PROJECTION INEQUALITIES FOR ANTICHAINS

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ABSTRACT. Let n be an integer with $n \geqslant 2$. A set $A \subseteq \mathbb{R}^n$ is called an *antichain* (resp. *weak antichain*) if it does not contain two distinct elements $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ satisfying $x_i \leqslant y_i$ (resp. $x_i < y_i$) for all $i \in \{1, \ldots, n\}$. We show that the Hausdorff dimension of a weak antichain A in the n-dimensional unit cube $[0,1]^n$ is at most $n-1$ and that the $(n-1)$-dimensional Hausdorff measure of A is at most n, which are the best possible bounds. This result is derived as a corollary of the following *projection inequality*, which may be of independent interest: The $(n-1)$-dimensional Hausdorff measure of a (weak) antichain $A \subseteq [0,1]^n$ cannot exceed the sum of the $(n-1)$-dimensional Hausdorff measures of the n orthogonal projections of A onto the facets of the unit n-cube containing the origin. For the proof of this result we establish a discrete variant of the projection inequality applicable to weak antichains in $\mathbb{Z}^n$ and combine it with ideas from geometric measure theory.

§1. INTRODUCTION

Sperner's theorem [20], a cornerstone of extremal set theory, determines for each positive integer n the maximal size of an antichain in the power set of an n-element set and describes the extremal configurations. In the statement that follows, $[n]$ denotes the set $\{1, \ldots, n\}$ of the first n natural numbers and $A \subseteq \mathcal{P}([n])$ is said to be an *antichain* if $x \nsubseteq y$ holds for any distinct $x, y \in A$.

Theorem 1.1 (Sperner). *If $n \geqslant 1$ is an integer and $A \subseteq \mathcal{P}([n])$ is an antichain, then $|A| \leqslant \binom{n}{\lfloor n/2 \rfloor}$. Equality holds if and only if for some $\ell \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$ the set A is the collection of all ℓ-element subsets of $[n]$.*

Sperner's fundamental result has been generalised in various ways and gave rise to a substantial body of future developments both within extremal set theory and beyond (see [1, 10]).

Observe that via characteristic vectors the power set $\mathcal{P}([n])$ can be identified with the set $\{0,1\}^n$ of n-dimensional 0-1-vectors. Moreover, for any two subsets x and y of $[n]$ we have $x \subseteq y$ if and only if the characteristic vector of x is coordinate-wise at most the characteristic vector of y. Therefore, Sperner's theorem can be reformulated as a statement about $\{0,1\}^n$ equipped with the product partial ordering. It seems natural to ask what happens when one replaces $\{0,1\}^n$ by the n-dimensional unit cube $[0,1]^n$.

Let us fix the following notation for discussing such situations. Given two n-tuples $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ in $\mathbb{R}^n$, we write $\mathbf{x} \leqslant \mathbf{y}$ if $x_i \leqslant y_i$ for all $i \in [n]$. Moreover, if $\mathbf{x} \leqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ we write $\mathbf{x} < \mathbf{y}$, while $\mathbf{x} \prec \mathbf{y}$ has the stronger meaning that $x_i < y_i$ holds for all $i \in [n]$. A set $A \subseteq \mathbb{R}^n$ is called an *antichain* (resp. *weak antichain*) if it does not contain two elements $\mathbf{x}$ and $\mathbf{y}$ satisfying $\mathbf{x} < \mathbf{y}$ (resp. $\mathbf{x} \prec \mathbf{y}$). So every antichain is also a weak antichain.

In order to get some deeper insight first we replace the unit cube $[0,1]^n$ by its discretization $D_m^n = \left\{\frac{0}{m}, \frac{1}{m}, \ldots, \frac{m-1}{m}\right\}^n$, where m is a fixed positive integer. De Bruijn et al. [6] proved that the sets $A_{n,\ell} = \left\{\mathbf{x} \in D_m^n \colon \sum_{i=1}^n x_i = \ell\right\}$, where $\ell \in \left\{\lfloor n(m-1)/2 \rfloor, \lceil n(m-1)/2 \rceil\right\}$, are maximum antichains. Using chains of the form

$$\cdots < (x_1, x_2, \ldots, x_n) < \left(x_1 + \tfrac{1}{m}, x_2 + \tfrac{1}{m}, \ldots, x_n + \tfrac{1}{m}\right) < \cdots$$

it is easy to show that the set $W_n = \{\mathbf{x} \in D_m^n \colon x_i = 0 \text{ for some } i\}$ is a maximum weak antichain. Note that $|A_{n,\ell}| = O\left(m^n/\sqrt{n}\right)$ as $n \to \infty$ and that $|W_n| = m^n - (m-1)^n$, whence $|A_{n,\ell}|/|W_n| = O(1/\sqrt{n})$ as $n \to \infty$.

Now we come back to the unit cube $[0,1]^n$. Obviously, the set

$$A_n^\star = \left\{\mathbf{x} \in [0,1]^n \colon \sum_{i=1}^n x_i = n/2\right\}$$

is an antichain and the set

$$W_n^\star = \left\{\mathbf{x} \in [0,1]^n \colon x_i = 0 \text{ for some } i\right\}$$

is a weak antichain in $[0,1]^n$. In view of its similarity with the previous extremal configurations one might expect them to have an interesting maximality property. Questions addressing the extremality of (weak) antichains in $[0,1]^n$ become meaningful as soon as one agrees on an (outer) *measure* on $[0,1]^n$ that allows us to compare any two different candidates. The first measure on $[0,1]^n$ that usually comes to mind is the n-dimensional Lebesgue measure. However, the antichain $A_n^\star$ (and also the weak antichain $W_n^\star$) is null with respect to this measure and the following result due to the first author [9] shows that, actually, all other antichains in $[0,1]^n$ are null in this sense as well.

Theorem 1.2. *If $c > 0$ and A is a Lebesgue measurable subset of $[0,1]^n$ that does not contain two elements $\mathbf{x} \leqslant \mathbf{y}$ with $\sum_{i=1}^{n}(y_i - x_i) \geqslant c$, then the Lebesgue measure of A cannot exceed the Lebesgue measure of the optimal set*

$$A(c) = \left\{ \mathbf{x} \in [0,1]^n \colon \frac{n-c}{2} \leqslant \sum_{i=1}^{n} x_i < \frac{n+c}{2} \right\}.$$

As a matter of fact, the antichain $A_n^\star$ and the weak antichain $W_n^\star$ are not only null with respect to the Lebesgue measure, but they also have the intuitively stronger property of being $(n-1)$-dimensional. One may thus wonder

 (*1*) whether every antichain and weak antichain in $[0,1]^n$ is at most $(n-1)$-dimensional
 (*2*) and if so, whether $A_n^\star$ and $W_n^\star$ are in a natural sense the "largest" $(n-1)$-dimensional antichain resp. weak antichain in $[0,1]^n$.

The perhaps most natural measure theoretic concepts for making these questions precise are Hausdorff dimension and Hausdorff measure, so let us briefly recall their definitions. If U is a non-empty subset of $\mathbb{R}^n$, we denote its *diameter* by $\mathrm{diam}(U)$. For real numbers $s \geqslant 0$, $\delta > 0$ and for $A \subseteq \mathbb{R}^n$ we write

$$\mathcal{H}_\delta^s(A) = \alpha_s \inf \left\{ \sum_{i \in \mathbb{N}} \mathrm{diam}(U_i)^s \colon A \subseteq \bigcup_{i \in \mathbb{N}} U_i \text{ and } \mathrm{diam}(U_i) \leqslant \delta \text{ for every } i \in \mathbb{N} \right\},$$

where the normalisation factor $\alpha_s = \frac{\pi^{s/2}}{2^s \Gamma(s/2+1)}$ denotes the volume of the s-dimensional sphere of radius $\frac{1}{2}$. Its presence ensures that if $s = n$ and $\{U_i \colon i \in \mathbb{N}\}$ is a collection of mutually disjoint balls, then the right side agrees with the total volume of these balls. For later use we remark that these quantities $\mathcal{H}_\delta^s(A)$ are very robust under the addition of various regularity properties that can be imposed on the sets U_i. For instance, one could insist that these sets need to be closed (see e.g. [2, p.4]).

Evidently for fixed s and A, the value of $\mathcal{H}_\delta^s(A)$ increases as δ decreases and thus the limit

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(A),$$

called the *s-dimensional Hausdorff measure of A*, exists. It is well-known that $\mathcal{H}^n$ agrees on $\mathbb{R}^n$ with the n-dimensional Lebesgue outer measure (see e.g. [11, p.87]). In particular, $\mathcal{H}^n([0,1]^n) = 1$.

The *Hausdorff dimension* of A, denoted by $\dim_H A$, is defined by

$$\dim_H(A) = \inf \{s \colon \mathcal{H}^s(A) = 0\}.$$

One checks easily that $s < \dim_H(A)$ implies $\mathcal{H}^s(A) = \infty$, while for $s > \dim_H(A)$ one has $\mathcal{H}^s(A) = 0$. Therefore, for fixed A the only value of s for which $\mathcal{H}^s(A)$ can have a "non-trivial" value is $s = \dim_H(A)$. We refer to [2, 11, 12] for legible textbooks on the topic.

Let us now return to our problems (1) and (2). Our main result does indeed imply that every weak antichain (and hence also every antichain) $A \subseteq [0,1]^n$ satisfies $\dim_H(A) \leqslant n-1$. The second question, however, has a negative answer concerning antichains, but a positive answer concerning weak antichains. Notice that $W_n^\star$ is a union of n facets of the unit n-cube, wherefore $\mathcal{H}^{n-1}(W_n^\star) = n$.

