t-1)}_t \trianglelefteq \mathbb{G}^{(t-1)}_{t+1} \trianglelefteq \cdots \trianglelefteq \mathbb{G}^{(t-1)}_m \trianglelefteq \mathbb{G}^{(t-1)}, \tag{11.59a}$$

$$1, \boldsymbol{\alpha}^{(t-1)}_t, \Theta^{(t-1)}_{t+1}, \ldots, \Theta^{(t-1)}_m, \tag{11.59b}$$

$$\{\mathbb{Q}^{(t-1)}_{2i-1}, \mathbb{P}^{(t-1)}_{2r}, |\boldsymbol{\beta}^{(t-1)}_{2i-1}, \boldsymbol{\alpha}^{(t-1)}_{2r}\}^{l,k}_{i=(t/2)+1,r=t/2} \tag{11.59c}$$

when $t$ is even. In the case of an odd $t$ we get

$$1 \trianglelefteq \mathbb{Q}^{(t-1)}_t \trianglelefteq \mathbb{G}^{(t-1)}_{t+1} \trianglelefteq \cdots \trianglelefteq \mathbb{G}^{(t-1)}_m \trianglelefteq \mathbb{G}^{(t-1)}, \tag{11.60a}$$

$$1, \boldsymbol{\beta}^{(t-1)}_t, \Theta^{(t-1)}_{t+1}, \ldots, \Theta^{(t-1)}_m, \tag{11.60b}$$

$$\{\mathbb{Q}^{(t-1)}_{2i-1}, \mathbb{P}^{(t-1)}_{2r}, |\boldsymbol{\beta}^{(t-1)}_{2i-1}, \boldsymbol{\alpha}^{(t-1)}_{2r}\}^{l,k}_{i=(t+1)/2+1,r=(t+1)/2} \tag{11.60c}$$

Furthermore, the groups $I, J, T$ and $U$ are reduced to the groups $\mathbb{I}^{(t-1)}, \mathbb{J}^{(t-1)}, \mathbb{T}^{(t-1)}$ and $\mathbb{U}^{(t-1)}$, where $\mathbb{T}^{(t-1)} = \mathbb{P}^{(t-1),*}_{2k} \rtimes \mathbb{I}^{(t-1)}$, and $\mathbb{I}^{(t-1)}$ is the image of $\widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}^{(t-1)}_{2k-1,2k})$ in $\mathrm{Aut}(\mathbb{P}^{(t-1),*}_{2k})$. Similarly, $\mathbb{U}^{(t-1)} = \mathbb{Q}^{(t-1),*}_{2l-1} \rtimes \mathbb{J}^{(t-1)}$ where $\mathbb{J}^{(t-1)}$ is the image of $\widehat{\mathbb{P}}^{(2t-1)}(\boldsymbol{\alpha}^{(t-1)}_{2l-2,2l-1})$ in $\mathrm{Aut}(\mathbb{Q}^{(t-1),*}_{2l-1})$. These groups satisfy

**Theorem 11.61.** *Any faithful linear limit of* $(\mathbb{T}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}^{(t-1),*}_{2k}, \boldsymbol{\alpha}^{(t-1),*}_{2k})$ *or the quintuple* $(\mathbb{U}^{(t-1)}, \mathbb{S}^{(t-1)}, \boldsymbol{\zeta}^{(t-1)}, \mathbb{Q}^{(t-1),*}_{2l-1}, \boldsymbol{\beta}^{(t-1),*}_{2l-1})$ *is also a faithful linear limit of* $(T, R, \eta, P^*_{2k}, \alpha^*_{2k})$ *or* $(U, S, \zeta, Q^*_{2l-1}, \beta^*_{2l-1})$*, respectively.*

Furthermore for the groups $\mathbb{I}^{(t-1)}$ and $\mathbb{J}^{(t-1)}$ we have

**Theorem 11.62.**

$$\mathbb{J}^{(t-1)} = \textit{ Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \textit{ in } \mathrm{Aut}(\mathbb{Q}^{(t-1),*}_{2l-1}) \ ,$$

$$\mathbb{I}^{(t-1)} = \textit{ Image of } \widehat{Q}(\beta_{2k-1,1k}) \textit{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t-1),*}) \ .$$

At the inductive step (the $t$-th step), either we perform a series of $\mathbf{B}^{(t-1)}(\boldsymbol{\alpha}_t^{(t-1)})$-invariant reductions,if $t$ is even, or a series of $\mathbf{A}^{(t-1)}(\boldsymbol{\beta}_t^{(t-1)})$-invariant reductions, if $t$ is odd. The systems (11.59) and (11.60) reduce to

$$1 \trianglelefteq \mathbb{P}_t^{(t)} \trianglelefteq \mathbb{G}_{t+1}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_m^{(t)} \trianglelefteq \mathbb{G}^{(t)}, \tag{11.63a}$$

$$1, \boldsymbol{\alpha}_t^{(t)}, \Theta_{t+1}^{(t)}, \ldots, \Theta_m^{(t)}, \tag{11.63b}$$

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)}, |\boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t/2)+1,r=t/2}^{l,k} \tag{11.63c}$$

when $t$ is even, and

$$1 \trianglelefteq \mathbb{Q}_t^{(t)} \trianglelefteq \mathbb{G}_{t+1}^{(t)} \trianglelefteq \cdots \trianglelefteq \mathbb{G}_m^{(t)} \trianglelefteq \mathbb{G}^{(t)}, \tag{11.64a}$$

$$1, \boldsymbol{\beta}_t^{(t)}, \Theta_{t+1}^{(t)}, \ldots, \Theta_m^{(t)}, \tag{11.64b}$$

$$\{\mathbb{Q}_{2i-1}^{(t)}, \mathbb{P}_{2r}^{(t)}, |\boldsymbol{\beta}_{2i-1}^{(t)}, \boldsymbol{\alpha}_{2r}^{(t)}\}_{i=(t+1)/2+1,r=(t+1)/2}^{l,k} \tag{11.64c}$$

when $t$ is odd.

**Case 1: $t$ is even**

Suppose first that $t$ is even. Then the groups $\mathbb{I}^{(t-1)}, \mathbb{J}^{(t-1)}, \mathbb{T}^{(t-1)}$ and $\mathbb{U}^{(t-1)}$ (for the system (11.59)), reduce to the groups $\mathbb{I}^{(t)}, \mathbb{J}^{(t)}, \mathbb{T}^{(t)}$ and $\mathbb{U}^{(t)}$, (for the system (11.63)), so that Theorem 10.105 holds. (Observe we can use Theorem 10.105 as the reductions we used to get (11.63) were $\mathbf{B}^{(t-1)}(\boldsymbol{\alpha}_t^{(t-1)})$-invariant.) In particular we get

$$\mathbb{Q}_{2l-1}^{(t-1),*} \text{ is naturally isomorphic to } \mathbb{Q}_{2l-1}^{(t),*}, \tag{11.65a}$$

$$\text{this isosmorphism sends the image of } \widehat{\mathbb{P}}^{(t-1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t-1)})$$

$$\text{in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t-1),*}) \text{ onto that of } \widehat{\mathbb{P}}^{(t)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*})\text{,i.e., } \mathbb{J}^{(t-1)} \cong \mathbb{J}^{(t)}, \tag{11.65b}$$

$$\mathbb{U}^{(t-1)} \cong \mathbb{U}^{(t)}, \tag{11.65c}$$

$$\text{Image of } \widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) = \text{ Image of } \widehat{\mathbb{Q}}^{(t)}(\boldsymbol{\beta}_{2k-1,2k}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}). \tag{11.65d}$$

Similar to (11.53), the above implies

$$\text{Image of } \widehat{\mathbb{P}}^{(t-1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \ = \mathbb{J}^{(t)} = \text{ Image of } \widehat{\mathbb{P}}^{(t)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}). \tag{11.66}$$

This, along with the fact that $\mathbb{S}^{(t)} \cong \mathbb{S}^{(t-1)}$ (see (11.30c)), implies that

$$(\mathbb{U}^{(t-1)}, \mathbb{S}^{(t-1)}, \boldsymbol{\zeta}^{(t-1)}, \mathbb{Q}_{2l-1}^{(t-1),*}, \boldsymbol{\beta}_{2l-1}^{(t-1),*}) \cong (\mathbb{U}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{Q}_{2l-1}^{(t),*}, \boldsymbol{\beta}_{2l-1}^{(t),*}). \tag{11.67}$$

From now on we identify these two quintuples. Observe that $(\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_{2k}^{(t),*}, \boldsymbol{\alpha}_{2k}^{(t),*})$ is a $\mathbb{G}^{(t-1)}$-associate faithful linear limit of $(\mathbb{T}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_{2k}^{(t-1),*}, \boldsymbol{\alpha}_{2k}^{(t-1),*})$, (see (10.107)). Hence, Proposition 10.108 implies that any faithful linear limit of $(\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_{2k}^{(t),*}, \boldsymbol{\alpha}_{2k}^{(t),*})$ is also a faithful linear limit of $(\mathbb{T}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_{2k}^{(t-1),*}, \boldsymbol{\alpha}_{2k}^{(t-1),*})$. This, along with (11.67) and Theorem 11.56 implies

**Theorem 11.68.** *Any faithful linear limit of the quintuple* $(\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_{2k}^{(t),*}, \boldsymbol{\alpha}_{2k}^{(t),*})$ *or the quintuple* $(\mathbb{U}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{Q}_{2l-1}^{(t),*}, \boldsymbol{\beta}_{2l-1}^{(t),*})$ *is also a faithful linear limit of the quintuple* $(T, R, \eta, P_{2k}^{*}, \alpha_{2k}^{*})$ *or* $(U, S, \zeta, Q_{2l-1}^{*}, \beta_{2l-1}^{*})$*, respectively.*

So Theorem 11.61 holds for the inductive $t$-th step, in the case of an even $t$. In addition, Theorem 11.62 is still valid for an even $t$, i.e.,

**Theorem 11.69.**

$$\mathbb{J}^{(t)} = \textit{ Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \textit{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \ , \tag{11.70a}$$

$$\mathbb{I}^{(t)} = \textit{ Image of } \widehat{Q}(\beta_{2k-1,2k}) \textit{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) \ . \tag{11.70b}$$

*Proof.* For the proof of (11.70a), note that Theorem 11.62, along with the fact that the section $\mathbb{Q}_{2l-1}^{(t),*}$ of $\mathbb{Q}_{2l-1}^{(t-1),*}$ is isomorphic to $\mathbb{Q}_{2l-1}^{(t-1),*}$(see (11.30a)), implies that

$$\text{Image of } \widehat{\mathbb{P}}^{(t-1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \ = \ \text{Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \ .$$

This, along with (11.66), implies

$$\begin{aligned} \mathbb{J}^{(t)} &= \ \text{Image of } \widehat{\mathbb{P}}^{(t-1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \\ &= \ \text{Image of } \widehat{P}(\alpha_{2l-2,2l-1}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) \ . \end{aligned}$$

Hence (11.70a) follows.

As far as (11.70b) is concerned, first observe that

$$\begin{aligned} \mathbb{I}^{(t)} &= \ \text{Image of } \widehat{\mathbb{Q}}^{(t)}(\boldsymbol{\beta}_{2k-1,2k}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) \\ &= \ \text{Image of } \widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) \qquad \text{by (11.65d).} \end{aligned} \tag{11.71}$$

On the other hand, Theorem 11.62 implies that

$$\mathbb{I}^{(t-1)} = \ \text{Image of } \widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t-1),*}) \ = \ \text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t-1),*}) \ . \tag{11.72}$$

But $\mathbb{P}^{(t),*}$ is a section of $\mathbb{P}^{(t-1),*}$. Hence (11.72) implies that

$$\text{Image of } \widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) \ = \ \text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}) \ .$$

This, along with (11.71), implies the desired equation in (11.70a). □

**Case 2:** $t$ **is odd** We can work similarly in the case of an odd $t$, to prove that Theorems 11.68 and 11.69 are still valid. This time we use Theorem 10.76 on the system 11.64, as the reductions we used to get 11.64 were $\mathbf{A}^{(t-1)}(\boldsymbol{\beta}_t^{(t-1)})$-invariant. Note that the analogue to (11.65) in this case is

$$\mathbb{P}_{2k}^{(t-1),*} \text{ is naturally isomorphic to } \mathbb{P}_{2k}^{(t),*}, \tag{11.73a}$$

$$\begin{gathered}\text{this isosmorphism sends the image of } \widehat{\mathbb{Q}}^{(t-1)}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1)}) \\ \text{in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t-1),*}) \text{ onto that of } \widehat{\mathbb{Q}}^{(t)}(\boldsymbol{\beta}_{2k-1,1k}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),*}), \text{i.e., } \mathbb{I}^{(t-1)} \cong \mathbb{I}^{(t)}\end{gathered} \tag{11.73b}$$

$$\mathbb{T}^{(t-1)} \cong \mathbb{T}^{(t)}, \tag{11.73c}$$

$$\text{Image of } \widehat{\mathbb{P}}^{(t-1)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t-1)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}) = \text{ Image of } \widehat{\mathbb{P}}^{(t)}(\boldsymbol{\alpha}_{2l-2,2l-1}^{(t)}) \text{ in } \mathrm{Aut}(\mathbb{Q}_{2l-1}^{(t),*}). \tag{11.73d}$$

As with the even case (note that this time $\mathbb{R}^{(t)} \cong \mathbb{R}^{(t-1)}$), we have

$$(\mathbb{T}^{(t-1)}, \mathbb{R}^{(t-1)}, \boldsymbol{\eta}^{(t-1)}, \mathbb{P}_{2k}^{(t-1),*}, \boldsymbol{\alpha}_{2k}^{(t-1),*}) \cong (\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_{2k}^{(t),*}, \boldsymbol{\alpha}_{2k}^{(t),*}). \tag{11.74}$$

Furthermore, Proposition 10.79 implies that any faithful linear limit of $(\mathbb{U}^{(t)}, \mathbb{S}^{(t)}, \boldsymbol{\zeta}^{(t)}, \mathbb{Q}_{2k}^{(t),*}, \boldsymbol{\beta}_{2k}^{(t),*})$ is also a faithful linear limit of $(\mathbb{U}^{(t-1)}, \mathbb{S}^{(t-1)}, \boldsymbol{\zeta}^{(t-1)}, \mathbb{Q}_{2k}^{(t-1),*}, \boldsymbol{\beta}_{2k}^{(t-1),*})$. These are enough for the proof of Theorem 11.68, when $t$ is odd.

