ON THE BUSEMANN-PETTY PROBLEM ABOUT CONVEX, CENTRALLY SYMMETRIC BODIES IN $\mathbb{R}^n$

MICHAEL PAPADIMITRAKIS

§1. *Introduction.* Let A and B be two compact, convex sets in $\mathbb{R}^n$, each symmetric with respect to the origin 0. L is any $(n-1)$-dimensional subspace. In 1956 H. Busemann and C. M. Petty (see [**6**]) raised the question: Does $\text{vol}(A \cap L) < \text{vol}(B \cap L)$ for every L imply $\text{vol}(A) < \text{vol}(B)$? The answer in case $n = 2$ is affirmative in a trivial way. Also in 1953 H. Busemann (see [**4**]) proved that if A is any ellipsoid the answer is affirmative. In fact, as he observed in [**5**], the answer is still affirmative if A is an ellipsoid with 0 as center of symmetry and B is any compact set containing 0.

The first breakthrough was in 1975 (see [**8**]) when D. G. Larman and C. A. Rogers took $B = B_n$, the unit ball in $\mathbb{R}^n$ and proved that, if $n \geq 12$, there exist A's which are arbitrarily small perturbations of B and which give a negative answer to the problem. Their proof is not constructive and uses probabilistic reasoning.

In 1988, K. Ball (see [**2**]) proved that, if $n \geq 10$, $B = B_n$ and A an appropriate dilation of $[-1, 1]^n$ give a negative answer. Also in 1990, A. Giannopoulos (see [**7**]) proved that, if $n \geq 7, B = B_n$ and A a cylinder of the form $\{(x_1, \ldots, x_n): x_1^2 + \ldots + x_{n-1}^2 \leq a^2, |x_n| \leq b\}$ (for a certain choice of a, b) provide a negative answer.

In 1990, J. Bourgain (see [**3**]) proves non-constructively that, if $n \geq 7$ and $B = B_n$, there are arbitrarily small perturbations A of B giving a negative answer. He also proves that, if $n = 3$ and $B = B_3$ then for every small perturbation A of B the answer is affirmative.

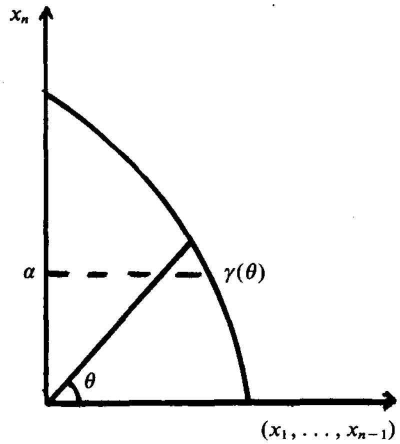
Observe that the only constructions giving a negative answer are those of K. Ball and A. Giannopoulos with $B = B_n$ but in both cases A is not a small perturbation of B.

In this paper I will construct B (not B_n) and small perturbations A of B which give negative answer for $n = 5, 6$. Thus the problem is still open for $n = 3, 4$.

§2. *Constructions.* I will consider the following type of solids. Let a curve be given in polar coordinates $(\theta, r(\theta)), 0 \leq \theta \leq \pi/2$ such that $r(\theta)$ is a continuous function of θ and $r(\theta) \sin \theta$ is a decreasing and concave function of $r(\theta) \cos \theta$. Consider the set:

$$A = \{(x_1, \ldots, x_n): |x_n| \leq \varphi(\sqrt{x_1^2 + \ldots + x_{n-1}^2})\},$$

[MATHEMATIKA, **39** (1992), 258-266]



where $s=\varphi(t)$ is defined by

$$t=r(\theta)\cos\theta, \qquad s=r(\theta)\sin\theta.$$

Then A is a compact convex set with 0 as center of symmetry.

Obviously

$$\operatorname{vol}(A)=2V_{n-1}\int_0^{\pi/2} r^{n-1}(\theta)\cos^{n-1}\theta d(r(\theta)\sin\theta),$$

where V_{n-1} is the $(n-1)$-dimensional volume of B_{n-1}. After integration by parts:

$$\operatorname{vol}(A)=2V_{n-1}\frac{n-1}{n}\int_0^{\pi/2} r^n(\theta)\cos^{n-2}\theta d\theta.$$

Now, if L is any $(n-1)$-dimensional subspace of $\mathbb{R}^n$, $\operatorname{vol}(A\cap L)$ is uniquely determined by θ, the angle of L and the $x_n=0$ subspace.