In order to prove the optimality of $W_n^\star$ we establish a more general result, which we call the *projection inequality*. It asserts that the $\mathcal{H}^{n-1}$-measure of a weak antichain $A \subseteq [0,1]^n$ is at most the sum of the $\mathcal{H}^{n-1}$-measures of the orthogonal projections of A to the n facets of the unit cube containing the origin. Such projections are just deleting a fixed coordinate. When n is clear from the context and $i \in [n]$ we write $\pi_i \colon \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ for the projection defined by

$$\pi_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Our main result on weak antichains in the unit n-cube reads as follows.

Theorem 1.3. *If A is a weak antichain in $[0,1]^n$, then*

$$\mathcal{H}^{n-1}(A) \leqslant \sum_{i=1}^n \mathcal{H}^{n-1}\big(\pi_i(A)\big).$$

In particular,

$$\mathcal{H}^{n-1}(A) \leqslant n \quad and \quad \dim_H(A) \leqslant n-1. \tag{1.1}$$

In order to find "better" antichains than $A_n^\star$ one can start with the "best" weak antichain $W_n^\star$ and deform it slightly to obtain an antichain. This can be done in a polyhedral way, but here we present a "smooth" way: Consider the hypersurface

$$A_p = \{\mathbf{x} \in [0,1]^n \colon \|\mathbf{x}\|_p = 1\}$$

as $p \to \infty$. One can easily verify that the ℓ^p-unit ball $B_p = \{\mathbf{x} \in \mathbb{R}^n \colon \|\mathbf{x}\|_p \leqslant 1\}$ converges with respect to the Hausdorff metric to the ℓ^∞-unit ball as $p \to \infty$. Moreover, it is well-known ([19, p.219]) that if a sequence of convex bodies K_i converges to a convex body K with respect to the Hausdorff metric, then $\mathcal{H}^{n-1}(\partial K_i) \to \mathcal{H}^{n-1}(\partial K)$. For these reasons we have $\mathcal{H}^{n-1}(A_p) \to n$ as $p \to \infty$.

Corollary 1.4. *For every positive integer n we have*

$$\sup\big\{\mathcal{H}^{n-1}(A) \colon A \subseteq [0,1]^n \text{ is an antichain}\big\} = n.$$

Note that $\mathcal{H}^1(A_2^\star) = \sqrt{2} < 2$ and $\mathcal{H}^2(A_3^\star) = 3\sqrt{3}/4 < 3$, and in general one can easily check that $\mathcal{H}^{n-1}(A_n^\star) < n$ and thus $A_n^\star$ is indeed not optimal.

Obviously, for every antichain $A \subseteq \mathbb{R}^n$, the restriction of π_i to A is injective. We emphasise, however, that the projection inequality does not remain true if the antichain-condition is relaxed to an injectivity-condition. Indeed, Foran [13, p.813] constructed bijective functions from $[0, 1]$ onto $[0, 1]$ whose graphs have arbitrary large $\mathcal{H}^1$-measure.

One of the two central ideas in our approach is to discretise Theorem 1.3. In particular, a basic tool for the proof of Theorem 1.3 is the following discrete variant of the projection inequality replacing $\mathcal{H}^{n-1}$ by the counting measure.

Theorem 1.5. *If A is a finite weak antichain in $\mathbb{Z}^n$, then*

$$|A| \leqslant \sum_{i=1}^{n} |\pi_i(A)| \,.$$

Our article is motivated by the idea that several combinatorial statements have continuous counterparts. This is a rather old idea, which dates back at least to the 70s, and since its conception many results have been reported in a "measurable" setting (see e.g. [4, 5, 9, 15, 16]) or in a "vector space" setting (see e.g. [14, 17]).

Organisation. In Section 2 we prove Theorem 1.5. We use this result together with some ideas pertaining to geometric measure theory in Section 3 for proving the special case of Theorem 1.3 where A is an antichain. This section might be the technically most demanding part of the article and we defer an outline of the argument to Subsection 3.1. In Section 4 we complete the proof of Theorem 1.3 by reducing it to the special case that A is an antichain. The concluding remarks in Section 5 describe several problems for future research.

§2. The discrete projection inequality

The key observation on which the proof of Theorem 1.5 relies is that every weak antichain in $\mathbb{Z}^n$ can be partitioned into n parts such that for each $i \in [n]$ the projection $\pi_i(\cdot)$ is injective on the i-th part.

Proof of Theorem 1.5. Let $A \subseteq \mathbb{Z}^n$ be a weak antichain. Define recursively disjoint subsets $A_1, \ldots, A_n$ of A as follows. If for some $i \in [n]$ the sets $A_1, \ldots, A_{i-1}$ have just been constructed, set $B_i = A \smallsetminus (A_1 \cup \ldots \cup A_{i-1})$ and let A_i be the set of points in B_i whose i-th coordinates are minimal. In other words, A_i is defined so as to contain those points $(x_1, \ldots, x_n) \in B_i$ for which there exists no integer $y_i < x_i$ such that $(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \in B_i$.

We contend that the set $B_{n+1} = A \smallsetminus (A_1 \cup \ldots \cup A_n)$ is empty. Indeed, assume indirectly that some point $\mathbf{x} = (x_1, \ldots, x_n)$ belongs to B_{n+1}. Using the definitions of $A_n, \ldots, A_1$ in

this order we find integers $y_n < x_n, \ldots, y_1 < x_1$ such that $(x_1, \ldots, x_{i-1}, y_i, \ldots, y_n) \in A_i$ for every $i \in [n]$. In particular, the point $\mathbf{y} = (y_1, \ldots, y_n)$ is in A and satisfies $\mathbf{y} \prec \mathbf{x}$, contrary to A being a weak antichain. This proves that $A = A_1 \cup A_2 \cup \ldots \cup A_n$ is a partition.

Now obviously for every $i \in [n]$ the projection π_i is injective on A_i, whence

$$|A_i| = |\pi_i(A_i)| \leqslant |\pi_i(A)| \,.$$

We conclude that

$$|A| = \sum_{i=1}^{n} |A_i| \leqslant \sum_{i=1}^{n} |\pi_i(A)| \,,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

§3. ANTICHAINS

3.1. **Overview.** This section deals with a special case of our main theorem, where rather than weak antichains we only consider antichains. So explicitly we aim at the following result.

Proposition 3.1. *If A is an antichain in $[0,1]^n$, then*

$$\mathcal{H}^{n-1}(A) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) \,.$$

As the proof of this estimate is somewhat involved, we would like to devote the present subsection to a discussion of our basic strategy. An obvious approach is the following discretisation procedure: Take a large natural number m, cut the unit cube into m^n smaller cubes of side length $\frac{1}{m}$ and keep track which of these cubes intersect A. This situation can be encoded as a weak antichain in $[m]^n$, to which the discrete projection inequality (Theorem 1.5) applies. On first sight one might hope that in the limit $m \to \infty$ this argument would yield Proposition 3.1. But when working out the details, one discovers that one looses a constant factor which depends on the dimension n, but not on the antichain A itself.

Lemma 3.2. *For every positive integer n there exists a constant $D > 0$ such that every weak antichain $A \subseteq [0,1]^n$ satisfies*

$$\mathcal{H}^{n-1}(A) \leqslant D \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) \,.$$

To aid the reader's orientation we remark that in Subsection 3.2 we will show this estimate for

$$D = n^{(n-1)/2} \cdot \alpha_{n-1} = \sqrt{\frac{2}{\pi^2 n}} \cdot \left(\frac{\pi e}{2}\right)^{n/2} (1 + o(1)) \,, \tag{3.1}$$

but the precise value of D will be rather immaterial to what follows.

A completely different and more analytical approach to Proposition 3.1 starts from the following observation. Suppose that $f \colon [0,1]^{n-1} \longrightarrow [0,1]$ is decreasing in each coordinate and sufficiently smooth, and that we want to study the antichain

$$A = \left\{ (\mathbf{x}, f(\mathbf{x})) \colon \mathbf{x} \in [0,1]^{n-1} \right\}.$$

Denoting the partial derivatives of f by $D_1 f, \ldots, D_{n-1} f$ one checks easily that

$$\mathcal{H}^{n-1}(A) = \int_{[0,1]^{n-1}} \sqrt{1 + |D_1 f(\mathbf{x})|^2 + \ldots + |D_{n-1} f(\mathbf{x})|^2} \, \mathrm{d}\mathbf{x}$$

and

$$\mathcal{H}^{n-1}\big(\pi_i(A)\big) = \int_{[0,1]^{n-1}} \big|D_i f(\mathbf{x})\big| \mathrm{d}\mathbf{x} \qquad \text{for all } i \in [n-1],$$

for which reason the projection inequality for A follows from

$$\sqrt{1 + |D_1 f(\mathbf{x})|^2 + \ldots + |D_{n-1} f(\mathbf{x})|^2} \leqslant 1 + |D_1 f(\mathbf{x})| + \ldots + |D_{n-1} f(\mathbf{x})|$$

and from $\pi_n(A) = [0,1]^{n-1}$.