The proof of Theorem 11.69 in the case of an odd $t$, is in the same spirit as the one we have already given for even $t$, so we omit it.

We can now give the proofs of Theorems 11.8, 11.18 and 11.33.

## 11.3 The $t = 1$ case

In this section we handle the $t = 1$ case, i.e., we give the proof of Theorem 11.8. For that we need a result stated as Theorem 11.76 below, which is interesting on its own.

We use the same notation about symplectic modules as the one introduced in Section 10.2. The "magic" theorem of this chapter is Theorem 3.2 in [1]. An equivalent form of this theorem we give here as Theorem 11.75.

**Theorem 11.75.** *Suppose that $F$ is a finite field of odd characteristic $p$, that $G$ is a finite $p$-solvable group, that $H$ is a subgroup of $p$-power index in $G$, that $B$ is an anisotropic symplectic $FG$-module and that $C$ is an $FG$-submodule of $B$. Then the $G$-invariant symplectic form on $B$ restricts to a $G$-invariant symplectic form on $C$. If $C$, with this form, restricts to a hyperbolic symplectic $FH$-module $C|_H$, then $C = 0$.*

*Proof.* Since $B$ is symplectic and $FG$-anisotropic, so is its $FG$-submodule $C$. Theorem 3.2 of [1], applied to $C$, tells us that $C$ is $FG$-hyperbolic if $C|_H$ is $FH$-hyperbolic. In that case $C$ is both $FG$-anisotropic and $FG$-hyperbolic. So it must be 0. □

After these preliminaries we can prove the first important theorem of this chapter.

**Theorem 11.76.** *Assume that $G$ is a $p,q$-group, where $p$ and $q$ are distinct odd primes, and that $N, M$ are normal subgroups of $G$. Let $M = P \times S$ and $N = P \rtimes Q$, where $P$ is a $p$-group, and $S, Q$ are $q$-groups with $S \leq Q$. Assume that the center $Z(P)$ of $P$ is maximal among the abelian $G$-invariant subgroups of $P$. Let $\chi, \alpha, \beta$ and $\zeta$ be irreducible characters of $G, P, S$ and $Z(P)$ respectively that satisfy*

$$\chi \in \mathrm{Irr}(G|\alpha \times \beta) \text{ and } \alpha \in \mathrm{Irr}(P|\zeta), \tag{11.77a}$$

$$\zeta \text{ is a faithful } G\text{-invariant character of } Z(P), \tag{11.77b}$$

$$G(\beta) = G, \tag{11.77c}$$

$$\chi \text{ is a monomial character of } G \text{ with } \chi(1)_q = \beta(1), \tag{11.77d}$$

*where $\chi(1)_q$ denotes the $q$-part of the integer $\chi(1)$. Then $Q$ centralizes $P$.*

*Proof.* First observe that $S, P$ are normal subgroups of $G$, as they are characteristic subgroups of $M \trianglelefteq G$. The existence of the $G$-invariant faithful character $\zeta \in \mathrm{Irr}(Z(P))$ implies that $Z(P)$ is a cyclic central subgroup of $G$, i.e., $Z(P) \leq Z(G)$. Furthermore, the fact that $Z(P)$ is maximal among the abelian subgroups of $P$ that are normal in $G$ implies that every characteristic abelian subgroup of $P$ is contained in $Z(P)$ and thus is cyclic. Hence P. Hall's theorem implies that either $P$ is an abelian group or it is the central product

$$P = T \odot Z(P), \tag{11.78a}$$

where $T$ is an extra special $p$-group of exponent $p$ and

$$T \cap Z(P) = Z(T). \tag{11.78b}$$

Note that the group $T$ is unique, as $T = \Omega(P)$.

In the case that $P = Z(P)$ is an abelian group, Theorem 11.76 holds trivially, as $P = Z(P) \leq Z(G)$ is centralized by $G$. Thus we may assume that $P > Z(P)$ and (11.78) holds.

Since $\chi$ is monomial, there exists a subgroup $H$ of $G$, and a linear character $\lambda \in \mathrm{Lin}(H)$ that induces $\chi = \lambda^G$. Then the product $HS$ forms a subgroup of $G$. Furthermore,

**Claim 11.79.** *$|G : HS|$ is a $p$-number, and $(\lambda^{HS})|_S = \beta$.*

*Proof.* Indeed, as $S \trianglelefteq G$, Clifford's theory, along with (11.77c), implies that

$$\chi|_S = m \cdot \beta,$$

for some integer $m \geq 0$. Hence $\deg(\chi) = m \deg(\beta)$. Since $\chi(1)_q = \beta(1)$, we have that $m$ is a power of $p$. As $H \leq HS \leq G$, the induced character $\lambda^{HS}$ lies in $\mathrm{Irr}(HS)$ and induces $(\lambda^{HS})^G = \lambda^G = \chi$. So

$$\deg(\lambda^{HS}) \cdot |G : HS| = \deg(\chi) = m \deg(\beta).$$

Clifford's theorem also implies that $\lambda^{HS}|_S = s\beta$, for some integer $s$. As $\deg(\lambda^{HS}) = |HS : H| = |S : H \cap S|$ we get that both $\deg(\lambda^{HS}|_S)$ and $s$ are $q$-numbers. But

$$s \deg(\beta)|G : HS| = \deg(\lambda^{HS}) \cdot |G : HS| = m \deg(\beta),$$

with $m$ a $p$-number. Hence $s = 1$, while $|G : HS|$ is a power of $p$. This completes the proof of the claim. □

The fact that $\lambda \in \mathrm{Lin}(H)$ induces irreducibly to $G$ implies that the center, $Z(G)$, of $G$ is a subgroup of $H$. This, along with the fact that $Z(P) \leq Z(G)$, implies

$$Z(P) \leq Z(G) \leq H. \tag{11.80}$$

Let $E := [P, Q]$. Then $E$ is a characteristic subgroup of $N$ and thus a normal subgroup of $G$. Even more, we have

**Claim 11.81.** *$E = [P, Q]$ is an abelian group.*

*Proof.* Suppose not. Then $E$ is a non-abelian normal subgroup of $G$ contained in $P = T \cdot Z(P)$. As $Z(P) \leq Z(G)$ (by (11.80)),we have $E = [P, Q] = [T, Q] \leq T$. Furthermore, $Z(E)$ is an abelian normal subgroup of $G$, contained in $T \leq P$. Hence $Z(E)$ is contained in $T \cap Z(P) = Z(T)$. As $E$ is non-abelian and $Z(T)$ has order $p$, we conclude that $Z(E) = Z(T) \leq Z(P) \leq Z(G)$. Therefore $E = [T, Q]$ is an extra special subgroup of $T$ of exponent $p$, whose center is central in $G$. Hence the group $E$ satisfies condition (4.3a) in [1]. In addition, $Q$ is a $p'$-subgroup of $G$ such that $QE$ is normal in $G$ (as $P = [P, Q]C(Q \text{ in } P)$ and thus $G = EN(Q \text{ in } G)$). Clearly the commutator subgroup $[E, Q] = [[P, Q], Q]$ coincides with $E = [P, Q]$. Hence (4.3b) in [1] holds with $Q$ here, in the place of $K$ there.

As the index of $HS$ in $G$ is a power of $p$, and $PQ = N$ is a normal subgroup of $G$, we conclude that $HS$ contains a $q$-Sylow subgroup of $PQ$. Hence $HS$ contains a $P$-conjugate of $Q$. Therefore, we may replace $H$ and $\lambda$ by some $P$-conjugates, and assume that $HS$ contains $Q$.

The subgroup $H \cap (E \times S)$ of $E \times S$ is equal to $(H \cap E) \times (H \cap S)$, since $|E|$ and $|S|$ are relatively prime. Note that $S$ centralizes $E$, as the latter is a subgroup of $P$. This implies that

$$HS \cap (E \times S) = (H \cap (E \times S)) \cdot S = (H \cap E) \times S.$$

Hence $H \cap E = HS \cap E$. Thus $H \cap E$ is a normal subgroup of $HS$. Furthermore, the restriction $\lambda|_{H \cap E}$ of $\lambda$ to $H \cap E$, is a linear character of $H \cap E$ that is clearly $H$-invariant. It is also $S$-invariant, as $S$ centralizes $E \geq H \cap E$. Hence $\lambda|_{H \cap E}$ is $HS$-invariant. We conclude that the restriction of the irreducible character $\lambda^{HS}$ of $HS$ to $H \cap E$ is a multiple of the linear character $\lambda|_{H \cap E}$. Of course the irreducible character $\lambda^{HS}$ of $HS$ induces irreducibly to $\chi \in \mathrm{Irr}(G)$, and lies above a non-trivial character of $Z(E)$ (as $Z(E) \leq Z(P)$ and $\zeta \in \mathrm{Irr}(Z(P))$ is non-trivial). Hence we can apply Lemma (4.4) and its Corollary (4.8) of [1], using $HS$ here in the place of $H$ there, and $\lambda^{HS}$ here in the place of $\phi$ there. We conclude that $HS \cap E = H \cap E$ is a maximal abelian subgroup of $E$.

Let $\bar{P} := P/Z(P)$. Then $\bar{P}$ is a symplectic $\mathbb{Z}_p(G)$-module. According to the hypotheses of the theorem, $Z(P)$ is the maximal abelian $G$-invariant subgroup of $P$. Hence $\bar{P}$ is an anisotropic $\mathbb{Z}_p(G)$-module. If $\bar{E}$ is the image of $E$ in $\bar{P}$, i.e., $\bar{E} \cong E/Z(E)$, then $\bar{E}$ is a symplectic $\mathbb{Z}_p(G)$-submodule of $\bar{P}$, as $E$ is normal in $G$. Furthermore, $\bar{E}$ is $\mathbb{Z}_p(HS)$-hyperbolic as $HS \cap E$ is a maximal abelian $HS$-invariant subgroup of $E$. Since the index $[G : HS]$ is a power of $p$, Theorem 11.75 forces $\bar{E}$ to be trivial. Hence $E = Z(E)$ is abelian, and the claim follows. □

Now $E = [P, Q]$ is an abelian subgroup of $P$ normal in $G$. According to the hypotheses of the theorem, $Z(P)$ is the maximal such subgroup. Therefore $1 \leq [P, Q] \leq Z(P) \leq Z(G)$. So $Q$ centralizes $[P, Q]$, which implies that $[P, Q, Q] = 1$ and thus $[P, Q] = [P, Q, Q] = 1$.

This completes the proof of the theorem. □

**Lemma 11.82.** *Assume that $G$ is a $p, q$-group, where $p$ and $q$ are distinct odd primes. Let $M = P \times S$ be a normal subgroup of $G$, where $P$ is a $p$-group and $S$ is a $q$-group. Assume further that $\beta$ is a $G$-invariant irreducible character of $S$ that can be extended to a $q$-Sylow subgroup $Q$ of $G$. Let $\alpha \in \mathrm{Irr}(P)$ be a $Q$-invariant character of $P$. Then there exists an irreducible character $\chi \in \mathrm{Irr}(G|\alpha \times \beta)$ with $\chi(1)_q = \beta(1)$.*

*Proof.* Let $A/S$ be a $p$-Sylow subgroup of $G/S$. Then the $q$-special character $\beta \in \mathrm{Irr}(S)$ can be extended to $A$, as $(|A : S|, \beta(1)o(\beta)) = 1$, (see Corollary 8.16 in [12]). As $\beta$ is also extendible to a $q$-Sylow subgroup of $G$, we conclude that it is extendible to $G$ (see Corollary 11.31 in [12]). Let $\beta^e$ be an extension of $\beta$ to $G$.

The irreducible character $\alpha$ of $P$ is $Q$-invariant. Hence Proposition 21.5 in [18] implies that the canonical extension $\alpha^e$ of $\alpha$ to $P \rtimes Q$ is the unique $p$-special character of $P \rtimes Q$ lying above $\alpha$. Furthermore, the index of the group $P \rtimes Q$ in $G$ is a $p$-number, as $Q$ is a $q$-Sylow subgroup of $G$. Therefore, the same proposition implies that any character of $G$ above $\alpha^e$ is $p$-special. Let $\Psi \in \mathrm{Irr}(G|\alpha^e)$ be such.

As $\Psi$ is a $p$-special character, while $\beta^e$ is a $q$-special character of $G$, Theorem 21.6 in [18] implies that the product $\chi := \Psi \cdot \beta^e$ is an irreducible character of $G$. Obviously $\chi$ lies above $\alpha \times \beta$, while $\chi(1)_q = \beta(1)$, as $\Psi(1)$ is a $p$-number. So the lemma follows. □

We can now complete the proof of the $t = 1$ case.

*Proof of Theorem 11.8.* According to Remark 11.5, the character $\boldsymbol{\beta}_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant, while $\mathbb{S}^{(1)} = Z(\mathbb{G}_1^{(1)})$ is maximal among the abelian $\mathbb{G}^{(1)}$-invariant subgroups of $\mathbb{G}_1^{(1)}$. Furthermore, $\mathbb{G}_1^{(1)}$ is a $q$-group (as a section of $G_1 = Q_1$), while $\mathbb{G}_2^{(1)}/\mathbb{G}_1^{(1)}$ is a $p$-group. Hence $\mathbb{G}_2^{(1)} = \mathbb{P}_2^{(1)} \ltimes \mathbb{G}_1^{(1)}$, where $\mathbb{P}_2^{(1)}$ is a $p$-Sylow subgroup of $\mathbb{G}_2^{(1)} = \mathbb{G}_2^{(1)}(\boldsymbol{\beta}^{(1)})$.