If $\theta=0$ then $\operatorname{vol}(A\cap L)=V_{n-1}r(0)^{n-1}$.

If $\theta>0$ then the intersection of $A\cap L$ with any $x_n=\alpha$ hyperplane is nonempty only if $|\alpha|\leq r(\theta)\sin\theta$ and then this intersection is an $(n-2)$-dimensional ball of radius

$$r(\varphi)\sqrt{\cos^2\varphi-\sin^2\varphi\cot^2\theta} \qquad \text{where} \qquad \alpha=r(\varphi)\sin\varphi, \quad |\varphi|\leq\theta.$$

Therefore, if $\theta>0$

$$\operatorname{vol}(A\cap L)=2V_{n-2}\int_0^{\theta} r^{n-2}(\varphi)(\cos^2\varphi-\sin^2\varphi\cot^2\theta)^{(n-2)/2}d\left(\frac{r(\varphi)\sin\varphi}{\sin\theta}\right)$$

which after integration by parts becomes

$$\operatorname{vol}(A\cap L)=2V_{n-2}\frac{n-2}{n-1}\cdot\frac{1}{\sin\theta}\int_0^{\theta} r^{n-1}(\varphi)\cos^{n-3}\varphi(1-\tan^2\varphi\cot^2\theta)^{(n-4)/2}d\varphi.$$

I will use the notation $R(\theta)$ for the integral in the last formula of vol $(A \cap L)$. Therefore the question of Busemann, Petty for this type of solid becomes. Do there exist two functions $r_j(\theta), j=1,2$, of $\theta \in [0, \pi/2]$ such that:

(i) $r_j(\theta)$ is continuous and $r_j(\theta) \sin\theta$ is a decreasing and concave function of $r_j(\theta)\cos\theta$;

(ii) $r_1(0) \leq r_2(0)$ and, for every $0<\theta\leq \pi/2$, $R_1(\theta)\leq R_2(\theta)$; but

(iii) $\int_0^{\pi/2} r_1^n(\theta)\cos^{n-2}\theta d\theta > \int_0^{\pi/2} r_2^n(\theta)\cos^{n-2}\theta d\theta$?

Case $n=6$. The idea is to invert the transform

$$R(\theta)=\int_0^\theta r^5(\varphi)\cos^3\varphi(1-\tan^2\varphi\cot^2\theta)d\varphi,$$

and then to choose $R(\theta)$ in a way that when we perform some negative variation $\delta R(\theta)$ the resulting variation in the volume integral $\int_0^{\pi/2} r^6(\theta)\cos^4\theta d\theta$ is positive.

The following change of notation is convenient

$$x=\tan\theta,\quad y=\tan\varphi,\quad \varphi(x)=r(\theta)\cos\theta,\quad f(x)=\varphi^5(x),\quad F(x)=R(\theta).$$

Then

$$F(x)=\int_0^x f(y)\left(1-\frac{y^2}{x^2}\right)dy=\int_0^x f(y)dy-\frac{1}{x^2}\int_0^x f(y)y^2dy,$$

and, by taking $g(x)=\int_0^x f(y)dy$,

$$F(x)=\frac{2}{x^2}\int_0^x yg(y)dy,$$

and finally

$$f(x)=\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\left(\frac{1}{2}x^2F(x)\right)\right). \tag{1}$$

The necessary conditions of continuity and concavity become

(a) $\varphi(x)$ and $x\varphi(x)$ continuous in $[0,+\infty]$, and

(b) $x\varphi(x)$ is a decreasing and concave function of $\varphi(x)$. Or, equivalently: $\varphi+x\varphi'\geq 0$, $2(\varphi')^2\geq \varphi\varphi''$.