In general we need to look at antichains of the form

$$A = \left\{ (\mathbf{x}, f(\mathbf{x})) \colon \mathbf{x} \in B \right\}, \tag{3.2}$$

where $B = \pi_n(A) \subseteq [0,1]^{n-1}$ is arbitrary and $f \colon B \longrightarrow [0,1]$ is only known to be decreasing in each coordinate, but not necessarily smooth. The question to what extent arguments that work well for smooth functions can be extended to more general scenarios lies at the very heart of a mathematical area known as *geometric measure theory*. In Subsection 3.4 we shall use some methods from this subject in order to generalise the previous idea as follows.

Lemma 3.3. *Given an antichain $A \subseteq [0,1]^n$ and $\delta > 0$ there exists a Borel set $B \subseteq [0,1]^{n-1}$ such that*

(i) $\mathcal{H}^{n-1}(B) > 1 - \delta$

(ii) *and the set $A' = A \cap \pi_n^{-1}(B)$ satisfies $\mathcal{H}^{n-1}(A') \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A')\big) + \delta$.*

Now roughly speaking one may hope to prove Proposition 3.1 (up to an arbitrarily small additive error) by using Lemma 3.3 in each coordinate direction, each time cutting out a substantial piece of A whose $\mathcal{H}^{n-1}$-measure can be estimated quite efficiently by part (ii). There will remain a "small" left-over part of A, which can then be handled by means of Lemma 3.2.

There is one final technical hurdle one needs to overcome when pursuing such a plan. The problem is that, in general, Hausdorff measure is only known to be subadditive. So when one attempts to prove the projection inequality for an antichain A by splitting it into two pieces, handling both pieces separately, and adding up the results, one can get into trouble with the right sides. We shall use the following lemma for getting around this difficulty.

Lemma 3.4. *Let $A \subseteq [0,1]^n$ be an antichain and let $[0,1]^{n-1} = B' \cup B''$ be a partition of the $(n-1)$-dimensional unit cube into Borel sets. Setting $A' = A \cap \pi_n^{-1}(B')$ and $A'' = A \cap \pi_n^{-1}(B'')$ we have*

$$\mathcal{H}^{n-1}\big(\pi_i(A)\big) = \mathcal{H}^{n-1}\big(\pi_i(A')\big) + \mathcal{H}^{n-1}\big(\pi_i(A'')\big)$$

for every $i \in [n]$.

We conclude the present subsection by showing that our three lemmas do indeed imply Proposition 3.1. The proofs of the lemmas themselves are deferred to the three subsections that follow. But it will be convenient to prove Lemma 3.4 in Subsection 3.3 before we turn our attention to Lemma 3.3 in Subsection 3.4.

Proof of Proposition 3.1 assuming Lemma 3.2–3.4. Fix a dimension $n \geqslant 1$ as well as some $\delta > 0$ and let D be the number provided by Lemma 3.2. We call a subset $I \subseteq [n]$ *good* if every antichain $A \subseteq [0,1]^n$ with $\mathcal{H}^{n-1}(\pi_i(A)) \leqslant \delta$ for all $i \in I$ satisfies

$$\mathcal{H}^{n-1}(A) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) + (nD + n - |I|)\delta\,.$$

As a consequence of Lemma 3.2 we know that the set $[n]$ is good. Hence there exists a minimal good set $I \subseteq [n]$. Assume for the sake of contradiction that $I \neq \varnothing$. By permuting the coordinates if necessary we may suppose that $n \in I$. Consider any antichain $A \subseteq [0,1]^n$ with

$$\mathcal{H}^{n-1}\big(\pi_i(A)\big) \leqslant \delta \quad \text{for all } i \in I \smallsetminus \{n\}\,. \tag{3.3}$$

By Lemma 3.3 there exists a Borel set B' with $\mathcal{H}^{n-1}(B') > 1 - \delta$ such that the set $A' = A \cap \pi_n^{-1}(B')$ satisfies

$$\mathcal{H}^{n-1}(A') \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A')\big) + \delta\,. \tag{3.4}$$

Set $B'' = [0,1]^{n-1} \smallsetminus B'$ and $A'' = A \cap \pi_n^{-1}(B'')$. Notice that $\mathcal{H}^{n-1}(\pi_i(A'')) \leqslant \delta$ for all $i \in I$. This is because for $i \in I \smallsetminus \{n\}$ we have

$$\mathcal{H}^{n-1}\big(\pi_i(A'')\big) \leqslant \mathcal{H}^{n-1}\big(\pi_i(A)\big) \leqslant \delta\,,$$

while for $i = n$ the first property of B' yields

$$\mathcal{H}^{n-1}\big(\pi_n(A'')\big) \leqslant \mathcal{H}^{n-1}(B'') = 1 - \mathcal{H}^{n-1}(B') \leqslant \delta\,.$$

Since I is good, it follows that

$$\mathcal{H}^{n-1}(A'') \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A'')\big) + (nD + n - |I|)\delta\,.$$

Adding (3.4) and taking Lemma 3.4 into account, we deduce

$$
\begin{aligned}
\mathcal{H}^{n-1}(A) \;&\leqslant\; \mathcal{H}^{n-1}(A') + \mathcal{H}^{n-1}(A'') \\
&\leqslant\; \sum_{i=1}^{n} \big(\mathcal{H}^{n-1}\big(\pi_i(A')\big) + \mathcal{H}^{n-1}\big(\pi_i(A'')\big)\big) + (nD + n - |I|)\delta + \delta \\
&=\; \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) + (nD + n - |I \smallsetminus \{n\}|)\delta\,.
\end{aligned}
$$

As this argument applies to any antichain A with (3.3), we have thereby shown that the set $I \smallsetminus \{n\}$ is good as well, contrary to the minimality of I.

This contradiction proves that $I = \varnothing$ is good, or in other words that for every antichain $A \subseteq [0,1]^n$ the estimate

$$\mathcal{H}^{n-1}(A) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) + n(D + 1)\delta$$

holds. In the limit $\delta \to 0$ this proves Proposition 3.1. $\qquad\square$

3.2. **Discretisation.** This subsection deals with the proof of Lemma 3.2. Let us recall the following well-known continuity property of the Hausdorff measure (see [18, Theorem 1.4], the remarks that follow, and [18, Corollary 4.5]).

Fact 3.5. *If $m \geqslant k \geqslant 1$ are integers, $E_1 \subseteq E_2 \subseteq \cdots \subseteq \mathbb{R}^m$ are arbitrary sets, and $E = \bigcup_{\ell \geqslant 1} E_\ell$, then*

$$\mathcal{H}^k(E) = \lim_{\ell \to \infty} \mathcal{H}^k(E_\ell)\,. \qquad\square$$

The discretisation procedure by means of which we shall establish Lemma 3.2 works as follows. Fix a large positive integer m the size of which determines how fine our approximation is going to be. Let $[0,1] = I_1 \cup \ldots \cup I_m$ be a partition of the unit interval into m consecutive intervals of length $\frac{1}{m}$. It is somewhat immaterial how we proceed with the boundary points $\frac{1}{m}, \ldots, \frac{m-1}{m}$, but for definiteness we set

$$
I_j = \begin{cases} \left[\frac{j-1}{m}, \frac{j}{m}\right) & \text{if } j \in [m-1], \\ \left[\frac{m-1}{m}, 1\right] & \text{if } j = m. \end{cases}
$$

This gives rise to the partition

$$[0,1]^n = \bigsqcup_{\mathbf{d} \in [m]^n} C(\mathbf{d})$$

of the n-dimensional unit cube into m^n subcubes defined by

$$C(\mathbf{d}) = I_{d_1} \times \cdots \times I_{d_n}$$

for every $\mathbf{d} = (d_1, \ldots, d_n) \in [m]^n$.

For any subset $W \subseteq [0,1]^n$ (not necessarily an antichain) we set

$$G_m(W) = \big\{ \mathbf{d} \in [m]^n \colon C(\mathbf{d}) \cap W \neq \varnothing \big\}$$

and observe that the disjoint union

$$H_m(W) = \bigsqcup_{\mathbf{d} \in G_m(W)} C(\mathbf{d}) \tag{3.5}$$

is a superset of W. Conversely, every point in $H_m(W)$ has at most the distance $\sqrt{n}/m$, the common diameter of our small cubes, from an appropriate point in W.

Let us introduce some useful notation for such situations. Given $S \subseteq \mathbb{R}^n$ and a point $\mathbf{x} \in \mathbb{R}^n$, we set $\mathrm{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{s}\| \colon \mathbf{s} \in S\}$, where $\| \cdot \|$ denotes the Euclidean norm. For a given positive real number δ and $S \subseteq \mathbb{R}^n$ the δ-*neighbourhood* of S is defined by

$$S^{(\delta)} = \{\mathbf{x} \in \mathbb{R}^n \colon \mathrm{dist}(\mathbf{x}, S) \leqslant \delta\} \, . \tag{3.6}$$

Summarising the above discussion, we have

$$W \subseteq H_m(W) \subseteq W^{(\sqrt{n}/m)}$$

for every $W \subseteq [0,1]^n$ and every $m \in \mathbb{N}$. In the special case where W is closed we have $W = \bigcap_{m \geqslant 1} W^{(\sqrt{n}/m)}$ and, consequently, Fact 3.5 yields

$$\lim_{m \to \infty} \frac{|G_m(W)|}{m^n} = \mathcal{H}^n(W) \, . \tag{3.7}$$

Lemma 3.6. *If $A_1, \ldots, A_n \subseteq [0,1]^{n-1}$ are closed sets and $A \subseteq [0,1]^n$ is a weak antichain with $\pi_i(A) \subseteq A_i$ for all $i \in [n]$, then*

$$\mathcal{H}^{n-1}(A) \leqslant D \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}(A_i) \, ,$$

where D denotes the constant introduced in (3.1).