We can now apply Theorem 11.76, with the roles of $p$ and $q$ interchanged. So we work with $\mathbb{G}^{(1)}, \mathbb{G}_2^{(1)}, \mathbb{G}_1^{(1)}, \mathbb{S}^{(1)}$ and $\mathbb{P}_2^{(1)}$ in the place of $G, N, M = P, Z(P)$ and $Q$, respectively. As $S$ there, we take the trivial group here. We also use $\boldsymbol{\beta}_1^{(1)} = \Theta_1^{(1)}$ and $\boldsymbol{\zeta}^{(1)}$ here, in the place of $\alpha$ and $\zeta$ there. Of course $\beta \in \mathrm{Irr}(S)$ there, is the trivial character here. Clearly, $\boldsymbol{\zeta}^{(1)}$ is a faithful $\mathbb{G}^{(1)}$-invariant character of $\mathbb{S}^{(1)} = Z(\mathbb{G}_1^{(1)})$. Thus (11.77b) holds. Also (11.77c) holds trivially as $1 \in \mathrm{Irr}(1)$ is $\mathbb{G}^{(1)}$-invariant.

We are missing a monomial character of $\mathbb{G}^{(1)}$ that will play the role of $\chi \in \mathrm{Irr}(G)$ there. For that we will use some baby $\pi$-special theory. Indeed, the character $\boldsymbol{\beta}_1^{(1)}$ is $q$-special, as a character of the $q$-group $\mathbb{G}_1^{(1)}$. Furthermore, $\mathbb{G}_1^{(1)}$ is a normal subgroup of $\mathbb{G}^{(1)}$, while $\boldsymbol{\beta}_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant. Hence Corollary 4.8 in [7], implies that there exists a $q$-special character $\Theta^{(1)} \in \mathrm{Irr}(\mathbb{G})$ that lies above $\boldsymbol{\beta}_1^{(1)}$, and thus above $\boldsymbol{\zeta}^{(1)}$. According to Proposition 11.7, the character $\Theta^{(1)}$ is monomial. Now all the hypothesis of Theorem 11.76 are satisfied. We conclude that $\mathbb{P}_2^{(1)}$ centralizes $\mathbb{G}_1^{(1)}$. Hence $\mathbb{G}_2^{(1)} = \mathbb{P}_2^{(1)} \ltimes \mathbb{G}_1^{(1)} = \mathbb{P}_2^{(1)} \times \mathbb{G}_1^{(1)}$. This completes the proof of Theorem 11.8. $\square$

## 11.4 The $t = 2$ case

In this section we handle the $t = 2$ case. So we ultimate prove that the group $\mathbb{G}_3^{(2)}$ is nilpotent.

For the rest of the section we assume that $n \geq 3$. We also fix the subsystem

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq G, \tag{11.83a}$$

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{3} \tag{11.83b}$$

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{2,1} \tag{11.83c}$$

of the system (11.1). (Note that this is the $m = 3$ case for (11.44). So $k = 1$, $P_{2k}^* = P_2$ and $\beta_{2k-1,2k} = \beta_{1,2}$.) We also pick and fix the groups $\widehat{Q}$ and $\widehat{P}$, for the above system, so that Theorems 8.13 and 8.15 hold. This way the groups $T$ and $U$ are also fixed.

According to (11.45), after the first set of reductions the above system reduces to

$$1 = \mathbb{G}_0^{(1)} \trianglelefteq \mathbb{G}_1^{(1)} \trianglelefteq \mathbb{G}_2^{(1)} \trianglelefteq \mathbb{G}_3^{(1)} \trianglelefteq \mathbb{G}^{(1)} \tag{11.84a}$$

$$\{\Theta_i^{(1)} \in \mathrm{Irr}(\mathbb{G}_i^{(1)})\}_{i=0}^{3} \tag{11.84b}$$

$$\{\mathbb{Q}_{2i-1}^{(1)}, \mathbb{P}_{2r}^{(1)} | \boldsymbol{\beta}_{2i-1}^{(1)}, \boldsymbol{\alpha}_{2r}^{(1)}\}_{i=1,r=0}^{2,1} \tag{11.84c}$$

According to Remark 11.5, the character $\boldsymbol{\beta}_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant. Furthermore, Theorem 11.8 implies that the group $\mathbb{G}_2^{(1)}$ is nilpotent. Hence $\mathbb{Q}_{1,2}^{(1)} = C(\mathbb{P}_2^{(1)} \text{ in } \mathbb{Q}_1^{(1)}) = \mathbb{Q}_1^{(1)}$. So the character $\boldsymbol{\beta}_{1,2}^{(1)}$ coincides with $\boldsymbol{\beta}_1^{(1)}$ (its $\mathbb{P}_2^{(1)}$-Glauberman correspondent). Thus $\mathbb{G}^{(1)}(\boldsymbol{\beta}_{1,2}^{(1)}) = \mathbb{G}^{(1)}(\boldsymbol{\beta}_1^{(1)}) = \mathbb{G}^{(1)}$. In particular, for the $q$-Sylow subgroup of $\mathbb{G}^{(1)}(\boldsymbol{\alpha}^{(2)})$ we have $\widehat{\mathbb{Q}}^{(1)}(\boldsymbol{\beta}_{1,2}^{(1)}) = \widehat{\mathbb{Q}}^{(1)}$. This, along with (11.47), implies (note that $k = 1$)

$$\text{Image of } \widehat{Q}(\beta_{1,2}) \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(1)}) \ = \ \text{Image of } \widehat{\mathbb{Q}}^{(1)} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(1)}). \tag{11.85}$$

Furthermore, (11.49) holds.

For the next set of reductions, we first have to shift the system (11.84) (see (11.50)), and then we perform the $\mathbf{B}^{(1)}(\boldsymbol{\alpha}_2^{(1)})$-invariant reductions. We end up with the system (11.51), which for $m = 3$ (i.e., the case here) gives

$$1 \trianglelefteq \mathbb{P}_2^{(2)} \trianglelefteq \mathbb{G}_3^{(2)} \trianglelefteq \mathbb{G}^{(2)}, \tag{11.86a}$$

$$1, \boldsymbol{\alpha}_2^{(2)}, \Theta_3^{(2)} \tag{11.86b}$$

$$\{\mathbb{Q}_3^{(2)}, \mathbb{P}_2^{(2)} | \boldsymbol{\beta}_3^{(2)}, \boldsymbol{\alpha}_2^{(2)}\} \tag{11.86c}$$

Note that, as $\mathbb{Q}_1^{(1)}$ is a normal subgroup of $\mathbb{G}^{(1)}$ that centralizes $\mathbb{P}_2^{(1)}$, Remark 10.7 implies that the group $\mathbb{Q}_1^{(1)}$ maps isomorphically to a normal subgroup $\mathbb{Q}_1^{(2)}$ of $\mathbb{G}^{(2)}$ that centralizes $\mathbb{P}_2^{(2)}$. In addition, its irreducible character $\boldsymbol{\beta}_1^{(1)} \in \mathrm{Irr}(\mathbb{Q}_1^{(1)})$ maps to $\boldsymbol{\beta}_1^{(2)} \in \mathrm{Irr}(\mathbb{Q}_1^{(2)})$. The fact that $\boldsymbol{\beta}_1^{(1)}$ is $\mathbb{G}^{(1)}$-invariant, while $\mathbb{G}^{(2)}$ is a section of $\mathbb{G}^{(1)}$, implies that $\boldsymbol{\beta}_1^{(2)}$ is $\mathbb{G}^{(2)}$-invariant, and thus $\widehat{\mathbb{Q}}^{(2)}$-invariant. This, along with Theorem 11.57, implies

$$\text{Image of } \widehat{\mathbb{Q}}^{(2)} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2)}) \;=\; \text{Image of } \widehat{Q}(\beta_{1,2}) \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2)}). \tag{11.87a}$$

Even more, Theorem 11.56 implies

$$\begin{gathered}\text{Any faithful linear limit of } (\mathbb{T}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)}) \text{ is also a} \\ \text{faithful linear limit of } (T, R, \eta, P_2, \alpha_2).\end{gathered} \tag{11.87b}$$

According to Remark 11.13, the character $\boldsymbol{\alpha}_2^{(2)}$ is $\mathbb{G}^{(2)}$-invariant. Hence the $q$-Sylow subgroup $\widehat{\mathbb{Q}}^{(2)}$ of $\mathbb{G}^{(2)}(\boldsymbol{\alpha}_2^{(2)})$ satisfies

$$\widehat{\mathbb{Q}}^{(2)} \in \mathrm{Syl}_q(\mathbb{G}^{(2)}). \tag{11.88}$$

Having fixed the system (11.83), and the groups $\widehat{Q}$ and $T$, we can get a new system, via Corollary 7.29. That is, we get a new character tower

$$1 = \chi_0^\nu, \chi_1^\nu, \chi_2^\nu \tag{11.89a}$$

for the series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G, \tag{11.89b}$$

along with a triangular set

$$\{1 = P_0^\nu, P_2^\nu, Q_1^\nu | 1 = \alpha_0^\nu, \alpha_2^\nu, \beta_1^\nu\} \tag{11.89c}$$

and a $q$-Sylow subgroup $\widehat{Q^\nu}$ of $G(\alpha_2^\nu)$, so that (7.30) holds. In particular, we have

$$P_2 = P_2^\nu = P_2^{\nu,*} \text{ and } \alpha_2 = \alpha_2^\nu = \alpha_2^{\nu,*}, \tag{11.90a}$$

$$\beta_{1,2}^\nu \text{ extends to } \widehat{Q} = \widehat{Q^\nu}. \tag{11.90b}$$

Observe that the character $\beta_{1,2}^\nu$ is $q$-special, as it is an irreducible character of the $q$-group $Q_{1,2}^\nu$. Hence the fact that it extends to a $q$-Sylow subgroup of $G(\alpha_2^\nu)$ implies (see Corollary 11.31 in [12]) that it extends to $G(\alpha_2^\nu)$. As expected, we write $I^\nu$ for the image of $\widehat{Q}^\nu(\beta_{1,2}^\nu) = \widehat{Q}^\nu$ $\mathrm{Aut}(P_2^{*,\nu})$. Therefore (11.90) implies that $I^\nu$ is the image of $\widehat{Q}$ in $\mathrm{Aut}(P_2)$.

Assume we perform the reduction procedure described in Section 11.1 for this new system. We assume that $m = n = 2$ here. We keep the same notation as earlier, with the addition of the superscript $\nu$ to any group that refers to the new system. So we first start with $\mathbf{A}^\nu(\beta_1^\nu)$-invariant linear reductions of the quintuple $(G, S^\nu, \zeta^\nu, G_1^\nu, \chi_1^\nu)$, where $S^\nu = 1 = S$ and $\zeta^\nu = 1 = \zeta$. Furthermore, $G_1^\nu = G_1$ and $\chi_1^\nu = \beta_1^\nu$. Let

$$(\mathbb{G}^{(1),\nu}, \mathbb{S}^{(1),\nu}, \boldsymbol{\zeta}^{(1),\nu}, \mathbb{G}^{(1),\nu}, \boldsymbol{\beta}_1^{(1),\nu}) \tag{11.91}$$

be an $\mathbf{A}^\nu(\beta_1^\nu)$-invariant faithful limit of $(G, S^\nu, \zeta^\nu, G_1^\nu, \chi_1^\nu)$. Note that, even though we start with $S^\nu = S$ and $G_1^\nu = G_1$, the limit groups $\mathbb{S}^{(1),\nu}, \mathbb{G}_1^{(1),\nu}$ and $\mathbb{S}^{(1)}, \mathbb{G}_1^{(1)}$ are not the same, as the reductions we perform may be different. At this point the system (11.89) has been reduced to

$$1 = \mathbb{G}_0^{(1),\nu} \trianglelefteq \mathbb{G}_1^{(1),\nu} \trianglelefteq \mathbb{G}_2^{(1),\nu} \trianglelefteq \mathbb{G}^{(1),\nu}, \tag{11.92a}$$

$$1, \boldsymbol{\beta}_1^{(1),\nu}, \Theta_2^{(1),\nu} \tag{11.92b}$$

$$\{1 = \mathbb{P}_0^{(1),\nu}, \mathbb{P}_2^{(1),\nu}, \mathbb{Q}_1^{(1),\nu} | 1 = \boldsymbol{\alpha}_0^{(1),\nu}, \boldsymbol{\alpha}_2^{(1),\nu}, \boldsymbol{\beta}_1^{(1),\nu}\} \tag{11.92c}$$

where, (according to Theorem 11.8), the limit group $\mathbb{G}_2^{(1),\nu}$ is nilpotent, i.e., $\mathbb{G}_2^{(1),\nu} = \mathbb{P}_2^{(1),\nu} \times \mathbb{Q}_1^{(1),\nu}$. So

$$\mathbb{Q}_1^{(1),\nu} = \mathbb{Q}_{1,2}^{(1),\nu},$$

and the limit character $\boldsymbol{\beta}_1^{(1),\nu}$ coincides with its $\mathbb{P}_2^{(1),\nu}$-Glauberman correspondent, i.e.

$$\boldsymbol{\beta}_1^{(1),\nu} = \boldsymbol{\beta}_{1,2}^{(1),\nu}.$$

According to Theorem 10.77, the character $\boldsymbol{\beta}_{1,2}^{(1),\nu} = \boldsymbol{\beta}_1^{(1),\nu}$ extends to $\widehat{\mathbb{Q}}^{(1),\nu}(\boldsymbol{\beta}_{1,2}^{(1),\nu})$ , as the character $\beta_{1,2}^{\nu}$ extends to $\widehat{Q} = \widehat{Q^{\nu}}$. But $\boldsymbol{\beta}_1^{(1),\nu}$ is $\mathbb{G}^{(1),\nu}$-invariant by Remark 11.5. Hence $\boldsymbol{\beta}_1^{(1),\nu}$ extends to $\widehat{\mathbb{Q}}^{(1),\nu}$. Even more, in this case we have $\mathbb{P}_{2k}^{(1),\nu,*} = \mathbb{P}_2^{(1),\nu,*} = \mathbb{P}_2^{(1),\nu}$. Therefore, if $\mathbb{I}^{(1),\nu}$ denotes the image of $\widehat{\mathbb{Q}}^{(1),\nu}(\boldsymbol{\beta}_{1,2}^{(1),\nu})$ in $\mathrm{Aut}(\mathbb{P}_2^{(1),\nu,*})$, the above equations, along with (11.47) for the new system, imply

$$\text{Image of } (\widehat{Q} = \widehat{Q^{\nu}}) \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(1),\nu}) \; = \mathbb{I}^{(1),\nu} = \; \text{Image of } \widehat{\mathbb{Q}}^{(1),\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(1),\nu}). \tag{11.93}$$

Furthermore, (11.4) implies that $\mathbb{R}^{(1),\nu} \trianglelefteq \mathbb{P}_2^{(1),\nu}$, while its irreducible character $\boldsymbol{\eta}^{(1),\nu}$ lies under $\boldsymbol{\alpha}_2^{(1),\nu}$.