Suppose we perform a small negative variation δF to F. Then by (1) the resulting variation in f is

$$\delta f=\left(\frac{1}{x}\left(\frac{1}{2}x^2\delta F\right)'\right)' \tag{2}$$

and the variation in the volume integral $V=\int_0^{\pi/2} r^6(\theta)\cos^4\theta d\theta=\int_0^\infty f^{6/5}(x)dx$ is

$$\partial V=\tfrac{6}{5}\int_0^\infty f^{1/5}(x)\delta f(x)dx+\int_0^\infty f^{6/5}(x)O\left(\left(\frac{df}{f}\right)^2\right)dx.$$

I require that δF and hence also δf be $\equiv 0$ outside some interval (a, b) $0<a<b<\infty$.

$$\delta V=\tfrac{6}{5}\int_0^\infty \varphi\cdot\delta f dx+O((\delta f)^2)$$

and I need only $\int_0^\infty \varphi\cdot\delta f dx>0$.

Using (2) and integration by parts

$$\int_0^\infty \varphi\cdot\delta f dx=\tfrac{1}{2}\int_0^\infty x^2\delta F\cdot\left(\frac{\varphi'}{x}\right)' dx.$$

I will construct φ so that it satisfies the necessary continuity and concavity conditions (a), (b) above. Also the (b) inequalities will be strict in an interval $(a, 1), 0<a<1$. Furthermore, $(\varphi'/x)'<0$ in $(a, 1)$.

This will enable me to take sufficiently small $\delta F\leq 0$ with $\delta F<0$ in a subinterval of $(a, 1)$ and $\delta F\equiv 0$ outside $(a, 1)$ and prove my claim.

Such a φ is given by

$$\varphi(x)=\begin{cases}\tfrac{1}{2}(3-a), & 0\leq x\leq a,\\ \tfrac{1}{2}(3-a)-\tfrac{1}{2}(x-a)^2/(1-a), & a\leq x\leq 1,\\ 1/x, & 1\leq x\leq+\infty.\end{cases}$$

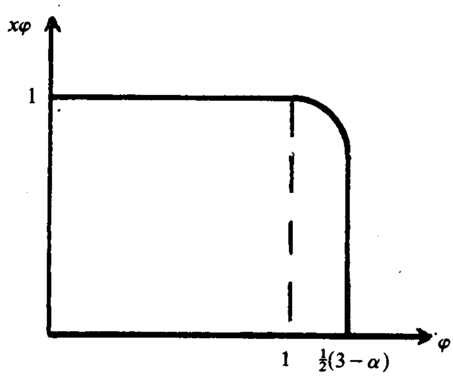
Remark. (a) The graph of $x\varphi=r(\theta)\sin\theta$ versus $\varphi=r(\theta)\cos\theta$ looks as in the picture. The corresponding solid is close to a cylinder, the type of solid used in [7].

(b) The sphere does not work since $\varphi(x)=\cos\theta$ and an elementary calculation shows $(\varphi'/x)'=3\cos^4\theta\sin\theta>0$.

Case $n=5$. Now $R(\theta)=\int_0^\theta r^4(\varphi)\cos^2\varphi\sqrt{1-\tan^2\varphi\cot^2\theta}\,d\varphi$.

The idea is the same as in case $n=6$ but the details are more complicated.

$$x=\tan\theta,\quad \varphi(x)=r(\theta)\cos\theta,\quad f(x)=\varphi^4(x),\quad F(x)=R(\theta).$$



Then

$$F(x)=\int_0^x f(y)\sqrt{1-\frac{y^2}{x^2}}dy.$$

Using $s=x^2$, $t=y^2$, $G(s)=2xF(x)$, $g(t)=1/xf(x)$ we find

$$2G'(s)=\int_0^s g(t)\frac{dt}{\sqrt{s-t}},$$

$$2G'(s)-\pi f(0)=\int_0^s\left(g(t)-\frac{f(0)}{\sqrt{t}}\right)\frac{dt}{\sqrt{s-t}}.$$

By the well-known Abel's inversion formula (see [9])

$$g(s)-\frac{f(0)}{\sqrt{s}}=\frac{2}{\pi}\int_0^s G''(s)\frac{dt}{\sqrt{s-t}},$$

which finally gives

$$f(x)-f(0)=\frac{2}{\pi}x\int_0^x\left(\frac{1}{y}(yF)'\right)'\frac{dy}{\sqrt{x^2-y^2}}.$$