Proof. Fix $\delta > 0$, consider an arbitrary positive integer $m \geqslant \sqrt{n}/\delta$, and set $B = G_m(A)$. The covering $A \subseteq \bigcup_{\mathbf{d} \in B} C(\mathbf{d})$ of A uses $|B|$ cubes of diameter $\sqrt{n}/m \leqslant \delta$ and thus we have

$$\mathcal{H}^{n-1}_\delta(A) \leqslant \alpha_{n-1} \cdot |B| \cdot \left(\frac{\sqrt{n}}{m}\right)^{n-1} \overset{(3.1)}{=\joinrel=} \frac{D|B|}{m^{n-1}} \, .$$

Since A is a weak antichain in $[0,1]^n$, the set B is a weak antichain in $\mathbb{Z}^n$ and Theorem 1.5 discloses

$$|B| \leqslant \sum_{i=1}^{n} |\pi_i(B)|\,.$$

Obviously, $\pi_i(B) \subseteq G_m(A_i)$ for every $i \in [n]$ (where the operator G_m is applied to the $(n-1)$-dimensional unit cube). Hence

$$\mathcal{H}_\delta^{n-1}(A) \leqslant D \cdot \sum_{i=1}^{n} \frac{|\pi_i(B)|}{m^{n-1}} \leqslant D \cdot \sum_{i=1}^{n} \frac{|G_m(A_i)|}{m^{n-1}}$$

and by (3.7) we obtain in the limit $m \to \infty$ that

$$\mathcal{H}_\delta^{n-1}(A) \leqslant D \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}(A_i)\,.$$

Finally $\delta \to 0$ yields the desired result. $\square$

Now a standard application of Fact 3.5 leads to Lemma 3.2.

Proof of Lemma 3.2. Let $\varepsilon > 0$ be arbitrary. Pick for every $i \in [n]$ a sequence $(C_{i,k})_{k \geqslant 1}$ of closed subsets of $[0,1]^{n-1}$ such that

$$\sum_{k} \mathcal{H}^{n-1}(C_{i,k}) \leqslant \mathcal{H}^{n-1}\big(\pi_i(A)\big) + \varepsilon \tag{3.8}$$

and $\pi_i(A) \subseteq \bigcup_k C_{i,k}$.

Now consider an arbitrary $\ell \geqslant 1$. Setting

$$A_i^{(\ell)} = \bigcup_{k \in [\ell]} C_{i,k}$$

for every $i \in [n]$ and

$$A^{(\ell)} = \bigcap_{i \in [n]} \pi_i^{-1}\big(A_i^{(\ell)}\big)$$

we deduce from Lemma 3.6 that

$$\mathcal{H}^{n-1}(A \cap A^{(\ell)}) \leqslant D \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(A_i^{(\ell)}\big)\,.$$

Using $A \subseteq \bigcup_{\ell \geqslant 1} A^{(\ell)}$, Fact 3.5, and (3.8) we obtain in the limit $\ell \to \infty$ that

$$\mathcal{H}^{n-1}(A) \leqslant D \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) + nD\varepsilon\,.$$

As $\varepsilon > 0$ was arbitrary, this proves Lemma 3.2. $\square$

Corollary 3.7. *If $A \subseteq [0,1]^n$ is a weak antichain, then $\mathcal{H}^{n-1}(A) \leqslant Dn$ is finite and, consequently, $\mathcal{H}^n(A) = 0$.* $\square$

3.3. **The decomposition Lemma.** Let an n-dimensional antichain $A \subseteq [0,1]^n$ be given, where $n \geqslant 2$. Since no two points in A can agree in their first $n-1$ coordinates, there exists a function $f_A \colon \pi_n(A) \longrightarrow [0,1]$ such that

$$A = \left\{(\mathbf{x}, f_A(\mathbf{x})) \colon \mathbf{x} \in \pi_n(A)\right\}.$$

The fact that A is indeed an antichain is equivalent to $f(\mathbf{x}) > f(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$ are in $\pi_n(A)$. It is often convenient to extend this function f_A in a monotonicity preserving way to the whole $(n-1)$-dimensional unit cube. To this end one defines $\hat{f}_A \colon [0,1]^{n-1} \longrightarrow [0,1]$ by

$$\hat{f}_A(\mathbf{x}) = \inf\left\{f_A(\mathbf{a}) \colon \mathbf{a} \in \pi_n(A) \text{ and } \mathbf{a} \leqslant \mathbf{x}\right\} \tag{3.9}$$

for all $\mathbf{x} \in [0,1]^{n-1}$, where, in this context, $\inf(\varnothing) = 1$. By the aforementioned fact on f_A we have $\hat{f}_A(\mathbf{x}) = f_A(\mathbf{x})$ for all $\mathbf{x} \in \pi_n(A)$. Moreover, if $\mathbf{x} \leqslant \mathbf{y}$ are in $[0,1]^{n-1}$, then $f(\mathbf{x}) \geqslant f(\mathbf{y})$. We shall refer to $\hat{f}_A$ as the *function associated with the antichain A*.

More generally, we call a function $f \colon [0,1]^{n-1} \longrightarrow [0,1]$ *order-reversing*, if we have $f(\mathbf{x}) \geqslant f(\mathbf{y})$ whenever $\mathbf{x} \leqslant \mathbf{y}$. So for instance the function $\hat{f}_A$ associated with an antichain A has just been observed to be order-reversing. We will need the following properties of such functions proved in [7].

Lemma 3.8. *If $f \colon [0,1]^{n-1} \longrightarrow [0,1]$ is order-reversing, then it is measurable in the sense that preimages of Borel sets are Lebesgue measurable. Moreover, f is almost everywhere differentiable.*

Let us record an easy consequence.

Fact 3.9. *Given an antichain $A \subseteq [0,1]$ with associated function $\hat{f}_A \colon [0,1]^{n-1} \longrightarrow [0,1]$ and $c \in [0,1]$, the set*

$$L = \left\{(x_1, \ldots, x_{n-1}) \in [0,1]^{n-1} \colon x_{n-1} < \hat{f}_A(x_1, \ldots, x_{n-2}, c)\right\}$$

is Lebesgue measurable.

Proof. By Lemma 3.8 it suffices to check that the characteristic function $\mathbf{1}_L$ of L is order-reversing. So let $(\mathbf{x}, x_{n-1}) \leqslant (\mathbf{y}, y_{n-1})$ be given, where $\mathbf{x}, \mathbf{y} \in [0,1]^{n-2}$. We need to prove that

$$\mathbf{1}_L(\mathbf{x}, x_{n-1}) \geqslant \mathbf{1}_L(\mathbf{y}, y_{n-1}).$$

If $(\mathbf{y}, y_{n-1}) \notin L$ this is clear, so we may suppose that $(\mathbf{y}, y_{n-1}) \in L$. Since $\hat{f}_A$ is order-reversing, it follows that

$$x_{n-1} \leqslant y_{n-1} < \hat{f}_A(\mathbf{y}, c) \leqslant \hat{f}_A(\mathbf{x}, c),$$

which in turn implies that $(\mathbf{x}, x_{n-1}) \in L$. $\qquad\square$

Now we are ready for the proof of Lemma 3.4, which will occupy the remainder of this subsection.

Proof of Lemma 3.4. For $i = n$ this follows from the Lebesgue measurability of the set B' and from the fact that in $\mathbb{R}^{n-1}$ the $(n-1)$-dimensional Hausdorff outer measure coincides with the $(n-1)$-dimensional Lebesgue outer measure. So without loss of generality we may henceforth assume that $i = n - 1$.

Define the set-function $\nu \colon \wp([0,1]^{n-1}) \longrightarrow [0,1]$ by setting

$$\nu(E) = \mathcal{H}^{n-1}\big(\pi_{n-1}(\pi_n^{-1}(E) \cap A)\big)$$

for every $E \subseteq [0,1]^{n-1}$. In other words, if $\hat{f}_A \colon [0,1]^{n-1} \longrightarrow [0,1]$ is the function associated with the antichain A, then $\nu(E) = \mathcal{H}^{n-1}(F_E)$, where

$$F_E = \big\{(x_1, \ldots, x_{n-2}, \hat{f}_A(x_1, \ldots, x_{n-1})) \colon (x_1, \ldots, x_{n-1}) \in E \cap \pi_n(A)\big\}.$$

Notice that ν is an outer measure. We will show later that B' is ν-measurable. This will imply that

$$\nu\big(\pi_n(A)\big) = \nu\big(\pi_n(A) \cap B'\big) + \nu\big(\pi_n(A) \cap B''\big),$$

which is equivalent to

$$\mathcal{H}^{n-1}\big(\pi_{n-1}(A)\big) = \mathcal{H}^{n-1}\big(\pi_{n-1}(A')\big) + \mathcal{H}^{n-1}\big(\pi_{n-1}(A'')\big),$$

and the result will follow.