To perform the next set of reductions on (11.92), we have to shift it first (see (11.50) with $m = 2$). So we get

$$1 \trianglelefteq \mathbb{P}_2^{(1),\nu} \trianglelefteq \mathbb{G}^{(1),\nu}, \tag{11.94a}$$

$$1, \boldsymbol{\alpha}_2^{(1),\nu} \tag{11.94b}$$

$$\{\mathbb{P}_2^{(1),\nu} | \boldsymbol{\alpha}_2^{(1),\nu}\} \tag{11.94c}$$

Now we can take a $\mathbf{B}^{(1),\nu}(\boldsymbol{\alpha}_2^{(1),\nu})$-invariant faithful linear limit

$$(\mathbb{G}^{(2),\nu}, \mathbb{R}^{(2),\nu}, \boldsymbol{\eta}^{(2),\nu}, \mathbb{P}_2^{(2),\nu}, \boldsymbol{\alpha}_2^{(2),\nu}), \tag{11.95}$$

of the quintuple $(\mathbb{G}^{(1),\nu}, \mathbb{R}^{(1),\nu} = 1, \boldsymbol{\eta}^{(1),\nu} = 1, \mathbb{P}_2^{(1),\nu}, \boldsymbol{\alpha}_2^{(1),\nu})$. This way (11.94) is reduced to

$$1 \trianglelefteq \mathbb{P}_2^{(2),\nu} \trianglelefteq \mathbb{G}^{(2),\nu}, \tag{11.96a}$$

$$1, \boldsymbol{\alpha}_2^{(2),\nu} \tag{11.96b}$$

$$\{\mathbb{P}_2^{(2),\nu} | \boldsymbol{\alpha}_2^{(2),\nu}\} \tag{11.96c}$$

Then, according to Remark 11.13, the center $\mathbb{R}^{(2),\nu}$ of $\mathbb{P}_2^{(2),\nu}$ is maximal among the abelian $\mathbb{G}^{(2),\nu}$-

invariant subgroups of $\mathbb{P}_2^{(2),\nu}$, while $\boldsymbol{\alpha}_2^{(2),\nu}$ is $\mathbb{G}^{(2),\nu}$-invariant. Hence

$$\mathbb{G}^{(2),\nu}(\boldsymbol{\alpha}_2^{(2),\nu}) = \mathbb{G}^{(2),\nu}(\boldsymbol{\alpha}_2^{(2),\nu,*}) = \mathbb{G}^{(2),\nu}. \tag{11.97}$$

Thus the $q$-Sylow subgroup $\widehat{\mathbb{Q}}^{(2),\nu}$ of $\mathbb{G}^{(2),\nu}(\boldsymbol{\alpha}_2^{(2),\nu})$ is actually a $q$-Sylow subgroup of $\mathbb{G}^{(2),\nu}$. Furthermore, Remark 10.7 implies that the normal subgroup $\mathbb{Q}_1^{(1),\nu}$ of $\mathbb{G}^{(1),\nu}$, that centralizes $\mathbb{P}_2^{(1),\nu}$, maps isomorphically to a normal subgroup $\mathbb{Q}_1^{(2),\nu}$ of the limit group $\mathbb{G}^{(2),\nu}$. So

$$\mathbb{Q}_1^{(1),\nu} \cong \mathbb{Q}_1^{(2),\nu} \trianglelefteq \mathbb{G}^{(2),\nu}. \tag{11.98}$$

Therefore the group $M := \mathbb{P}_2^{(2),\nu} \times \mathbb{Q}_1^{(2),\nu}$ is a normal subgroup of $\mathbb{G}^{(2),\nu}$. Under the isomorphism in (11.98), the character $\boldsymbol{\beta}_1^{(1),\nu}$ of $\mathbb{Q}_1^{(1),\nu}$ maps to the character $\boldsymbol{\beta}_1^{(2),\nu} \in \mathrm{Irr}(\mathbb{Q}_1^{(2),\nu})$. Note that $\mathbb{Q}_1^{(2),\nu}$ is the faithful linear limit of $\mathbb{Q}_1^{(1),\nu}$, under the $\mathbf{B}^{(1),\nu}(\boldsymbol{\alpha}_2^{(1),\nu})$-invariant linear reductions we perform. In addition, $\boldsymbol{\beta}_1^{(2),\nu}$ is the faithful linear limit of $\boldsymbol{\beta}_1^{(1),\nu}$ and $M = \mathbb{P}_2^{(2),\nu} \times \mathbb{Q}_1^{(2),\nu}$ that of $\mathbb{P}_2^{(1),\nu} \times \mathbb{Q}_1^{(1),\nu} = \mathbb{G}_2^{(1),\nu}$. The character $\boldsymbol{\beta}_1^{(2),\nu}$ is $\mathbb{G}^{(2),\nu}$-invariant as $\boldsymbol{\beta}_1^{(1),\nu}$ is $\mathbb{G}^{(1),\nu}$-invariant. Hence $\boldsymbol{\beta}_1^{(2),\nu}$ is $\widehat{\mathbb{Q}}^{(2),\nu}$-invariant. This, along with Theorem 11.57, implies

$$\text{Image of } \widehat{\mathbb{Q}}^{(2),\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2),\nu}) \;=\; \text{Image of } \widehat{Q}^{\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2),\nu}). \tag{11.99a}$$

Furthermore, Theorem 11.56 implies for the new system

$$\begin{gathered}\text{Any faithful linear limit of } (\mathbb{T}^{(2),\nu}, \mathbb{R}^{(2),\nu}, \boldsymbol{\eta}^{(2),\nu}, \mathbb{P}_2^{(2),\nu}, \boldsymbol{\alpha}_2^{(2),\nu}) \text{ is also a}\\ \text{faithful linear limit of } (T^{\nu}, R^{\nu}, \eta^{\nu}, P_2^{\nu}, \alpha_2^{\nu}).\end{gathered} \tag{11.99b}$$

(Observe that (11.99) is the analogue of (11.87) for the new system (11.89).) Note that $R^{\nu} = 1 = R$ while $\eta^{\nu} = 1 = \eta$. Furthermore, $P_2^{\nu} = P_2$ and $\alpha_2^{\nu} = \alpha^{\nu}$, by (11.90a). In addition, the image $I^{\nu}$ of $\widehat{Q^{\nu}}(\beta_{1,2}^{\nu})$ in $\mathrm{Aut}(P_2^{\nu})$ equals the image of $\widehat{Q}$ in $\mathrm{Aut}(P_2^{\nu})$, as $\widehat{Q} = \widehat{Q^{\nu}} = \widehat{Q^{\nu}}(\beta_{1,2}^{\nu})$ by (11.90b). Hence the image $I$ of $\widehat{Q}(\beta_{1,2})$ in $\mathrm{Aut}(P_2)$, is a subgroup of $I^{\nu}$. So $T^{\nu} = P_2^{\nu} \rtimes I^{\nu} = P_2 \rtimes I^{\nu}$ contains $T = P_2 \rtimes I$, while $T \geq P_2^{\nu} = P_2$. This, along with Remark 10.5, Definition 10.6 and (11.99), implies that

$$\begin{gathered}\text{If } (\mathbb{T}^{\nu}, \mathbb{R}^{\nu}, \eta^{\nu}, \mathbb{P}_2^{\nu}, \alpha_2^{\nu}) \text{ is a faithful linear limit of } (\mathbb{T}^{(2),\nu}, \mathbb{R}^{(2),\nu}, \boldsymbol{\eta}^{(2),\nu}, \mathbb{P}_2^{(2),\nu}, \boldsymbol{\alpha}_2^{(2),\nu})\\ \text{then } (\mathbb{T}^{\nu} \cap T, \mathbb{R}^{\nu}, \eta^{\nu}, \mathbb{P}_2^{\nu}, \alpha_2^{\nu}) \text{ is a faithful linear limit of } (T, R, \eta, P_2, \alpha_2).\end{gathered} \tag{11.99c}$$

We give the rest of the proof in a series of steps

**Step 1.** *The character $\boldsymbol{\beta}_1^{(2),\nu}$ is $\mathbb{G}^{(2),\nu}$-invariant, and extends to a q-Sylow subgroup of $\mathbb{G}^{(2),\nu}$.*

*Proof.* The group $\widehat{\mathbb{Q}}^{(2),\nu}$ is both a faithful invariant limit of $\widehat{\mathbb{Q}}^{(1),\nu}$, and isomorphic to the latter group. Similarly, $\mathbb{Q}_1^{(2),\nu}$ is both a faithful invariant limit of, and isomorphic to, $\mathbb{Q}_1^{(1),\nu}$. In addition, $\boldsymbol{\beta}_1^{(2),\nu}$ is the faithful linear limit of $\boldsymbol{\beta}_1^{(1),\nu}$. Hence the fact that $\boldsymbol{\beta}_1^{(1),\nu}$ is $\mathbb{G}^{(1),\nu}$-invariant implies that $\boldsymbol{\beta}_1^{(2),\nu}$ is $\mathbb{G}^{(2),\nu}$-invariant. Furthermore, as $\boldsymbol{\beta}_1^{(1),\nu}$ extends to $\widehat{\mathbb{Q}}^{(1),\nu}$, we conclude that $\boldsymbol{\beta}_1^{(2),\nu}$ extends to $\widehat{\mathbb{Q}}^{(2),\nu}$, that is a $q$-Sylow subgroup of $\mathbb{G}^{(2),\nu}$. So the first step follows. □

**Step 2.** *There exists a monomial character $\Theta^{(2),\nu} \in \mathrm{Irr}(\mathbb{G}^{(2),\nu})$ lying above $\boldsymbol{\alpha}_2^{(2),\nu} \times \boldsymbol{\beta}_1^{(2),\nu}$ and satisfying $\Theta^{(2),\nu}(1)_q = \boldsymbol{\beta}_1^{(2),\nu}(1)$.*

*Proof.* As $\boldsymbol{\beta}_1^{(2),\nu}$ extends to a $q$-Sylow subgroup of $\mathbb{G}^{(2),\nu}$, while $\boldsymbol{\alpha}_2^{(2),\nu}$ is $\mathbb{G}^{(2),\nu}$-invariant, and $\mathbb{P}_2^{(2),\nu} \times \mathbb{Q}_1^{(2),\nu}$ is a normal subgroup of $\mathbb{G}^{(2),\nu}$, Lemma 11.82 implies the existence of an irreducible character $\Theta^{(2),\nu} \in \mathrm{Irr}(\mathbb{G}^{(2),\nu})$ that lies above $\boldsymbol{\alpha}_2^{(2),\nu} \times \boldsymbol{\beta}_1^{(2),\nu}$ and satisfies $\Theta^{(2),\nu}(1)_q = \boldsymbol{\beta}_1^{(2),\nu}(1)$. It suffices to show that $\Theta^{(2),\nu}$ is monomial. The character $\boldsymbol{\beta}_1^{(1),\nu}$ lies above $\boldsymbol{\zeta}^{(1),\nu}$ (see (11.91)), hence its faithful linear limit $\boldsymbol{\beta}_1^{(2),\nu}$ lies above the faithful linear limit $\boldsymbol{\zeta}^{(2),\nu}$ of $\boldsymbol{\zeta}^{(1),\nu}$, (see (11.15) and the following remarks for the definition of $\boldsymbol{\zeta}^{(2),\nu}$). In addition, $\boldsymbol{\alpha}_2^{(2),\nu}$ lies above $\boldsymbol{\eta}^{(2),\nu}$ (see (11.95)). Hence $\Theta^{(2),\nu} \in \mathrm{Irr}(\mathbb{G}^{(2),\nu})$ lies above $\boldsymbol{\eta}^{(2),\nu} \times \boldsymbol{\zeta}^{(2),\nu} \in \mathrm{Irr}(\mathbb{R}^{(2),\nu} \times \mathbb{S}^{(2),\nu})$. Therefore, Proposition 11.17 implies that $\Theta^{(2),\nu}$ is monomial. $\square$

**Step 3.**

$$\textit{Image of } \widehat{Q} \cap G_3 \textit{ in } \mathrm{Aut}(\mathbb{P}_2^{(2),\nu}) \ = 1.$$