Suppose that we perform a small non-positive variation δF to $F(x)$, which is $\equiv 0$ outside an interval (a, b) with $0<a<b<\infty$. Then, by also taking $\delta f(0)=0$,

$$\delta f(x)=\frac{2}{\pi}x\int_0^x\left(\frac{1}{y}(y\delta F)'\right)'\frac{dy}{\sqrt{x^2-y^2}}. \tag{3}$$

The corresponding variation in the volume integral

$$V=\int_0^{\pi/2} r^5(\theta)\cos^3\theta d\theta=\int_0^\infty f^{5/4}(x)dx$$

is

$$\delta V=\tfrac{5}{4}\int_0^\infty f^{1/4}(x)\delta f(x)dx+\int_0^\infty f^{5/4}(x)O\left(\left(\frac{\delta f}{f}\right)^2\right)dx. \tag{4}$$

Now we must guarantee that $\delta f/f$ is uniformly small in $[0, +\infty]$. Since $\delta F\equiv 0$ in $[0, a]$ (3) gives that $\delta f\equiv 0$ in $[0, a]$. Now as $x\to+\infty$

$$\delta f(x)=\frac{2}{\pi}x\int_a^b\left(\frac{1}{y}(y\delta F)'\right)'\frac{dy}{\sqrt{x^2-y^2}},$$

$$\delta f(x)=\frac{2}{\pi}\int_a^b\left(\frac{1}{y}(y\delta F)'\right)'\left[1+\frac{1}{2}\frac{y^2}{x^2}+\frac{3}{8}\frac{y^4}{x^4}+O\left(\frac{y^6}{x^6}\right)\right]dy,$$

$$\delta f(x)=\frac{6}{\pi x^4}\int_0^\infty y^2\delta F(y)dy+O\left(\frac{1}{x^6}\right), \tag{5}$$

where O is small if δF is small. But $x\varphi(x)=r(\theta)\sin\theta$ is bounded from below as $x\to+\infty$. Therefore, $\delta f/f=\delta f/\varphi^4$ is bounded as $x\to\infty$ and so uniformly bounded in $[0,\infty)$. It is also uniformly small if δF is small.

From (4) we need $\int_0^\infty \varphi(x)\delta f(x)>0$ in order to have a positive variation in V.

Using (3) and integration by parts the last integral becomes

$$\frac{2}{\pi}\int_0^\infty \delta F(y)y^2\int_y^\infty\left(\frac{\varphi'}{x}\right)'\frac{dx}{\sqrt{x^2-y^2}}\,dy. \tag{6}$$

In order not to interrupt the line of thought I will prove this in the supplement.

To produce a counterexample to the Busemann–Petty problem we need a φ such that:

(a) φ and $x\varphi$ are continuous on $[0,+\infty]$;

(b) $x\varphi'+\varphi\geqslant 0$ and $2(\varphi')^2\geqslant\varphi\varphi''$; and

(c) $$\int_y^\infty\left(\frac{\varphi'}{x}\right)'\frac{dx}{\sqrt{x^2-y^2}}<0$$

in some interval (a,b), $0<a<b<+\infty$.

Because then we take $\delta F\equiv 0$ outside some subinterval of (a,b) and $\delta F<0$ and small otherwise. We have to make sure though that $\varphi+\delta\varphi$ satisfies (b). We achieve this as follows. Let $0<a<1$ and $1<c<2$. Let $d=\frac{1}{2}(2-c)/(1-a)$.

$$\varphi(x)=\begin{cases}1+d(1-a)^2, & 0\leqslant x\leqslant a,\\ 1+d(1-a)^2-d(x-a)^2, & a\leqslant x\leqslant 1,\\ (c/x)-(c-1)/x^2, & 1\leqslant x\leqslant+\infty.\end{cases}$$

Then $\varphi(1)=1$, $\varphi'(1)=c-2<0$ and $x\varphi'+\varphi>0$, $2(\varphi')^2>\varphi\varphi''$ in $(a,1)$. Also, in $(1,+\infty)$,

$$x\varphi'+\varphi=\frac{c-1}{x^2},\qquad 2(\varphi')^2-\varphi\varphi''=\frac{2(c-1)^2}{x^6}.$$