Thus it remains to show that all Borel sets B' are ν-measurable. It is well known that the sigma algebra of Borel subsets of $[0,1]^{n-1}$ is generated by the closed half-spaces bounded by hyperplanes which are orthogonal to the coordinate axes. Therefore it suffices to establish that for all $c \in [0,1]$ and $i \in [n-1]$ the set

$$B_i(c) = \big\{(x_1, \ldots, x_{n-1}) \colon x_i \leqslant c\big\}$$

is ν-measurable. This means that for each test set $E \subseteq [0,1]^{n-1}$ we need to prove (see [3, Proposition 1.5.11]) that

$$\nu(E) = \nu\big(E \cap B_i(c)\big) + \nu\big(E \smallsetminus B_i(c)\big). \tag{3.10}$$

In case $i \in [n-2]$ this rewrites as

$$\mathcal{H}^{n-1}(F_E) = \mathcal{H}^{n-1}\big(F_E \cap B_i(c)\big) + \mathcal{H}^{n-1}\big(F_E \smallsetminus B_i(c)\big)$$

and follows from the measurability of $B_i(c)$. Thus we may suppose $i = n - 1$ from now on.

Setting

$$F_E^- = \big\{(x_1, \ldots, x_{n-2}, \hat{f}_A(x_1, \ldots, x_{n-1})) \colon (x_1, \ldots, x_{n-1}) \in E \cap \pi_n(A) \text{ and } x_{n-1} \leqslant c\big\},$$

and

$$F_E^+ = \left\{ (x_1, \ldots, x_{n-2}, \hat{f}_A(x_1, \ldots, x_{n-1})) \colon (x_1, \ldots, x_{n-1}) \in E \cap \pi_n(A) \text{ and } x_{n-1} > c \right\}$$

we can reformulate (3.10) as

$$\mathcal{H}^{n-1}(F_E) = \mathcal{H}^{n-1}(F_E^-) + \mathcal{H}^{n-1}(F_E^+) \,. \tag{3.11}$$

Now by Fact 3.9 the set

$$L = \left\{ (x_1, \ldots, x_{n-1}) \in [0,1]^{n-1} \colon x_{n-1} < \hat{f}_A(x_1, \ldots, x_{n-2}, c) \right\}$$

is Lebesgue measurable, whence

$$\mathcal{H}^{n-1}(F_E) = \mathcal{H}^{n-1}(F_E \smallsetminus L) + \mathcal{H}^{n-1}(F_E \cap L) \,. \tag{3.12}$$

Moreover, the set

$$N = \left\{ (x_1, \ldots, x_{n-2}, \hat{f}_A(x_1, \ldots, x_{n-2}, c)) \colon x_1, \ldots, x_{n-2} \in [0,1] \right\}$$

is a weak antichain in $[0,1]^{n-1}$, so by Corollary 3.7 we have $\mathcal{H}^{n-1}(N) = 0$.

Together with the inclusions

$$(F_E \smallsetminus L) \smallsetminus N \subseteq F_E^- \subseteq F_E \smallsetminus L$$

$$\text{and} \qquad F_E \cap L \subseteq F_E^+ \subseteq (F_E \cap L) \cup N \,,$$

which follow from the fact that $\hat{f}_A$ is order-reversing, this shows

$$\mathcal{H}^{n-1}(F_E^-) = \mathcal{H}^{n-1}(F_E \smallsetminus L) \quad \text{and} \quad \mathcal{H}^{n-1}(F_E^+) = \mathcal{H}^{n-1}(F_E \cap L) \,.$$

Therefore (3.12) implies (3.11). $\qquad \square$

3.4. **Proof of Lemma 3.3.** There are two issues that need to be addressed when transferring the proof of the projection inequality for smooth antichains sketched in Subsection 3.1 to the general case. Starting with the representation (3.2) of a given antichain A with an order-reversing function f, we need to deal with the fact that B may fail to be measurable and, moreover, with the possible non-differentiability of f. It turns out that these two points can be handled separately from each other and we start by giving an argument that applies to the case where f is linear and B may be arbitrary.

Lemma 3.10. *If* $L \colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ *is a linear function,* $B \subseteq [0,1]^{n-1}$ *is arbitrary and*

$$S = \left\{ (\mathbf{x}, L(\mathbf{x})) \colon \mathbf{x} \in B \right\},$$

then

$$\mathcal{H}^{n-1}(S) \leqslant \mathcal{H}^{n-1}(B) + \sum_{i=1}^{n-1} \mathcal{H}^{n-1}\big(\pi_i(S)\big) \,.$$

Proof. Let L be given by

$$(x_1, \ldots, x_{n-1}) \longmapsto \sum_{i=1}^{n-1} c_i x_i \, .$$

For $i \in [n-1]$ the map: $B \longrightarrow \pi_i(S)$ given by

$$(x_1, \ldots, x_{n-1}) \longmapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, c_1 x_1 + \ldots + c_{n-1} x_{n-1})$$

and the map: $B \longrightarrow S$ given by

$$(x_1, \ldots, x_{n-1}) \longmapsto (x_1, \ldots, x_{n-1}, c_1 x_1 + \ldots + c_{n-1} x_{n-1})$$

are linear and surjective. Since the $(n-1)$-dimensional Hausdorff measure of B agrees with the $(n-1)$-dimensional Lebesgue outer measure of B, we have (see e.g. [11, p.114]) $H^{n-1}\big(\pi_i(S)\big) = |c_i| \cdot \mathcal{H}^{n-1}(B)$ and $\mathcal{H}^{n-1}(S) = \big(1 + \sum_{i=1}^{n-1} c_i^2\big)^{1/2} \mathcal{H}^{n-1}(B)$. Thus it remains to remark

$$\big(1 + \sum_{i=1}^{n-1} c_i^2\big)^{1/2} \leqslant 1 + \sum_{i=1}^{n-1} |c_i| \, ,$$

which is clear. $\qquad\square$

Recall that a function $f \colon F \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *Lipschitz with constant K* (or *K-Lipschitz* for short) if

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leqslant K \cdot \|\mathbf{x} - \mathbf{y}\| \text{ for all } \mathbf{x}, \mathbf{y} \in F \, .$$

We use several times the following well-known result concerning the s-dimensional Hausdorff measure (see [12, p.24]).

Lemma 3.11. *Let m and n be positive integers and let $F \subseteq \mathbb{R}^n$. If $f \colon F \longrightarrow \mathbb{R}^m$ is a K-Lipschitz function, then $\mathcal{H}^s(f(F)) \leqslant K^s \cdot \mathcal{H}^s(F)$.*

For the rest of this subsection we fix an antichain A in $[0,1]^n$ and a positive real number δ for which we would like to establish Lemma 3.3. Let $\hat{f}_A$ be the function associated with A (see (3.9)). If for some $\mathbf{x} \in (0,1)^{n-1}$ and $i \in [n-1]$ the i-th partial derivative of $\hat{f}_A$ at $\mathbf{x}$ exists, we denote it by $D_i \hat{f}_A(\mathbf{x})$. Furthermore, if a point $\mathbf{x}$ has the property that all partial derivatives $D_1 \hat{f}_A(\mathbf{x}), \ldots, D_{n-1} \hat{f}_A(\mathbf{x})$ exist, we define $L_{\mathbf{x}} \colon \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ to be the linear form given by

$$L_{\mathbf{x}}(v_1, \ldots, v_{n-1}) = \sum_{i=1}^{n-1} D_i \hat{f}_A(\mathbf{x}) v_i \, .$$

The Borel set B we need to exhibit will be a subset of a closed set $C \subseteq (0,1)^{n-1}$ on which $\hat{f}_A$ has some useful differentiability properties collected in the lemma that follows.

Lemma 3.12. *There exists a closed set $C \subseteq (0,1)^{n-1}$ such that*

(*i*) $\mathcal{H}^{n-1}(C) > 1 - \delta/2$;

(*ii*) *all partial derivatives of $\hat{f}_A$ exist and are continuous on C;*

(*iii*) *for every $\mathbf{x} \in C$, the function $\hat{f}_A$ is differentiable at $\mathbf{x}$ with the derivative $L_{\mathbf{x}}$;*

(*iv*) *the differentiability of $\hat{f}_A$ is uniform on C, i.e., for every $\eta > 0$ there exists an $\varepsilon > 0$ such that for all $\mathbf{a}, \mathbf{x} \in C$ with $\|\mathbf{a} - \mathbf{x}\| < \varepsilon$ we have*

$$\left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{a} - \mathbf{x}) \right| \leqslant \eta \|\mathbf{a} - \mathbf{x}\| \, .$$

Proof. Since the function $\hat{f}_A$ is order-reversing, Lemma 3.8 implies that it is almost everywhere differentiable. Hence there exists a measurable set $C_1 \subseteq (0,1)^{n-1}$, whose measure equals 1, such that for every $\mathbf{x} \in C_1$ all partial derivatives of $\hat{f}_A$ exist and, moreover, $\hat{f}_A$ is differentiable at $\mathbf{x}$ with the derivative $L_{\mathbf{x}}$. So by choosing $C \subseteq C_1$ later, we can ensure (*iii*) as well as the first part of (*ii*).

Next, by Lusin's theorem (see e.g. [3, Theorem 2.2.10]), there exists a closed set $C_2 \subseteq C_1$ with $\mathcal{H}^{n-1}(C_2) > 1 - \delta/4$ such that all partial derivatives $D_i \hat{f}_A$ exist and are continuous on C_2. So every $C \subseteq C_2$ satisfies (*ii*) as well.