*Proof.* Let $N$ be any normal subgroup of $\mathbb{G}^{(2),\nu}$ with $M = \mathbb{P}_2^{(2),\nu} \times \mathbb{Q}_1^{(2),\nu} \leq N \trianglelefteq \mathbb{G}^{(2),\nu}$, and $N/M$ a $q$-group. Then $N = \mathbb{P}_2^{(2),\nu} \rtimes Q$ for some $q$-group $Q$, with $\mathbb{Q}_1^{(2),\nu} \trianglelefteq Q$. Furthermore, as we have seen, $Z(\mathbb{P}_2^{(2),\nu}) = \mathbb{R}^{(2),\nu}$ is maximal among the abelian $\mathbb{G}^{(2),\nu}$-invariant subgroups of $\mathbb{P}_2^{(2),\nu}$, while $\boldsymbol{\alpha}_2^{(2),\nu} \in \mathrm{Irr}(\mathbb{P}_2^{(2),\nu})$ lies above the $\mathbb{G}^{(2),\nu}$-invariant faithful irreducible character $\boldsymbol{\eta}^{(2),\nu} \in \mathrm{Irr}(\mathbb{R}^{(2),\nu})$. This, along with Step 1 and 2, implies that we can apply Theorem 11.76 with the groups $\mathbb{G}^{(2),\nu}, \mathbb{P}_2^{(2),\nu}, \mathbb{Q}_1^{(2),\nu}, Q$ here, in the place of the groups $G, P, S, Q$ there, and the characters $\Theta^{(2)}, \boldsymbol{\alpha}_2^{(2),\nu}, \boldsymbol{\beta}_1^{(2),\nu}$ and $\boldsymbol{\eta}^{(2),\nu}$ here, in the place of $\chi, \alpha, \beta$ and $\zeta$ there. We conclude that any such normal subgroup $N$ of $\mathbb{G}^{(2),\nu}$ is nilpotent, i.e.,

$$\begin{gathered} Q \in \mathrm{Syl}_q(N) \text{ centralizes } \mathbb{P}_2^{(2),\nu} \in \mathrm{Syl}_p(N), \text{ whenever} \\ M \trianglelefteq N \trianglelefteq \mathbb{G}^{(2),\nu} \text{ with } N/M \text{ a } q\text{-group.} \end{gathered} \tag{11.100}$$

The group $G_3$ is a normal subgroup of $G$ that contains $G_1 = G_1^\nu$ and $G_2 = G_2^\nu$. Hence, see Remark 10.5, when the normal series (11.89b) reduces to (11.96a), the group $G_3$ reduces to a normal subgroup $\mathbb{G}_3^{(2),\nu}$ of $\mathbb{G}^{(2),\nu}$. Furthermore, $\mathbb{G}_3^{(2),\nu}/M$ is a $q$-group as $G_3/G_2$ is a $q$-group, and $M$ is the limit of $\mathbb{G}_2^{(1),\nu}$ and thus of $G_2$. As $\widehat{\mathbb{Q}}^{(2),\nu}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(2),\nu}$, we get that $\widehat{\mathbb{Q}}^{(2),\nu} \cap \mathbb{G}_3^{(2),\nu}$ is a $q$-Sylow subgroup of $\mathbb{G}_3^{(2),\nu}$. Hence (11.100) implies

$$\widehat{\mathbb{Q}}^{(2),\nu} \cap \mathbb{G}_3^{(2),\nu} \text{ centralizes } \mathbb{P}_2^{(2),\nu} \in \mathrm{Syl}_p(\mathbb{G}_3^{(2),\nu}).$$

This, along with (11.99a) implies

$$\text{Image of } \widehat{Q}^\nu \cap G_3 \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2),\nu}) \ = \ \text{Image of } \widehat{\mathbb{Q}}^{(2),\nu} \cap \mathbb{G}_3^{(2),\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2),\nu}) \ = 1.$$

As $\widehat{Q} = \widehat{Q}^\nu$, Step 3 follows. $\square$

Assume that $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_2, \boldsymbol{\alpha}_2)$ and $(\mathbb{T}^\nu, \mathbb{R}^\nu, \eta^\nu, \mathbb{P}_2^\nu, \alpha_2^\nu)$ are faithful linear limits of the quintuples $(\mathbb{T}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)})$ and $(\mathbb{T}^{(2),\nu}, \mathbb{R}^{(2),\nu}, \boldsymbol{\eta}^{(2),\nu}, \mathbb{P}_2^{(2),\nu}, \boldsymbol{\alpha}_2^{(2),\nu})$, respectively. Then according to (11.87) and (11.99) the quintuples $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_2, \boldsymbol{\alpha}_2)$ and $(\mathbb{T}^\nu \cap T, \mathbb{R}^\nu, \eta^\nu, \mathbb{P}_2^\nu, \alpha_2^\nu)$ are both faithful linear limits of $(T, R, \eta, P_2, \alpha_2)$. Hence Theorem 10.19 and Corollary 10.35 imply that $\mathbb{P}_2/\mathbb{R}$ and $\mathbb{P}_2^\nu/\mathbb{R}^\nu$ are isomorphic anisotropic symplectic $\mathbb{Z}_p(T(\alpha_2)/P_2)$-modules. The group $T = P_2 \rtimes I$ fixes $\alpha_2 \in \mathrm{Irr}(P_2)$, as $\widehat{Q}$ fixes $\alpha_2 = \alpha_2^*$ and $I$ is the image of $\widehat{Q}(\beta_{1,2})$ in $\mathrm{Aut}(P_2)$. Hence $T(\alpha_2)/P_2$ is naturally isomorphic to the $q$-group $I$. So

$$\mathbb{P}_2/\mathbb{R} \cong \mathbb{P}_2^\nu/\mathbb{R}^\nu \text{ as anisotropic symplectic } \mathbb{Z}_p(I)\text{-modules.} \tag{11.101}$$

In view of Step 3, the group $\widehat{Q} \cap G_3$ centralizes $\mathbb{P}_2^{(2),\nu}$. As $\mathbb{P}_2^\nu$ is a section of $\mathbb{P}_2^{(2),\nu}$, we conclude that $\widehat{Q}(\beta_{1,2}) \cap G_3$ centralizes $\mathbb{P}_2^\nu$. Hence it also centralizes the factor group $\mathbb{P}_2^\nu/\mathbb{R}^\nu$. As the latter factor group is isomorphic to $\mathbb{P}_2/\mathbb{R}$ as $I$-modules, and $I$ is the image of $\widehat{Q}(\beta_{1,2})$ in $\mathrm{Aut}(P_2) = \mathrm{Aut}(P_2^\nu)$, we conclude that $\widehat{Q}(\beta_{1,2}) \cap G_3$ also centralizes $\mathbb{P}_2/\mathbb{R}$. (The action of $I$ on the above two isomorphic factor groups is defined in Corollary 10.35.) If $\mathbb{G}_3^{(2)}$ denotes the limit group to which $G_3$ reduces, as $G$ reduces to $\mathbb{G}^{(2)}$, then (11.87a) implies

$$I_3 := \text{ Image of } \widehat{Q}(\beta_{1,2}) \cap G_3 \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2)}) = \text{ Image of } \widehat{\mathbb{Q}}^{(2)} \cap \mathbb{G}_3^{(2)} \text{ in } \mathrm{Aut}(\mathbb{P}_2^{(2)}). \tag{11.102}$$

Since $\mathbb{P}_2$ is a section of $\mathbb{P}_2^{(2)}$, the fact that $\widehat{Q}(\beta_{1,2}) \cap G_3$ centralizes $\mathbb{P}_2/\mathbb{R}$, implies

$$I_3 \text{ centralizes } \mathbb{P}_2/\mathbb{R}. \tag{11.103}$$

Let $V := \mathbb{P}_2^{(2)}/\mathbb{R}^{(2)}$. Then $V$ is anisotropic $\mathbb{Z}_p(\mathbb{G}^{(2)})$-module (by Remark 11.13). Thus $V$, when written additively, is the direct sum

$$V = V_1 \dotplus V_2, \tag{11.104a}$$

of the perpendicular $\mathbb{Z}_p(\mathbb{G}^{(2)})$-modules, $V_1 = C(\mathbb{G}_3^{(2)} \text{ in } V)$ and $V_2 = [V, \mathbb{G}_3^{(2)}]$. Note that, as $\widehat{\mathbb{Q}}^{(2)}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(2)}$, see (11.88), we get that $\widehat{\mathbb{Q}}_3^{(2)} = \widehat{\mathbb{Q}}^{(2)} \cap \mathbb{G}_3^{(2)}$ is a $q$-Sylow subgroup of $\mathbb{G}_3^{(2)}$. Thus $\mathbb{G}_3^{(2)} = \widehat{\mathbb{Q}}_3^{(2)} \ltimes \mathbb{P}_2^{(2)}$. We conclude that the direct summands in (11.104a) are

$$V_1 = C(\widehat{\mathbb{Q}}_3^{(2)} \text{ in } V) \text{ and } V_2 = [V, \widehat{\mathbb{Q}}_3^{(2)}]. \tag{11.104b}$$

Both $V_1$ and $V_2$ are anisotropic $\mathbb{Z}_p(\mathbb{G}^{(2)})$-modules.

As $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_2, \boldsymbol{\alpha}_2)$ is a faithful linear limit of $(\mathbb{T}^{(2)}, \mathbb{R}^{(2)}, \boldsymbol{\eta}^{(2)}, \mathbb{P}_2^{(2)}, \boldsymbol{\alpha}_2^{(2)})$, we have that $U := \mathbb{P}_2/\mathbb{R}$ is isomorphic to a factor subgroup of $V$. Furthermore, $U$ is isomorphic, as a symplectic $\mathbb{Z}_p(I)$-module, to the orthogonal direct sum $U = U_1 \dotplus U_2$, where $U_i$ is a limit module for $V_i$, for each $i = 1, 2$. In view of (11.104b) we have

$$U_1 = C(I_3 \text{ in } U) \text{ and } U_2 = [U, I_3], \tag{11.105}$$

where $I_3$ is the image of $\widehat{\mathbb{Q}}_3^{(2)}$ in $\mathrm{Aut}(\mathbb{P}_2^{(2)})$ (see (11.102)). In view of (11.103) we get $U_2 = 0$.

According to Remark 11.13, the center $\mathbb{R}^{(2)}$ of $\mathbb{P}_2^{(2)}$ is a cyclic central subgroup of $\mathbb{G}^{(2)}$, maximal among the abelian $\mathbb{G}^{(2)}$-invariant subgroups of $\mathbb{P}_2^{(2)}$. As $\mathbb{T}^{(2)} = \mathbb{P}_2^{(2)} \rtimes \mathbb{I}^{(2)}$, where $\mathbb{I}^{(2)}$ is the image of $\widehat{\mathbb{Q}}^{(2)}$ in $\mathrm{Aut}(\mathbb{P}_2^{(2)})$, we get that $\mathbb{R}^{(2)}$ is a central subgroup of $\mathbb{T}^{(2)}$. Even more, it is maximal among the characteristic abelian subgroups of $\mathbb{P}_2^{(2)}$. Thus Proposition 10.47 applies to the faithful linear limit $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_2, \boldsymbol{\alpha}_2)$. So $U$ is isomorphic to $W^\perp/W$ for some maximal $I$-invariant totally isotropic subspace $W$ of $V$. Then $W = W_1 \dotplus W_2$, where $W_i$ is a maximal totally isotropic $I$-invariant subspace of $V_i$, for $i = 1, 2$. But $U_2 = 0$. Hence $W_2^\perp$ must equal $W_2$. Thus $V_2$ contains a self perpendicular $I$-invariant subspace. We conclude that $V_2$ is hyperbolic as a $\mathbb{Z}_p(I)$-module, and so as a $\mathbb{Z}_p(\widehat{\mathbb{Q}}^{(2)})$-module. As $\widehat{\mathbb{Q}}^{(2)}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(2)}$, it has $p$-power index in $\mathbb{G}^{(2)}$. Since $V_2$ is an anisotropic symplectic $\mathbb{Z}_p(\mathbb{G}^{(2)})$-module, Theorem 3.2 in [1], implies that $V_2$ is both hyperbolic and anisotropic. Therefore $V_2$ is 0. So $V = V_1 = C(\widehat{\mathbb{Q}}_3^{(2)} \text{ in } V)$. Thus $\widehat{\mathbb{Q}}_3^{(2)}$ centralizes $\mathbb{P}_2^{(2)}/\mathbb{R}^{(2)}$. We conclude that the $q$-Sylow subgroup $\widehat{\mathbb{Q}}_3^{(2)}$ of $\mathbb{G}_3^{(2)}$ centralizes the $p$-Sylow subgroup $\mathbb{P}_2^{(2)}$ of the same group. Hence $\mathbb{G}_3^{(2)}$ is nilpotent, and Theorem 11.18 follows.

## 11.5 The general case

The aim in this section is to prove Theorem 11.33, i.e. to show that the group $\mathbb{G}_{t+1}^{(t)}$ is nilpotent. The ideas for the proof are already given in the $t = 2$ case, whose proof is a demonstration of the general argument. That's the reason we leave hidden some of the details of the general proof, already discussed in the previous section.