From (5) we get

$$\varphi+\delta\varphi=(f+\delta f)^{1/4}=\left[\varphi^4+\frac{6}{\pi x^4}\int_0^\infty y^2\delta F dy+O\left(\frac{1}{x^6}\right)\right]^{1/4},$$
$$\frac{\delta\varphi}{\varphi}=-\rho+\frac{k}{x}+O\left(\frac{1}{x^2}\right)=-\rho+\frac{k}{x}+T(x), \tag{7}$$

where

$$\rho = 1 - \left\{ 1 + \frac{6}{\pi c^4} \int_0^\infty y^2 \delta F dy \right\}^{1/4},$$

$$k = \frac{6(c-1)}{\pi c^5} \int_0^\infty y^2 \delta F dy \left\{ 1 + \frac{6}{\pi c^4} \int_0^\infty y^2 \delta F(y) dy \right\}^{-3/4}.$$

Since $\delta F \leqslant 0$, we get $\rho > 0, k < 0$. Also, ρ, k and the O in (7) are small if δF is small.

We can also prove that $T'(x) = O(1/x^3)$, $T''(x) = O(1/x^4)$. Using all this information we have from (7)

$$2((\varphi + \delta\varphi)')^2 - (\varphi + \delta\varphi)(\varphi + \delta\varphi)'' > 0 \qquad \text{and,}$$

$$x(\varphi + \delta\varphi)' + (\varphi + \delta\varphi) > 0 \qquad \text{as } x \to \infty.$$

Hence, if δF is small, $\varphi + \delta\varphi$ satisfies (b).

Next I will prove that

$$\int_a^\infty \left(\frac{\varphi'(x)}{x} \right)' \frac{dx}{\sqrt{x^2 - a^2}} < 0$$

which will give (c) for an interval around a.

$$\int_a^\infty \left(\frac{\varphi'}{x} \right)' \frac{dx}{\sqrt{x^2 - a^2}} = -\int_a^1 \frac{2ad}{x^2 \sqrt{x^2 - a^2}} dx + \text{bounded term}$$

$$= -\frac{2d}{a} \int_1^{1/a} \frac{dt}{t^2 \sqrt{t^2 - 1}} + \text{bounded term.}$$

If $a \to 0+$ then the last expression $\to -\infty$.

Exactly the same remarks apply as in case $n = 6$.

§3. *Supplement*

$$\frac{\pi}{2} \int_0^\infty \varphi(x) \delta f(x) dx$$

$$= \int_0^\infty x\varphi(x) \int_a^x \left(\frac{1}{y} (y\delta F)' \right)' \frac{dy}{\sqrt{x^2 - y^2}} dx$$

$$= \int_b^\infty \varphi(x) \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \left(\frac{x}{\sqrt{x^2-y^2}} - 1 - \frac{1}{2}\frac{y^2}{x^2}\right) dydx$$

$$+ \int_0^b \varphi(x) \int_0^x \left(\frac{1}{y}(y\delta F)'\right)' \frac{x}{\sqrt{x^2-y^2}}\, dydx$$

$$= \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_b^\infty \varphi(x) \left(\frac{x}{\sqrt{x^2-y^2}} - 1 - \frac{1}{2}\frac{y^2}{x^2}\right) dxdy$$

$$+ \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_y^b \varphi(x) \frac{x}{\sqrt{x^2-y^2}}\, dxdy$$

$$= \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_b^\infty \varphi(x) \left(\sqrt{x^2-y^2} - x + \frac{1}{2}\frac{y^2}{x}\right)' dxdy$$

$$+ \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_y^b \varphi(x)(\sqrt{x^2-y^2})' dxdy$$

$$= \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \left\{ -\varphi(b)\left(\sqrt{b^2-y^2} - b + \frac{1}{2}\frac{y^2}{b}\right) - \int_b^\infty \varphi'(x)\left(\sqrt{x^2-y^2} - x + \frac{1}{2}\frac{y^2}{x}\right) dx \right\} dy$$