Now define for every $m \in \mathbb{N}$ the measurable function $g_m \colon C_2 \longrightarrow \mathbb{R}$ by

$$g_m(\mathbf{x}) = \sup \left\{ \frac{|\hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{x}) - L_{\mathbf{x}}(\mathbf{a} - \mathbf{x})|}{\|\mathbf{a} - \mathbf{x}\|} : \mathbf{a} \in \left(\mathbb{Q} \cap (0,1) \right)^{n-1}, \, 0 < \|\mathbf{a} - \mathbf{x}\| < 1/m \right\} .$$

Since $\lim_{m \to \infty} g_m(\mathbf{x}) = 0$ holds for every $\mathbf{x} \in C_2$, Egoroff's theorem (see e.g. [3, Theorem 2.2.1]) implies that there exists a closed set $C \subseteq C_2$ with $\mathcal{H}^{n-1}(C) > 1 - \delta/2$ and such that $g_m \to 0$ holds uniformly on C. Such a set has the properties (*i*) and (*iv*) as well. $\qquad \square$

Throughout the remainder of this subsection, C denotes a set provided by the previous lemma. Set

$$K = \left(1 + \frac{\delta}{n} \right)^{1/(2n-2)} \tag{3.13}$$

and for $\mathbf{x} \in (0,1)^{n-1}$ and $\varepsilon > 0$ let

$$Q_\varepsilon(\mathbf{x}) = \left\{ \mathbf{y} \in \mathbb{R}^{n-1} : \|\mathbf{x} - \mathbf{y}\|_\infty < \varepsilon \right\}$$

be the ε-cube around $\mathbf{x}$. Next we intend to show for every $\mathbf{x} \in C$, that if $\varepsilon > 0$ is sufficiently small, then the projection inequality holds in an approximate form for $A \cap \pi_n^{-1}(C \cap Q_\varepsilon(\mathbf{x}))$ instead of A. Once this is known, a Vitali covering argument will allow us to combine many such cubes, so that the desired set B can be taken to be a disjoint union of several sets of the form $C \cap Q_\varepsilon(\mathbf{x})$. The definition that follows collects some properties of such cubes that will be useful for implementing this strategy.

Definition 3.13. Given $\mathbf{x} \in C$ and $\varepsilon > 0$ the ε-cube $Q = Q_\varepsilon(\mathbf{x})$ is said to be *nice* if it has the following properties:

(a) $Q \subseteq [0, 1]^{n-1}$.

(b) If $\mathbf{a}, \mathbf{b} \in Q \cap C$, then

$$\left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b}) \right|^2 \leqslant (K^2 - 1) \cdot \|\mathbf{a} - \mathbf{b}\|^2 + K^2 \cdot \left| L_{\mathbf{x}}(\mathbf{a} - \mathbf{b}) \right|^2 .$$

(c) If $\mathbf{a}, \mathbf{b} \in Q \cap C$, $i \in [n-1]$, and $D_i \hat{f}_A(\mathbf{x}) \neq 0$, then

$$\left| L_{\mathbf{x}}(\mathbf{a} - \mathbf{b}) \right|^2 \leqslant (K^2 - 1) \sum_{j \in [n-1] \smallsetminus \{i\}} |a_j - b_j|^2 + K^2 \cdot \left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b}) \right|^2 .$$

The following result shows that nice cubes determine parts of A, for which the projection inequality holds up to a multiplicative factor that is close to 1.

Lemma 3.14. *If $Q = Q_\varepsilon(\mathbf{x})$ is a nice cube and $A_Q = A \cap \pi_n^{-1}(Q \cap C)$, then*

$$\mathcal{H}^{n-1}(A_Q) \leqslant K^{2(n-1)} \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A_Q)\big) .$$

Proof. Observe that Definition 3.13(b) asserts that the map $\big(\mathbf{a}, L_{\mathbf{x}}(\mathbf{a})\big) \longmapsto \big(\mathbf{a}, \hat{f}_A(\mathbf{a})\big)$ from the set $S = \big\{(\mathbf{a}, L_{\mathbf{x}}(\mathbf{a})) \colon \mathbf{a} \in \pi_n(A_Q)\big\}$ onto the set A_Q is Lipschitz with constant K. Therefore Lemma 3.11 and Lemma 3.10 (applied with $\pi_n(A)$ and $L_{\mathbf{x}}$ here in place of B and L there) yield

$$\begin{aligned} \mathcal{H}^{n-1}(A_Q) &\leqslant K^{n-1} \cdot \mathcal{H}^{n-1}(S) \\ &\leqslant K^{n-1}\bigg(\mathcal{H}^{n-1}\big(\pi_n(A_Q)\big) + \sum_{i=1}^{n-1} \mathcal{H}^{n-1}\big(\pi_i(S)\big)\bigg) . \end{aligned}$$

So to conclude the proof it suffices to show

$$\mathcal{H}^{n-1}\big(\pi_i(S)\big) \leqslant K^{n-1} \mathcal{H}^{n-1}\big(\pi_i(A_Q)\big) \quad \text{for all } i \in [n-1]. \tag{3.14}$$

If $D_i \hat{f}_A(\mathbf{x}) = 0$, the set $\pi_i(S)$ is contained in an $(n-2)$-dimensional vector space and (3.14) is clear. On the other hand, if $D_i \hat{f}_A(\mathbf{x}) \neq 0$, then Definition 3.13($c$) implies that the map

$$\big(\mathbf{a}_i, \hat{f}_A(\mathbf{a})\big) \longmapsto \big(\mathbf{a}_i, L_{\mathbf{x}}(\mathbf{a})\big)$$

from $\pi_i(A_Q)$ to $\pi_i(S)$ is Lipschitz with constant K, which entails (3.14) in view of Lemma 3.11. $\square$

Next we show that nice cubes are ubiquitous.

Lemma 3.15. *Given $\mathbf{x} \in C$, the cube $Q_\varepsilon(\mathbf{x})$ is nice for every sufficiently small $\varepsilon > 0$.*

Proof. We verify for each of the three clauses in Defintion 3.13 separately that it holds for every sufficiently small $\varepsilon > 0$. Since $C \subseteq (0,1)^{n-1}$, this is immediate for (a). For (b), we put

$$\eta = \frac{K^2 - 1}{2K}$$

and let $\varepsilon > 0$ be sufficiently small. For arbitrary $\mathbf{a}, \mathbf{b} \in Q_\varepsilon(\mathbf{x}) \cap C$ Lemma 3.12 yields

$$
\begin{aligned}
\left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b}) \right| &\overset{(iv)}{\leqslant} |L_{\mathbf{a}}(\mathbf{a} - \mathbf{b})| + \eta \|\mathbf{a} - \mathbf{b}\| \\
&\leqslant |L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| + (\|L_{\mathbf{x}} - L_{\mathbf{a}}\| + \eta) \|\mathbf{a} - \mathbf{b}\| \\
&\overset{(ii)}{\leqslant} |L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| + 2\eta \|\mathbf{a} - \mathbf{b}\| .
\end{aligned}
$$

Now the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
\left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b}) \right|^2 &\leqslant \left(\frac{1}{K} \cdot K \cdot |L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| + \frac{2\eta}{\sqrt{K^2 - 1}} \cdot \sqrt{K^2 - 1} \cdot \|\mathbf{a} - \mathbf{b}\| \right)^2 \\
&\leqslant \left(\frac{1}{K^2} + \frac{4\eta^2}{K^2 - 1} \right) \cdot \left(K^2 \cdot |L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})|^2 + (K^2 - 1) \cdot \|\mathbf{a} - \mathbf{b}\|^2 \right) ,
\end{aligned}
$$

and, as the first factor is equal to 1 by the definition of η, this proves part (b) of Definition 3.13.

It remains to check (c). Without loss of generality, we may assume that $i = n - 1$ and $D_{n-1}f(\mathbf{x}) \neq 0$. Set

$$
\begin{aligned}
\lambda' &= \left(\sum_{i=1}^{n-2} D_i \hat{f}_A(\mathbf{x})^2 \right)^{1/2} , \\
\lambda'' &= \left| D_{n-1} \hat{f}_A(\mathbf{x}) \right| , \\
\xi &= \min \left\{ \frac{\lambda''}{2}, \frac{\lambda'' \sqrt{K^2 - 1}}{2(\lambda' + \lambda'')} \right\} , \\
\eta &= \frac{(K - 1)\xi}{2K} ,
\end{aligned}
$$

let $\varepsilon > 0$ be sufficiently small, and fix arbitrary points $\mathbf{a}, \mathbf{b} \in Q_\varepsilon(\mathbf{x}) \cap C$. Recall that we have to show

$$
|L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})|^2 \leqslant (K^2 - 1) \sum_{i=1}^{n-2} (a_i - b_i)^2 + K^2 \cdot \left| \hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b}) \right|^2 .
$$

To this end, it suffices to establish the following two implications:

(1) If $|L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| \geqslant \xi \cdot \|\mathbf{a} - \mathbf{b}\|$, then $|L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| \leqslant K |\hat{f}(\mathbf{a}) - \hat{f}(\mathbf{b})|$.