Assume the system (11.1) is fixed, with $n \geq t+1$. Assume further, using an inductive argument, that the groups $\mathbb{G}_{i+1}^{(i)}$ are nilpotent for all $i = 1, \dots, t-1$. Thus we can perform all the reductions described in Section 11.1, until we reach the group in question, i.e., the group $\mathbb{G}_{t+1}^{(t)}$. As the last step in the reductions depends on the parity of $t$, we first assume that $t = 2k$ is even. (As expected, we will see that it is enough to prove Theorem 11.33 in the even case.) In addition, we assume fixed the subsystem

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_{t+1} \trianglelefteq G, \tag{11.106a}$$

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{t+1}, \tag{11.106b}$$

$$\{Q_{2i-1}, P_{2r} | \beta_{2i-1}, \alpha_{2r}\}_{i=1,r=0}^{l,k}, \tag{11.106c}$$

where, in our case (that of an even $t$), $k = t/2$ and $l = t/2 + 1$. (Note that $k$ and $l$ are related to $t+1$ via (5.7).) Along with that system we pick and fix the groups $\widehat{Q}$ and $\widehat{P}$ so as to satisfy the conditions in Theorems 8.13 and 8.15. This way the groups $T$ and $U$ are also fixed. After $t$ steps of reductions the above system reduces to (see (11.63) with $m = t+1$)

$$1 \trianglelefteq \mathbb{P}_t^{(t)} \trianglelefteq \mathbb{G}_{t+1}^{(t)} \trianglelefteq \mathbb{G}^{(t)}, \tag{11.107a}$$

$$1, \boldsymbol{\alpha}_t^{(t)}, \Theta_{t+1}^{(t)} \tag{11.107b}$$

$$\{\mathbb{Q}_{t+1}^{(t)}, \mathbb{P}_t^{(t)} | \boldsymbol{\beta}_{t+1}^{(t)}, \boldsymbol{\alpha}_t^{(t)}\} \tag{11.107c}$$

In addition note that the group $\mathbb{Q}_{2k-1}^{(t)}$ and all the groups with indices smaller than $t = 2k$, have become trivial. (To be precise, all the $q$-groups $\mathbb{Q}_{2i-1}^{(t)}$, with indices $2i-1$ smaller than $2k$ have been dropped by repeated shifts of the original series, as they are normal subgroups of $\mathbb{G}^{(t)}$ that are contained in $\mathbb{Q}_{2k+1}^{(t)}$ and are centralized by $\mathbb{P}_{2k}^{(t)}$, while their characters $\boldsymbol{\beta}_{2i-1}^{(t)}$ are $\mathbb{G}^{(t)}$-invariant. The same holds for the $p$-groups $\mathbb{P}_{2j}^{(t)}$ with indices $2j$ smaller than $2k = t$. They are normal subgroups of $\mathbb{G}^{(t)}$ that are contained in $\mathbb{P}_{2k}^{(t)}$ and centralized by $\mathbb{Q}_{2k+1}^{(t)}$, while their characters are $\mathbb{G}^{(t)}$-invariant. This is the reason we drop them on the way.) So $\mathbb{P}_{2k}^{(t),*} = \mathbb{P}_{2k}^{(t)} = \mathbb{P}_t^{(t)}$. Furthermore the last group that is been dropped is the $q$-group $\mathbb{Q}_{2k-1}^{(t-1)}$. Note that $\mathbb{Q}_{2k-1}^{(t-1)}$ is centralized by $\mathbb{P}_{2k}^{(t-1)}$. In addition, the irreducible character $\boldsymbol{\beta}_{2k-1}^{(t-1)}$ is $\mathbb{G}^{(t-1)}$-invariant. After the last set of reductions is been performed, the group $\mathbb{Q}_{2k-1}^{(t-1)}$ maps isomorphically to a normal subgroup $\mathbb{Q}_{2k-1}^{(t)}$ of the limit group $\mathbb{G}^{(t)}$ that centralizes $\mathbb{P}_{2k}^{(t)}$, by Remark 10.7. In addition, the irreducible character $\boldsymbol{\beta}_{2k-1}^{(t-1)}$ of $\mathbb{Q}_{2k-1}^{(t-1)}$ maps, under the above isomorphism, to an irreducible character $\boldsymbol{\beta}_{2k-1}^{(t)}$ of $\mathbb{Q}_{2k-1}^{(t)}$. Note that $\boldsymbol{\beta}_{2k-1}^{(t)}$ is $\mathbb{G}^{(t)}$-invariant, as $\boldsymbol{\beta}_{2k-1}^{(t-1)}$ is $\mathbb{G}^{(t-1)}$-invariant and $\mathbb{G}^{(t)}$ is a section of $\mathbb{G}^{(t-1)}$. Even more, the centralizer $\mathbb{Q}_{2k-1,2k}^{(t)}$ of $\mathbb{P}_{2k}^{(t)}$ in $\mathbb{Q}_{2k-1}^{(t)}$ equals $\mathbb{Q}_{2k-1}^{(t)}$. Thus the character $\boldsymbol{\beta}_{2k-1,2k}^{(t)}$ coincides with $\boldsymbol{\beta}_{2k-1}^{(t)}$. So $\boldsymbol{\beta}_{2k-1,2k}^{(t)}$ is $\mathbb{G}^{(t)}$-invariant. This implies that $\widehat{\mathbb{Q}}^{(t)}(\boldsymbol{\beta}_{2k-1,2k}^{(t)}) = \widehat{\mathbb{Q}}^{(t)}$. Hence the image $\mathbb{I}^{(t)}$

of $\widehat{\mathbb{Q}}^{(t)}(\boldsymbol{\beta}^{(t)}_{2k-1,2k})$ in $\mathrm{Aut}(\mathbb{P}^{(t),*}_{2k})$ is reduced to

$$\mathbb{I}^{(t)} = \text{ Image of } \widehat{\mathbb{Q}}^{(t)} \text{ in } \mathrm{Aut}(\mathbb{P}^{(t)}_{2k}). \tag{11.108}$$

This, along with Theorem 11.69 and in particular (11.70b), implies

$$\text{Image of } \widehat{\mathbb{Q}}^{(t)} \text{ in } \mathrm{Aut}(\mathbb{P}^{(t)}_{2k}) \;=\; \text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \text{ in } \mathrm{Aut}(\mathbb{P}^{(t)}_{2k}). \tag{11.109a}$$

If $\mathbb{T}^{(t)} = \mathbb{P}^{(t),*}_{2k} \rtimes \mathbb{I}^{(t)} = \mathbb{P}^{(t)}_{2k} \rtimes \mathbb{I}^{(t)}$, then Theorem 11.68 implies

$$\begin{gathered}\text{Any faithful linear limit of } (\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}^{(t)}_{2k}, \boldsymbol{\alpha}^{(2)}_{2k}) \text{ is also a}\\ \text{faithful linear limit of } (T, R, \eta, P^*_{2k}, \alpha^*_{2k}).\end{gathered} \tag{11.109b}$$

Observe also that $\boldsymbol{\alpha}^{(t),*}_{2k} = \boldsymbol{\alpha}^{(t)}_{2k}$ is $\mathbb{G}^{(t)}$-invariant, by Remark 11.28, as $2k = t$. Hence the fact that $\widehat{\mathbb{Q}}^{(t)}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(t)}(\boldsymbol{\alpha}^{(t),*}_{2k})$ implies

$$\widehat{\mathbb{Q}}^{(t)} \in \mathrm{Syl}_q(\mathbb{G}^{(t)}). \tag{11.109c}$$

(Note that (11.109) is the analogue of (11.87) and (11.88) for the general case.)

According to Corollary 7.29 we get a new character tower

$$1 = \chi^\nu_0, \chi^\nu_1, \dots, \chi^\nu_t \tag{11.110a}$$

for the series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G_n = G, \tag{11.110b}$$

along with a triangular set

$$\{Q^\nu_{2i-1}, P^\nu_{2r} | \beta^\nu_{2i-1}, \alpha^\nu_{2r}\}^{k,k}_{i=1,r=0} \tag{11.110c}$$

and a $q$-Sylow subgroup $\widehat{Q^\nu}$ of $G(\alpha^{\nu,*}_{2k})$, so that (7.30) holds. In particular, we have

$$P^*_{2k} = P^{\nu,*}_{2k} \text{ and } \alpha^*_{2k} = \alpha^{\nu,*}_{2k}, \tag{11.111a}$$

$$\beta^\nu_{2k-1,2k} \text{ extends to } \widehat{Q} = \widehat{Q^\nu}. \tag{11.111b}$$

We proceed with the reductions described in Section 11.1 for the new system (11.110). We follow the same notation as that of Sections 11.1 and 11.2 with the addition of a supescript $\nu$ on anything that refers to the new system (11.110). So, we reduce the above new system, using first reductions that are $\mathbf{A}^\nu(\beta^\nu_1)$-invariant, then $\mathbf{B}^{(1),\nu}(\boldsymbol{\alpha}^{(1),\nu}_2)$-invariant that are followed by $\mathbf{A}^{(2),\nu}(\boldsymbol{\beta}^{(2),\nu}_3)$-invariant and so on. Hence all the conclusions of Section 11.2 are valid for the system (11.110). In particular, after $t$ steps of reductions, what is left of the system (11.110) is (see (11.63) with $m = t = 2k$)

$$1 \trianglelefteq \mathbb{P}^{(t),\nu}_{2k} \trianglelefteq \mathbb{G}^{(t),\nu}, \tag{11.112a}$$

$$1, \boldsymbol{\alpha}^{(t),\nu}_{2k} \tag{11.112b}$$

$$\{\mathbb{P}^{(t),\nu}_{2k} | \boldsymbol{\alpha}^{(t),\nu}_{2k}\} \tag{11.112c}$$

According to Remark 11.28 the character $\boldsymbol{\alpha}_{2k}^{(t),\nu,*} = \boldsymbol{\alpha}_{2k}^{(t),\nu}$ is $\mathbb{G}^{(t),\nu}$-invariant. Thus

$$\mathbb{G}^{(t),\nu}(\boldsymbol{\alpha}_{2k}^{(t),\nu,*}) = \mathbb{G}^{(t),\nu}(\boldsymbol{\alpha}_{2k}^{(t),\nu}) = \mathbb{G}^{(t),\nu}. \tag{11.113}$$

Note that the last group dropped after $t-1$ steps of reductions is the $q$-group $\mathbb{Q}_{t-1}^{(t-1),\nu}$, as according to the inductive hypothesis the last group proved to be nilpotent is $\mathbb{G}_t^{(t-1),\nu} = \mathbb{P}_t^{(t-1),\nu} \times \mathbb{Q}_{t-1}^{(t-1),\nu}$ (see (11.24)). It is clear that the centralizer $\mathbb{Q}_{2k-1,2k}^{(t-1),\nu}$ of $\mathbb{P}_{2k}^{(t-1),\nu} = \mathbb{P}_t^{(t-1),\nu}$ in $\mathbb{Q}_{2k-1}^{(t-1),\nu} = \mathbb{Q}_{t-1}^{(t-1),\nu}$ equals $\mathbb{Q}_{2k-1}^{(t-1),\nu}$. Furthermore, the irreducible character $\boldsymbol{\beta}_{2k-1,2k}^{(t-1),\nu}$ of $\mathbb{Q}_{2k-1,2k}^{(t-1),\nu}$ coincides with $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$. In addition, repeated applications (at every step of reductions) of Theorems 10.77 and 10.95 imply that the character $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$ extends to $\widehat{\mathbb{Q}}^{(t-1),\nu}(\boldsymbol{\beta}_{2k-1}^{(t-1),\nu})$, (where $\widehat{\mathbb{Q}}^{(t-1),\nu}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(t-1),\nu}(\boldsymbol{\alpha}_{2k}^{(t-1),\nu})$), as the character $\beta_{2k-1,2k}^{\nu}$ extends to $\widehat{Q^{\nu}}(\beta_{2k-1,2k}^{\nu}) = \widehat{Q^{\nu}} = \widehat{Q}$ (see (11.111)). But the character $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu} = \boldsymbol{\beta}_{t-1}^{(t-1),\nu}$ is $\mathbb{G}^{(t-1),\nu}$-invariant, according to the inductive hypothesis (see (11.24c)). Hence

$$\boldsymbol{\beta}_{2k-1}^{(t-1),\nu} \in \operatorname{Irr}(\mathbb{Q}_{2k-1}^{(t-1),\nu}) \text{ extends to } \widehat{\mathbb{Q}}^{(t-1),\nu}. \tag{11.114}$$

Each reduction in the $t$-th set of reductions is $\mathbf{B}^{(t-1),\nu}(\boldsymbol{\alpha}_t^{(t-1),\nu})$-invariant (see the comments following (11.24)). After this set of reductions is performed, the normal subgroup $\mathbb{Q}_{2k-1}^{(t-1),\nu}$ of $\mathbb{G}^{(t-1),\nu}$ that centralizes $\mathbb{P}_{2k}^{(t-1),\nu}$, maps isomorphically to a normal subgroup $\mathbb{Q}_{2k-1}^{(t),\nu}$ of the limit group $\mathbb{G}^{(t),\nu}$, by Remark 10.7. Under this isomorphism the character $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$ of $\mathbb{Q}_{2k-1}^{(t-1),\nu}$ maps to the character $\boldsymbol{\beta}_{2k-1}^{(t),\nu} \in \operatorname{Irr}(\mathbb{Q}_{2k-1}^{(t),\nu})$. So

$$\begin{aligned} \mathbb{Q}_{2k-1}^{(t-1),\nu} \cong \mathbb{Q}_{2k-1}^{(t),\nu} \trianglelefteq \mathbb{G}^{(t),\nu}, \text{ and} \\ \boldsymbol{\beta}_{2k-1}^{(t-1),\nu} \to \boldsymbol{\beta}_{2k-1}^{(t),\nu}. \end{aligned} \tag{11.115}$$

(Observe this is the analogue of (11.98).) Clearly $\boldsymbol{\beta}_{2k-1}^{(t),\nu}$ is $\mathbb{G}^{(t),\nu}$-invariant as $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$ is $\mathbb{G}^{(t-1),\nu}$-invariant, and $\mathbb{G}^{(t),\nu}$ is a section of $\mathbb{G}^{(t-1),\nu}$. According to Theorem 10.105 (see its first part), under this last set of $\mathbf{B}^{(t-1),\nu}(\boldsymbol{\alpha}_t^{(t-1),\nu})$-invariant reductions, the $q$-Sylow subgroup $\widehat{\mathbb{Q}}^{(t),\nu}$ of $\mathbb{G}^{(t),\nu}(\boldsymbol{\alpha}_{2k}^{(t),\nu,*})$ satisfies

$$\widehat{\mathbb{Q}}^{(t),\nu}(\boldsymbol{\beta}_{2k-1,2k}^{(t),\nu}) \cong \widehat{\mathbb{Q}}^{(t-1),\nu}(\boldsymbol{\beta}_{2k-1,2k}^{(t-1),\nu}).$$