$$+ \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \left\{ \varphi(b)\sqrt{b^2-y^2} - \int_y^b \varphi'(x)\sqrt{x^2-y^2}\,dx \right\} dy$$

$$= -\int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_b^\infty \varphi'(x)\left(\sqrt{x^2-y^2} - x + \frac{1}{2}\frac{y^2}{x}\right) dxdy$$

$$- \int_0^b \left(\frac{1}{y}(y\delta F)'\right)' \int_y^b \varphi'(x)\sqrt{x^2-y^2}\,dxdy$$

$$= \int_0^b \frac{1}{y}(y\delta F)' \int_b^\infty \varphi'(x)\left(-\frac{y}{\sqrt{x^2-y^2}} + \frac{y}{x}\right) dxdy$$

$$- \int_0^b \frac{1}{y}(y\delta F)' \int_y^b \varphi'(x)\frac{y}{\sqrt{x^2-y^2}} dxdy$$

$$= \int_0^b (y\delta F)' \int_b^\infty \frac{\varphi'(x)}{x}\left(1 - \frac{x}{\sqrt{x^2-y^2}}\right) dxdy$$

$$- \int_0^b (y\delta F)' \int_y^b \frac{\varphi'(x)}{x} \cdot \frac{x}{\sqrt{x^2-y^2}}\, dxdy$$

$$= \int_0^b (y\delta F)' \left\{ -\frac{\varphi'(b)}{b}(b - \sqrt{b^2-y^2}) - \int_b^\infty \left(\frac{\varphi'(x)}{x}\right)'(x - \sqrt{x^2-y^2})dx \right\} dy$$

$$- \int_0^b (y\delta F)' \left\{ \frac{\varphi'(b)}{b}\sqrt{b^2-y^2} - \int_y^b \left(\frac{\varphi'(x)}{x}\right)' \sqrt{x^2-y^2}dx \right\} dy$$

$$= - \int_0^b (y\delta F)' \int_b^\infty \left(\frac{\varphi'(x)}{x}\right)'(x - \sqrt{x^2-y^2})dxdy$$

$$+ \int_0^b (y\delta F)' \int_y^b \left(\frac{\varphi'(x)}{x}\right)' \sqrt{x^2-y^2}dxdy$$

$$= \int_0^\infty y\delta F \int_y^\infty \left(\frac{\varphi'(x)}{x}\right)' \frac{y}{\sqrt{x^2-y^2}}\, dxdy.$$

Acknowledgement. I thank Prof. S. Pichorides for the many discussions we had about this problem.

References

1. K. M. Ball. Cube slicing in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 97 (1986), 465-473.
2. K. M. Ball. Some remarks on the geometry of convex sets. *Geometric Aspects of Functional Analysis, LNM* 1317 (Springer, 1988), 224-231.
3. J. Bourgain. On the Busemann-Petty problem for perturbations of the ball. *Geom. Funct. Anal.*, 1 (1991), 1-13.
4. H. Busemann. Volumes in terms of concurrent cross-sections. *Pacific J. Math.*, 3 (1953), 1-12.
5. H. Busemann. Volumes and areas of cross-sections. *Amer. Math. Monthly*, 67 (1960), 248-250 and 671.
6. H. Busemann and C. M. Petty. Problems on convex bodies. *Math. Scand.*, 4 (1956), 88-94.
7. A. Giannopoulos. A note on a problem of H. Busemann and C. M. Petty concerning sections of convex symmetric bodies. *Mathematika*, 37 (1990), 239-244.
8. D. G. Larman and C. A. Rogers. The existence of a centrally symmetric convex body with central sections which are unexpectedly small. *Mathematika*, 22 (1975), 164-175.
9. F. Riesz and B. S. Nagy. *Functional Analysis* (Ungar, New York, 1955).

Dr. M. Papadimitrakis,
Department of Mathematics,
Washington University,
Campus Box 1146,
One Brookings Drive,
St. Louis, MO 63130-4899,
U.S.A.

52A20: *CONVEX AND DISCRETE GEOMETRY; General convexity; Convex sets in n-dimensions.*

Received on the 8th of November, 1991.