(2) If $|L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| \leqslant \xi \cdot \|\mathbf{a} - \mathbf{b}\|$, then $|L_{\mathbf{x}}(\mathbf{a} - \mathbf{b})| \leqslant \sqrt{K^2 - 1} \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|$,

where $\mathbf{a}_{n-1} = (a_1, \ldots, a_{n-2})$ and $\mathbf{b}_{n-1}$ is defined analogously. For the proof of (1) we observe that, similarly as before, Lemma 3.12 yields

$$|\hat{f}_A(\mathbf{a}) - \hat{f}_A(\mathbf{b})| \overset{(iv)}{\geqslant} |L_\mathbf{a}(\mathbf{a} - \mathbf{b})| - \eta\|\mathbf{a} - \mathbf{b}\|$$

$$\geqslant |L_\mathbf{x}(\mathbf{a} - \mathbf{b})| - (\|L_\mathbf{x} - L_\mathbf{a}\| + \eta) \cdot \|\mathbf{a} - \mathbf{b}\|$$

$$\overset{(ii)}{\geqslant} |L_\mathbf{x}(\mathbf{a} - \mathbf{b})| - 2\eta\|\mathbf{a} - \mathbf{b}\|.$$

Moreover, the definitions of ξ and η imply

$$2\eta\|\mathbf{a} - \mathbf{b}\| = \frac{K-1}{K} \cdot \xi \cdot \|\mathbf{a} - \mathbf{b}\| \leqslant \frac{K-1}{K} \cdot |L_\mathbf{x}(\mathbf{a} - \mathbf{b})|,$$

so that altogether we arrive at the desired estimate

$$|\hat{f}(\mathbf{a}) - \hat{f}(\mathbf{b})| \geqslant \frac{1}{K} \cdot |L_\mathbf{x}(\mathbf{a} - \mathbf{b})|.$$

Proceeding with (2) we set $c_i = D_i\hat{f}_A(\mathbf{x})$ for every $i \in [n-1]$ and $\mathbf{c} = (c_1, \ldots, c_{n-1})$. Thus $\lambda' = \|\mathbf{c}_{n-1}\|$, $\lambda'' = |c_{n-1}|$, and $L_\mathbf{x}(\mathbf{a} - \mathbf{b}) = \mathbf{c} \cdot (\mathbf{a} - \mathbf{b})$. Owing to the Cauchy-Schwarz inequality, the triangle inequality, and the assumption of (2) we have

$$\lambda'' \cdot |a_{n-1} - b_{n-1}| - \lambda' \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\| = |c_{n-1}(a_{n-1} - b_{n-1})| - \|\mathbf{c}_{n-1}\|\|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|$$

$$\leqslant |c_{n-1}(a_{n-1} - b_{n-1})| - |\mathbf{c}_{n-1} \cdot (\mathbf{a}_{n-1} - \mathbf{b}_{n-1})|$$

$$\leqslant |\mathbf{c} \cdot (\mathbf{a} - \mathbf{b})|$$

$$\leqslant \xi \cdot \|\mathbf{a} - \mathbf{b}\|$$

$$\leqslant \xi \cdot |a_{n-1} - b_{n-1}| + \xi \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|.$$

Since $\xi \leqslant \frac{\lambda''}{2}$, this leads to

$$\lambda'' \cdot |a_{n-1} - b_{n-1}| \leqslant (2\lambda' + \lambda'') \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|,$$

wherefore

$$\|\mathbf{a} - \mathbf{b}\| \leqslant \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\| + |a_{n-1} - b_{n-1}| \leqslant \frac{2(\lambda' + \lambda'')}{\lambda''} \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|.$$

Hence we have indeed

$$|L_\mathbf{x}(\mathbf{a} - \mathbf{b})| \leqslant \xi \cdot \|\mathbf{a} - \mathbf{b}\| \leqslant \sqrt{K^2 - 1} \cdot \|\mathbf{a}_{n-1} - \mathbf{b}_{n-1}\|,$$

which concludes the proof. $\qquad\square$

Given a measurable set $S \subseteq \mathbb{R}^d$, we shall say that a family $\mathcal{V}$ of open d-dimensional cubes forms a *Vitali covering* of S if for every $\mathbf{x} \in S$ and $\varepsilon > 0$ there is a cube $Q \in \mathcal{V}$ with $\mathbf{x} \in Q$ and $\mathrm{diam}(Q) < \varepsilon$. Recall that by *Vitali's Covering Theorem* (see e.g. [8, p.164])

in such a situation there is a countable subset $\mathcal{U} \subseteq \mathcal{V}$ such that the members of $\mathcal{U}$ are mutually disjoint and $S \smallsetminus \bigcup \mathcal{U}$ is null with respect to the Lebesgue measure.

Proof of Lemma 3.3. By Lemma 3.15 the collection of nice cubes forms a Vitali covering of C. Therefore Vitali's Covering Theorem yields countably many mutually disjoint nice cubes that cover C except for a null set. In view of the compactness of C and Lemma 3.12(i) this leads to finitely many mutually disjoint nice cubes, say $Q^{(1)}, \ldots, Q^{(N)}$, such that the Borel set

$$B = \bigcup_{k \in [N]} \left(C \cap Q^{(k)} \right)$$

satisfies $\mathcal{H}^{n-1}(B) > 1 - \delta$. It remains to show

$$\mathcal{H}^{n-1}(A') \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\left(\pi_i(A') \right) + \delta\,, \tag{3.15}$$

where $A' = A \cap \pi_n^{-1}(B)$.

Setting

$$A^{(k)} = A \cap \pi_n^{-1}\left(C \cap Q^{(k)} \right)$$

for every $k \in [N]$, we infer

$$\mathcal{H}^{n-1}(A^{(k)}) \leqslant K^{2(n-1)} \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\left(\pi_i(A^{(k)}) \right) \tag{3.16}$$

from Lemma 3.14.

Next we observe that due to

$$A' = \bigcup_{k \in [N]} A^{(k)}$$

a repeated application of Lemma 3.4 reveals

$$\sum_{k=1}^{N} \mathcal{H}^{n-1}\left(\pi_i(A^{(k)}) \right) = \mathcal{H}^{n-1}\left(\pi_i(A') \right)$$

for every $i \in [n]$. Therefore, by summing (3.16) over k we obtain

$$\begin{aligned}
\mathcal{H}^{n-1}(A') &\leqslant \sum_{k=1}^{N} \mathcal{H}^{n-1}\left(A^{(k)} \right) \\
&\leqslant K^{2(n-1)} \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\left(\pi_i(A') \right) \\
&\leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\left(\pi_i(A') \right) + n\left(K^{2(n-1)} - 1 \right)
\end{aligned}$$

and our choice of K in (3.13) leads to the desired estimate (3.15). $\qquad\square$

§4. Proof of the main Theorem

Let $n \geqslant 2$ be fixed throughout this section. We begin by describing a construction that allows us to "approximate" a given n-dimensional weak antichain with arbitrary "accuracy" by an antichain.

It will be convenient to write $S(\mathbf{x}) = \sum_{i=1}^{n} x_i$ for $\mathbf{x} \in \mathbb{R}^n$. Moreover, for every $\varepsilon \in \left(0, \frac{1}{2n}\right)$ we let $f_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ denote the linear transformation

$$(x_1, \ldots, x_n) \longmapsto \left(x_1 - \varepsilon S(\mathbf{x}), \ldots, x_n - \varepsilon S(\mathbf{x})\right),$$

and set

$$L_\varepsilon = \frac{1}{\sqrt{1 - 2n\varepsilon}} \,. \tag{4.1}$$

One checks easily that

$$S(f_\varepsilon(\mathbf{x})) = (1 - n\varepsilon)S(\mathbf{x}) \tag{4.2}$$

and, consequently, f_ε is invertible. Let us also note that f_ε maps $[0, 1]^n$ into $[-1, 1]^n$.

Fact 4.1. *Let $\varepsilon \in \left(0, \frac{1}{2n}\right)$.*

   (i) The inverse f_ε^{-1} is L_ε-Lipschitz.

   (ii) If $A \subseteq [0, 1]^n$ is a weak antichain, then $f_\varepsilon(A)$ is an antichain.

Proof. For part (i) it is enough to verify

$$\|f_\varepsilon(\mathbf{x}) - f_\varepsilon(\mathbf{y})\|^2 \geqslant (1 - 2n\varepsilon)\|\mathbf{x} - \mathbf{y}\|^2$$

for any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. In terms of $\mathbf{a} = \mathbf{x} - \mathbf{y}$ this rewrites as

$$\sum_{i=1}^{n} \left(a_i - \varepsilon S(\mathbf{a})\right)^2 \geqslant (1 - 2n\varepsilon) \sum_{i=1}^{n} a_i^2,$$

i.e.,

$$2n\varepsilon \sum_{i=1}^{n} a_i^2 \geqslant (2\varepsilon - n\varepsilon^2)S(\mathbf{a})^2 \,.$$

This follows from the fact that the Cauchy-Schwarz inequality yields the even stronger estimate

$$n \cdot \sum_{i=1}^{n} a_i^2 \geqslant S(\mathbf{a})^2 \,.$$

Now assume that contrary to (ii) we have a weak antichain $A \subseteq [0, 1]^n$ and two distinct points $f_\varepsilon(\mathbf{x}), f_\varepsilon(\mathbf{y}) \in f_\varepsilon(A)$ with $f_\varepsilon(\mathbf{x}) \leqslant f_\varepsilon(\mathbf{y})$. Using $S(f_\varepsilon(\mathbf{x})) < S(f_\varepsilon(\mathbf{y}))$ and (4.2) we obtain $S(\mathbf{x}) < S(\mathbf{y})$. So for every $i \in [n]$ the assumption

$$x_i - \varepsilon S(\mathbf{x}) \leqslant y_i - \varepsilon S(\mathbf{y})$$

yields

$$x_i \leqslant y_i + \varepsilon(S(\mathbf{x}) - S(\mathbf{y})) < y_i \,.$$

But $\mathbf{x} \prec \mathbf{y}$ contradicts A being a weak antichain. $\square$

Later it will be useful to know that in the situation of Fact 4.1(ii) the projection inequality (as in Proposition 3.1) applies to $f_\varepsilon(A) \subseteq [-1, 1]^n$. This is because the homothety from $[-1, 1]^n$ onto $[0, 1]^n$ sends $f_\varepsilon(A) \subseteq [-1, 1]^n$ onto an antichain in $[0, 1]^n$ and the $\mathcal{H}^{n-1}$-measure gets rescaled by a factor of 2^{n-1} under this map.