We conclude that

$$\widehat{\mathbb{Q}}^{(t),\nu} \cong \widehat{\mathbb{Q}}^{(t-1),\nu}, \tag{11.116}$$

as $\boldsymbol{\beta}_{2k-1,2k}^{(t-1),\nu} = \boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$ and $\boldsymbol{\beta}_{2k-1,2k}^{(t),\nu} = \boldsymbol{\beta}_{2k-1}^{(t),\nu}$ are $\mathbb{G}^{(t-1),\nu}$- and $\mathbb{G}^{(t),\nu}$-invariant respectively. Note also that (11.113) implies

$$\widehat{\mathbb{Q}}^{(t),\nu}(\boldsymbol{\beta}_{2k-1,2k}^{(t),\nu}) = \widehat{\mathbb{Q}}^{(t),\nu} \in \operatorname{Syl}_q(\mathbb{G}^{(t),\nu}). \tag{11.117}$$

We can know easily prove

**Step 1.** *The character $\boldsymbol{\beta}_{2k-1}^{(t),\nu} \in \operatorname{Irr}(\mathbb{Q}_{2k-1}^{(t),\nu})$ is $\mathbb{G}^{(t),\nu}$-invariant and extends to $\widehat{\mathbb{Q}}^{(t),\nu} \in \operatorname{Syl}_q(\mathbb{G}^{(t),\nu})$.*

*Proof.* Follows immediately from (11.114)-(11.117). ☐

Of course the last set of reductions send $\mathbb{P}_{2k}^{(t-1),\nu}$ to $\mathbb{P}_{2k}^{(t),\nu}$. Thus the group $M := \mathbb{P}_{2k}^{(t),\nu} \times \mathbb{Q}_{2k-1}^{(t),\nu}$ is a normal subgroup of $\mathbb{G}^{(t),\nu}$. (The group $M$ is the image of $\mathbb{G}_t^{(t-1),\nu} \trianglelefteq \mathbb{G}^{(t-1),\nu}$ in $\mathbb{G}^{(t),\nu}$ under the last set of reductions.)

Completing the list of the general properties about the new system (11.112), we remark that if $\mathbb{I}^{(t),\nu}$ denotes the image of $\widehat{\mathbb{Q}}^{(t),\nu}(\boldsymbol{\beta}_{2k-1,2k}^{(t),\nu}) = \widehat{\mathbb{Q}}^{(t),\nu}$ in $\mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu,*}) = \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu})$, then Theorem 11.69 implies

$$\mathbb{I}^{(t),\nu} = \text{ Image of } \widehat{\mathbb{Q}}^{(t),\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu}) \ = \text{ Image of } \widehat{Q}^{\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu}). \tag{11.118}$$

Furthermore, Theorem 11.68 implies

$$\begin{gathered}\text{Any faithful linear limit of } (\mathbb{T}^{(t),\nu}, \mathbb{R}^{(t),\nu}, \boldsymbol{\eta}^{(t),\nu}, \mathbb{P}_{2k}^{(t),\nu}, \boldsymbol{\alpha}_{2k}^{(t),\nu}) \text{ is also a} \\ \text{faithful linear limit of } (T^{\nu}, R^{\nu}, \eta^{\nu}, P_{2k}^{\nu,*}, \alpha_{2k}^{\nu,*}).\end{gathered} \tag{11.119}$$

(We remind the reader the definition of $\mathbb{T}^{(t),\nu}$ as the product $\mathbb{T}^{(t),\nu} = \mathbb{P}_{2k}^{(t),\nu} \rtimes \mathbb{I}^{(t),\nu}$.) Of course $R^{\nu} = 1 = R$ and $\eta^{\nu} = 1 = \eta$. In addition, $P_{2k}^{\nu,*} = P_{2k}^{*}$ and $\alpha_{2k}^{\nu,*} = \alpha_{2k}^{*}$, by (11.111). Note also that the image $I$ of $\widehat{Q}(\beta_{2k-1,2k})$in $\mathrm{Aut}(P_{2k}^{*})$ is a subgroup of the image $I^{\nu}$ of $\widehat{Q^{\nu}}(\beta_{2k-1,2k}^{\nu})$ in $\mathrm{Aut}(P_{2k}^{\nu,*}) = \mathrm{Aut}(P_{2k}^{*})$, as $\widehat{Q^{\nu}}(\beta_{2k-1,2k}^{\nu}) = \widehat{Q^{\nu}} = \widehat{Q}$. Hence $T = P_{2k}^{*} \rtimes I$ is a subgroup of $T^{\nu} = P_{2k}^{\nu,*} \rtimes I^{\nu} = P_{2k}^{*} \rtimes I^{\nu}$. This, along with Remark 10.5, Definition 10.6 and (11.5), implies

$$\begin{gathered}\text{If } (\mathbb{T}^{\nu}, \mathbb{R}^{\nu}, \eta^{\nu}, \mathbb{P}_{2k}^{\nu}, \alpha_{2k}^{\nu}) \text{ is a faithful linear limit of } (\mathbb{T}^{(t),\nu}, \mathbb{R}^{(t),\nu}, \boldsymbol{\eta}^{(t),\nu}, \mathbb{P}_{2k}^{(t),\nu,*}, \boldsymbol{\alpha}_{2k}^{(t),\nu,*}) \\ \text{then } (\mathbb{T}^{\nu} \cap T, \mathbb{R}^{\nu}, \eta^{\nu}, \mathbb{P}_{2k}^{\nu}, \alpha_{2k}^{\nu}) \text{ is a faithful linear limit of } (T, R, \eta, P_{2k}^{*}, \alpha_{2k}^{*}).\end{gathered} \tag{11.120}$$

The next two steps will complete the proof of the theorem, and are identical to those proved in the case $t = 2$.

**Step 2.** *There exists a monomial character* $\Theta^{(t),\nu} \in \mathrm{Irr}(\mathbb{G}^{(t),\nu})$ *lying above* $\boldsymbol{\alpha}_{2k}^{(t),\nu} \times \boldsymbol{\beta}_{2k-1}^{(t),\nu}$ *and satisfying* $\Theta^{(t),\nu}(1)_q = \boldsymbol{\beta}_{2k-1}^{(t),\nu}(1)$.

*Proof.* As $\boldsymbol{\beta}_{2k-1}^{(t),\nu}$ extends to a $q$-Sylow subgroup of $\mathbb{G}^{(t),\nu}$, while $\boldsymbol{\alpha}_{2k}^{(t),\nu}$ is $\mathbb{G}^{(t),\nu}$-invariant, and $M = \mathbb{P}_{2k}^{(t),\nu} \times \mathbb{Q}_{2k-1}^{(t),\nu}$ is a normal subgroup of $\mathbb{G}^{(t),\nu}$, Lemma 11.82 implies the existence of an irreducible character $\Theta^{(t),\nu} \in \mathrm{Irr}(\mathbb{G}^{(t),\nu})$ that lies above $\boldsymbol{\alpha}_{2k}^{(t),\nu} \times \boldsymbol{\beta}_{2k-1}^{(t),\nu}$ and satisfies $\Theta^{(t),\nu}(1)_q = \boldsymbol{\beta}_{2k-1}^{(t),\nu}(1)$. It suffices to show that $\Theta^{(t),\nu}$ is monomial. The character $\boldsymbol{\beta}_{2k-1}^{(t-1),\nu}$ lies above $\boldsymbol{\zeta}^{(t-1),\nu}$ by (11.24b). Hence its faithful linear limit $\boldsymbol{\beta}_{2k-1}^{(t),\nu}$ lies above the faithful linear limit $\boldsymbol{\zeta}^{(t),\nu}$ of $\boldsymbol{\zeta}^{(t-1),\nu}$, (for the definition of $\boldsymbol{\zeta}^{(t),\nu}$ see (11.30)). In addition, $\boldsymbol{\alpha}_{2k}^{(t),\nu}$ lies above $\boldsymbol{\eta}^{(t),\nu}$ (see Proposition 11.31, with $t = 2k$ even). Hence $\Theta^{(t),\nu} \in \mathrm{Irr}(\mathbb{G}^{(t),\nu})$ lies above $\boldsymbol{\eta}^{(t),\nu} \times \boldsymbol{\zeta}^{(t),\nu} \in \mathrm{Irr}(\mathbb{R}^{(t),\nu} \times \mathbb{S}^{(t),\nu})$. Therefore, Proposition 11.32 implies that $\Theta^{(t),\nu}$ is monomial. ☐

**Step 3.**

$$\textit{Image of } \widehat{Q} \cap G_{t+1} \textit{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu}) \ = 1.$$

*Proof.* Let $N$ be any normal subgroup of $\mathbb{G}^{(t),\nu}$ with $M = \mathbb{P}_{2k}^{(t),\nu} \times \mathbb{Q}_{2k-1}^{(t),\nu} \trianglelefteq N \trianglelefteq \mathbb{G}^{(t),\nu}$, and $N/M$ a $q$-group. Then $N = \mathbb{P}_{2k}^{(t),\nu} \rtimes Q$ for some $q$-group $Q$, with $\mathbb{Q}_{2k-1}^{(t),\nu} \trianglelefteq Q$. Furthermore, as we have seen in Remark 11.28, $Z(\mathbb{P}_{2k}^{(t),\nu}) = \mathbb{R}^{(t),\nu}$ is maximal among the abelian $\mathbb{G}^{(t),\nu}$-invariant subgroups of $\mathbb{P}_{2k}^{(t),\nu}$. Furthermore, $\boldsymbol{\alpha}_{2k}^{(t),\nu} \in \mathrm{Irr}(\mathbb{P}_{2k}^{(t),\nu})$ lies above the $\mathbb{G}^{(t),\nu}$-invariant faithful irreducible character $\boldsymbol{\eta}^{(t),\nu} \in \mathrm{Irr}(\mathbb{R}^{(t),\nu})$, by Proposition 11.31. This, along with Steps 1 and 2, implies that we can apply Theorem 11.76 with the groups $\mathbb{G}^{(t),\nu}, \mathbb{P}_{2k}^{(t),\nu}, \mathbb{Q}_{2k-1}^{(t),\nu}, Q$ here, in the place of the groups $G, P, S, Q$

there, and the characters $\Theta^{(t)}, \boldsymbol{\alpha}_{2k}^{(t),\nu}, \boldsymbol{\beta}_{2k-1}^{(t),\nu}$ and $\boldsymbol{\eta}^{(t),\nu}$ here, in the place of $\chi, \alpha, \beta$ and $\zeta$ there. We conclude that any such normal subgroup $N$ of $\mathbb{G}^{(2),\nu}$ is nilpotent, i.e.,

$$\begin{gathered} Q \in \mathrm{Syl}_q(N) \text{ centralizes } \mathbb{P}_{2k}^{(t),\nu} \in \mathrm{Syl}_p(N), \text{ whenever} \\ M \trianglelefteq N \trianglelefteq \mathbb{G}^{(t),\nu} \text{ with } N/M \text{ a } q\text{-group.} \end{gathered} \tag{11.121}$$

The group $G_{t+1}$ is a normal subgroup of $G$ that contains $G_i = G_i^\nu$ for all $i = 1, \dots, t$. Hence, by Remark 10.5, the normal series (11.110b) reduces to (11.112a), the group $G_{t+1}$ reduces to a normal subgroup $\mathbb{G}_{t+1}^{(t),\nu}$ of $\mathbb{G}^{(t),\nu}$. Furthermore, $\mathbb{G}_{t+1}^{(t),\nu}/M$ is a $q$-group as $G_{t+1}/G_t$ is a $q$-group, and $M$ is the limit of $\mathbb{G}_t^{(t-1),\nu}$ and thus of $G_t$. As $\widehat{\mathbb{Q}}^{(t),\nu}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(t),\nu}$, we get that $\widehat{\mathbb{Q}}^{(t),\nu} \cap \mathbb{G}_{t+1}^{(t),\nu}$ is a $q$-Sylow subgroup of $\mathbb{G}_{t+1}^{(t),\nu}$. Hence (11.121) implies

$$\widehat{\mathbb{Q}}^{(t),\nu} \cap \mathbb{G}_{t+1}^{(t),\nu} \text{ centralizes } \mathbb{P}_{2k}^{(t),\nu} \in \mathrm{Syl}_p(\mathbb{G}_{t+1}^{(t),\nu}).$$

This, along with (11.118) implies

$$\text{Image of } \widehat{Q}^\nu \cap G_{t+1} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu}) \ = \ \text{Image of } \widehat{\mathbb{Q}}^{(t),\nu} \cap \mathbb{G}_{t+1}^{(t),\nu} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t),\nu}) \ = 1.$$