Proof of Theorem 1.3. We divide the argument into three steps.

Part I. Suppose first that A is a compact set. Let $\varepsilon \in \left(0, \frac{1}{2n}\right)$ be arbitrary and recall that by Fact 4.1(ii) the projection inequality applies to $f_\varepsilon(A)$. In combination with Lemma 3.11 and Fact 4.1(i) we obtain

$$\mathcal{H}^{n-1}(A) \leqslant L_\varepsilon^{n-1} \cdot \mathcal{H}^{n-1}\big(f_\varepsilon(A)\big) \leqslant L_\varepsilon^{n-1} \cdot \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(f_\varepsilon(A))\big) \,.$$

Therefore it suffices to prove for every $i \in [n]$ that

$$\liminf_{\varepsilon \to 0} \mathcal{H}^{n-1}\big(\pi_i(f_\varepsilon(A))\big) \leqslant \mathcal{H}^{n-1}\big(\pi_i(A)\big) \,. \tag{4.3}$$

Fix $i \in [n]$. Since $\|\pi_i(f_\varepsilon(\mathbf{a})) - \pi_i(\mathbf{a})\| \leqslant n^2 \varepsilon$ holds for all $\mathbf{a} \in A$ and $\varepsilon > 0$, we have

$$\pi_i\big(f_\varepsilon(A)\big) \subseteq \pi_i(A)^{(n^2 \varepsilon)} \,,$$

where the notation is as in (3.6). Thus a complementary variant of Fact 3.5 yields

$$\liminf_{\varepsilon \to 0} \mathcal{H}^{n-1}\big(\pi_i(f_\varepsilon(A))\big) \leqslant \liminf_{\varepsilon \to 0} \mathcal{H}^{n-1}\big(\pi_i(A)^{(n^2 \varepsilon)}\big) \leqslant \mathcal{H}^{n-1}\left(\bigcap_{\varepsilon > 0} \pi_i(A)^{(n^2 \varepsilon)}\right) \,.$$

As the compactness of A implies $\bigcap_{\varepsilon > 0} \pi_i(A)^{(n^2 \varepsilon)} = \pi_i(A)$, we thereby arrive at (4.3).

Part II. Next we treat the more general case that A is $\mathcal{H}^{n-1}$-measurable. For every compact $K \subseteq A \cap (0, 1)^n$ the result of the first part entails

$$\mathcal{H}^{n-1}(K) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(K)\big) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big) \,. \tag{4.4}$$

Since $\mathcal{H}^{n-1}(A)$ is finite by Corollary 3.7, the measurability of A implies (see e.g. [11, Theorem 1.7 and Theorem 1.8(ii)]) that

$$\mathcal{H}^{n-1}(A) = \sup\big\{\mathcal{H}^{n-1}(K) \colon K \text{ is a compact subset of } A\big\} \,.$$

So the desired result follows from (4.4).

Part III. Assume finally that A is an arbitrary weak antichain. We claim that the closure $\bar{A}$ of A is likewise a weak antichain. Otherwise there existed two distinct points

$\mathbf{a}, \mathbf{b} \in \bar{A}$ such that $\mathbf{a} \prec \mathbf{b}$. Observe that there are sufficiently small neighbourhoods $U(\mathbf{a})$ and $U(\mathbf{b})$ of $\mathbf{a}$ and $\mathbf{b}$ respectively, such that $\mathbf{c} \prec \mathbf{d}$ holds for all $\mathbf{c} \in U(\mathbf{a})$ and all $\mathbf{d} \in U(\mathbf{b})$. Since $U(\mathbf{a})$ and $U(\mathbf{b})$ necessarily contain points of A, we get a contradiction to the fact that A is a weak antichain. This proves that $\bar{A}$ is indeed a weak antichain.

Now by the Borel regularity of the Hausdorff measure, there exists for every $i \in [n]$ a Borel set $B_i \supseteq \pi_i(A)$ such that $\mathcal{H}^{n-1}(\pi_i(A)) = \mathcal{H}^{n-1}(B_i)$. Applying the result of the previous step to the measurable weak antichain

$$A^\star = \bar{A} \cap \bigcap_{i=1}^{n} \pi_i^{-1}(B_i),$$

we obtain

$$\begin{aligned}
\mathcal{H}^{n-1}(A) \;\leqslant\; & \mathcal{H}^{n-1}(A^\star) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A^\star)\big) \\
\leqslant\; & \sum_{i=1}^{n} \mathcal{H}^{n-1}(B_i) = \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\pi_i(A)\big),
\end{aligned}$$

as required. $\qquad\square$

§5. CONCLUDING REMARKS

5.1. **The discrete case.** For $n \geqslant 2$ the only case where equality holds in Theorem 1.5 occurs when $A = \varnothing$. This gives rise to the following question.

Problem 5.1. *Determine an optimal lower bound $g(n, m)$ on the gap $\sum_{i=1}^{n} |\pi_i(A)| - |A|$ as A varies over weak antichains in $\mathbb{Z}^n$ of size m.*

Let us mention that a slightly improved version of the argument presented in Section 2 yields

$$g(n, m) \geqslant n - 1$$

for all integers $m, n \geqslant 1$. This can be easily seen using $|A_i| < |\pi_i(A)|$ if $A_i = \varnothing$ and $\pi_i(A_{i-1}) \neq \pi_i(A_i)$ and consequently $|\pi_i(A_i)| < |\pi_i(A)|$ if $A_i \neq \varnothing$ and $i \geqslant 2$.

5.2. **Supremum vs. Maximum.** In connection with Corollary 1.4 one may wonder for which values of n the supremum is attained. For instance for $n = 1$ any one-point antichain in $[0, 1]$ has $\mathcal{H}^0$-measure 1 and a more sophisticated construction mentioned below shows that for $n = 2$ the supremum is a maximum as well. We believe that actually such antichains exist in all dimensions.

Conjecture 5.2. For every $n \geqslant 1$ there is an antichain in $[0, 1]^n$ whose $(n-1)$-dimensional Hausdorff measure equals n.

The aforementioned planar example exploits the known fact (see [13, p. 810]) that the 1-dimensional Hausdorff measure of the graph of a decreasing function $f\colon [0,1] \longrightarrow [0,1]$ is at most 2 and that this bound is attained by *singular functions* (i.e., strictly decreasing functions whose derivative equals zero almost everywhere). It remains to observe that the graph of a singular function is an antichain in $[0,1]^2$.

5.3. **Skewed projections of weak antichains.** So far we focused on inequalities for the orthogonal projections of weak antichains, but we believe that, actually, more general statements hold. In order to be more precise, we need some additional notation.

Given $A \subseteq [0,1]^n$ and $i \in [n]$, we set

$$\underline{A}_i = A \cap \left\{ (x_1, \ldots, x_n) \in [0,1]^n \colon x_i = \min\{x_1, \ldots, x_n\} \right\}.$$

Moreover, let for $i \in [n]$ the *skewed projections* $\Delta_i \colon \underline{A}_i \longrightarrow [0,1]^{n-1}$ be defined by

$$(x_1, \ldots, x_n) \longmapsto (x_1 - x_i, \ldots, x_{i-1} - x_i, x_{i+1} - x_i, \ldots, x_n - x_i).$$

Notice that the skewed projections restricted to a weak antichain are injective.

Conjecture 5.3. If $A \subseteq [0,1]^n$ is a weak antichain, then

$$\mathcal{H}^{n-1}(A) \leqslant \sum_{i=1}^{n} \mathcal{H}^{n-1}\big(\Delta_i(\underline{A}_i)\big).$$

Let us note that, if true, this would furnish a different proof of $\mathcal{H}^{n-1}(A) \leqslant n$. As a final result, we verify the validity of this conjecture when $n = 2$.

Theorem 5.4. *Conjecture 5.3 is true when $n = 2$.*

Proof. Since $\underline{A}_1$ is a weak antichain, it follows that Δ_1 is a bijection from $\underline{A}_i$ onto its image. Given two numbers $a, b \in \Delta_1(\underline{A}_1)$ with $a < b$ their inverse images under Δ_1 are of the form $\Delta_1^{-1}(a) = (x, y)$ and $\Delta_1^{-1}(b) = (x - \delta, y + \varepsilon)$ for some $x, y \in [0,1]$ and $\delta, \varepsilon \geqslant 0$. Owing to

$$\|\Delta_1^{-1}(a) - \Delta_1^{-1}(b)\| = \sqrt{\delta^2 + \varepsilon^2} \leqslant \delta + \varepsilon = |a - b|$$

the map Δ_1^{-1} is Lipschitz with constant 1 and Lemma 3.11 yields

$$\mathcal{H}^1(\underline{A}_1) \leqslant \mathcal{H}^1\big(\Delta_1(\underline{A}_1)\big).$$

Applying the same reasoning to $\Delta_2 \colon \underline{A}_2 \longrightarrow [0,1]$ we infer

$$\mathcal{H}^1(\underline{A}_2) \leqslant \mathcal{H}^1\big(\Delta_2(\underline{A}_2)\big).$$

From these two inequalities we conclude

$$\mathcal{H}^1(A) \leqslant \mathcal{H}^1\big(\Delta_1(A_1)\big) + \mathcal{H}^1\big(\Delta_2(A_2)\big),$$

as required. $\square$

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