As $\widehat{Q} = \widehat{Q}^\nu$, Step 3 follows. □

We continue as in the case $t = 2$. So assume that $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_{2k}, \boldsymbol{\alpha}_{2k})$ and $(\mathbb{T}^\nu, \mathbb{R}^\nu, \eta^\nu, \mathbb{P}_{2k}^\nu, \alpha_{2k}^\nu)$ are faithful linear limits of the quintuples $(\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}_{2k}^{(t)}, \boldsymbol{\alpha}_{2k}^{(t)})$ and $(\mathbb{T}^{(t),\nu}, \mathbb{R}^{(t),\nu}, \boldsymbol{\eta}^{(t),\nu}, \mathbb{P}_{2k}^{(t),\nu}, \boldsymbol{\alpha}_{2k}^{(t),\nu})$, respectively. Then $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_{2k}, \boldsymbol{\alpha}_{2k})$ and $(\mathbb{T}^\nu \cap T, \mathbb{R}^\nu, \eta^\nu, \mathbb{P}_{2k}^\nu, \alpha_{2k}^\nu)$ are both faithful linear limits of $(T, R, \eta, P_{2k}^*, \alpha_{2k}^*)$. Hence, by Theorem 10.19,

$$\mathbb{P}_{2k}/\mathbb{R} \cong \mathbb{P}_{2k}^\nu/\mathbb{R}^\nu,$$

as anisotropic symplectic $\mathbb{Z}_p(T(\alpha_{2k}^*)/P_{2k}^*)$-modules. But $T = P_{2k}^* \rtimes I$ fixes $\alpha_{2k}^*$, as $I$ is the image of $\widehat{Q}(\beta_{2k-1,2k})$ in $\mathrm{Aut}(P_{2k}^*)$ and the group $\widehat{Q} \in G(\alpha_{2k}^*)$ fixes $\alpha_{2k}^*$. So $T(\alpha_{2k}^*)/P_{2k}^*$ is naturally isomorphic to $I$. We conclude that

$$\mathbb{P}_{2k}/\mathbb{R} \cong \mathbb{P}_{2k}^\nu/\mathbb{R}^\nu \text{ as anisotropic symplectic } \mathbb{Z}_p(I)\text{-modules.} \tag{11.122}$$

In view of Step 3, the group $\widehat{Q} \cap G_{t+1}$ centralizes $\mathbb{P}_{2k}^{(t),\nu}$ and thus also centralizes its section $\mathbb{P}_{2k}^\nu$. This, along with (11.122) and the definition of $I$, implies that $\widehat{Q} \cap G_{t+1}$ centralizes $\mathbb{P}_{2k}/\mathbb{R}$. Let $\mathbb{G}_{t+1}^{(t)}$ denote the limit group to which $G_{t+1}$ reduces, as $G$ reduces to $\mathbb{G}^{(t)}$. Then (11.109a) implies

$$I_{t+1} := \ \text{Image of } \widehat{Q}(\beta_{2k-1,2k}) \cap G_{t+1} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t)}) \ = \ \text{Image of } \widehat{\mathbb{Q}}^{(t)} \cap \mathbb{G}_{t+1}^{(t)} \text{ in } \mathrm{Aut}(\mathbb{P}_{2k}^{(t)}). \tag{11.123}$$

The fact that $\widehat{Q}(\beta_{2k-1,2k}) \cap G_{t+1}$ centralizes $\mathbb{P}_{2k}/\mathbb{R}$, while $\mathbb{P}_{2k}$ is a section of $\mathbb{P}_{2k}^{(t)}$, implies

$$I_{t+1} \text{ centralizes } \mathbb{P}_{2k}/\mathbb{R}. \tag{11.124}$$

This time we define $V := \mathbb{P}_{2k}^{(t)}/\mathbb{R}^{(t)}$. So $V$ is an anisotropic $\mathbb{Z}_p(\mathbb{G}^{(t)})$-module. Thus $V$, when written additively, is the direct sum

$$V = V_1 \dotplus V_2, \tag{11.125a}$$

of the perpendicular $\mathbb{Z}_p(\mathbb{G}^{(t)})$-modules,

$$V_1 = C(\mathbb{G}^{(t)}_{t+1} \text{ in } V) = C(\widehat{\mathbb{Q}}^{(t)}_{2k+1} \text{ in } V) \tag{11.125b}$$

$$V_2 = [V, \mathbb{G}^{(t)}_{t+1}] = [V, \widehat{\mathbb{Q}}^{(t)}_{2k+1}], \tag{11.125c}$$

where $\widehat{\mathbb{Q}}^{(t)}_{2k+1} = \widehat{\mathbb{Q}}^{(t)} \cap \mathbb{G}^{(t)}_{2k+1}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(t)}_{2k+1} = \mathbb{G}^{(t)}_{t+1}$, as $\widehat{\mathbb{Q}}^{(t)}$ is a $q$-Sylow subgroup of $\mathbb{G}^{(t)}$.

As $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_{2k}, \boldsymbol{\alpha}_{2k})$ is a faithful linear limit of $(\mathbb{T}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{P}^{(t)}_{2k}, \boldsymbol{\alpha}^{(t)}_{2k})$, we have that $U := \mathbb{P}_{2k}/\mathbb{R}$ is isomorphic to a section of $V$. Furthermore, $U$ is isomorphic as a symplectic $\mathbb{Z}_p(I)$-module, to the orthogonal direct sum $U = U_1 \dotplus U_2$, where $U_i$ is a limit module for $V_i$, for each $i = 1, 2$. In view of (11.125b) we have

$$U_1 = C(I_{t+1} \text{ in } U) \text{ and } U_2 = [U, I_{t+1}], \tag{11.126}$$

where $I_{t+1}$ is the image of $\widehat{\mathbb{Q}}^{(t)}_{2k+1}$ in $\mathrm{Aut}(\mathbb{P}^{(t)}_{2k})$ (see (11.123)). In view of (11.124) we get $U_2 = 0$.

According to Remark 11.28, the center $\mathbb{R}^{(t)}$ of $\mathbb{P}^{(t)}_{2k}$ is a cyclic central subgroup of $\mathbb{G}^{(t)}$, maximal among the characteristic abelian subgroups of $\mathbb{P}^{(t)}_{2k}$. So $\mathbb{R}^{(t)}$ is also a central subgroup of $\mathbb{T}^{(t)} = \mathbb{P}^{(t)}_{2k} \rtimes \mathbb{I}^{(t)}$, as $\mathbb{I}^{(t)}$ is the image of $\widehat{\mathbb{Q}}^{(t)} \leq \mathbb{G}^{(t)}$ in $\mathrm{Aut}(\mathbb{P}^{(t)}_{2k})$. Thus Proposition 10.47 applies to the faithful linear limit $(\mathbb{T}, \mathbb{R}, \boldsymbol{\eta}, \mathbb{P}_{2k}, \boldsymbol{\alpha}_{2k})$. So $U$ is isomorphic to $W^{\perp}/W$ for some maximal $I$-invariant totally isotropic subspace $W$ of $V$. Then $W = W_1 \dotplus W_2$, where $W_i$ is a maximal totally isotropic $I$-invariant subspace of $V_i$, for $i = 1, 2$. But $U_2 = 0$. Hence $W_2^{\perp}$ must equal $W_2$. Thus $V_2$ contains a self perpendicular $I$-invariant subspace. We conclude that $V_2$ is hyperbolic as both a $\mathbb{Z}_p(I)$- and a $\mathbb{Z}_p(\widehat{\mathbb{Q}}^{(t)})$-module. The group $\widehat{\mathbb{Q}}^{(t)}$ has $p$-power index in $\mathbb{G}^{(t)}$, since it is a $q$-Sylow subgroup of the latter group. Since $V_2$ is an anisotropic symplectic $\mathbb{Z}_p(\mathbb{G}^{(t)})$-module, Theorem (3.2) in [1], implies that $V_2$ is both hyperbolic and anisotropic. Therefore $V_2$ is 0. So $V = V_1 = C(\widehat{\mathbb{Q}}^{(t)}_{2k+1} \text{ in } V)$. Thus $\widehat{\mathbb{Q}}^{(t)}_{2k+1}$ centralizes $\mathbb{P}^{(t)}_{2k+1}/\mathbb{R}^{(t)}$. We conclude that the $q$-Sylow subgroup $\widehat{\mathbb{Q}}^{(t)}_{2k+1}$ of $\mathbb{G}^{(t)}_{2k+1} = \mathbb{G}^{(t)}_{t+1}$ centralizes the $p$-Sylow subgroup $\mathbb{P}^{(t)}_{2k}$ of the same group. Hence $\mathbb{G}^{(t)}_{t+1}$ is nilpotent. So Theorem 11.33 follows in the case of an even $t$.

The proof for an odd $t$ follows immediately from the already proved case of an even $t$. Indeed, assume that $t$ is odd. Then we can adjoin a trivial group and character at the bottom of (11.106) so that $t$ becomes even. We now interchange $p$ and $q$, and apply the already proved result. (Note that the normal series (11.106a) becomes $1 \trianglelefteq 1 = H_1 \trianglelefteq G_1 = H_2 \trianglelefteq G_2 = H_3 \trianglelefteq \cdots \trianglelefteq G_{t+1} = H_{t+2} \trianglelefteq G$, so that $1 = H_1$ is assumed to be the first $p$-group of order $p^0 = 1$, while $G_1/1 = H_2/H_1$ is a $q$-group and $H_{i+1}/H_i$ is either a $q$-group if $i$ is odd, or a $p$-group if $i$ is even, for all $i = 1, \ldots, t+1$.)

Hence Theorem 11.33 follows.

## 11.6 Corollaries

Below we list a series of corollaries following Theorem 11.33.

**Corollary 11.127.** *Let $G$ be a finite $p^a q^b$-monomial group, for some odd primes $p$ and $q$. Assume that $N$ is a normal subgroup of $G$ and that $\psi$ is an irreducible character of $N$. Consider the linear quintuple $(G, 1, 1, N, \psi)$. Then there exists a faithful linear limit $(\mathbb{G}, \mathbb{A}, \boldsymbol{\Phi}, \mathbb{N}, \boldsymbol{\Psi})$ of $(G, 1, 1, N, \psi)$ such that $\mathbb{N}$ is a nilpotent group.*

Observe that this is Theorem 1 of the introduction.

*Proof.* As $G$ is a solvable group and $N$ is normal subgroup of $G$, we can form a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_t \trianglelefteq G_{t+1} = N \trianglelefteq G_{t+2} \trianglelefteq \cdots \trianglelefteq G_n = G, \tag{11.128}$$

such that $G_i$ is a normal subgroup of $G$, while the order of $G_{i+1}/G_i$ is a power of a prime, for all $i = 0, 1, \dots, t$. Furthermore, without any loss of generality we can assume that $G_{i+1}/G_i$ is $p$-group if $i$ is odd and a $q$-group if $i$ is even, for all $i = 0, 1, \dots, t$. We also form recursively a character tower

$$\{\chi_i \in \mathrm{Irr}(G_i)\}_{i=0}^{t+1}, \tag{11.129}$$

for (11.128), so that $\chi_{t+1} = \psi$ and $\chi_i$ is any irreducible character of $G_i$ that lies under $\chi_{i+1} \in \mathrm{Irr}(G_{i+1})$, for all $i = 0, 1, \dots, t$. In addition, we fix a representative of the unique conjugacy class of triangular sets that corresponds to the above character tower, along with a Sylow system for $G$ so that (8.4) holds.

We proceed with the series of reductions described in Section 11.1. So after $t$ steps, we reach the limit groups $\mathbb{G}^{(t)} = \mathbb{G}_n^{(t)}, \mathbb{G}_{t+1}^{(t)}$ and $\mathbb{R}^{(t)}, \mathbb{S}^{(t)}$ along with their limit characters $\Theta_n^{(t)}, \Theta_{t+1}^{(t)}$ and $\boldsymbol{\eta}^{(t)}$ and $\boldsymbol{\zeta}^{(t)}$. According to Theorem 11.33 the group $\mathbb{G}_{t+1}^{(t)}$ is nilpotent. Furthermore, Remark 11.29 implies that any faithful linear limit $(\mathbb{G}, \mathbb{A}, \boldsymbol{\Phi}, \mathbb{N}, \boldsymbol{\Psi})$ of $(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{G}_{t+1}^{(t)}, \Theta_{t+1}^{(t)})$ is also a faithful linear limit of $(G, 1, 1, G_{t+1}, \chi_{t+1}) = (G, 1, 1, N, \psi)$. Hence if we take $(\mathbb{G}, \mathbb{A}, \boldsymbol{\Phi}, \mathbb{N}, \boldsymbol{\Psi})$ to be any faithful linear limit of $(\mathbb{G}^{(t)}, \mathbb{R}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbb{G}_{t+1}^{(t)}, \Theta_{t+1}^{(t)})$, then $\mathbb{N}$ is nilpotent as a section of $\mathbb{G}_{t+1}^{(t)}$. So Corollary 11.127 follows. □

As any nilpotent group is monomial, Proposition 10.18, along Corollary 11.127 applied to any irreducible character $\psi$ of $N$, easily implies

**Corollary 11.130.** *Let $G$ be a finite $p^a q^b$-monomial group, for some odd primes $p$ and $q$. Assume that $N$ is a normal subgroup of $G$. Then $N$ is a monomial group.*

Observe that this is Theorem 2 of the Introduction.

If, in addition, we take $N = G$ and $\chi \in \mathrm{Irr}(G)$, then Corollary 11.127 implies the existence of a faithful linear limit $(\mathbb{G}, \mathbb{A}, \boldsymbol{\Phi}, \mathbb{G}, \boldsymbol{\Psi})$ of $(G, 1, 1, G, \chi)$ so that $\mathbb{G}$ is nilpotent. In view of Corollary 10.9 the group $\mathbb{A} = Z(\mathbb{G})$ is maximal among the abelian $\mathbb{G}$-invariant subgroups of $\mathbb{G}$. As $\mathbb{G}$ is nilpotent this forces $Z(\mathbb{G}) = \mathbb{A} = \mathbb{G}$. Hence $\boldsymbol{\Psi} = \boldsymbol{\Phi} \in \mathrm{Irr}(\mathbb{A})$ is linear. We conclude

**Corollary 11.131.** *Let $G$ be an odd order monomial $p^a q^b$-group and let $\chi \in \mathrm{Irr}(G)$. Then there exists a faithful linear limit $\boldsymbol{\Psi}$ of $\chi$ such that $\boldsymbol{\Psi}(1) = 1$, i.e., $\boldsymbol{\Psi}$ is a linear character.*

Hence Theorem 3 follows.

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